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Research article

A class of bi-univalent functions subordinate to the β-Gregory function with an application

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Abstract: In this investigation, a generalized form of the Gregory function, called the β-Gregory function, is derived. This function includes several well-known analytic functions as special cases. We then consider a new symmetric class of analytic and bi-univalent functions related to the β-Gregory function. The geometric and analytic properties of this class are investigated, and inclusion relationships with other known subclasses are established under suitable sufficient conditions. Furthermore, we derive accurate estimates of the coefficients of this class and discuss the associated Fekete-Szegö inequality, showing how the results obtained contribute to generalizing and improving several previous works in bi-univalent functions.

Keywords: β-Gregory coefficients; bi-univalent functions; Fekete-Szegö inequality
Mathematics Subject Classification: 30C45, 30C50

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1. Introduction and preliminaries

We denote by the symbol $\mathcal{A}$ the main set that includes all the analytic functions h defined as

$$h(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j \zeta^j \tag{1.1}$$

in the open unit disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ (see [1]).

We also refer to the class of Carathéodory function $\mathbb{P}$ as $\mathfrak{p}(\zeta)$ with $\Re(\mathfrak{p}(\zeta)) > 0$, and which has the following form:

$$\mathfrak{p}(\zeta) = 1 + \sum_{j=1}^{\infty} \mathfrak{p}_j \zeta^j. \tag{1.2}$$

The class $\mathcal{E}$ indicates Schwarz function $\varpi(\zeta)$, which hold the cases $\varpi(0) = 0, |\varpi(\zeta)| < 1$, and

$$\varpi(\zeta) = \sum_{j=1}^{\infty} s_j \zeta^j. \tag{1.3}$$

The analytic function $h(\zeta)$ is subordinated to another analytic function $q(\zeta)$ if a Schwarz function $\varpi(\zeta)$ exists such that $h(\zeta) = q(\varpi(\zeta))$ for all $\varpi \in \mathcal{E}$, denoted by $h(\zeta) \prec q(\zeta)$. Moreover, if the function $q(\zeta)$ is univalent in $\mathcal{E}$, the relation $h(\zeta) \prec q(\zeta)$ is equivalent to the conditions $h(0) = q(0)$ and the inclusion of the image (see [2, 3])

$$h(\mathcal{E}) \subset q(\mathcal{E}). \tag{1.4}$$

Sakaguchi [4] established the symmetric class of starlike function as follows:

$$\Re\left(\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)}\right) > 0, \quad (h \in \mathcal{A}, \zeta \in \mathbb{U}).$$

Moreover, Selvaraj and Selvakumaran [5] provided the symmetric class of convex function as

$$\Re\left(\frac{(\zeta h'(\zeta))'}{h'(\zeta) + h'(-\zeta)}\right) > 0, \quad (h \in \mathcal{A}, \zeta \in \mathbb{U}).$$

Nasr and Aouf [6] derived the following class of starlike functions

$$\Re\left(1 - \frac{1}{u}\left(\frac{\zeta h'(\zeta)}{h(\zeta)} - 1\right)\right) > 0, \quad (h \in \mathcal{A}, u \in \mathbb{C}\backslash\{0\}; \zeta \in \mathbb{U}).$$

Selvaraj et al. [7] introduced a class of starlike function related to the carathéodory function $\mathfrak{p}(\zeta)$ as follows:

$$1 - \frac{1}{u}\left(\frac{\zeta h'(\zeta)}{h(\zeta)} - 1\right) \prec \mathfrak{p}(\zeta), \quad (h \in \mathcal{A}, u \in \mathbb{C}\backslash\{0\}; \zeta \in \mathbb{U}).$$

Furthermore, Ravichandran [8] investigated the following class of starlike functions concerning to symmetric points:

$$\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)} \prec \mathfrak{p}(\zeta).$$

Khan et al. [9] studied the class of symmetric functions subordinated to Gregory coefficients as below:

$$\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)} \prec \mathcal{G}(\zeta) = \frac{\zeta}{\log(\zeta + 1)}.$$

Arif et al. [10] provided symmetric classes of Janowski function as follows:

$$1 + \frac{1}{u}\left(\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)} - 1\right) \prec \frac{1 + P_1\zeta}{1 + P_2\zeta}, \quad (u \in \mathbb{C}\backslash\{0\}; \zeta \in \mathbb{U})$$

and

$$1+\frac{1}{u}\left(\frac{2(\zeta h'(\zeta))'}{(h(\zeta)-h(-\zeta))'}-1\right)\prec\frac{1+P_1\zeta}{1+P_2\zeta},\quad (u\in\mathbb{C}\backslash\{0\};\zeta\in\mathbb{U}),$$

where $\frac{1+P_1\zeta}{1+P_2\zeta}$ $(-1\leqq P_2<P_1\leqq 1)$ is the Janowski function, see [11, 12].

The inverse of analytic-univalent function is given as

$$h^{-1}(h(\zeta))=\zeta,\quad (\zeta\in\mathbb{U})$$

and

$$h(h^{-1}(\omega))=\omega\quad \left(|\omega|<r_0(h);r_0(h)\geq\frac{1}{4}\right).$$

The Taylor form of h^{-1} is stated as

$$h^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots.$$

The function h is known as bi-univalent of the family Σ if it is an analytic function and both $h(\zeta)$ and $h^{-1}(\zeta)$ are univalent in $\mathbb{U}$. For further properties of this class of functions, see references [13–15]. Bilal et al. [16] studied Mathieu series and derived a class of Pascu analytic functions, deriving results related to the Fekete-Szegö inequality, radius property, and paritial sums.

Researchers have offered several estimates for the a_2 coefficient when studying the Σ family. Brannan and Clooney [17] suggested a value of no more than $a_2<\sqrt{2}$, while Levin [18] improved this estimate to $a_2<1.51$. Later, Netanyahu [19] demonstrated that $\max a_2=3/4$. Despite these findings, finding general limits for a_j $(j\geqq 3)$ and above remains unresolved.

Employing Jackson calculus, Kamble and Shrigan [20, 21] derived novelty classes of bi-univalent functions, focusing on finding limits for the differentiation coefficients $|a_2|$ and $|a_3|$. Meanwhile, Yusuf et al. [15] addressed the Fekete-Szegö inequality within certain classes. Srivastava et al. [22] investigated the second determinant of Hankel for different classes of this type of function. It is worth noting that the Fekete-Szegö inequality, introduced by Fekete and Szegö [23], is a fundamental problem related to the coefficients of functions belonging to the family Σ. The Fekete-Szegö inequality is given by

$$|a_3-\epsilon a_2^2|\leqslant 1+2e^{-2\epsilon/(1-\epsilon)},\quad \epsilon\in\mathbb{R}.$$

In recent years, numerous studies have focused on developing classes of analytic and bivalent functions in diverse ways. Tayyah et al. [24] linked Ma-Minda functions to a geometric domain, while Hadi et al. [25] studied biunivalent functions using Lucas polynomial and obtained coefficient estimates. Hadi et al. [26] also presented a class of convex harmonic functions using a generalization of the Mittag-Leffler function. Atshan et al. [27] investigated Fekete-Szegö inequalities using special multipliers, and Tayyah et al. [28] obtained sharp bounds for classes related to the exponential function. El-Ityan et al. [29] also investigated a third-order Henkel determinant within a geometric framework.

Gregory coefficients are also known by several other names, including inverse logarithmic numbers, Bernoulli numbers of the second kind, and Cauchy numbers of the first kind (see [30, 31]). These coefficients represent a sequence of decreasing rational numbers that alternate in sign. They naturally arise in the Maclaurin series of the function $\frac{\zeta}{\log(\zeta+1)}$, which can be expressed as follows:

$$\mathcal{G}(\zeta) = \frac{\zeta}{\log(\zeta+1)} = \sum_{j=0}^{\infty} \mathrm{g}_j \zeta^j = 1 + \frac{1}{2}\zeta - \frac{1}{12}\zeta^2 + \frac{1}{24}\zeta^3 + \cdots, \quad \zeta \in \mathbb{U}.$$

It is convergent because the function $\frac{\zeta}{\log(\zeta+1)}$ is analytic about zero, and its nearest isolate point is located at $\zeta = -1$ at a distance of 1, so the radius of convergence is 1, and therefore the series converges for every $|\zeta| < 1$.

The coefficients derived from this function are attributed to James Gregory, who first presented them in 1670 in his research on numerical integration [32]. These coefficients are important tools in advanced mathematics, appearing when writing certain functions, such as the natural logarithm, as infinite series. Their significance lies in their effective role in improving methods for approximating complex functions, making them highly valuable in applications of physics, engineering, and computer science. These coefficients also contribute to the relationship between discrete and continuous sequences, which helps in a deeper understanding of convergence concepts and the analytical behavior of functions. Thus, their impact extends to both theoretical and applied aspects, despite their precise nature. Since then, these coefficients have garnered continuous attention across various periods, with numerous mathematicians studying and analyzing them, contributing to their development and deepening the understanding of their properties. This has resulted in their prominent place in the work of many researchers [31, 33, 34]. Even today, these coefficients remain highly regarded in modern mathematical fields.

We present a generalization of Gregory coefficients by extending their formulation to a framework based on a real coefficient, thus enabling a more comprehensive analytical structure. This generalization is linked to a symmetric class with a complex parameter to highlight the impact of this coefficient on the geometric and analytical properties of functions. Gregory coefficients represent a special case that is recovered when the coefficient is fixed to appropriate real values, thus confirming the consistency and originality of the generalization.

Now, we propose β-Gregory coefficients as

$$\mathcal{G}_\beta(\zeta) = 1 + \beta\left(\frac{\zeta}{\log(\zeta+1)} - 1\right), \quad (0 < \beta \leqq 1; \zeta \in \mathbb{U}). \tag{1.5}$$

The generating function χ_j is as follows:

$$\mathcal{G}_\beta(\zeta) = 1 + \beta\left(\frac{\zeta}{\log(\zeta+1)} - 1\right) = \sum_{j=0}^{\infty} \chi_j \zeta^j = 1 + \chi_1\zeta - \chi_2\zeta^2 + \chi_3\zeta^3 + \cdots, \tag{1.6}$$

where

$$\chi_1 = \frac{\beta}{2},\ \chi_2 = \frac{\beta}{12},\ \chi_3 = \frac{\beta}{24}, \cdots.$$

Now, since

$$\frac{\zeta}{\log(\zeta+1)} = \sum_{j=0}^{\infty} c_j\zeta^j \Longrightarrow \frac{\zeta}{\log(\zeta+1)} - 1 = \sum_{j=1}^{\infty} c_j\zeta^j.$$

Then,

$$1+\beta\left(\frac{\zeta}{\log(\zeta+1)}-1\right)=1+\beta\sum_{j=1}^{\infty}c_j\zeta^j=1+\sum_{j=1}^{\infty}\beta c_j\zeta^j=\sum_{j=0}^{\infty}\gamma_j\zeta^j,$$

where $\gamma_0 = 1$ and $\gamma_j = \beta c_j$.

Therefore,

(1) β-Gregory coefficients related to β-Bernoulli of the second kind, when $\chi_j = \frac{\gamma_j}{j!}$.

(2) If $\beta = 1$, β-Gregory coefficients can be reduced to Gregory coefficients.

The analytic function $\mathcal{G}_\beta(\zeta)$ defined in $\mathbb{U}$ fulfills the conditions Ma-Minda function $\phi(\zeta)$ (see [35]). We now illustrate that the function $\mathcal{G}_\beta(\zeta)$ possesses a set of fundamental geometric and analytic properties.

(1) $\mathcal{G}_\beta(0) = 1$ and $\mathcal{G}'_\beta(0) = \frac{\beta}{2} > 0,\ (0 < \beta \leqq 1)$.

(2) For $\mathcal{G}_\beta(\zeta)$, we attain by choosing $u = \log(\zeta + 1)$ that

$$\begin{aligned}W(\zeta) &:= \Re\left(\frac{\zeta\mathcal{G}'_\beta(\zeta)}{\mathcal{G}_\beta(\zeta)-1}\right)=\Re\left(\frac{\zeta\log(\zeta+1)-\frac{\zeta^2}{\zeta+1}}{\left(\frac{\zeta}{\log(\zeta+1)}-1\right)\log^2(\zeta+1)}\right)=\Re\left(\frac{\zeta u-\frac{\zeta^2}{\zeta+1}}{\left(\frac{\zeta}{u}-1\right)u^2}\right)\\ &=\Re\left(\frac{\zeta u-\frac{\zeta^2}{\zeta+1}}{(\zeta-u)u}\right)=\Re\left(\frac{\zeta-\frac{\zeta^2}{(\zeta+1)u}}{(\zeta-u)}\right)=\Re\left(\frac{P(\zeta)}{\zeta-\log(\zeta+1)}\right).\end{aligned}$$

Since $\Re(\log(\zeta + 1)) > 0$ and $\Re(\zeta - \log(\zeta + 1)) > 0$ for all $|\zeta| < 1$, thus

$$W(\zeta)=\Re\left(\frac{\zeta\log(\zeta+1)-\frac{\zeta^2}{\zeta+1}}{\left(\frac{\zeta}{\log(\zeta+1)}-1\right)\log^2(\zeta+1)}\right)>0\quad(|\zeta|<1).$$

Therefore, the function is univalent starlike concerning to 1 (see Figure 1).

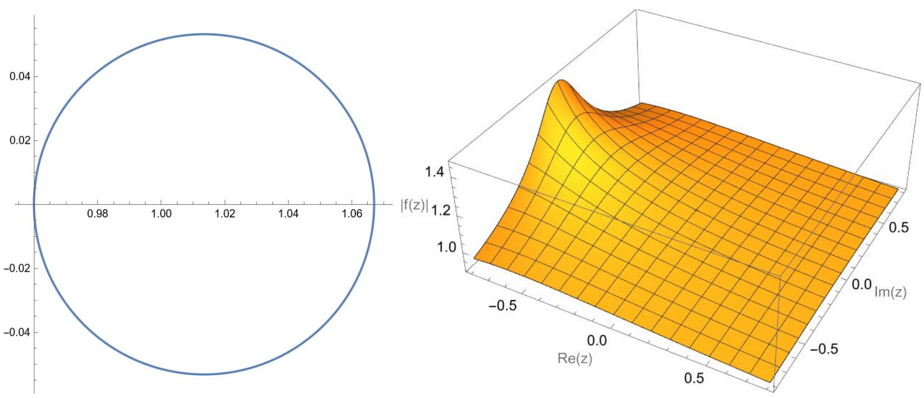

Figure 1. The image of $W(\zeta)$.

(3) From Figure 2, it becomes clear that the function $\mathcal{G}_\beta(\zeta)$ is symmetric about the x-axis.

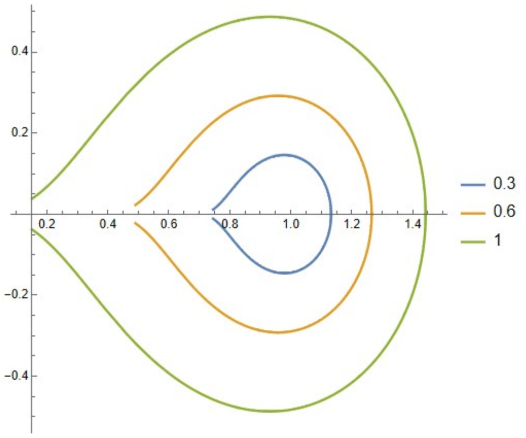

Figure 2. The image of $\mathbb{U}$ under $\mathcal{G}_\beta(\zeta)$ with different values of β.

Proposition 1 (Inclusion Property)**.** *Consider the analytic functions*

$$\Omega_1(\zeta) = \frac{2(2+\zeta)}{4+\zeta} = 1 + \frac{z}{4} - \frac{z^2}{16} + \frac{z^3}{64} - \frac{z^4}{256} + \cdots ,$$

$$\Omega_2(\zeta) = \sqrt{1+\frac{\zeta}{2}} = 1 + \frac{z}{4} - \frac{z^2}{32} + \frac{z^3}{128} - \frac{5z^4}{2048} + \cdots ,$$

$$\Omega_3(\zeta) = 1 + \frac{z}{3} - \frac{z^3}{9},$$

$$\Omega_4(\zeta) = 1 + \frac{8}{27}\zeta - \frac{2\zeta^5}{33},$$

$$\Omega_5(\zeta) = 1 + \frac{1}{5}\zeta - \frac{1}{15}\zeta^3.$$

It is evident that the familiar functions $\Omega_s(\zeta)$ satisfy the normalization condition $\Omega_s(0) = 1$, and the unit disk image $\mathbb{U}$ under transformation Ω_s lies within the range $\mathcal{G}_\beta(\mathbb{U})$ for each $s = 1, 2, \cdots, 5$. Since the function $\mathcal{G}_\beta(\mathbb{U})$ is univalent, the subordination relationship must be $\Omega_s(\zeta) \prec \mathcal{G}_\beta(\zeta)$, by using the subordination principle in (1.4) *and as shown in Figure 3.*

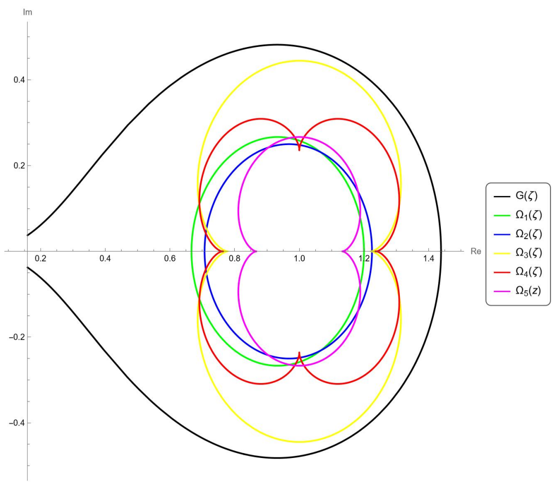

Figure 3. Plot boundaries of $\Omega_s(\mathbb{U})$ under $\mathcal{G}_\beta(\mathbb{U})$.

Given the increasing significance of the Gregory function in many recent studies, this manuscript proposes a new expansion of the Gregory coefficient function, which falls under the family of analytic functions with Ma-Minda properties. The inclusion properties of this function are also verified with several other families of analytic functions. Furthermore, we consider a new class of bi-univalent symmetric functions, including several relevant special cases. Based on this class, we derive upper limits for the coefficients $|a_2|$ and $|a_3|$, which in turn lead to the corresponding Fekete-Szegö inequality.

Definition 1.1. *The class $\mathcal{S}^*(\mathcal{G}_\beta)$ contains the function $h(\zeta)$ in* (1.1) *if*

$$\mathscr{L}^{\upsilon}_{\varpi,\varsigma}(h(\zeta)) = 1 + \frac{1}{\upsilon}\left(\frac{2\zeta h'(\zeta) + 2\varsigma\zeta^2 h''(\zeta)}{(1-\varsigma)(h(\zeta) - h(-\zeta)) + \varsigma\zeta(h'(\zeta) + h'(-\zeta))} - 1\right) \prec \mathcal{G}_\beta(\zeta), \tag{1.7}$$

where $0 \leqq \varsigma \leqq 1$, $\upsilon \in \mathbb{C}\backslash\{0\}$ and $\mathcal{G}_\beta(\zeta)$ is the generating of β-Gregory function in (1.5).

We now consider the subsequent class of bi-univalent functions.

Definition 1.2. *A function $h(\zeta)$, as in* (1.1)*, belongs to the class $\mathcal{BS}^*(\mathcal{G}_\beta)$ if*

$$1 + \frac{1}{\upsilon}\left(\frac{2\zeta h'(\zeta) + 2\varsigma\zeta^2 h''(\zeta)}{(1-\varsigma)(h(\zeta) - h(-\zeta)) + \varsigma\zeta(h'(\zeta) + h'(-\zeta))} - 1\right) \prec \mathcal{G}_\beta(\zeta) \tag{1.8}$$

and

$$1 + \frac{1}{\upsilon}\left(\frac{2\omega\hbar'(\omega) + 2\varsigma\omega^2 \hbar''(\omega)}{(1-\varsigma)(\hbar(\omega) - \hbar(-\omega)) + \varsigma\omega(\hbar'(\omega) + \hbar'(-\omega))} - 1\right) \prec \mathcal{G}_\beta(\omega), \tag{1.9}$$

with the assumption $0 \leqq \varsigma \leqq 1$, $\upsilon \in \mathbb{C}\backslash\{0\}$, and $\hbar = h^{-1}$.

Assuming the class of function $\mathcal{BS}^*(\mathcal{G}_\beta)$ defined above, assigning specific values of υ, ς, and β results in a number of special cases, which we discuss below.

Case 1: For $\upsilon = 1$, we get

$$\frac{2\zeta h'(\zeta) + 2\varsigma\zeta^2 h''(\zeta)}{(1-\varsigma)(h(\zeta) - h(-\zeta)) + \varsigma\zeta(h'(\zeta) + h'(-\zeta))} \prec \mathcal{G}_\beta(\zeta)$$

and

$$\frac{2\omega\hbar'(\omega) + 2\varsigma\omega^2 \hbar''(\omega)}{(1-\varsigma)(\hbar(\omega) - \hbar(-\omega)) + \varsigma\omega(\hbar'(\omega) + \hbar'(-\omega))} \prec \mathcal{G}_\beta(\omega).$$

Case 2: For $\upsilon = 1/2, \varsigma = 0$, we have

$$1 + 2\left(\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)} - 1\right) \prec \mathcal{G}_\beta(\zeta)$$

and

$$1 + 2\left(\frac{2\omega\hbar'(\omega)}{\hbar(\omega) - \hbar(-\omega)} - 1\right) \prec \mathcal{G}_\beta(\omega).$$

Case 3: For $\upsilon = 1, \varsigma = 0$, we have

$$\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)} \prec \mathcal{G}_\beta(\zeta)$$

and

$$\frac{2\omega\hbar'(\omega)}{\hbar(\omega)-\hbar(-\omega)} \prec \mathcal{G}_\beta(\omega).$$

Case 4: If $\upsilon = 1 = \varsigma$, then

$$\frac{2\zeta h'(\zeta)+2\zeta^2 h''(\zeta)}{\zeta(h'(\zeta)+h'(-\zeta))} \prec \mathcal{G}_\beta(\zeta)$$

and

$$\frac{2\omega\hbar'(\omega)+2\omega^2\hbar''(\omega)}{\omega(\hbar'(\omega)+\hbar'(-\omega))} \prec \mathcal{G}_\beta(\omega).$$

Case 5: If $\upsilon = \varsigma = 1$ and $\beta = 1$, then

$$\frac{2\zeta h'(\zeta)+2\zeta^2 h''(\zeta)}{\zeta(h'(\zeta)+h'(-\zeta))} \prec \mathcal{G}(\zeta)$$

and

$$\frac{2\omega\hbar'(\omega)+2\omega^2\hbar''(\omega)}{\omega(\hbar'(\omega)+\hbar'(-\omega))} \prec \mathcal{G}(\omega).$$

Case 6: If $\upsilon = e^{-ib}\cos(b)$ $(|b| < \frac{\pi}{2})$, $\varsigma = 0$, and $\beta = 1$, then

$$1+\frac{e^{ib}}{\cos(b)}\left(\frac{2\zeta h'(\zeta)}{h(\zeta)-h(-\zeta)}-1\right) \prec \mathcal{G}(\zeta)$$

and

$$1+\frac{e^{ib}}{\cos(b)}\left(\frac{2\omega\hbar'(\omega)}{\hbar(\omega)-\hbar(-\omega)}-1\right) \prec \mathcal{G}(\omega).$$

Case 7: If $\upsilon = e^{-ib}\cos(b)$ $(|b| < \frac{\pi}{2})$, $\varsigma = 0$, $\beta = 1$, and $h(-\zeta) = -h(\zeta)$, then

$$1+\frac{e^{ib}}{\cos(b)}\left(\frac{\zeta h'(\zeta)}{h(\zeta)}-1\right) \prec \mathcal{G}_\beta(\zeta)$$

and

$$1+\frac{e^{ib}}{\cos(b)}\left(\frac{\omega\hbar'(\omega)}{\hbar(\omega)}-1\right) \prec \mathcal{G}_\beta(\omega).$$

Example 1.1. *It is clear that the class $\mathcal{BS}^*(\mathcal{G}_\beta)$ in* (1.7) *contains two analytic functions $h_1(\zeta) = \frac{\zeta}{1-\varpi\zeta}$ and $h_2(\zeta) = 1+\kappa sinh(\zeta)$, hence*

$$\mathscr{L}^{\upsilon}_{\varpi,\varsigma}(h_1(\zeta)) \prec \mathcal{G}_\beta(\zeta),\ \ (|\varpi| < 1)$$

and

$$\mathscr{L}^{\upsilon}_{\varpi,\varsigma}(h_2(\zeta)) \prec \mathcal{G}_\beta(\zeta)\ (0 < \kappa < 0.35).$$

The following lemmas are fundamental tools for analyzing the estimation of initial coefficients and the Fekete-Szegö inequality in this study.

Lemma 1.1. *[36] Let $\mathfrak{p} \in \mathbb{P}$ be a Carathéodory function as in* (1.2). *Then, the result $|\mathfrak{p}_j| \leqq 2$ is sharp for $h(\zeta) = \frac{1+\zeta}{1-\zeta}$.*

Lemma 1.2. *[[37], Lemma 7, p.2] If $e, t \in R$ and $\zeta_1, \zeta_2 \in \mathbb{C}$, letting $|\zeta_1| < E$ and $|\zeta_2| < E$, then*

$$|(e+t)\zeta_1 + (e-t)\zeta_2| \le \begin{cases} 2|e|E, & for \quad |e| \ge |t| \\ 2|t|E, & for \quad |e| \le |t| \end{cases}. \tag{1.10}$$

2. Essential findings

This section begins by concluding the initial limits of the Taylor coefficients, which are the key to deriving Fekete-Szegö inequality. These estimations represent a pivotal step in determining the final form of the inequality, as well as in defining its analytic limits within the class under study.

Theorem 2.1. *If the function $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$, then*

$$|a_2| \leqq min\{Q_1, Q_2\}$$

and

$$|a_3| \leqq min\{L_1, L_2\},$$

where

$$Q_1 := \sqrt{\frac{3\beta|\upsilon|^2}{4|(3\beta(2\varsigma+1)\upsilon + 14(\varsigma+1)^2)|}},\ Q_2 := \frac{\beta|\upsilon|}{4(\varsigma+1)}$$

and

$$L_1 := \frac{\beta|\upsilon|}{8(2\varsigma+1)} + \frac{1}{4}Q_2^2,\ L_2 := \frac{\beta|\upsilon|}{8(2\varsigma+1)} + Q_1^2.$$

Proof. Since $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$, there are Schwarz functions η, δ with $\eta(0) = \delta(0) = 0$ and $|\eta(\zeta)| < 1$, $|\delta(\omega)| < 1$, which fulfills the following:

$$K_1 := 1 + \frac{1}{\upsilon}\left(\frac{2\zeta h'(\zeta) + 2\varsigma\zeta^2 h''(\zeta)}{(1-\varsigma)(h(\zeta) - h(-\zeta)) + \varsigma\zeta(h'(\zeta) + h'(-\zeta))} - 1\right) = \mathcal{G}_\beta(\eta(\zeta)) \tag{2.1}$$

$$K_2 := 1 + \frac{1}{\upsilon}\left(\frac{2\omega\hbar'(\omega) + 2\varsigma\omega^2\hbar''(\omega)}{(1-\varsigma)(\hbar(\omega) - \hbar(-\omega)) + \varsigma\omega(\hbar'(\omega) + \hbar'(-\omega))} - 1\right) = \mathcal{G}_\beta(\delta(\omega)). \tag{2.2}$$

Subsequently, the Carathéodory functions $\mathfrak{p}(\zeta), \tau(\omega)$ in $\mathbb{P}$ give

$$\mathfrak{p}(\zeta) = \frac{1+\eta(\zeta)}{1-\eta(\zeta)} = \mathfrak{p}_1\zeta + \mathfrak{p}_2\zeta^2 + \mathfrak{p}_3\zeta^3 + \cdots$$

and

$$\tau(\omega) = \frac{1+\delta(\omega)}{1-\delta(\omega)} = \tau_1\omega + \tau_2\omega^2 + \tau_3\omega^3 + \cdots.$$

The series of functions $\eta(\zeta)$ and $\delta(\omega)$ as the following:

$$\begin{aligned} \eta(\zeta) &= \frac{\mathfrak{p}(\zeta)-1}{\mathfrak{p}(\zeta)+1} = \frac{1}{2}\mathfrak{p}_1\zeta + \frac{1}{2}\left(\mathfrak{p}_2 - \frac{1}{2}\mathfrak{p}_1^2\right)\zeta^2 + \cdots, \\ \delta(\omega) &= \frac{\tau(\omega)-1}{\tau(\omega)+1} = \frac{1}{2}\tau_1\omega + \frac{1}{2}\left(\tau_2 - \frac{1}{2}\tau_1^2\right)\omega^2 + \cdots. \end{aligned} \tag{2.3}$$

From Eqs (2.1) and (2.2), we obtain the series $\mathcal{G}_\beta(\eta(\zeta))$ and $\mathcal{G}_\beta(\delta(\omega))$ as follows:

$$\begin{aligned} 1+\beta\left(\frac{\eta(\zeta)}{\log(\eta(\zeta)+1)}-1\right) &= 1+\frac{1}{4}\beta\mathfrak{p}_1\zeta+\frac{1}{48}\beta\left(12\mathfrak{p}_2-7\mathfrak{p}_1^2\right)\zeta^2 \\ &\quad+\frac{1}{192}\beta\left(17\mathfrak{p}_1^3-56\mathfrak{p}_2\mathfrak{p}_1+48\mathfrak{p}_3\right)\zeta^3+\cdots, \\ 1+\beta\left(\frac{\delta(\omega)}{\log(\delta(\omega)+1)}-1\right) &= 1+\frac{1}{4}\beta\tau_1\omega+\frac{1}{48}\beta\left(12\tau_2-7\tau_1^2\right)\omega^2 \\ &\quad+\frac{1}{192}\beta\left(17\tau_1^3-56\tau_2\tau_1+48\tau_3\right)\omega^3+\cdots. \end{aligned} \tag{2.4}$$

The series of K_1, K_2 in Eqs (2.1) and (2.2) lead to

$$\begin{aligned} K_1 &= 1+\frac{2(\varsigma+1)}{\upsilon}a_2\zeta+\frac{2(2\varsigma+1)}{\upsilon}a_3\zeta^2+\cdots, \\ K_2 &= 1-\frac{2(\varsigma+1)}{\upsilon}a_2\omega+\frac{2(2\varsigma+1)}{\upsilon}[2a_2^2-a_3]\omega^2+\cdots. \end{aligned} \tag{2.5}$$

Comparing expansions (2.4) and (2.5), we get

$$\frac{2(\varsigma+1)}{\upsilon}a_2=\frac{1}{4}\beta\mathfrak{p}_1, \tag{2.6}$$

$$\frac{2(2\varsigma+1)}{\upsilon}a_3=\frac{1}{48}\beta\left(12\mathfrak{p}_2-7\mathfrak{p}_1^2\right), \tag{2.7}$$

$$-\frac{2(\varsigma+1)}{\upsilon}a_2=\frac{1}{4}\beta\tau_1, \tag{2.8}$$

$$\frac{2(2\varsigma+1)}{\upsilon}[2a_2^2-a_3]=\frac{1}{48}\beta\left(12\tau_2-7\tau_1^2\right). \tag{2.9}$$

From the fact $\mathfrak{p}_1=-\tau_1$, Eqs (2.6) and (2.8) yield

$$\frac{8(\varsigma+1)^2}{\upsilon^2}a_2^2=\frac{1}{16}\beta^2(\mathfrak{p}_1^2+\tau_1^2). \tag{2.10}$$

The sum of Eqs (2.7) and (2.9) yields

$$\frac{4(2\varsigma+1)}{\upsilon}a_2^2=\beta\left(\frac{1}{4}(\mathfrak{p}_2+\tau_2)-\frac{7}{48}\left(\mathfrak{p}_1^2+\tau_1^2\right)\right). \tag{2.11}$$

Substituting the value of $(\mathfrak{p}_1^2+\tau_1^2)$ in (2.11) produces

$$\left\{\frac{4\left(3\beta(2\varsigma+1)\upsilon+14(\varsigma+1)^2\right)}{3\upsilon^2}\right\}a_2^2=\frac{1}{4}\beta(\mathfrak{p}_2+\tau_2), \tag{2.12}$$

hence

$$a_2=\sqrt{\frac{3\beta\upsilon^2(\mathfrak{p}_2+\tau_2)}{16\left(3\beta(2\varsigma+1)\upsilon+14(\varsigma+1)^2\right)}}.$$

We conclude from Lemma 1.1 that

$$|a_2| \leqq \sqrt{\frac{3\beta|\upsilon|^2}{4|\left(3\beta(2\varsigma+1)\upsilon+14(\varsigma+1)^2\right)|}} =: Q_1.$$

The other constraint for the coefficient a_2 is directly calculated by

$$|a_2| \leqq \frac{\beta|\upsilon|}{4(\varsigma+1)} =: Q_2.$$

Thus,

$$|a_2| \leqq min\{Q_1, Q_2\}.$$

Subtracting (2.9) from (2.7) with $\mathfrak{p}_1^2 = \tau_1^2$, we find that

$$\frac{4(2\varsigma+1)}{\upsilon}\left(a_3 - a_2^2\right) = \frac{1}{4}\beta\left(\mathfrak{p}_2 - \tau_2\right). \tag{2.13}$$

Solving the equation to find a_3 and using the previous quadratic terms, we obtain

$$a_3 = \frac{\beta\upsilon}{16(2\varsigma+1)}(\mathfrak{p}_2 - \tau_2) + a_2^2. \tag{2.14}$$

Now, using (2.10), we deduce that

$$a_3 = \frac{\beta\upsilon}{16(2\varsigma+1)}(\mathfrak{p}_2 - \tau_2) + \frac{\beta^2\upsilon^2}{128(\varsigma+1)^2}(\mathfrak{p}_1^2 + \tau_1^2).$$

Using absolute values with inequalities $|\mathfrak{p}_1^2 + \tau_1^2| \leq 2$ and $|\mathfrak{p}_2 - \tau_2| \leq 2$, according to Lemma 1.2, we obtain

$$|a_3| \leqq \frac{\beta|\upsilon|}{8(2\varsigma+1)} + \frac{\beta^2|\upsilon|^2}{64(\varsigma+1)^2} := L_1$$

and from (2.12), we get

$$|a_3| \leqq \frac{\beta|\upsilon|}{8(2\varsigma+1)} + \frac{3\beta|\upsilon|^2}{4|\left(3\beta(2\varsigma+1)\upsilon+14(\varsigma+1)^2\right)|} := L_2.$$

This completes the proof. □

If $\upsilon = 1$, we have the subsequent finding.

Corollary 2.1. *For $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$, we have*

$$|a_2| \leqq min\left\{\hat{Q}_1, \hat{Q}_2\right\}$$

and

$$|a_3| \leqq min\left\{\hat{L}_1, \hat{L}_2\right\},$$

where

$$\hat{Q}_1 := \sqrt{\frac{3\beta}{4\left(3\beta(2\varsigma+1)+14(\varsigma+1)^2\right)}},\ \hat{Q}_2 := \frac{\beta}{4(\varsigma+1)}$$

and

$$\hat{L}_1 := \frac{\beta}{8(2\varsigma+1)} + \frac{1}{4}\hat{Q}_2^2,\ \hat{L}_2 := \frac{\beta}{8(2\varsigma+1)} + \frac{1}{2}\hat{Q}_1^2.$$

For $\upsilon = \upsilon_1 = e^{-ib}\cos(b)$ ($|b| < \frac{\pi}{2}$), we obtain Corollary 2.2 as below.

Corollary 2.2. *If the function* $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$*, then*

$$|a_2| \leqq min\{l_1, l_2\}$$

and

$$|a_3| \leqq min\{A_1, A_2\},$$

where

$$l_1 := \sqrt{\frac{3\beta|\upsilon_1|^2}{4|(3\beta(2\varsigma+1)\upsilon_1 + 14(\varsigma+1)^2)|}},\ l_2 := \frac{\beta|\upsilon_1|}{4(\varsigma+1)}$$

and

$$A_1 := \frac{\beta|\upsilon_1|}{8(2\varsigma+1)} + \frac{1}{4}l_2^2,\ A_2 := \frac{\beta|\upsilon_1|}{8(2\varsigma+1)} + l_1^2.$$

For $\upsilon = 1/2, \varsigma = 0$, the below result can be obtained.

Theorem 2.2. *For* $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$*, we have*

$$|a_2| \leqq min\left\{\frac{1}{8}\beta, \sqrt{\frac{3\beta^2}{24\beta+224}}\right\} = \sqrt{\frac{3\beta^2}{24\beta+224}}$$

and

$$|a_3| \leqq min\left\{\frac{\beta}{16} + \frac{\beta^2}{256}, \frac{\beta}{16} + \frac{3\beta^2}{24\beta+224}\right\} = \frac{\beta}{16} + \frac{\beta^2}{256},$$

where $0 < \beta \leqq 1$.

Proof. According to the assumption $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$ along with $\upsilon = 1/2, \varsigma = 0$, it follows that

$$1 + 2\left(\frac{2\zeta h'(\zeta)}{h(\zeta) - h(-\zeta)} - 1\right) = \mathcal{G}_\beta(\eta(\zeta))$$

and

$$1 + 2\left(\frac{2\omega\hbar'(\omega)}{\hbar(\omega) - \hbar(-\omega)} - 1\right) = \mathcal{G}_\beta(\delta(\omega)).$$

Comparing with (2.4), we obtain that

$$\begin{aligned} 4a_2 &= \frac{1}{4}\beta\mathfrak{p}_1, \\ 4a_3 &= \frac{1}{48}\beta\left(12\mathfrak{p}_2 - 7\mathfrak{p}_1^2\right), \\ -4a_2 &= \frac{1}{4}\beta\tau_1, \\ 4[2a_2^2 - a_3] &= \frac{1}{48}\beta\left(12\tau_2 - 7\tau_1^2\right). \end{aligned}$$

Following the technique derived in Theorem 2.1, we arrive at the probable result. □

For $\upsilon = 1 = \varsigma$, the below result can be obtained.

Corollary 2.3. *Letting $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$, we obtain*

$$|a_2| \leqq \sqrt{\frac{3\beta^2}{36\beta + 224}}$$

and

$$|a_3| \leqq \frac{\beta}{24} + \frac{\beta^2}{256},$$

where $0 < \beta \leqq 1$.

If $\upsilon = 1 = \varsigma$ and $\beta = 1$, then

Corollary 2.4. *For $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_1)$, we get*

$$|a_2| \leqq \sqrt{\frac{3}{260}} = 0.107$$

and

$$|a_3| \leqq \frac{35}{768} = 0.045.$$

Corollary 2.5. *[9] In the univalent functions, if $\upsilon = 1$ and $\varsigma = 0$, the initial coefficients and Fekete-Szegö inequality are given by*

$$|a_2| \leqq 1/2,$$

$$|a_3| \leqq 1/6$$

and

$$|a_3 - \epsilon a_2^2| \leqq 1/6 \max\left\{1, \frac{1}{24}|4 + 9\epsilon|\right\}.$$

3. Fekete-Szegö inequality

In the following, we derive the Fekete-Szegö inequality for the class of symmetric bi-univalent functions related to the β-Gregory function.

Theorem 3.1. *Let $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$. If $\epsilon \in \mathbb{R}$, then*

$$\left|a_3 - \epsilon a_2^2\right| \leqq \frac{\beta|\upsilon|}{8} \begin{cases} \frac{4}{2\varsigma+1}, & |1-\epsilon| \leqq \frac{|(3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2)|}{3|\upsilon|(2\varsigma+1)} \\ \frac{12|1-\epsilon||\upsilon|}{|3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2|}, & |1-\epsilon| \geqq \frac{|(3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2)|}{3|\upsilon|(2\varsigma+1)} \end{cases}.$$

Proof. We infer from (2.12) and (2.14) that

$$\begin{aligned} a_3 - \epsilon a_2^2 &= \frac{\beta\upsilon}{16(2\varsigma+1)}(\mathfrak{p}_2 - \tau_2) + (1-\epsilon)a_2^2 \\ &= \frac{\beta\upsilon\,(\mathfrak{p}_2 - \tau_2)}{16(2\varsigma+1)} + \frac{(1-\epsilon)\left(3\beta\upsilon^2\,(\mathfrak{p}_2 + \tau_2)\right)}{16\left(3\beta\upsilon(2\varsigma+1) + 14(\varsigma+1)^2\right)} \end{aligned}$$

$$= \frac{\beta\upsilon}{16}\left[\frac{(1-\epsilon)(3\upsilon(\mathrm{p}_2+\tau_2))}{(3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2)}+\frac{\mathrm{p}_2-\tau_2}{2\varsigma+1}\right]$$
$$= \frac{\beta\upsilon}{16}\left[\left(Q^{\varsigma;\epsilon}_{\beta,\upsilon}+\frac{1}{2\varsigma+1}\right)p_2+\left(Q^{\varsigma;\epsilon}_{\beta,\upsilon}-\frac{1}{2\varsigma+1}\right)\tau_2\right],$$

where

$$Q^{\varsigma;\epsilon}_{\beta,\upsilon} := \frac{3(1-\epsilon)\upsilon}{3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2}.$$

Taking the modulus value, we get

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta|\upsilon|}{8}\left|\left(Q^{\varsigma;\epsilon}_{\beta,\upsilon}+\frac{1}{2\varsigma+1}\right)+\left(Q^{\varsigma;\epsilon}_{\beta,\upsilon}-\frac{1}{2\varsigma+1}\right)\right|.$$

Employing Lemma 1.2 yields

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta|\upsilon|}{8}\begin{cases}\frac{4}{2\varsigma+1}, & 0 \leqq |Q^{\varsigma;\epsilon}_{\beta,\upsilon}| \leqq \frac{1}{2\varsigma+1}\\ 4|Q^{\varsigma;\epsilon}_{\beta,\upsilon}|, & |Q^{\varsigma;\epsilon}_{\beta,\upsilon}| \geqq \frac{1}{2\varsigma+1}\end{cases}.$$

Consequently,

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta|\upsilon|}{8}\begin{cases}\frac{4}{2\varsigma+1}, & |1-\epsilon| \leqq \frac{4|(3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2)|}{12|\upsilon|(2\varsigma+1)}\\ \frac{12|1-\epsilon||\upsilon|}{|3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2|}, & |1-\epsilon| \geqq \frac{4|(3\beta\upsilon(2\varsigma+1)+14(\varsigma+1)^2)|}{12|\upsilon|(2\varsigma+1)}\end{cases},$$

which is analogous to the accurate outcome. □

For the following cases with the aid of assumption $h(\zeta) \in \mathcal{BS}^*(\mathcal{G}_\beta)$ and $\epsilon \in \mathbb{R}$, we derive the Fekete-Szegö inequality.

Corollary 3.1. *Case 1: Taking $\upsilon = 1$, we have*

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta}{8}\begin{cases}\frac{4}{2\varsigma+1}, & |1-\epsilon| \leqq \frac{4(3\beta(2\varsigma+1)+14(\varsigma+1)^2)}{12(2\varsigma+1)}\\ \frac{12|1-\epsilon|}{3\beta(2\varsigma+1)+14(\varsigma+1)^2}, & |1-\epsilon| \geqq \frac{4(3\beta(2\varsigma+1)+14(\varsigma+1)^2)}{12(2\varsigma+1)}\end{cases}.$$

Case 2: Taking $\upsilon = \upsilon_1 = e^{-ib}\cos(b)$, we get

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta|\upsilon_1|}{8}\begin{cases}\frac{4}{2\varsigma+1}, & |1-\epsilon| \leqq \frac{|(3\beta\upsilon_1(2\varsigma+1)+14(\varsigma+1)^2)|}{3|\upsilon_1|(2\varsigma+1)}\\ \frac{12|1-\epsilon||\upsilon_1|}{|3\beta\upsilon_1(2\varsigma+1)+14(\varsigma+1)^2|}, & |1-\epsilon| \geqq \frac{|(3\beta\upsilon_1(2\varsigma+1)+14(\varsigma+1)^2)|}{3|\upsilon_1|(2\varsigma+1)}\end{cases}.$$

Case 3: Taking $\upsilon = 1/2$ and $\varsigma = 0$, we get

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta}{16}\begin{cases}4, & |1-\epsilon| \leqq \frac{(3\beta+14)}{3}\\ \frac{12|1-\epsilon|}{3\beta+14}, & |1-\epsilon| \geqq \frac{(3\beta+14)}{3}\end{cases}.$$

Case 4: Taking $\upsilon = \varsigma = 1$, we infer

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{\beta}{8}\begin{cases}\frac{4}{3}, & |1-\epsilon| \leqq \frac{(9\beta+56)}{9}\\ \frac{12|1-\epsilon|}{9\beta+56}, & |1-\epsilon| \geqq \frac{(9\beta+56)}{9}\end{cases}.$$

Case 5: Taking $\upsilon = \varsigma = \beta = 1$, we infer

$$\left|a_3-\epsilon a_2^2\right| \leqq \frac{1}{8}\begin{cases}\frac{4}{3}, & |1-\epsilon| \leqq \frac{65}{9}\\ \frac{12|1-\epsilon|}{65}, & |1-\epsilon| \geqq \frac{65}{9}\end{cases}.$$

4. Concluding and valuable remark

This research presented a new generalization of the Gregory function, the known β-Gregory function, along with an investigation of its associated containment properties. To derive the Fekete-Szegö inequality when considering the coefficients e and t as real numbers, Zaprawa [37] was adopted as an effective tool for estimating the inequality's limits. Based on this, new classes of symmetric analytic functions, associated with the β-Gregory coefficients, were introduced. Accurate upper limits for the second and third coefficients were also obtained, which were subsequently used to derive the corresponding Fekete-Szegö inequality. The results demonstrate that this work can serve as a foundation for future research and applications, including the study of Henkel determinants, differential subordination, and harmonic functions.

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Sarem H. Hadi: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing–original draft preparation, Writing–review and editing, Supervision, Project administration, Funding acquisition; Yahea Hashem Saleem: Conceptualization, Methodology, Formal analysis, Resources, Writing–original draft preparation, Project administration, Funding acquisition; Abdullah Alatawi: Methodology, Software, Validation, Formal analysis, Resources, Writing–review and editing, Funding acquisition; Maslina Darus: Conceptualization, Methodology, Software, Resources, Writing–original draft preparation, Writing–review and editing, Funding acquisition; Alina Alb Lupaş: Methodology, Software, Formal analysis, Investigation, Resources, Writing–original draft preparation, Writing–review and editing, Visualization, Supervision, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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