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Research article

A Banach fixed–point approach to Salem's nonhomogeneous integral equation related to the Riemann hypothesis

Irshad Ayoob and Nabil Mlaiki*

Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

* **Correspondence:** Email: nmlaiki@psu.edu.sa.

Abstract: Salem's equivalence of the Riemann hypothesis asserts that the hypothesis is true if and only if $g(t) = 0$ is the only nontrivial bounded measurable solution (for fixed $1/2 < r < 1$) of the following integral equation:

$$\int_0^\infty \frac{t^{r-1}g(t)}{e^{xt}+1}\,dt = 0, \qquad x > 0.$$

Recent works have investigated this integral equation and shown that various classes of bounded measurable functions, subject to suitable growth assumptions, cannot furnish counterexamples to Salem's criterion. These approaches typically employ Mellin inversion, Widder–Lambert–type transforms, or distribution-theoretic methods. The aim of this paper is to study the associated nonhomogeneous Salem-type equation

$$f(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1}f(t)}{1+e^{xt}}\,dt, \qquad x > 0,$$

where g is a given measurable function, $r \in (1/2, 1)$, and $\lambda \in \mathbb{R}$, from the viewpoint of fixed–point theory. We show that the corresponding integral operator does not map the space of bounded measurable functions into itself on $(0, \infty)$. We therefore introduce a suitable weighted complete function space in order to apply Banach's contraction principle. The nonhomogeneous equation is important because it is a more general form of Salem's equation. As we shall see, its analysis allows one to identify precisely why Banach's contraction principle succeeds for the nonhomogeneous equation in an appropriate weighted space, yet does not directly settle Salem's original integral equation. This paper suggests that possible future directions may include the use of generalized metric spaces and generalized contraction mappings to study Salem's integral equation.

Keywords: Riemann hypothesis; Riemann zeta function; integral equations; fixed–point; Banach fixed–point theorem; generalized metric space
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1. Introduction

The Riemann zeta function is initially defined for $\Re(s) > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

and it is well–known that $\zeta(s)$ extends meromorphically to the entire complex plane with a single simple pole at $s = 1$. This continuation is obtained through the classical functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big)\Gamma(1 - s)\, \zeta(1 - s),$$

which relates values of $\zeta(s)$ across the critical line $\Re(s) = \frac{1}{2}$. In the half-plane $\Re(s) > 1$, the function has no zeros; its trivial zeros occur at the negative even integers. All remaining zeros lie in the critical strip $0 < \Re(s) < 1$. The distribution of these nontrivial zeros forms the basis of the classical Riemann hypothesis:

Conjecture 1.1 (Riemann hypothesis)**.** *All nontrivial zeros of the Riemann zeta function lie on the line* $\Re(s) = \frac{1}{2}$*; equivalently, each such zero satisfies* $s = \frac{1}{2} + it$ *for some* $t \in \mathbb{R}$*.*

Since Riemann's seminal work, a vast number of equivalent formulations of the hypothesis have been discovered (see, for instance, the compilations in [1–3]). Several of these equivalences arise from integral equations, including notable contributions by Wang [4], Levinson [5], Volchkov [6], Salem [7], Balazard–Saias–Yor [8], and Sekatskii–Beltraminelli–Merlini [9]. Among these, the reformulation due to Salem has recently attracted significant attention because of its analytic simplicity.

Salem's integral equation reformulation of the Riemann hypothesis may be stated as follows.

Theorem 1.2 (Salem's criterion)**.** *Fix* $1/2 < r < 1$*. Then, the Riemann hypothesis holds if and only if the integral equation*

$$\int_0^{\infty} \frac{t^{r-1} g(t)}{1 + e^{xt}}\, dt = 0, \qquad x > 0 \tag{1.1}$$

admits no bounded measurable solutions except the trivial function $g \equiv 0$*.*

The natural problem therefore arises, for which classes of functions can one guarantee that a bounded measurable solution of (1.1) must vanish? This question has been investigated by several authors. Yakubovich [10] initiated a detailed analysis using Mellin transform techniques, and González and Negrín [11, 12] obtained sharper uniqueness results by means of Mellin inversion formulae and Widder–Lambert-type integral transforms.

One of their principal results [12] (Corollary 3.3) is the following.

Theorem 1.3. *Let* $g : \mathbb{R}_+ \to \mathbb{R}$ *be bounded and measurable, and assume that* $g(t) = O(t^{1/2})$ *as* $t \to 0^+$*. If* g *satisfies Salem's equation* (1.1) *for some* $1/2 < r < 1$*, then* $g(t) = 0$ *almost everywhere on* $\mathbb{R}_+$*.*

Two further results proved by the same authors in [11] (Corollaries 2.4 and 2.5) require additional regularity or integrability conditions and yield the following uniqueness theorems.

Theorem 1.4. *Suppose that* g *is bounded and measurable, satisfies Salem's equation* (1.1) *for all* $x > 0$ *and all* $1/2 < r < 1$*, and that* $t^{b-1} g(t) \in L^1(\mathbb{R}_+)$ *for some* $b < 0$*. If, in addition,* g *is continuous on* $\mathbb{R}_+$ *and of bounded variation in a neighborhood of each point, then* $g \equiv 0$*.*

Theorem 1.5. *Let g be a bounded measurable function satisfying* (1.1) *for all* $x > 0$ *and all* $1/2 < r < 1$. *Suppose there exists* $b < 0$ *such that*

$$t^{b-1}g(t) \in L^1(\mathbb{R}_+) \quad \textit{and} \quad t^{b-\frac{1}{2}}g(t) \in L^2(\mathbb{R}_+).$$

Then, $g(t) = 0$ *almost everywhere on* $\mathbb{R}_+$.

A complementary result using methods from distribution theory was more recently established by Negrín and Maan [13]. Their uniqueness theorem applies to compactly supported functions and may be phrased as follows.

Theorem 1.6. *Let g be bounded and measurable with compact support contained in* $\mathbb{R}_+$. *If g satisfies the integral equation* (1.1) *for some* $1/2 < r < 1$, *then* $g(t) = 0$ *almost everywhere on* $\mathbb{R}_+$.

Theorems 1.3–1.6 (together with the additional uniqueness results above) illustrate the strength of Salem's integral formulation; that is, under reasonable growth conditions on g, the only possible solution is the trivial one. Consequently, these results systematically exclude broad classes of functions from serving as potential counterexamples to Salem's equivalence of the Riemann hypothesis.

It is natural to ask whether Salem's criterion can be studied by mathematical tools other than the integral transform methods that have dominated much of the recent literature. We note that the original proof of Salem's criterion is rooted in Wiener's theorems and the theory of Fourier transforms. From this perspective, it is more natural to investigate Salem's equivalence within the broader framework of integral equations and operator theory; rather than solely through methods of analytic number theory.

One possible approach is to use fixed–point theory, which provides a natural and well-established framework for studying existence and uniqueness questions for integral and differential equations. Motivated by this viewpoint, it is natural to consider the corresponding nonhomogeneous Salem-type integral equation. We avoid the use of integral transform techniques and distribution theory, and instead adopt a fixed–point theoretic approach to analyze this associated nonhomogeneous problem.

The study of the nonhomogeneous equation is particularly natural because it is a more general form of Salem's equation, and as we shall see, its analysis clarifies the precise functional analytic behavior behind the problem. More specifically, it allows one to determine why Banach's contraction principle can be successfully applied to the nonhomogeneous equation in a suitable weighted space, whereas the same approach does not directly resolve Salem's original equation. For Salem's original integral equation, we also indicate possible directions for future research, including the use of generalized contractions and generalized metric spaces.

More precisely, this paper addresses the following three questions.

Question 1.7. *Can we apply Banach's fixed–point theorem to the study of the following nonhomogeneous Salem-type integral equation on the space of bounded measurable functions* ($f, g : [0,\infty) \to \mathbb{R}$)*:*

$$f(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0, \tag{1.2}$$

where g is a prescribed bounded measurable function, $r \in (1/2, 1)$, *and* $\lambda \in \mathbb{R}$*?*

Question 1.8. *Can we apply Banach's fixed–point theorem to the study of the following homogeneous Salem-type integral equation on the space of bounded measurable functions* ($f : [0,\infty) \to \mathbb{R}$)*:*

$$f(x) = \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0, \tag{1.3}$$

where $r \in (1/2, 1)$*, and* $\lambda \in \mathbb{R}$*?*

Question 1.9. *Can we apply Banach's fixed–point theorem to the study of the original Salem's integral equation*

$$0 = \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0, \tag{1.4}$$

again in a Banach space of bounded measurable functions?

The purpose of this paper is to give a precise answer to Questions 1.7–1.9.

Henceforth, we refer to Eq (1.2) as the associated nonhomogeneous equation, Eq (1.3) as the associated homogeneous equation, and Eq (1.4) as the degenerate equation or Salem's original equation.

The paper is organized as follows. In Section 2, we prove that the integral operator associated with the Salem nonhomogeneous integral equation (1.2) fails to map the space of bounded measurable functions into itself on $(0, \infty)$ under the usual supremum norm. This obstruction leads us to introduce a suitable weighted complete function space, and Proposition 2.12 shows that this space is in fact a Banach space. We also include several illustrative examples of functions belonging to this space.

In Subsection 3.1, we answer Question 1.7 affirmatively in the form of Theorem 3.2 and then present an explicit example of the nonhomogeneous integral equation (1.2) in Theorem 3.4. More precisely, Theorem 3.2 establishes that Banach's fixed–point theorem can be applied to obtain existence and uniqueness for the nonhomogeneous integral equation in the weighted Banach space.

In Subsection 3.2, we study the corresponding homogeneous equation (1.3) associated with the same operator. In particular, Theorem 3.5 shows that, in our weighted function space, Banach's fixed–point theorem implies that the only solution of this homogeneous equation is the trivial one.

Finally, in Subsection 3.3, we explain why Salem's original integral equation does not fall directly within the scope of Banach's fixed–point theorem. In the conclusion, we suggest possible directions for future research, including the investigation of Salem's integral equation in the framework of generalized metric spaces and generalized contractive or expansive mappings.

First of all, let us recall the classical Banach fixed–point theorem [14].

Theorem 1.10 (Banach fixed–point theorem)**.** *Let* (X, d) *be a complete metric space, and let* $T : X \to X$ *be a mapping for which there exists a constant* $q \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq q\, d(x, y) \quad \text{for all } x, y \in X. \tag{1.5}$$

Then:

1) *There exists a unique point* $x^* \in X$ *such that*

$$Tx^* = x^*. \tag{1.6}$$

2) *For every initial point* $x_0 \in X$*, the sequence of Picard iterates*

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \ldots, \tag{1.7}$$

converges in X *to the fixed–point* x^**.*

3) Moreover, one has the quantitative estimate

$$d(x_n, x^*) \leq \frac{q^n}{1-q}\, d(x_1, x_0), \qquad n \in \mathbb{N}. \tag{1.8}$$

Banach's fixed–point theorem has a wide range of applications, notably in the theory of nonlinear functional equations and integral equations. In particular, if an integral operator T can be shown to map a complete metric space of functions into itself and to satisfy a contraction estimate of the form (1.5), then Theorem 1.10 yields existence, uniqueness, and a constructive iterative scheme for solutions of the fixed–point equation $Tf = f$. In the context of (1.2), this turns the non–homogeneous Salem-type integral equation into a genuine fixed–point problem on a suitable Banach space, and allows one to obtain explicit convergence of Picard iterates. By contrast, Salem's original integral equation (1.4) raises more delicate questions about the kernel and injectivity of the associated integral transform and cannot be handled by a straightforward contraction argument alone, as we shall see in the next sections.

2. Preliminaries

The answer to Question 1.7 is affirmative; that is, we can apply Banach's fixed–point theorem to establish existence and uniqueness for the Salem-type nonhomogeneous integral equation. However, it is important to stress that on the whole half-line $(0, \infty)$, one cannot work in the usual Banach space of bounded measurable functions with the usual supremum norm. More precisely, if we consider

$$X_\infty = \Big\{ f : (0, \infty) \to \mathbb{R} \text{ measurable } : \ \|f\|_\infty = \sup_{x>0} |f(x)| < \infty \Big\}, \tag{2.1}$$

then the integral operator associated with Salem's nonhomogeneous integral equation (1.2), given by,

$$(Tf)(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0, \tag{2.2}$$

does not, in general, map X_∞ into itself. In other words, Tf need not be bounded on $(0, \infty)$, even when f is bounded.

We now exhibit a very simple test function $f \in X_\infty$ for which Tf is explicitly unbounded on $(0, \infty)$. Fix $r \in (1/2, 1)$, and choose

$$f(t) = 1 \quad \text{for all } t > 0, \tag{2.3}$$

together with $g \equiv 0$. Clearly $f \in X_\infty$, and $\|f\|_\infty = 1$. For this choice, we have

$$(Tf)(x) = \lambda \int_0^\infty \frac{t^{r-1}}{1 + e^{xt}}\, dt, \qquad x > 0. \tag{2.4}$$

We can evaluate the integral in (2.4) exactly by the change of variables

$$u = xt, \qquad t = \frac{u}{x}, \qquad dt = \frac{du}{x}. \tag{2.5}$$

Substituting (2.5) into (2.4) yields

$$(Tf)(x) = \lambda \int_0^\infty \frac{(u/x)^{r-1}}{1 + e^u} \frac{du}{x} \tag{2.6}$$

$$= \lambda x^{-r} \int_0^\infty \frac{u^{r-1}}{1+e^u}\, du. \tag{2.7}$$

We denote the (finite, positive) constant

$$C_r = \int_0^\infty \frac{u^{r-1}}{1+e^u}\, du. \tag{2.8}$$

The convergence and positivity of C_r follow from a split of the integral at $u = 1$, using the estimates,

$$\frac{u^{r-1}}{1+e^u} \le u^{r-1} \quad (0 < u \le 1), \qquad \frac{u^{r-1}}{1+e^u} \le u^{r-1}e^{-u} \quad (u \ge 1),$$

and the fact that $r - 1 > -1$.

Thus, the identity (2.7) becomes

$$(Tf)(x) = \lambda C_r\, x^{-r}, \qquad x > 0. \tag{2.9}$$

Next, we show that T given by (2.9) is unbounded. From (2.9), we immediately obtain

$$|Tf(x)| = |\lambda|\, C_r\, x^{-r}, \qquad x > 0. \tag{2.10}$$

Because $r > 0$, we have

$$\lim_{x\to 0^+} x^{-r} = +\infty, \tag{2.11}$$

and therefore,

$$\sup_{x>0} |Tf(x)| = \infty. \tag{2.12}$$

In particular, Tf is not a bounded function on $(0, \infty)$, and hence,

$$Tf \notin X_\infty \qquad \text{even though } f \in X_\infty. \tag{2.13}$$

This shows that, on the whole half-line $(0, \infty)$, the operator T in (2.2) does not map the Banach space $(X_\infty, \|\cdot\|_\infty)$ into itself, already for the very simple constant test function $f \equiv 1$. Consequently, one cannot apply the Banach fixed–point theorem directly in the usual sup–norm setting.

This motivates the introduction of a new weighted space (defined below) X_r with norm $\|f\|_{r/2}$, which is tailored to absorb the growth of the integral term near $x = 0$ and under which the operator T becomes well-defined and contractive.

2.1. *New weighted function space*

Fix a parameter $r \in \left(\frac{1}{2}, 1\right)$. For a measurable function $f : (0, \infty) \to \mathbb{R}$, we define the weighted norm

$$\|f\|_{r/2} = \sup_{x>0} x^{r/2}\, |f(x)| \in [0, \infty] \tag{2.14}$$

and the associated weighted space

$$X_r = \left\{ f : (0, \infty) \to \mathbb{R} \text{ measurable } : \ \|f\|_{r/2} < \infty \right\}. \tag{2.15}$$

We equip X_r with the norm $\|\cdot\|_{r/2}$ and the induced metric

$$d(f, g) = \|f - g\|_{r/2}, \qquad f, g \in X_r. \tag{2.16}$$

Remark 2.1. *By definition, $f \in X_r$ if and only if the function*

$$h(x) = x^{r/2} f(x), \qquad x > 0 \tag{2.17}$$

is bounded on $(0, \infty)$. *In particular, for every $f \in X_r$, one has*

$$|f(x)| \leq \|f\|_{r/2}\, x^{-r/2} \quad \textit{for all } x > 0. \tag{2.18}$$

We present some examples that belong to the weighted space X_r.

Example 2.2. *The function $f(x) = x^{-r/2}$ belongs to X_r.*

Proof. We compute

$$x^{r/2}|f(x)| = x^{r/2}x^{-r/2} = 1 \qquad \text{for all } x > 0.$$

Thus, $\|f\|_{r/2} = 1$; hence, $f \in X_r$. □

Example 2.3. *The function $f(x) = \dfrac{1}{1+x}$ belongs to X_r.*

Proof. For $x \in (0, 1]$, we have $x^{r/2}|f(x)| \leq x^{r/2} \leq 1$. For $x \geq 1$, we estimate

$$x^{r/2}|f(x)| = \frac{x^{r/2}}{1+x} \leq \frac{x^{r/2}}{x} = x^{r/2-1} \leq 1,$$

because $r/2 < 1$. Hence, $\|f\|_{r/2} \leq 1$, so $f \in X_r$. □

Example 2.4. *The function $f(x) = e^{-x}$ belongs to X_r.*

Proof. For $0 < x \leq 1$, we have $x^{r/2}|e^{-x}| \leq x^{r/2} \leq 1$. For $x \geq 1$, the factor e^{-x} dominates all powers of x; hence, $x^{r/2}e^{-x} \leq 1$. Therefore, $\|f\|_{r/2} \leq 1$, so $f \in X_r$. □

Example 2.5. *The oscillatory function $f(x) = x^{-r/2}\cos(\log x)$ belongs to X_r.*

Proof. Because $|\cos(\log x)| \leq 1$ for all $x > 0$,

$$x^{r/2}|f(x)| = x^{r/2}x^{-r/2}|\cos(\log x)| \leq 1.$$

Thus, $\|f\|_{r/2} \leq 1$ and $f \in X_r$. □

Example 2.6. *The function $f(x) = \dfrac{\sin x}{1 + x^{1/2}}$ belongs to X_r.*

Proof. because $|\sin x| \leq 1$, for $0 < x \leq 1$, we have

$$x^{r/2}|f(x)| \leq x^{r/2} \leq 1.$$

For $x \geq 1$,

$$x^{r/2}|f(x)| \leq \frac{x^{r/2}}{x^{1/2}} = x^{r/2-1/2},$$

and $r/2 - 1/2 < 0$ implies this expression is bounded above by 1. Hence, $\|f\|_{r/2} \leq 1$, so $f \in X_r$. □

We now present functions that do not belong to X_r.

Example 2.7. *The constant function* $f(x) = 1$ *does not belong to* X_r*.*

Proof. We compute

$$x^{r/2}|f(x)| = x^{r/2} \to \infty \qquad \text{as } x \to \infty.$$

Hence, $\|f\|_{r/2} = \infty$, so $f \notin X_r$. □

Example 2.8. *The power function* $f(x) = x^{\alpha}$ *with* $\alpha > 0$ *does not belong to* X_r*.*

Proof. We have

$$x^{r/2}|f(x)| = x^{r/2+\alpha} \to \infty \qquad \text{as } x \to \infty,$$

so $\|f\|_{r/2} = \infty$. Thus, $f \notin X_r$. □

Example 2.9. *The power function* $f(x) = x^{-\beta}$ *with* $\beta > r/2$ *does not belong to* X_r*.*

Proof. Near $x = 0$,

$$x^{r/2}|f(x)| = x^{r/2-\beta} \to \infty \qquad \text{because } r/2 - \beta < 0.$$

Thus, $\|f\|_{r/2} = \infty$, and $f \notin X_r$. □

Example 2.10. *The exponentially growing function* $f(x) = e^{x}$ *does not belong to* X_r*.*

Proof. As $x \to \infty$,

$$x^{r/2}e^{x} \to \infty;$$

hence, $\|f\|_{r/2} = \infty$, so $f \notin X_r$. □

Example 2.11. *The slowly decaying function* $f(x) = \dfrac{1}{\log(1+x)}$ *does not belong to* X_r*.*

Proof. For large x,

$$x^{r/2}|f(x)| \sim \frac{x^{r/2}}{\log x} \to \infty;$$

hence, the weighted norm diverges, and $f \notin X_r$. □

The next result shows that $(X_r, \|\cdot\|_{r/2})$ is a Banach space.

Proposition 2.12. *The space* $(X_r, \|\cdot\|_{r/2})$ *is a Banach space.*

Proof. First, we show that $\|\cdot\|_{r/2}$ is a norm on X_r.

Let $f, g \in X_r$, and $\alpha \in \mathbb{R}$. From the definition (2.14), we have the following:

(i) *Non-negativity and definiteness:* It is clear that $\|f\|_{r/2} \geq 0$ for all $f \in X_r$. If $\|f\|_{r/2} = 0$, then $x^{r/2}|f(x)| = 0$ for all $x > 0$ so

$$|f(x)| = 0 \quad \text{for all } x > 0; \tag{2.19}$$

hence, $f(x) = 0$ for all $x > 0$. That is, f is the zero function in X_r. Conversely, if $f = 0$, then $\|f\|_{r/2} = 0$ by (2.14).

(ii) *Homogeneity:* For $\alpha \in \mathbb{R}$,

$$\|\alpha f\|_{r/2} = \sup_{x>0} x^{r/2}|\alpha f(x)| = |\alpha| \sup_{x>0} x^{r/2}|f(x)| = |\alpha|\,\|f\|_{r/2}. \tag{2.20}$$

(iii) *Triangle inequality:* For $f, g \in X_r$,

$$x^{r/2}|(f+g)(x)| \le x^{r/2}(|f(x)| + |g(x)|) \le x^{r/2}|f(x)| + x^{r/2}|g(x)|, \tag{2.21}$$

for all $x > 0$. Taking the supremum over $x > 0$, we obtain

$$\|f+g\|_{r/2} = \sup_{x>0} x^{r/2}|(f+g)(x)| \le \sup_{x>0} x^{r/2}|f(x)| + \sup_{x>0} x^{r/2}|g(x)| = \|f\|_{r/2} + \|g\|_{r/2}. \tag{2.22}$$

This proves that $\|\cdot\|_{r/2}$ is a norm on X_r.

Next, we show that $(X_r, \|\cdot\|_{r/2})$ is a complete space.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X_r, \|\cdot\|_{r/2})$. By definition of Cauchy, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{r/2} = \sup_{x>0} x^{r/2}|f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } n, m \ge N. \tag{2.23}$$

Fix $x > 0$. From (2.23), we obtain

$$x^{r/2}|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{r/2} < \varepsilon \quad \text{for all } n, m \ge N. \tag{2.24}$$

Because $x^{r/2} > 0$ is fixed, (2.24) implies that

$$|f_n(x) - f_m(x)| < \varepsilon\, x^{-r/2} \quad \text{for all } n, m \ge N. \tag{2.25}$$

Thus, for each fixed $x > 0$, the real sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. As $\mathbb{R}$ is complete, there exists a limit

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x > 0. \tag{2.26}$$

We have thereby defined a function $f : (0, \infty) \to \mathbb{R}$ pointwise.

(a) Measurability of f. Each f_n is measurable. Because the pointwise limit of measurable functions is measurable, it follows that f is measurable on $(0, \infty)$.

(b) Finiteness of the weighted norm of f. For every $x > 0$ and every $n \in \mathbb{N}$, we have

$$x^{r/2}|f_n(x)| \le \|f_n\|_{r/2}. \tag{2.27}$$

Because (f_n) is Cauchy in X_r, there exists $n_0 \in \mathbb{N}$ and $M > 0$ such that

$$\|f_n\|_{r/2} \le M \quad \text{for all } n \ge n_0. \tag{2.28}$$

Let $x > 0$ be fixed. Taking the limit $n \to \infty$ in (2.27) and using (2.26), we obtain

$$x^{r/2}|f(x)| = \lim_{n\to\infty} x^{r/2}|f_n(x)| \le M. \tag{2.29}$$

Because (2.29) holds for every $x > 0$, it follows that

$$\|f\|_{r/2} = \sup_{x>0} x^{r/2}|f(x)| \le M < \infty. \tag{2.30}$$

Thus, $f \in X_r$.

(c) Convergence in the norm. We now prove that $f_n \to f$ in $\|\cdot\|_{r/2}$. Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that (2.23) holds for all $n, m \geq N$. For such $n \geq N$ and any fixed $x > 0$, letting $m \to \infty$ in (2.24) and using (2.26), we get

$$x^{r/2}|f_n(x) - f(x)| \leq \varepsilon \quad \text{for all } x > 0. \tag{2.31}$$

Taking the supremum over $x > 0$ in (2.31) yields

$$\|f_n - f\|_{r/2} = \sup_{x>0} x^{r/2}|f_n(x) - f(x)| \leq \varepsilon \quad \text{for all } n \geq N. \tag{2.32}$$

Hence, $f_n \to f$ in $(X_r, \|\cdot\|_{r/2})$.

Because every Cauchy sequence in X_r converges (in $\|\cdot\|_{r/2}$) to a limit in X_r, the space $(X_r, \|\cdot\|_{r/2})$ is complete. This completes the proof of Proposition 2.12. □

We now answer Question 1.7 and consider the Salem–type nonhomogeneous integral equation,

$$f(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0, \tag{2.33}$$

where $r \in (1/2, 1)$ is fixed, $\lambda \in \mathbb{R}$ is a constant, and $g : (0, \infty) \to \mathbb{R}$ is a given measurable function.

We define an associated operator by

$$(Tf)(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0. \tag{2.34}$$

We have already shown (see 2.13) that the integral operator T is not a self-map on the Banach space of bounded measurable functions equipped with the supremum norm. However, in the next section, we will show that for suitable λ, the operator T defined in 2.34 is a contraction on X_r, and hence has a unique fixed–point which is the unique solution of (2.33) in X_r.

3. Main results

3.1. Answer to Question 1.7

We introduce the constant

$$B_r = \int_0^\infty \frac{u^{\frac{r}{2}-1}}{1 + e^u}\, du. \tag{3.1}$$

We first verify that B_r is finite and positive.

Lemma 3.1. *Let* $r \in (1/2, 1)$ *and* B_r *be given by* (3.1). *Then,*

$$0 < B_r < \infty. \tag{3.2}$$

Proof. The integrand in (3.1) is non-negative, so $B_r \geq 0$. Moreover, because the denominator is $1 + e^u > 0$ for all $u > 0$, and the numerator is positive, $B_r > 0$.

To show finiteness, we split the integral at $u = 1$:

$$B_r = \int_0^1 \frac{u^{\frac{r}{2}-1}}{1 + e^u}\, du + \int_1^\infty \frac{u^{\frac{r}{2}-1}}{1 + e^u}\, du =: I_1 + I_2. \tag{3.3}$$

For $0 < u \leq 1$, we have $1 + e^u \geq 1$; hence,

$$0 \leq \frac{u^{\frac{r}{2}-1}}{1+e^u} \leq u^{\frac{r}{2}-1}. \tag{3.4}$$

Because $r \in (1/2, 1)$, we have

$$\frac{r}{2} - 1 > -1. \tag{3.5}$$

Therefore, the function $u^{\frac{r}{2}-1}$ is integrable on $(0, 1]$, and hence,

$$I_1 \leq \int_0^1 u^{\frac{r}{2}-1}\, du < \infty. \tag{3.6}$$

For $u \geq 1$, we have $1 + e^u \geq e^u$, so

$$0 \leq \frac{u^{\frac{r}{2}-1}}{1+e^u} \leq u^{\frac{r}{2}-1} e^{-u}. \tag{3.7}$$

The function $u^{\frac{r}{2}-1} e^{-u}$ is integrable on $[1, \infty)$ because

$$\int_1^\infty u^{\frac{r}{2}-1} e^{-u}\, du \leq \int_0^\infty u^{\frac{r}{2}-1} e^{-u}\, du < \infty, \tag{3.8}$$

and the last integral is a standard gamma-type integral, finite for all exponents larger than -1, which holds by (3.5). Hence,

$$I_2 \leq \int_1^\infty u^{\frac{r}{2}-1} e^{-u}\, du < \infty. \tag{3.9}$$

Combining (3.3), (3.6), and (3.9), we obtain $B_r < \infty$. Together with $B_r > 0$, this proves (3.2). □

We now state and prove the main theorem.

Theorem 3.2. *Let* $r \in (1/2, 1)$, *and define* X_r *as in* (2.15) *with norm* $\|\cdot\|_{r/2}$ *given by* (2.14). *Assume that* $g \in X_r$, *and let* B_r *be given by* (3.1). *Suppose that the parameter* $\lambda \in \mathbb{R}$ *satisfies*

$$|\lambda| B_r < 1. \tag{3.10}$$

Then, the operator T *defined in* (2.34) *maps* X_r *into itself and is a strict contraction on* $(X_r, \|\cdot\|_{r/2})$. *Consequently, the integral equation* (2.33) *has a unique solution* $f \in X_r$.

Proof. We show that T maps X_r into X_r.

Let $f \in X_r$. Then, by (2.18),

$$|f(t)| \leq \|f\|_{r/2}\, t^{-r/2} \quad \text{for all } t > 0. \tag{3.11}$$

For each fixed $x > 0$, we consider the integral

$$I_f(x) = \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt. \tag{3.12}$$

Using (3.11), we obtain the bound

$$|I_f(x)| \leq \|f\|_{r/2} \int_0^\infty \frac{t^{r-1}t^{-r/2}}{1+e^{xt}}\,dt = \|f\|_{r/2} \int_0^\infty \frac{t^{\frac{r}{2}-1}}{1+e^{xt}}\,dt. \tag{3.13}$$

We now evaluate the last integral by a change of variables:

$$u = xt, \qquad t = \frac{u}{x}, \qquad dt = \frac{du}{x}. \tag{3.14}$$

Substituting (3.14) into (3.13) gives

$$\int_0^\infty \frac{t^{\frac{r}{2}-1}}{1+e^{xt}}\,dt = \int_0^\infty \frac{(u/x)^{\frac{r}{2}-1}}{1+e^{u}}\,\frac{du}{x} \tag{3.15}$$

$$= x^{-\frac{r}{2}} \int_0^\infty \frac{u^{\frac{r}{2}-1}}{1+e^{u}}\,du \tag{3.16}$$

$$= x^{-\frac{r}{2}} B_r, \tag{3.17}$$

where B_r is the finite positive constant from Lemma 3.1.

Combining (3.13) and (3.17), we obtain

$$|I_f(x)| \leq \|f\|_{r/2}\, x^{-\frac{r}{2}} B_r \quad \text{for all } x > 0. \tag{3.18}$$

Now, by (2.34) and (3.12), we have

$$(Tf)(x) = g(x) + \lambda I_f(x). \tag{3.19}$$

Multiplying (3.19) by $x^{r/2}$ and using the triangle inequality, we get

$$x^{r/2}|Tf(x)| \leq x^{r/2}|g(x)| + |\lambda|\, x^{r/2}|I_f(x)|. \tag{3.20}$$

Using (3.18) in (3.20), we find

$$x^{r/2}|Tf(x)| \leq x^{r/2}|g(x)| + |\lambda|\, x^{r/2}(\|f\|_{r/2}\, x^{-r/2} B_r) = x^{r/2}|g(x)| + |\lambda| B_r \|f\|_{r/2}. \tag{3.21}$$

Taking the supremum over $x > 0$ in (3.21), we obtain

$$\|Tf\|_{r/2} = \sup_{x>0} x^{r/2}|Tf(x)| \leq \sup_{x>0} x^{r/2}|g(x)| + |\lambda| B_r \|f\|_{r/2} = \|g\|_{r/2} + |\lambda| B_r \|f\|_{r/2}. \tag{3.22}$$

Because $g \in X_r$, the quantity $\|g\|_{r/2}$ is finite, and (3.22) shows that $\|Tf\|_{r/2} < \infty$. Thus,

$$Tf \in X_r \quad \text{for all } f \in X_r. \tag{3.23}$$

Moreover, Tf is measurable. Indeed, the function

$$(x,t) \mapsto \frac{t^{r-1} f(t)}{1+e^{xt}}$$

is measurable on $(0,\infty)^2$, and for each fixed $x > 0$, the estimate (3.18) shows that the defining integral is absolutely convergent. Hence, the map

$$x \mapsto \int_0^\infty \frac{t^{r-1} f(t)}{1+e^{xt}}\,dt$$

is measurable, and therefore, so is Tf.

Next, we show that T is a contraction on $(X_r, \|\cdot\|_{r/2})$.

Let $f, h \in X_r$. Consider the difference

$$(Tf)(x) - (Th)(x) = \lambda \int_0^\infty \frac{t^{r-1}(f(t)-h(t))}{1+e^{xt}}\,dt, \qquad x > 0. \tag{3.24}$$

Using the bound analogous to (3.11), namely

$$|f(t) - h(t)| \le \|f-h\|_{r/2}\, t^{-r/2} \quad \text{for all } t > 0, \tag{3.25}$$

we obtain

$$\begin{aligned} |(Tf)(x) - (Th)(x)| &\le |\lambda|\, \|f-h\|_{r/2} \int_0^\infty \frac{t^{r-1} t^{-r/2}}{1+e^{xt}}\,dt \\ &= |\lambda|\, \|f-h\|_{r/2} \int_0^\infty \frac{t^{\frac{r}{2}-1}}{1+e^{xt}}\,dt. \end{aligned} \tag{3.26}$$

As in (3.15)–(3.17), the same change of variables $u = xt$ yields

$$\int_0^\infty \frac{t^{\frac{r}{2}-1}}{1+e^{xt}}\,dt = x^{-\frac{r}{2}} B_r, \tag{3.27}$$

and therefore,

$$|(Tf)(x) - (Th)(x)| \le |\lambda| B_r\, \|f-h\|_{r/2}\, x^{-\frac{r}{2}} \quad \text{for all } x > 0. \tag{3.28}$$

Multiplying (3.28) by $x^{r/2}$ gives

$$x^{r/2} |(Tf)(x) - (Th)(x)| \le |\lambda| B_r\, \|f-h\|_{r/2}, \tag{3.29}$$

and taking the supremum over $x > 0$ yields

$$\|Tf - Th\|_{r/2} = \sup_{x>0} x^{r/2} |(Tf)(x) - (Th)(x)| \le |\lambda| B_r\, \|f-h\|_{r/2}. \tag{3.30}$$

Let

$$q = |\lambda| B_r. \tag{3.31}$$

By the hypothesis (3.10), we have

$$0 \le q < 1. \tag{3.32}$$

Thus, (3.30) shows that T is a strict contraction on the complete metric space (X_r, d) with d given by (2.16).

By Proposition 2.12, the space $(X_r, \|\cdot\|_{r/2})$ is complete. Also, the operator $T : X_r \to X_r$ is a contraction with Lipschitz constant $q < 1$ as in (3.32). Therefore, by the Banach fixed–point theorem, there exists a unique function $f \in X_r$ such that

$$Tf = f. \tag{3.33}$$

Unwinding the definition (2.34), the fixed–point condition (3.33) is equivalent to

$$f(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0; \tag{3.34}$$

that is, f is a solution of the integral equation (2.33).

The uniqueness of the fixed–point in X_r implies uniqueness of the solution $f \in X_r$ of (2.33). This completes the proof of Theorem 3.2. □

Remark 3.3. *The Banach fixed–point theorem also yields a constructive procedure to approximate the unique solution $f \in X_r$. Given any initial function $f_0 \in X_r$, define the Picard iterates*

$$f_{n+1} = Tf_n, \qquad n = 0, 1, 2, \ldots. \tag{3.35}$$

Then, $(f_n)_{n\in\mathbb{N}}$ converges in $\|\cdot\|_{r/2}$ to the unique fixed–point f, and one has the error estimate

$$\|f_n - f\|_{r/2} \le \frac{q^n}{1-q}\, \|f_1 - f_0\|_{r/2}, \qquad n \in \mathbb{N}, \tag{3.36}$$

where $q = |\lambda| B_r < 1$ is given by (3.31) *and* (3.32).

In the next result, we will illustrate the application of Theorem 3.2 with an explicit example.

Theorem 3.4. *Fix $r \in (1/2, 1)$, and let $B_r > 0$ be the constant defined in* (3.1). *Consider the integral equation*

$$f(x) = g(x) + \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0 \tag{3.37}$$

with the explicit choice

$$g(x) = x^{-r/2}, \qquad x > 0 \tag{3.38}$$

and a parameter $\lambda \in \mathbb{R}$ satisfying

$$|\lambda| B_r < 1. \tag{3.39}$$

Then:

1) The unique solution $f \in X_r$ of (3.37) *is given explicitly by*

$$f(x) = \frac{1}{1 - \lambda B_r}\, x^{-\frac{r}{2}}, \qquad x > 0. \tag{3.40}$$

2) If $(f_n)_{n\ge 0}$ is the Picard iteration defined by

$$f_0(x) = g(x) = x^{-\frac{r}{2}}, \quad f_{n+1} = Tf_n, \qquad n = 0, 1, 2, \ldots, \tag{3.41}$$

where T *is the operator in* (2.34)*, then each* f_n *is of the form*

$$f_n(x) = a_n x^{-\frac{r}{2}}, \qquad x > 0 \tag{3.42}$$

with coefficients

$$a_n = \frac{1 - (\lambda B_r)^{n+1}}{1 - \lambda B_r}, \qquad n \geq 0, \tag{3.43}$$

and

$$\lim_{n\to\infty} a_n = \frac{1}{1 - \lambda B_r}. \tag{3.44}$$

Consequently, $f_n \to f$ *in* X_r*, and*

$$\|f_n - f\|_{r/2} = \frac{|\lambda B_r|^{n+1}}{|1 - \lambda B_r|} \leq \frac{|\lambda|^{n+1} B_r^{n+1}}{1 - |\lambda| B_r} \xrightarrow[n\to\infty]{} 0. \tag{3.45}$$

Proof. First, we verify that $g \in X_r$.

Recall that X_r is defined in (2.15) with the norm

$$\|f\|_{r/2} = \sup_{x>0} x^{r/2} |f(x)|. \tag{3.46}$$

For g given by (3.38), we compute

$$\|g\|_{r/2} = \sup_{x>0} x^{r/2} |g(x)| = \sup_{x>0} x^{r/2} x^{-r/2} = \sup_{x>0} 1 = 1. \tag{3.47}$$

Thus, $g \in X_r$, and the hypotheses of Theorem 3.2 are satisfied under the smallness condition (3.39) on λ.

Construction of the exact solution: We look for a solution $f \in X_r$ of (3.37) of the form

$$f(x) = A\, x^{-r/2}, \qquad x > 0 \tag{3.48}$$

for some constant $A \in \mathbb{R}$.

For such an f, we compute the integral term

$$\begin{aligned} \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt &= A \int_0^\infty \frac{t^{r-1} t^{-r/2}}{1 + e^{xt}}\, dt \\ &= A \int_0^\infty \frac{t^{\frac{r}{2}-1}}{1 + e^{xt}}\, dt. \end{aligned} \tag{3.49}$$

Using the change of variables

$$u = xt, \qquad t = \frac{u}{x}, \qquad dt = \frac{du}{x}, \tag{3.50}$$

we obtain

$$\int_0^\infty \frac{t^{\frac{r}{2}-1}}{1 + e^{xt}}\, dt = \int_0^\infty \frac{(\frac{u}{x})^{\frac{r}{2}-1}}{1 + e^{u}}\, \frac{du}{x}$$

$$
\begin{aligned}
&= x^{-\frac{r}{2}} \int_0^\infty \frac{u^{\frac{r}{2}-1}}{1+e^u}\,du \\
&= x^{-\frac{r}{2}} B_r,
\end{aligned}
\tag{3.51}
$$

where B_r is the finite positive constant from Lemma 3.1.

Substituting (3.51) into (3.49) gives

$$
\int_0^\infty \frac{t^{r-1} f(t)}{1+e^{xt}}\,dt = A\, x^{-\frac{r}{2}} B_r. \tag{3.52}
$$

Now, insert (3.38), (3.48), and (3.52) into (3.37):

$$
\begin{aligned}
A\, x^{-\frac{r}{2}} &= x^{-\frac{r}{2}} + \lambda (A\, x^{-\frac{r}{2}} B_r) \\
&= x^{-\frac{r}{2}} (1 + \lambda A B_r).
\end{aligned}
\tag{3.53}
$$

Because $x^{-\frac{r}{2}} > 0$ for all $x > 0$, (3.53) is equivalent to the scalar equation

$$
A = 1 + \lambda A B_r. \tag{3.54}
$$

Rewriting (3.54) as

$$
A(1 - \lambda B_r) = 1 \tag{3.55}
$$

and using $|\lambda| B_r < 1$ so that $1 - \lambda B_r \neq 0$, we obtain

$$
A = \frac{1}{1 - \lambda B_r}. \tag{3.56}
$$

Thus, the function

$$
f(x) = \frac{1}{1 - \lambda B_r}\, x^{-\frac{r}{2}}, \qquad x > 0, \tag{3.57}
$$

belongs to X_r and is a solution of (3.37). Uniqueness of the solution in X_r follows from Theorem 3.2. This proves (1) and identifies the explicit exact solution.

Construction of the solution via Picard iterations using Remark 3.3: We now recover the same solution (3.57) by the Picard iteration associated with the operator T in (2.34). Choose the initial function

$$
f_0(x) = g(x) = x^{-\frac{r}{2}}, \qquad x > 0. \tag{3.58}
$$

Note that we can choose any initial function living in X_r, for example, any choice from the Examples 2.2–2.6. All the choices will lead to the unique solution (3.57).

With the above choice of $f_0(x)$, we define inductively

$$
f_{n+1} = T f_n, \qquad n = 0, 1, 2, \ldots. \tag{3.59}
$$

We claim by induction that each iterate f_n is of the form

$$
f_n(x) = a_n x^{-\frac{r}{2}}, \qquad x > 0 \tag{3.60}
$$

for some sequence of real coefficients $(a_n)_{n\geq 0}$.

Base step: For $n = 0$, (3.60) holds with $a_0 = 1$ by (3.58).

Induction step: Assume that for some $n \in \mathbb{N}$, we have

$$f_n(x) = a_n x^{-\frac{r}{2}} \quad \text{for all } x > 0. \tag{3.61}$$

Then, by (2.34) and the computation (3.52) (with A replaced by a_n), we obtain

$$\begin{aligned} f_{n+1}(x) = (Tf_n)(x) &= g(x) + \lambda \int_0^\infty \frac{t^{r-1} f_n(t)}{1 + e^{xt}}\, dt \\ &= x^{-\frac{r}{2}} + \lambda a_n x^{-\frac{r}{2}} B_r \\ &= (1 + \lambda B_r a_n) x^{-\frac{r}{2}}. \end{aligned} \tag{3.62}$$

Thus, (3.60) holds for $n + 1$ with

$$a_{n+1} = 1 + \lambda B_r a_n. \tag{3.63}$$

The recursion (3.63) is a linear first-order difference equation. Let us set

$$q = \lambda B_r. \tag{3.64}$$

Then, (3.63) becomes

$$a_{n+1} = 1 + q a_n, \qquad a_0 = 1. \tag{3.65}$$

We first find a fixed–point a_* of the affine map $a \mapsto 1 + qa$. Solving

$$a_* = 1 + q a_* \tag{3.66}$$

gives

$$a_* = \frac{1}{1-q} = \frac{1}{1 - \lambda B_r}. \tag{3.67}$$

Subtracting a_* from both sides of (3.65) yields

$$a_{n+1} - a_* = q a_n + 1 - a_* = q a_n + 1 - \frac{1}{1-q} = q a_n - \frac{q}{1-q} = q(a_n - a_*). \tag{3.68}$$

Thus,

$$a_{n+1} - a_* = q(a_n - a_*), \tag{3.69}$$

and by induction,

$$a_n - a_* = q^n (a_0 - a_*) \quad \text{for all } n \geq 0. \tag{3.70}$$

Using $a_0 = 1$ and (3.67), we obtain

$$a_0 - a_* = 1 - \frac{1}{1-q} = \frac{1 - q - 1}{1-q} = -\frac{q}{1-q}, \tag{3.71}$$

and therefore,

$$a_n = a_* + q^n (a_0 - a_*) = \frac{1}{1-q} - q^n \frac{q}{1-q} = \frac{1 - q^{n+1}}{1-q}. \tag{3.72}$$

Recalling (3.64), we may rewrite (3.72) as

$$a_n = \frac{1 - (\lambda B_r)^{n+1}}{1 - \lambda B_r}, \qquad n \geq 0. \tag{3.73}$$

Because $|q| = |\lambda| B_r < 1$ by (3.39), we have

$$\lim_{n\to\infty} a_n = \frac{1}{1 - \lambda B_r} = A, \tag{3.74}$$

where A is exactly the coefficient in (3.56). In particular,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} a_n x^{-\frac{r}{2}} = A x^{-\frac{r}{2}} = f(x), \qquad x > 0. \tag{3.75}$$

Convergence in the norm $\|\cdot\|_{r/2}$: We now compute the error in the weighted norm:

$$\begin{aligned}
\|f_n - f\|_{r/2} &= \sup_{x>0} x^{r/2} |f_n(x) - f(x)| \\
&= \sup_{x>0} x^{r/2} \left|(a_n - A) x^{-\frac{r}{2}}\right| \\
&= |a_n - A| \\
&= |q|^n |a_0 - a_*| \quad \text{by (3.70) and (3.71)} \\
&= |q|^n \left|\frac{q}{1-q}\right| \\
&= \frac{|q|^{n+1}}{|1-q|}.
\end{aligned} \tag{3.76}$$

In particular,

$$\|f_n - f\|_{r/2} \leq \frac{|q|^{n+1}}{1 - |q|} = \frac{|\lambda|^{n+1} B_r^{n+1}}{1 - |\lambda| B_r}. \tag{3.77}$$

Because $|q| = |\lambda| B_r < 1$, the right-hand side of (3.77) tends to 0 as $n \to \infty$. This proves that

$$\|f_n - f\|_{r/2} \xrightarrow[n\to\infty]{} 0; \tag{3.78}$$

that is the Picard iterates f_n converge to f in the Banach space X_r. This establishes (2) and completes the proof of the theorem.

In Figure 1, we present an example to illustrate the results of Theorem 3.4.

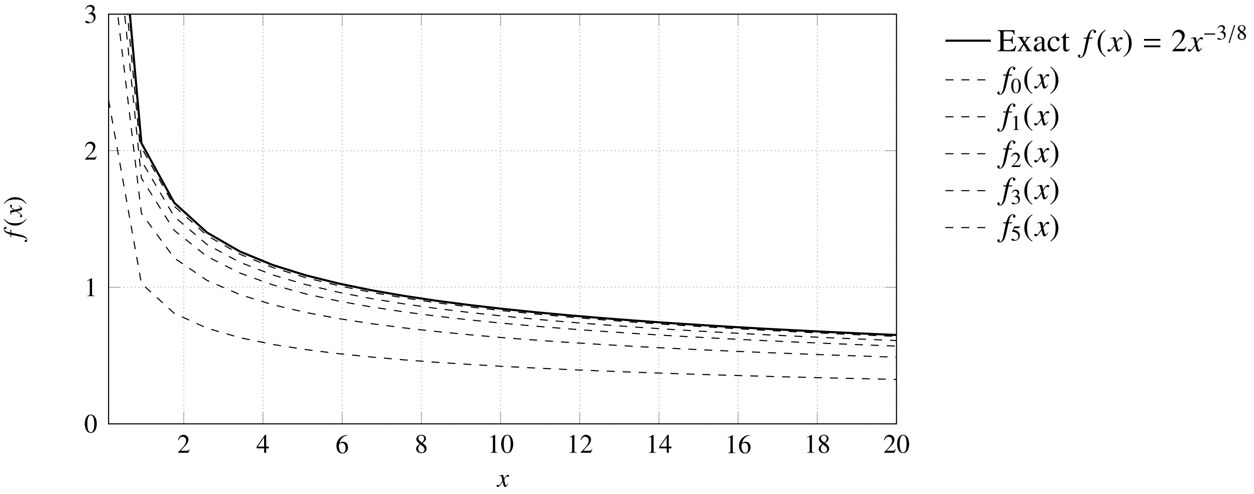

Figure 1. Plot of the exact solution $f(x) = 2x^{-3/8}$ and several Picard iterates $f_n(x) = a_n x^{-3/8}$ with $r = \frac{3}{4}$, where the contraction constant for the operator is $q = |\lambda| B_r = \frac{1}{2}$. The iterates converge geometrically to the unique fixed–point established in Theorem 3.4.

□

3.2. *Answer to Question 1.8*

The next result discusses the solution of the homogeneous Salem type integral equation given in 2.33 in the weighted function space X_r.

Theorem 3.5. *In Theorem 3.2, suppose that*

$$g(x) \equiv 0 \quad \textit{for all } x > 0. \tag{3.79}$$

Then, the operator $T : X_r \to X_r$ *defined in* (2.34) *reduces to*

$$(Tf)(x) = \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0. \tag{3.80}$$

Under the same condition,

$$|\lambda| B_r < 1, \tag{3.81}$$

Theorem 3.2 asserts that T *has a unique fixed–point in* X_r.
On the other hand, it is immediate from (3.80) *that the zero function*

$$f(x) \equiv 0, \qquad x > 0 \tag{3.82}$$

satisfies

$$Tf = f. \tag{3.83}$$

By the uniqueness part of Theorem 3.2, this implies that

$$f \equiv 0 \tag{3.84}$$

is the only solution of the homogeneous integral equation

$$f(x) = \lambda \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0, \tag{3.85}$$

in the space X_r.

3.3. Answer to Question 1.9

The original motivation for this work was to understand whether one may apply Banach's contraction principle directly to Salem's integral equation (1.4) from Question 1.9 so as to conclude that $f \equiv 0$ is the unique solution in the function space X_r.

Let us write

$$(Kf)(x) = \int_0^\infty \frac{t^{r-1} f(t)}{1 + e^{xt}}\, dt, \qquad x > 0.$$

Then, Salem's original equation (1.4) is simply

$$(Kf)(x) = 0, \qquad x > 0. \tag{3.86}$$

A natural auxiliary operator is

$$(\widetilde{T} f)(x) = f(x) + \lambda (Kf)(x) \tag{3.87}$$

for some fixed $\lambda \in \mathbb{R}$. Using the same estimate as in (3.30), we obtain

$$\|Kf - Kh\|_{r/2} \leq B_r \, \|f - h\|_{r/2},$$

and therefore,

$$\begin{aligned} \|\widetilde{T} f - \widetilde{T} h\|_{r/2} &\leq \|f - h\|_{r/2} + |\lambda| \, \|Kf - Kh\|_{r/2} \\ &\leq (1 + |\lambda| B_r) \, \|f - h\|_{r/2}. \end{aligned} \tag{3.88}$$

Because $1 + |\lambda| B_r \geq 1$, the operator $\widetilde{T}$ is not, in general, a contraction on X_r. Hence, the contraction argument that underpins Theorem 3.2 does not directly resolve (3.86).

Although the zero function $f(x) \equiv 0$ is always a solution of (3.86), the question of whether nontrivial solutions exist is a question about the kernel, or equivalently the injectivity, of the integral operator K on X_r. That issue lies beyond the scope of the Banach fixed–point argument. However, in the conclusion, we indicate a possible future research direction based on generalized metric spaces.

4. Conclusions

In this paper, we have studied the nonhomogeneous integral equation associated with Salem's integral equation by employing a fixed–point framework. Our results demonstrate that Banach's contraction principle provides an elegant and powerful tool for establishing the existence and uniqueness of solutions when the prescribed function g belongs to the admissible function class X_r.

Although our initial motivation stemmed from the degenerate equation (3.86), the present results apply effectively only to the associated nonhomogeneous and homogeneous equations. The associated operator gives rise to a genuine fixed-point problem in appropriate weighted Banach spaces. In this framework, the operator exhibits stability and contractive properties that guarantee the existence, uniqueness, and iterative construction of solutions for each prescribed $g \in X_r$. The degenerate case, namely Salem's original integral equation, remains substantially more delicate and requires additional investigation, because it does not fit directly into the classical Banach contraction framework developed in this paper.

The Banach contraction principle has served as a cornerstone for a wide range of generalizations, developed primarily along two major directions:

(1) by enlarging the class of underlying spaces, and
(2) by relaxing or modifying the contraction requirement itself.

With regard to the second direction, one finds several important extensions, including Kannan–type contractions [15], Chatterjea type contractions [16], Reich–type contractions [17], Hardy–Rogers–type contractions [18], and as many other modified contractive conditions and related fixed–point theorems [19]. Comprehensive treatments of such generalized contractions may be found in [20–23]. On the other hand, the fixed–point literature also contains many generalized metric structures such as $b-$metric spaces, double–controlled metric spaces, double–composed metric spaces, $M-$metric spaces, and other spaces that extend the usual metric space definition.

These developments suggest a natural continuation of the present work. A potential direction for future investigation is to study whether the operator $\widetilde{T}$ defined in (3.87), or more fundamentally, the integral operator K associated with (3.86), can be treated successfully within the framework of generalized metric spaces, generalized contractions, or even expansive-type mappings. Such an approach may provide a more suitable setting for Salem's original equation and could potentially open a new line of research complementary to the existing integral transform methods.

Author contributions

Irshad Ayoob: Conceptualization, Methodology, Writing-original draft, and formal analysis. Nabil Mlaiki: Conceptualization, Methodology, Writing-original draft, and formal analysis. All authors agree to take responsibility for the content and conclusions of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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