Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 12094–12107.
DOI: 10.3934/math.2026496
Received: 16 January 2026
Revised: 07 April 2026
Accepted: 16 April 2026
Published: 29 April 2026

Research article

A novel modified one-step iterative method for common fixed points of three nonexpansive mappings

Athar Abbas[1], **Somayya Komal**[2], **Hafiz Fukhar-ud-din**[1], **Muhammad Aqeel Ahmad Khan**[1] and **Muhammad Jabir Khan**[3,*]

[1] Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan
[2] Department of Mathematics, Faculty of Sciences, University of Mianwali, Punjab, Pakistan
[3] School of Artificial Intelligence and Computer Science, Nantong University, Nantong 226019, China

* **Correspondence:** Email: jabirkhan.uos@gmail.com.

Abstract: This paper proposes a novel modified one-step iterative scheme for approximating common fixed points of three nonexpansive mappings in uniformly convex Banach spaces. The proposed scheme extends the classical one-step iteration associated with two mappings, which can be recovered as a particular case by an appropriate choice of the third mapping. Weak convergence of the generated sequence is established under both the Kadec-Klee property (KKP) and the Opial condition, while strong convergence is obtained by assuming a modified condition (B). Moreover, we introduce a new Condition (P) tailored for three mappings, which reduces to the well-known Condition (S) in the case of two mappings. The results presented here broaden and unify several existing contributions in the theory of common fixed point approximation.

Keywords: modified one-step iteration; common fixed point approximation; Kadec-Klee property; condition (P); uniformly convex Banach spaces; convergence
Mathematics Subject Classification: 46B20, 47H10, 47H09, 47J25

1. Introduction

Suppose that $\mathcal{W}$ is a real Banach space and $\mathcal{M} \neq \emptyset$ is a convex subset of $\mathcal{W}$. We call a mapping $\Phi : \mathcal{M} \to \mathcal{M}$ nonexpansive whenever it fulfills the following condition:

$$\|\Phi\zeta - \Phi\xi\| \leq \|\zeta - \xi\| \quad \text{for all } \zeta, \xi \in \mathcal{M}.$$

The study of fixed points, points $\zeta \in \mathcal{M}$ such that $\Phi\zeta = \zeta$, represents a fundamental area of research in nonlinear analysis due to its widespread applications in various fields, including optimization theory, image processing, and signal recovery [3]. Iterative approaches to fixed point approximation have constituted a core line of research in this field for many years.

The simplest iterative approach, known as Picard iteration, defined by $\zeta_{\nu+1} = \Phi\zeta_\nu$, often fails to converge even weakly for nonexpansive mappings [12]. This limitation prompted the development of more sophisticated iterative techniques. Mann [14] introduced the following iteration scheme:

$$\zeta_{\nu+1} = (1-\alpha_\nu)\zeta_\nu + \alpha_\nu\Phi\zeta_\nu,$$

where $\{\alpha_\nu\}$ is a sequence in $[0,1]$ with $\lim_{\nu\to\infty}\alpha_\nu = 0$ and $\sum_{\nu=1}^{\infty}\alpha_\nu = \infty$. While this method converges strongly for nonexpansive mappings on compact convex subsets, it encounters difficulties when applied to Lipschitz pseudocontractive mappings [4].

To address these challenges, Ishikawa [8] developed a two-step iterative process:

$$\begin{cases} \zeta_1 = \zeta \in \mathcal{M}, \\ \zeta_{\nu+1} = (1-\alpha_\nu)\zeta_\nu + \alpha_\nu\Phi\xi_\nu, \\ \xi_\nu = (1-\beta_\nu)\zeta_\nu + \beta_\nu\Phi\zeta_\nu, \qquad \forall \nu \in \mathbb{N} \end{cases} \tag{1.1}$$

where $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ are appropriately chosen sequences in $[0,1]$. This breakthrough inspired numerous extensions, particularly for finding common fixed points of multiple mappings. Several researchers [5,7,11,17,18] have investigated Ishikawa-type iterations for two mappings:

$$\begin{cases} \zeta_1 = \zeta \in \mathcal{M}, \\ \zeta_{\nu+1} = (1-\alpha_\nu)\zeta_\nu + \alpha_\nu\Theta\xi_\nu, \\ \xi_\nu = (1-\beta_\nu)\zeta_\nu + \beta_\nu\Phi\zeta_\nu, \qquad \forall \nu \in \mathbb{N}. \end{cases} \tag{1.2}$$

In 2007, Liu et al. [13] proposed an alternative two-step process:

$$\begin{cases} \zeta_1 = \zeta \in \mathcal{M}, \\ \zeta_{\nu+1} = (1-\alpha_\nu)\Theta\zeta_\nu + \alpha_\nu\Phi\xi_\nu, \\ \xi_\nu = (1-\beta_\nu)\Theta\zeta_\nu + \beta_\nu\Phi\zeta_\nu, \qquad \forall \nu \in \mathbb{N}. \end{cases} \tag{1.3}$$

Khan et al. [10] introduced an effective one-step iterative approach for a pair of nonexpansive mappings in 2009:

$$\begin{cases} \zeta_1 = \zeta \in \mathcal{M}, \\ \zeta_{\nu+1} = \alpha_\nu\Theta\zeta_\nu + (1-\alpha_\nu)\Phi\zeta_\nu, \quad \forall \nu \in \mathbb{N}. \end{cases} \tag{1.4}$$

Motivated by the work of Khan et al. [10], we introduce a novel modified iterative process for three nonexpansive mappings $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$, defined by

$$\begin{cases} \zeta_1 = \zeta \in \mathcal{M}, \\ \zeta_{\nu+1} = \alpha_\nu\Theta\zeta_\nu + \beta_\nu\Phi\zeta_\nu + (1-\alpha_\nu-\beta_\nu)\Psi\zeta_\nu, \quad \forall \nu \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ are sequences in $[0,1]$, with $\alpha_\nu + \beta_\nu \leq 1$, for all $\nu \in \mathbb{N}$. It is noteworthy that the iteration process (1.4) becomes a special case of our new scheme (1.5) when $\Psi = \Phi$, demonstrating the comprehensive nature of our approach.

We now recall some essential concepts that will be utilized throughout our analysis. A Banach space $\mathcal{W}$ is said to satisfy Opial's condition [15] if, for every sequence $\{\zeta_\nu\}$ in $\mathcal{W}$ converging weakly to some ζ, the following inequality is satisfied:

$$\limsup_{\nu\to\infty} \|\zeta_\nu - \zeta\| < \limsup_{\nu\to\infty} \|\zeta_\nu - \xi\|, \quad \forall \xi \in \mathcal{W}, \xi \neq \zeta.$$

This property is satisfied by Hilbert spaces and ℓ^p spaces for $1 < p < \infty$, but not by $L^p[0, 2\pi]$ spaces when $1 < p \neq 2$. A mapping $\Phi : \mathcal{M} \to \mathcal{W}$ is demiclosed with respect to $\xi \in \mathcal{W}$ if for every sequence $\{\zeta_\nu\}$ in $\mathcal{M}$ and $\zeta \in \mathcal{W}$, the conditions $\zeta_\nu \rightharpoonup \zeta$ and $\Phi\zeta_\nu \to \xi$ imply that $\zeta \in \mathcal{M}$ and $\Phi\zeta = \xi$.

The KKP in a Banach space $\mathcal{W}$ requires that if a sequence $\{\zeta_\nu\}$ converges weakly to ζ and $\|\zeta_\nu\| \to \|\zeta\|$, then $\{\zeta_\nu\}$ must converge strongly to ζ. Finite-dimensional Banach spaces and uniformly convex Banach spaces all have this property.

Now, we state some useful lemmas.

Lemma 1.1. *[16] Let* $\{\zeta_\nu\}$ *and* $\{\xi_\nu\}$ *be sequences in a uniformly convex Banach space* $\mathcal{W}$. *Suppose there is a real number* $\vartheta \geq 0$ *such that* $\limsup_{\nu\to\infty} \|\zeta_\nu\| \leq \vartheta$, $\limsup_{\nu\to\infty} \|\xi_\nu\| \leq \vartheta$, *and for a sequence* $\{t_\nu\}$ *in* $(0, 1)$, *we have* $\lim_{\nu\to\infty} \|t_\nu\zeta_\nu + (1 - t_\nu)\xi_\nu\| = \vartheta$. *Then,* $\lim_{\nu\to\infty} \|\zeta_\nu - \xi_\nu\| = 0$.

Lemma 1.2. *[1] Let* $\Phi : \mathcal{M} \to \mathcal{M}$ *be a nonexpansive mapping on a nonempty, closed, and convex subset* $\mathcal{M}$ *of a uniformly convex Banach space* $\mathcal{W}$. *Then, the mapping* $I - \Phi$ *is* demiclosed at zero.

Lemma 1.3. *[2] Let* $\Phi : \mathcal{M} \to \mathcal{M}$ *be a nonexpansive mapping on a nonempty bounded closed convex subset* $\mathcal{M}$ *of a uniformly convex Banach space* $\mathcal{W}$. *Then, there exists a strictly increasing continuous convex function* $\wp : [0, \infty) \to [0, \infty)$ *with* $\wp(0) = 0$ *such that for all* $\zeta, \xi \in \mathcal{M}$ *and any* $t \in [0, 1]$, *the following inequality is satisfied:*

$$\wp\left(\|\Phi(t\zeta + (1 - t)\xi) - (t\Phi\zeta + (1 - t)\Phi\xi)\|\right) \leq \|\zeta - \xi\| - \|\Phi\zeta - \Phi\xi\|,$$

where $\omega_w(\{\zeta_\nu\})$ represents the set of all weak subsequential limits of a sequence $\{\zeta_\nu\}$ in the space $\mathcal{W}$. The following lemma is adapted from Falset et al. [6].

Lemma 1.4. *[6] Let* $\{\zeta_\nu\}$ *be a bounded sequence in a uniformly convex Banach space* $\mathcal{W}$ *whose dual space* $\mathcal{W}^*$ *satisfies the KKP. For each* $t \in [0, 1]$ *and each* $e_1, e_2 \in \omega_w(\{\zeta_\nu\})$, *the limit*

$$\lim_{\nu\to\infty} \|t\zeta_\nu + (1 - t)e_1 - e_2\|$$

exists. Then $\omega_w(\{\zeta_\nu\})$ *consists of exactly one element.*

This paper is organized as follows: Section 2 deals with the preliminary concepts needed for the subsequent development. Section 3 establishes the convergence analysis of the proposed algorithm, while Section 4 provides a numerical experiment illustrating the main result of Section 3.

2. Preparatory lemmas

Throughout this section, we assume the following:

Assumption 1: Let $\emptyset \neq F = F(\Theta) \cap F(\Phi) \cap F(\Psi)$ represent the set of all common fixed points of nonexpansive mappings Θ, Φ, and Ψ.

Lemma 2.1. *Let $\mathcal{M}$ be a nonempty closed and convex subset of a normed space $\mathcal{W}$. Suppose $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$ are nonexpansive mappings and let $\{\zeta_\nu\}$ be a sequence generated by an iterative process of the form (1.5). Then for every $\zeta^* \in F \neq \emptyset$, the limit $\lim_{\nu\to\infty} \|\zeta_\nu - \zeta^*\|$ exists.*

Proof. Assume that $\zeta^* \in F$. Then, we have

$$\begin{aligned} \|\zeta_{\nu+1} - \zeta^*\| &= \|\alpha_\nu \Theta\zeta_\nu + \beta_\nu \Phi\zeta_\nu + (1-\alpha_\nu-\beta_\nu)\Psi\zeta_\nu - \zeta^*\| \\ &= \|\alpha_\nu(\Theta\zeta_\nu - \zeta^*) + \beta_\nu(\Phi\zeta_\nu - \zeta^*) + (1-\alpha_\nu-\beta_\nu)(\Psi\zeta_\nu - \zeta^*)\| \\ &\leq \alpha_\nu\|\Theta\zeta_\nu - \zeta^*\| + \beta_\nu\|\Phi\zeta_\nu - \zeta^*\| + (1-\alpha_\nu-\beta_\nu)\|\Psi\zeta_\nu - \zeta^*\| \\ &\leq \alpha_\nu\|\zeta_\nu - \zeta^*\| + \beta_\nu\|\zeta_\nu - \zeta^*\| + (1-\alpha_\nu-\beta_\nu)\|\zeta_\nu - \zeta^*\| \\ &= \|\zeta_\nu - \zeta^*\|. \end{aligned}$$

Thus, $\lim_{\nu\to\infty} \|\zeta_\nu - \zeta^*\|$ exists for all $\zeta^* \in F$. □

Khan et al. [10] used the condition $\|\zeta - \Theta\zeta\| \leq \|\Theta\zeta - \Phi\zeta\|$ for all $\zeta \in \mathcal{M}$. Call it Condition (S). We propose the condition: $\|\zeta - \Theta\zeta\| \leq \frac{1}{2}\Big[\|\Theta\zeta - \Phi\zeta\| + \|\Theta\zeta - \Psi\zeta\|\Big]$ for all $\zeta \in \mathcal{M}$, and call it Condition (P). It should be noted that Condition (P) is weaker than Condition (S) and that Condition (S) is a specific case of our suggested Condition (P). Three nonexpansive mappings that satisfy Condition (P) but not Condition (S) are shown in this example. This shows that these results may be more applicable to a broader class of mappings than the comparable results obtained by Khan et al. [10] and Liu et al. [13].

Example 2.1. Now let us define $\Theta, \Phi, \Psi : \mathbb{R} \to \mathbb{R}$ as follows: $\Theta\zeta = 1 + \frac{1}{2}(\zeta - 1)$, $\Phi\zeta = \frac{3}{10} + \frac{7}{10}\zeta$, $\Psi\zeta = 2 - \zeta, \quad \forall \zeta \in \mathbb{R}$.

Clearly, Θ, Φ, and Ψ are nonexpansive with a common fixed point 1. Moreover, $\|\zeta - \Theta\zeta\| \leq \frac{1}{2}\Big[\|\Theta\zeta - \Phi\zeta\| + \|\Theta\zeta - \Psi\zeta\|\Big]$ for all $\zeta \in \mathbb{R}$. In fact,

$$\begin{aligned} |\zeta - \Theta\zeta| &= |\zeta - (\frac{1}{2} + \frac{1}{2}\zeta)| \\ &= |\frac{1}{2}(\zeta - 1)| \\ &= \frac{1}{2}|\zeta - 1|, \end{aligned} \tag{2.1}$$

$$\begin{aligned} |\Theta\zeta - \Phi\zeta| &= |(\frac{1}{2} + \frac{1}{2}\zeta) - (\frac{3}{10} + \frac{7}{10}\zeta)| \\ &= |\frac{1}{5} - \frac{1}{5}\zeta| \\ &= |\frac{1}{5}(\zeta - 1)| \\ &= \frac{1}{5}|\zeta - 1|, \end{aligned} \tag{2.2}$$

and

$$|\Theta\zeta - \Psi\zeta| = |(\frac{1}{2} + \frac{1}{2}\zeta) - (2 - \zeta)|$$

$$\begin{aligned} &= |\frac{3}{2}(\zeta - 1)| \\ &= \frac{3}{2}|\zeta - 1|. \end{aligned} \tag{2.3}$$

Now, from (2.1)–(2.3) and Condition (P), we have

$$\begin{aligned} \frac{1}{2}|\zeta - 1| &\leq \frac{1}{2}\Big[\frac{1}{5}|\zeta - 1| + \frac{3}{2}|\zeta - 1|\Big] \\ &\leq \frac{1}{2}\Big[\frac{17}{10}|\zeta - 1|\Big] \\ &\leq \frac{17}{20}|\zeta - 1|. \end{aligned}$$

This implies that Condition (P) is satisfied. Moreover, from (2.1) and (2.2), we observe that Condition (S) is not satisfied.

Based on Condition (P), we now establish the following lemma.

Lemma 2.2. *Let $\mathcal{W}$ be a uniformly convex Banach space and $\mathcal{M} \neq \emptyset$ a closed convex subset of $\mathcal{W}$. Let $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$ be nonexpansive mappings and let $\{\zeta_\nu\}$ be a sequence defined by iterative scheme (1.5), where $\{\alpha_\nu\}, \{\beta_\nu\} \in [\delta, 1-\delta], \alpha_\nu + \beta_\nu \leq 1$, for all $\nu \in \mathbb{N}$, and $\delta \in (0, \frac{1}{2}]$. Assume that Condition (P) holds and $F \neq \emptyset$. Then,*

$$\lim_{\nu\to\infty} \|\Theta\zeta_\nu - \zeta_\nu\| = 0, \quad \lim_{\nu\to\infty} \|\Phi\zeta_\nu - \zeta_\nu\| = 0 \quad \text{and} \quad \lim_{\nu\to\infty} \|\Psi\zeta_\nu - \zeta_\nu\| = 0.$$

Proof. Let $\zeta^* \in F$. By Lemma 2.1, $\lim_{\nu\to\infty} \|\zeta_\nu - \zeta^*\|$ exists. Assume that $\lim_{\nu\to\infty} \|\zeta_\nu - \zeta^*\| = \tau$ for some $\tau \geq 0$. From nonexpansivity, we have

$$\limsup_{\nu\to\infty} \|\Theta\zeta_\nu - \zeta^*\| \leq \tau \tag{2.4}$$

Similarly,

$$\limsup_{\nu\to\infty} \|\Phi\zeta_\nu - \zeta^*\| \leq \tau \tag{2.5}$$

and

$$\limsup_{\nu\to\infty} \|\Psi\zeta_\nu - \zeta^*\| \leq \tau. \tag{2.6}$$

Now, rewrite $\zeta_{\nu+1}$ in (1.5) as

$$\zeta_{\nu+1} = \alpha_\nu\Theta\zeta_\nu + (1-\alpha_\nu)\left[\frac{\beta_\nu}{1-\alpha_\nu}\Phi\zeta_\nu + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}\Psi\zeta_\nu\right], \tag{2.7}$$

Assume that

$$\xi_\nu = \frac{\beta_\nu}{1-\alpha_\nu}\Phi\zeta_\nu + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}\Psi\zeta_\nu. \tag{2.8}$$

Now, taking

$$\|\xi_\nu - \zeta^*\| = \|\frac{\beta_\nu}{1-\alpha_\nu}\Phi\zeta_\nu + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}\Psi\zeta_\nu - \zeta^*\|$$

$$
\begin{aligned}
&= \left\|\frac{\beta_\nu}{1-\alpha_\nu}(\Phi\zeta_\nu - \zeta^*) + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}(\Psi\zeta_\nu - \zeta^*)\right\| \\
&\leq \frac{\beta_\nu}{1-\alpha_\nu}\|\Phi\zeta_\nu - \zeta^*\| + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}\|\Psi\zeta_\nu - \zeta^*\| \\
&\leq \frac{\beta_\nu}{1-\alpha_\nu}\|\zeta_\nu - \zeta^*\| + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}\|\zeta_\nu - \zeta^*\| \\
&\leq \|\zeta_\nu - \zeta^*\|. \qquad (2.9)
\end{aligned}
$$

Taking $\limsup_{\nu\to\infty}$ on both sides of (2.9), we have

$$\limsup_{\nu\to\infty} \|\xi_\nu - \zeta^*\| \leq \tau. \tag{2.10}$$

Further, $\lim_{\nu\to\infty} \|\zeta_{\nu+1} - \zeta^*\| = \tau$ gives that

$$\lim_{\nu\to\infty} \|\alpha_\nu(\Theta\zeta_\nu - \zeta^*) + (1-\alpha_\nu)(\xi_\nu - \zeta^*)\| = \tau. \tag{2.11}$$

Applying Lemma 1.1, we obtain

$$\lim_{\nu\to\infty} \|\Theta\zeta_\nu - \xi_\nu\| = 0. \tag{2.12}$$

By the definition of $\zeta_{\nu+1}$, we have

$$
\begin{aligned}
\|\zeta_{\nu+1} - \Theta\zeta_\nu\| &= \|\alpha_\nu\Theta\zeta_\nu + (1-\alpha_\nu)\xi_\nu - \Theta\zeta_\nu\| \\
&\leq (1-\alpha_\nu)\|\xi_\nu - \Theta\zeta_\nu\| \\
&\leq (1-\delta)\|\xi_\nu - \Theta\zeta_\nu\|. \qquad (2.13)
\end{aligned}
$$

Taking $\lim_{\nu\to\infty}$ on both sides of above inequality and from (2.12), we obtain

$$\lim_{\nu\to\infty} \|\zeta_{\nu+1} - \Theta\zeta_\nu\| = 0. \tag{2.14}$$

From the triangle inequality, we have $\|\zeta_{\nu+1}-\zeta^*\| \leq \|\zeta_{\nu+1}-\Theta\zeta_\nu\|+\|\Theta\zeta_\nu-\xi_\nu\|+\|\xi_\nu-\zeta^*\|$. Taking $\liminf_{\nu\to\infty}$ of the above inequality and from (2.12) and (2.14), we obtain

$$\tau \leq \liminf_{\nu\to\infty} \|\xi_\nu - \zeta^*\|, \tag{2.15}$$

from (2.10) and (2.15), we have

$$\lim_{\nu\to\infty} \|\xi_\nu - \zeta^*\| = \tau, \tag{2.16}$$

which gives that

$$\lim_{\nu\to\infty} \left\|\frac{\beta_\nu}{1-\alpha_\nu}(\Phi\zeta_\nu - \zeta^*) + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}(\Psi\zeta_\nu - \zeta^*)\right\| = \tau, \tag{2.17}$$

Using Lemma 1.1, we obtain

$$\lim_{\nu\to\infty} \|\Phi\zeta_\nu - \Psi\zeta_\nu\| = 0. \tag{2.18}$$

From the triangle inequality, we have

$$\|\Theta\zeta_\nu - \Phi\zeta_\nu\| \leq \|\Theta\zeta_\nu - \xi_\nu\| + \|\xi_\nu - \Phi\zeta_\nu\|$$

$$\begin{aligned} &\leq \|\Theta\zeta_\nu - \xi_\nu\| + \frac{(1-\alpha_\nu-\beta_\nu)}{1-\alpha_\nu}\|\Psi\zeta_\nu - \Phi\zeta_\nu\|. \\ &\leq \|\Theta\zeta_\nu - \xi_\nu\| + (1-\frac{\delta}{1-\delta})\|\Psi\zeta_\nu - \Phi\zeta_\nu\|. \end{aligned}$$

Taking $\lim_{\nu\to\infty}$ in the above inequality, from (2.12) and (2.18), we obtain

$$\lim_{\nu\to\infty}\|\Theta\zeta_\nu - \Phi\zeta_\nu\| = 0. \tag{2.19}$$

Again, from the triangle inequality, we have

$$\|\Theta\zeta_\nu - \Psi\zeta_\nu\| \leq \|\Theta\zeta_\nu - \Phi\zeta_\nu\| + \|\Phi\zeta_\nu - \Psi\zeta_\nu\|.$$

Taking $\lim_{\nu\to\infty}$ in the above inequality, from (2.18) and (2.19), we have

$$\lim_{\nu\to\infty}\|\Theta\zeta_\nu - \Psi\zeta_\nu\| = 0. \tag{2.20}$$

From Condition (P) and applying $\lim_{\nu\to\infty}$ as well as from (2.19) and (2.20), we obtain

$$\lim_{\nu\to\infty}\|\Theta\zeta_\nu - \zeta_\nu\| = 0. \tag{2.21}$$

Again, by using the triangle inequality,

$$\|\zeta_\nu - \Phi\zeta_\nu\| \leq \|\zeta_\nu - \Theta\zeta_\nu\| + \|\Theta\zeta_\nu - \Phi\zeta_\nu\|. \tag{2.22}$$

Taking $\lim_{\nu\to\infty}$ in the above inequality and from (2.19) and (2.21), we obtain

$$\lim_{\nu\to\infty}\|\Phi\zeta_\nu - \zeta_\nu\| = 0. \tag{2.23}$$

By the triangle inequality, we have

$$\|\zeta_\nu - \Psi\zeta_\nu\| \leq \|\zeta_\nu - \Theta\zeta_\nu\| + \|\Theta\zeta_\nu - \Psi\zeta_\nu\|. \tag{2.24}$$

Taking $\lim_{\nu\to\infty}$ in the above inequality and from (2.20) and (2.21), we obtain

$$\lim_{\nu\to\infty}\|\Psi\zeta_\nu - \zeta_\nu\| = 0. \tag{2.25}$$

□

Lemma 2.3. *Let $\mathcal{M} \neq \emptyset$ be a closed and convex subset of a uniformly convex Banach space $\mathcal{W}$. Suppose that $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$ are nonexpansive mappings and that $\{\zeta_\nu\}$ is defined by the iterative process (1.5). Then, for any $e_1, e_2 \in F$,*

$$\lim_{\nu\to\infty}\left\|t\zeta_\nu + (1-t)e_1 - e_2\right\| \quad \textit{exists for every } t \in [0,1].$$

Proof. According to Lemma 2.1, $\lim_{\nu\to\infty}\|\zeta_\nu - e\|$ exists for every $e \in F$, and so $\{\zeta_\nu\}$ is bounded. Therefore, there is a real number $\vartheta > 0$ such that $\{\zeta_\nu\} \subseteq \overline{\mathcal{B}_\vartheta(0)} \cap \mathcal{M}$, such that $\varsigma \neq \emptyset$ is a closed, bounded, and convex subset of $\mathcal{M}$. Select $\alpha_\nu(t) = \|t\zeta_\nu + (1-t)e_1 - e_2\|$. Notice that $\alpha_\nu(0) = \|e_1 - e_2\|$ and $\alpha_\nu(1) = \|\zeta_\nu - e_2\|$. Moreover, by Lemma 2.1, the limit $\lim_{\nu\to\infty}\alpha_\nu(1) = \lim_{\nu\to\infty}\|\zeta_\nu - e_2\|$ exists. Define $\varpi_\nu : \varsigma \to \varsigma$ for every $\nu \in \mathbb{N}$ by

$$\varpi_\nu\zeta = \alpha_\nu\Theta\zeta + \beta_\nu\Phi\zeta + (1 - \alpha_\nu - \beta_\nu)\Psi\zeta_\nu, \quad \text{for all } \zeta \in \varsigma.$$

It is easy to verify that

$$\|\varpi_\nu\zeta - \varpi_\nu\xi\| \leq \|\zeta - \xi\|, \quad \text{for all } \zeta, \xi \in \varsigma.$$

Set

$$\mathcal{R}_{\nu,\mu} = \varpi_{\nu+\mu-1}\varpi_{\nu+\mu-2}\ldots\varpi_\nu,$$

and

$$\eta_{\nu,\mu} = \|\mathcal{R}_{\nu,\mu}(t\zeta_\nu + (1-t)e_1) - (t\mathcal{R}_{\nu,\mu}\zeta_\nu + (1-t)e_1)\|, \quad \text{for all } \nu, \mu \in \mathbb{N}.$$

Then, it follows that $\|\mathcal{R}_{\nu,\mu}\zeta - \mathcal{R}_{\nu,\mu}\xi\| \leq \|\zeta - \xi\|$ for all $\zeta, \xi \in \varsigma$, $\mathcal{R}_{\nu,\mu}\zeta_\nu = \zeta_{\nu+\mu}$ and $\mathcal{R}_{\nu,\mu}e = e$ for all $e \in F$. By Lemma 1.3, there exists a strictly increasing continuous function $\wp : [0,\infty) \to [0,\infty)$ with $\wp(0) = 0$ such that

$$\begin{aligned}\wp(\eta_{\nu,\mu}) &\leq \|\zeta_\nu - e_1\| - \|\mathcal{R}_{\nu,\mu}\zeta_\nu - \mathcal{R}_{\nu,\mu}e_1\| \\ &= \|\zeta_\nu - e_1\| - \|\zeta_{\nu+\mu} - e_1\|.\end{aligned}$$

Since $\lim_{\nu\to\infty}\|\zeta_\nu - e\|$ exists for all $e \in F$, then $\wp(\eta_{\nu,\mu}) \to 0$ as $\nu \to \infty$. Hence, $\eta_{\nu,\mu} \to 0$ for all $\mu \in \mathbb{N}$ as $\nu \to \infty$. Finally, from the inequality

$$\begin{aligned}\alpha_{\nu+\mu}(t) &= \|t\zeta_{\nu+\mu} + (1-t)e_1 - e_2\| \\ &\leq \eta_{\nu,\mu} + \|\mathcal{R}_{\nu,\mu}(t\zeta_\nu + (1-t)e_1) - e_2\| \\ &= \eta_{\nu,\mu} + \|\mathcal{R}_{\nu,\mu}(t\zeta_\nu + (1-t)e_1) - \mathcal{R}_{\nu,\mu}e_2\| \\ &\leq \eta_{\nu,\mu} + \|t\zeta_\nu + (1-t)e_1 - e_2\| \\ &= \eta_{\nu,\mu} + \alpha_\nu(t),\end{aligned}$$

consequently

$$\limsup_{\mu\to\infty}\alpha_{\nu+\mu}(t) \leq \limsup_{\mu\to\infty}\eta_{\nu,\mu} + \alpha_\nu(t).$$

That is,

$$\limsup_{\mu\to\infty}\alpha_\mu(t) \leq \liminf_{\nu\to\infty}\alpha_\nu(t),$$

so $\lim_{\nu\to\infty}\|t\zeta_\nu + (1-t)e_1 - e_2\|$ exists for all $t \in [0,1]$. □

3. Weak and strong convergence

We now establish results for approximating common fixed points of the mappings Θ, Φ, and Ψ through the weak convergence of $\{\zeta_\nu\}$ defined in (1.5). Our first theorem employs the Opial condition to guarantee weak convergence, while the second utilizes the *KKP* to obtain weak convergence.

Theorem 3.1. *Let $\mathcal{M}, \Theta, \Phi, \Psi$, $\{\zeta_\nu\}$ be as in Lemma 2.2 and $\mathcal{W}$ be a uniformly convex Banach space that satisfies Opial's condition; then, $\{\zeta_\nu\}$ weakly converges to a point in $F \neq \emptyset$.*

Proof. Let $\zeta^* \in F$. Then, $\lim_{\nu\to\infty} \|\zeta_\nu - \zeta^*\|$ exists, as established in Lemma 2.1. We show that there is only one weak subsequential limit in F for $\{\zeta_\nu\}$. Given that $\{\zeta_\nu\}$ is bounded in a uniformly convex Banach space $\mathcal{W}$, $\{\zeta_{\nu_i}\}$ and $\{\zeta_{\nu_j}\}$ are two convergent subsequences of $\{\zeta_\nu\}$. Let the subsequences $\{\zeta_{\nu_i}\}$ and $\{\zeta_{\nu_j}\}$ have weak limits $e_1 \in \mathcal{M}$ and $e_2 \in \mathcal{M}$, respectively. From Lemma 2.2, $\lim_{\nu\to\infty} \|\zeta_\nu - \Theta\zeta_\nu\| = 0$ and $I - \Theta$ is demiclosed with respect to zero by Lemma 1.2. So, we obtain $\Theta e_1 = e_1$. Similarly, $\Phi e_1 = e_1$ and $\Psi e_1 = e_1$. Again, in the same way, we can show that $e_2 \in F$.

Now we establish the uniqueness. For this, assume that $e_1 \neq e_2$. According to Opial's condition, we obtain

$$\begin{aligned}
\lim_{\nu\to\infty} \|\zeta_\nu - e_1\| &= \lim_{\nu_i\to\infty} \|\zeta_{\nu_i} - e_1\| \\
&< \lim_{\nu_i\to\infty} \|\zeta_{\nu_i} - e_2\| \\
&= \lim_{\nu\to\infty} \|\zeta_\nu - e_2\| \\
&= \lim_{\nu_j\to\infty} \|\zeta_{\nu_j} - e_2\| \\
&< \lim_{\nu_j\to\infty} \|\zeta_{\nu_j} - e_1\| \\
&= \lim_{\nu\to\infty} \|\zeta_\nu - e_1\|.
\end{aligned}$$

This is inconsistent with the said assumption. Therefore, $\{\zeta_\nu\}$ weakly converges to a point in F. □

Theorem 3.2. *Assume that the dual $\mathcal{W}^*$ of a uniformly convex Banach space $\mathcal{W}$ possesses the KKP. Let $\mathcal{M}, \Theta, \Phi, \Psi$, $\{\zeta_\nu\}$ be as in Lemma 2.2. Then, $\{\zeta_\nu\}$ weakly converges to a point in $F \neq \emptyset$.*

Proof. Since $\{\zeta_\nu\}$ is bounded and $\mathcal{W}$ is reflexive, there exists a subsequence $\{\zeta_{\nu_i}\}$ of ζ_ν that converges weakly to some point $e \in \mathcal{M}$. From Lemma 2.2, we obtain

$$\lim_{i\to\infty} \|\zeta_{\nu_i} - \Theta\zeta_{\nu_i}\| = 0, \quad \lim_{i\to\infty} \|\zeta_{\nu_i} - \Phi\zeta_{\nu_i}\| = 0 \quad \text{and} \quad \lim_{i\to\infty} \|\zeta_{\nu_i} - \Psi\zeta_{\nu_i}\| = 0.$$

This implies that $e \in F$.

Assume that $\{\zeta_{\nu_k}\}$ is another subsequence of $\{\zeta_\nu\}$ that converges weakly to a point $e_0 \in \mathcal{M}$ in order to establish that the sequence $\{\zeta_\nu\}$ weakly converges to e. Then, by Lemmas 2.2, both e and $e_0 \in \varpi \cap F$, where $\varpi = \omega_w(\{\zeta_\nu\})$. By Lemma 2.3, the limit $\lim_{\nu\to\infty} \|t\zeta_\nu + (1-t)e - e_0\|$ exists for every $t \in [0,1]$. Lemma 1.4 then yields $e = e_0$. Hence, $\{\zeta_\nu\}$ converges weakly to $e \in F$. □

The following result establishes a strong convergence theorem in the context of real Banach spaces.

Theorem 3.3. *Let $\mathcal{W}$ be a real Banach space, $\mathcal{M}$ a closed convex subset of $\mathcal{W} \neq \emptyset$, and $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$ be nonexpansive mappings. Assume $\{\zeta_\nu\}$ is the sequence defined by (1.5). Then, $\{\zeta_\nu\}$ strongly converges to a point in $F \neq \emptyset$ if and only if*

$$\liminf_{\nu\to\infty} d(\zeta_\nu, F) = 0,$$

where $d(\zeta, F) = \inf\{\|\zeta - e\| : e \in F\}$.

Proof. Necessity is obvious. For sufficiency, assume that $\liminf_{\nu\to\infty} d(\zeta_\nu, F) = 0$. From Lemma 2.1 we have, for any $e \in F$,

$$\|\zeta_{\nu+1} - e\| \leq \|\zeta_\nu - e\|,$$

which implies

$$d(\zeta_{\nu+1}, F) \leq d(\zeta_\nu, F).$$

Thus, $\{d(\zeta_\nu, F)\}$ is nonincreasing and bounded below, so $\lim_{\nu\to\infty} d(\zeta_\nu, F)$ exists. By the hypothesis, this limit must be zero, i.e.,

$$\lim_{\nu\to\infty} d(\zeta_\nu, F) = 0.$$

we now prove that $\{\zeta_\nu\}$ is a Cauchy sequence in $\mathcal{M}$. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{\nu\to\infty} d(\zeta_\nu, F) = 0$, there exists an integer ν_0 such that

$$d(\zeta_\nu, F) < \frac{\varepsilon}{2} \quad \text{for all } \nu \geq \nu_0.$$

In particular,

$$\inf\{\|\zeta_{\nu_0} - e\| : e \in F\} < \frac{\varepsilon}{2},$$

so we can choose $e^* \in F$ satisfying

$$\|\zeta_{\nu_0} - e^*\| < \frac{\varepsilon}{2}.$$

For any $\mu, \nu \geq \nu_0$, we obtain

$$\begin{aligned}\left\|\zeta_{\nu+\mu} - \zeta_\nu\right\| &\leq \left\|\zeta_{\nu+\mu} - e^*\right\| + \|\zeta_\nu - e^*\| \\ &\leq 2\left\|\zeta_{\nu_0} - e^*\right\| \\ &< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.\end{aligned}$$

This implies that $\{\zeta_\nu\}$ is a Cauchy sequence in the closed subset $\mathcal{M}$ of the Banach space $\mathcal{W}$, and hence converges strongly to some point $\zeta^* \in \mathcal{M}$. Since $\lim_{\nu\to\infty} d(\zeta_\nu, F) = 0$ and F is closed, it follows that $\zeta^* \in F$. Consequently, $\{\zeta_\nu\}$ strongly converges to a point in F. □

Condition (A') was first introduced by Khan and Fukhar-ud-din [9], and in their later work, they offered an enhanced version.

We now introduce the following adaptation for the three mappings and call it Condition (B).

Let, $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$ be three mappings defined on a subset $\mathcal{M} \neq \emptyset$ of a Banach space. We say that Θ, Φ, and Ψ satisfy Condition (B) if there is a nondecreasing function $\sigma : [0,\infty) \to [0,\infty)$ with $\sigma(0) = 0$ and $\sigma(\vartheta) > 0$ for all $\vartheta \in (0,\infty)$, such that for every $\zeta \in \mathcal{M}$, either

$$\|\zeta - \Theta\zeta\| \geq \sigma(d(\zeta, F)) \quad \text{or} \quad \|\zeta - \Phi\zeta\| \geq \sigma(d(\zeta, F)) \quad \text{or} \quad \|\zeta - \Psi\zeta\| \geq \sigma(d(\zeta, F)).$$

We shall employ Condition (B) to establish the strong convergence of ζ_ν defined by (1.5). It is important to note that for nonexpansive mappings $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$, Condition (B) is strictly weaker than requiring the compactness of $\mathcal{M}$, making it applicable in broader settings.

Theorem 3.4. *Let $\mathcal{W}$ be a uniformly convex Banach space, $\mathcal{M}$ a nonempty, closed, and convex subset of $\mathcal{W}$, and let $\{\zeta_\nu\}$ be the sequence generated by (1.5). Let $\Theta, \Phi, \Psi : \mathcal{M} \to \mathcal{M}$ be nonexpansive mappings satisfying Condition (B). Then, $\{\zeta_\nu\}$ strongly converges to a point in $F \neq \emptyset$.*

Proof. By Lemma 2.1, the limit $\lim_{\nu\to\infty}\|\zeta_\nu - \zeta^*\|$ exists for every $\zeta^* \in F$. Let us denote this limit by $\tau \geq 0$. If $\tau = 0$, there is nothing to prove.

Assume now that $\tau > 0$. From the inequality $\|\zeta_{\nu+1} - \zeta^*\| \leq \|\zeta_\nu - \zeta^*\|$, valid for any $\zeta^* \in F$, we obtain

$$\inf_{\zeta^*\in F}\|\zeta_{\nu+1} - \zeta^*\| \leq \inf_{\zeta^*\in F}\|\zeta_\nu - \zeta^*\|,$$

which means $d(\zeta_{\nu+1}, F) \leq d(\zeta_\nu, F)$, and so $\lim_{\nu\to\infty} d(\zeta_\nu, F)$ exists.

By applying Condition (B), either

$$\lim_{\nu\to\infty}\sigma(d(\zeta_\nu, F)) \leq \lim_{\nu\to\infty}\|\zeta_\nu - \Theta\zeta_\nu\| = 0 \quad \text{for all } \nu,$$

or

$$\lim_{\nu\to\infty}\sigma(d(\zeta_\nu, F)) \leq \lim_{\nu\to\infty}\|\zeta_\nu - \Phi\zeta_\nu\| = 0 \quad \text{for all } \nu,$$

or

$$\lim_{\nu\to\infty}\sigma(d(\zeta_\nu, F)) \leq \lim_{\nu\to\infty}\|\zeta_\nu - \Psi\zeta_\nu\| = 0 \quad \text{for all } \nu.$$

Therefore, from all these cases, we have

$$\lim_{\nu\to\infty}\sigma(d(\zeta_\nu, F)) = 0.$$

Since σ is nondecreasing and $\sigma(\vartheta) > 0$ for every $\vartheta > 0$, it follows that

$$\lim_{\nu\to\infty} d(\zeta_\nu, F) = 0.$$

Suppose, for contradiction, that $\lim_{\nu\to\infty} d(\zeta_\nu, F) \neq 0$. Then, σ would remain strictly positive, contradicting that $\lim_{\nu\to\infty}\sigma(d(\zeta_\nu, F)) = 0$. Thus, the monotonicity and positivity of σ guarantee that $d(\zeta_\nu, F)$ tends to zero. □

4. Numerical results

Now, we present numerical results in the context of a uniformly convex Banach space to show the efficacy and applicability of our Algorithm 1.5. Define $\Theta, \Phi, \Psi : \mathbb{R} \to \mathbb{R}$ as follows: $\Theta\zeta = 1 + \frac{1}{2}(\zeta - 1)$, $\Phi\zeta = \frac{3}{10} + \frac{7}{10}\zeta$, and $\Psi\zeta = 2 - \zeta$, $\forall \zeta \in \mathbb{R}$, and are nonexpansive with common fixed point 1. Taking $\alpha_\nu = \frac{1}{(\nu+1)^{\frac{3}{2}}+2}$ and $\beta_\nu = \frac{1}{(\nu+2)^{\frac{6}{5}}+3}$, we now consider the following four cases for our numerical results:

$$\text{Case 1: } \xi_1 = 2,\ \nu = 15; \quad \text{Case 2: } \xi_1 = 3,\ \nu = 10;$$
$$\text{Case 3: } \xi_1 = 4,\ \nu = 20; \quad \text{Case 4: } \xi_1 = 5,\ \nu = 25.$$

The numerical computations shown in Figure 1 (cases (a) and (b)) and Figure 2 (cases (c) and (d)) illustrate the convergence behavior of Liu et al. algorithm [13], Khan et al. algorithm [10], and the proposed algorithm 1.5. In each case, all algorithms converge to the common fixed point 1, while the proposed algorithm 1.5 shows faster convergence as compared to the Liu et al. algorithm [13] and Khan et al. algorithm [10]. This demonstrates the improved efficiency of the proposed algorithm 1.5 under different initializations.

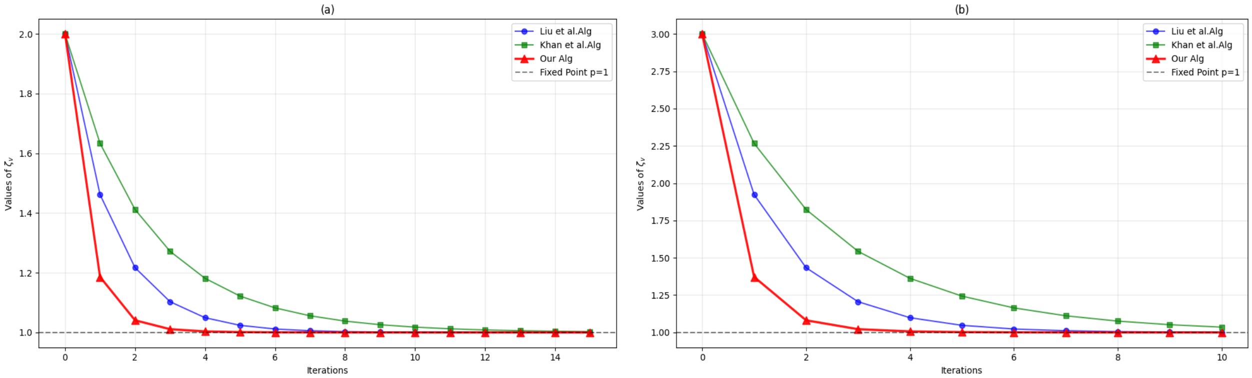

Figure 1. The convergence behavior of the Liu et al. algorithm [13], the Khan et al. algorithm [10], and the proposed algorithm for two different initial points: (a) $\zeta_1 = 2$, and (b) $\zeta_1 = 3$. All algorithms converge monotonically to the common fixed point 1. The proposed algorithm converges faster in both cases, reaching the fixed point in iteration 4, whereas the Liu et al. and Khan et al. algorithms require up to 8 or 9 iterations. This highlights the enhanced efficiency of the proposed algorithm.

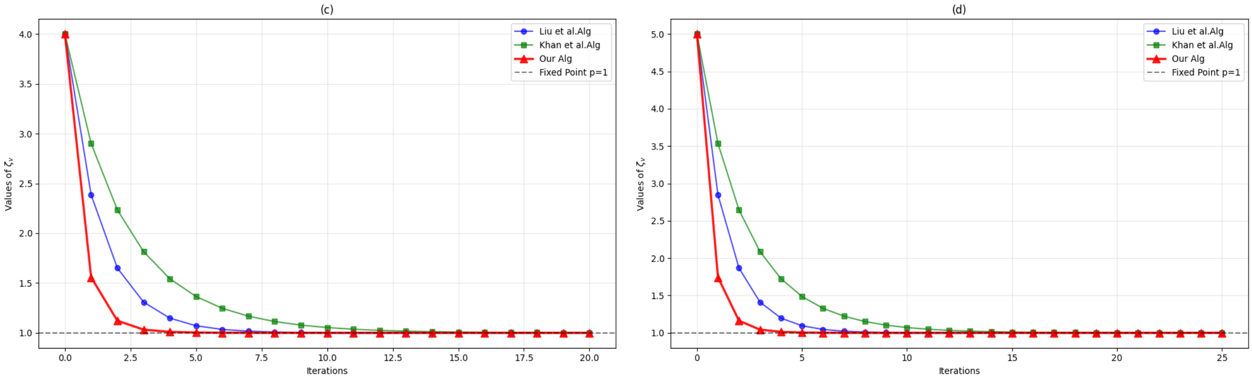

Figure 2. The convergence performance of the Liu et al. algorithm [13], the Khan et al. algorithm [10], and the proposed algorithm for two different initial points: (c) $\zeta_1 = 4$, and (d) $\zeta_1 = 5$. In both cases (c) and (d), our algorithm converges fast. This highlights the enhanced efficiency of the proposed algorithm.

5. Conclusions

This article presents a comprehensive study on the approximation of common fixed points for three nonexpansive mappings via a novel one-step iterative approach in uniformly convex Banach spaces. The main results establish both weak and strong convergence under appropriate geometric conditions. Specifically, weak convergence is obtained when the space satisfies Opial's condition or possesses the KKP in its dual, while strong convergence is guaranteed under a newly introduced Condition (B). The theoretical findings are further substantiated by a numerical example that demonstrates the practical implementation and behavior of the proposed algorithm (1.5). These results provide a meaningful generalization of existing convergence theorems from two to three mappings and contribute a versatile iterative framework for multi-operator fixed-point theory.

Author contributions

Athar Abbas: Writing–original draft, methodology, and validation; Somayya Komal: Investigation and data curation; Muhammad Jabir Khan: Editing and Funding acquisition; Hafiz Fukhar-ud-din: Supervision, Conceptualization, editing, and methodology; Muhammad Aqeel Ahmad Khan: Formal analysis and validation.

Use of Generative-AI tools declaration

The authors confirm that no generative AI tools were used in the writing, analysis, or preparation of this manuscript.

Acknowledgments

This research was funded by the Jiangsu Province Excellent Postdoctoral Program under grant No. (2024ZB879).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Z.*, **100** (1967), 201–225. https://doi.org/10.1007/BF01109805
2. R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, *Israel J. Math.*, **32** (1979), 107–116. https://doi.org/10.1007/BF02764907
3. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse probl.*, **20** (2004), 103–120. https://doi.org/10.1088/0266-5611/20/1/006
4. C. Chidume, S. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, *Proc. Amer. Math. Soc.*, **129** (2001), 2359–2363. https://doi.org/10.1090/S0002-9939-01-06009-9
5. G. Das, Fixed points of quasinonexpansive mappings, *Indian J. Pure Appl. Math.*, **17** (1986), 1263–1269.
6. J. G. Falset, W. Kaczor, T. Kuczumow and S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal. Theor.*, **43** (2001), 377–401. https://doi.org/10.1016/S0362-546X(99)00200-X
7. H. Fukhar-ud-din, S. H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, *J. Math. Anal. Appl.*, **328** (2007), 821–829. https://doi.org/10.1016/j.jmaa.2006.05.068

8. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.1090/S0002-9939-1974-0336469-5
9. S. H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, *Nonlinear Anal. Theor.*, **61** (2005), 1295–1301. https://doi.org/10.1016/j.na.2005.01.081
10. S. H. Khan, M. Abbas, A. R. Khan, Common fixed points of two nonexpansive mappings by a new one-step iteration process, *Iran. J. Sci. Technol. A.*, **33** (2009), 249–257.
11. S. H. Khan, W. Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, *Sci. Math. Jpn.*, **53** (2001), 143–148.
12. T. H. Kim, H. K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal. Theor.*, **61** (2005), 51–60. https://doi.org/10.1016/j.na.2004.11.011
13. Z. Liu, C. Feng, J. S. Ume, S. M. Kang, Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings, *Taiwan. J. Math.*, **11** (2007), 27–42.
14. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.2307/2032162
15. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597. https://doi.org/10.1090/S0002-9904-1967-11761-0
16. J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Aust. Math. Soc.*, **43** (1991), 153–159. https://doi.org/10.1017/S0004972700028884
17. W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, *J. Convex Anal.*, **5** (1998), 45–56.
18. D. J. Wen, Weak and strong convergence theorems of G-monotone nonexpansive mapping in Banach spaces with a graph, *Numer. Func. Anal. Opt.*, **40** (2019), 163–177. https://doi.org/10.1080/01630563.2018.1513031

AIMS Press