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Research article

Evaluations of some infinite series involving r-Stirling numbers and harmonic numbers

Xudong Chen*

School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China

* **Correspondence:** Email: xdxudongchen@126.com.

Abstract: In this paper, we study infinite series involving r-Stirling numbers of the first kind and harmonic numbers. As a result, we establish the integral expressions for three parametric r-Stirling series, and show that these series are ultimately reducible to alternating multiple zeta values (AMZVs). As applications, the explicit AMZV expressions for some parametric series are determined, including some results presented by Wang and Chen, and some results similar to those of Hoffman.

Keywords: infinite series identities; r-Stirling numbers of the first kind; harmonic numbers; alternating multiple zeta values
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1. Introduction

Infinite series involving Stirling numbers of the first kind $\left[{n \atop k}\right]$ and harmonic numbers $H_n^{(r)}$ are widely investigated in the literature, and they are closely related to the theory of multiple zeta values (MZVs).

The systematic study of the theory of MZVs can be traced back to the works of Hoffman [1,2] and Zagier [3]. In particular, let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of positive integers and define the alternating multiple zeta values (AMZVs) and generalized polylogarithms by

$$\zeta(\boldsymbol{s};\boldsymbol{\sigma}) := \sum_{n_1>\cdots>n_k\geq 1} \frac{\sigma_1^{n_1}\cdots\sigma_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}} \quad \text{and} \quad \mathrm{Li}_{\boldsymbol{s}}(z) := \sum_{n_1>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}, \tag{1.1}$$

respectively, where $z \in [-1, 1]$, $\boldsymbol{s} := (s_1, \ldots, s_k) \in \mathbb{N}^k$, and $\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_k) \in \{\pm 1\}^k$, with $(s_1, \sigma_1), (s_1, z) \neq (1, 1)$ to ensure the convergence. For the evaluation of alternating multiple zeta values (AMZVs) and generalized polylogarithms, readers may consult Au's Mathematica package [4, 5]. For convenience, the strings of exponents and signs of an AMZV can be merged into a single string so that the component in the jth position is s_j if $\sigma_j = +1$ and $\bar{s}_j$ if $\sigma_j = -1$; for example,

$\zeta(2,1;-1,1) = \zeta(\bar{2},1)$. For clarity, we adopt the Dirichlet eta function $\eta(s)$ in place of the AMZV notation $\zeta(\bar{s})$, that is $\zeta(\bar{s}) = \zeta(s;-1) = -\eta(s)$, where $\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$, with $\eta(1) = \ln(2)$. When $s_1 > 1$, setting $\sigma_j = 1$ for $j = 1,2,\ldots,k$ in $\zeta(\boldsymbol{s};\boldsymbol{\sigma})$ or $z = 1$ in $\mathrm{Li}_{\boldsymbol{s}}(z)$, we obtain the so-called MZVs, which are denoted by $\zeta(\boldsymbol{s})$. The partial sums (or truncated series) of MZVs $\zeta(\boldsymbol{s})$ are multiple harmonic sums (MHSs), defined by

$$\zeta_n(\boldsymbol{s}) := \sum_{n \geq n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} .$$

Replacing all the ">" by "≥" in the definition of $\zeta_n(\boldsymbol{s})$ gives the multiple harmonic star sums (MHSSs) $\zeta_n^{\star}(\boldsymbol{s})$. Note that when $k = 1$, we have $\zeta_n(s) = \zeta_n^{\star}(s) = H_n^{(s)}$, which are the harmonic numbers of order s. Setting $s = 1$ gives the classical harmonic numbers H_n, that is $H_n := H_n^{(1)}$.

In 2017, Hoffman [6] presented several general infinite series identities involving harmonic numbers by certain multivariate polynomials $\mathcal{P}_k$ and $\mathcal{Q}_l$. For example, for integers $k, l \geq 0$, he obtained

$$\sum_{n=0}^{\infty} \frac{\mathcal{P}_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) \mathcal{Q}_l(H_{n+1}, H_{n+1}^{(2)}, \ldots, H_{n+1}^{(l)})}{(n+1)^2} = \binom{k+l+1}{k+1} \zeta(k+l+2) . \tag{1.2}$$

It is known from [7, Lemmas 3.2 and 3.3] that the multivariate polynomials $\mathcal{P}_k$ and $\mathcal{Q}_l$ satisfy

$$\mathcal{P}_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)}) = \zeta_n(\{1\}_k) = \frac{1}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}, \tag{1.3}$$

$$\mathcal{Q}_l(H_n, H_n^{(2)}, \ldots, H_n^{(l)}) = \tfrac{1}{l!} Y_l(n) = \zeta_n^{\star}(\{1\}_l) , \tag{1.4}$$

where $\{1\}_k$ represents a sequence of k consecutive 1s, $Y_l(n) := Y_l(H_n, 1!H_n^{(2)}, \ldots, (l-1)!H_n^{(l)})$, and $Y_l(x_1, \ldots, x_l)$ are the exponential complete Bell polynomials (see [8, Section 3] and [9, Lemma 1]). Therefore, Eq (1.2) can be rewritten as

$$\sum_{n=1}^{\infty} \frac{\zeta_{n-1}(\{1\}_k) \zeta_n^{\star}(\{1\}_l)}{n^2} = \sum_{n=1}^{\infty} \begin{bmatrix} n \\ k+1 \end{bmatrix} \frac{\zeta_n^{\star}(\{1\}_l)}{n! \cdot n} = \binom{k+l+1}{k+1} \zeta(k+l+2) . \tag{1.5}$$

Various similar series involving Stirling numbers, harmonic numbers, and MHSSs $\zeta_n^{\star}(\{1\}_l)$ have been studied. In particular, Wang and Chen [10] and Xu [11, 12] established the integral expressions and AMZV expressions for several infinite series of this type, which were further used to evaluate Euler-type series. More results on Stirling series can be found in the papers by Kuba and Panholzer [7] and Kuba and Prodinger [13].

According to Broder [14] and Merris [15], the r-Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_r$ is defined as the number of permutations of the set $\{1, 2, \ldots, n+r\}$ having $k+r$ cycles, with the elements $1, 2, \ldots, r$ lying in distinct cycles. The corresponding generating function is

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{t^n}{n!} = \frac{1}{k!} \frac{(-\ln(1-t))^k}{(1-t)^r} . \tag{1.6}$$

By the definition, we have $\begin{bmatrix} n \\ k \end{bmatrix}_r = 0$ for $n < k$, and when $r = 0, 1$, the r-Stirling numbers reduce to the classical Stirling numbers: $\begin{bmatrix} n \\ k \end{bmatrix}_0 = \begin{bmatrix} n \\ k \end{bmatrix}$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$. In particular, $\begin{bmatrix} n \\ 1 \end{bmatrix}_r = n! h_n^{(r)}$, where $h_n^{(r)}$ is the nth hyperharmonic number defined by the recurrence relation

$$h_n^{(0)} = \frac{1}{n} \quad \text{and} \quad h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}, \qquad n, r \geq 1$$

(see Conway and Guy [16]).

There are some studies on infinite series involving hyperharmonic numbers. For example, Dil and Boyadzhiev [17] and Mező and Dil [18] investigated the linear Euler sums of hyperharmonic numbers of the form $\sigma_{r,l} = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^l}$. Some recent results on series involving hyperharmonic numbers can be found in [19–21]. In addition, infinite series involving r-Stirling numbers have also drawn considerable attention; see [22–24].

Inspired by the above works, in this paper, we study infinite series involving the r-Stirling numbers of the first kind, harmonic numbers and MHSSs $\zeta_n^\star(\{1\}_l)$. The paper is organized as follows. In Section 2, we introduce the following r-Stirling series:

$$\Omega_r(k, l; q) := \sum_{n=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{q^n}{n!(n+1)^l}, \qquad k \geq 0,\ l > r \geq 0,\ q \in [-1, 1],$$

which can be evaluated via recurrence relations given in [23, Section 2], and are closely related to the integral $\mathcal{I}(r, k, l; q) := \int_0^1 \frac{\ln^k(1-qt)\ln^l(t)}{(1-qt)^r} \mathrm{d}t$. In Sections 3 and 4, we establish identities that relate the following parametric r-Stirling series:

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{H_n^{(l+1)}}{2^n n!}, \qquad \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{\zeta_{n+1}^\star(\{1\}_l)}{2^n (n+1)!}, \qquad \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{\zeta_n^\star(\{1\}_l)}{2^n n \cdot n!},$$

to the integrals $\mathcal{I}(r, k, l; \frac{1}{2})$ and $\mathcal{I}(r, k, l; -1)$. These identities show that the above r-Stirling series can be expressed in terms of AMZVs via the results from Section 2. As applications, the AMZV expressions for some parametric series involving Stirling numbers, harmonic numbers, and hyperharmonic numbers can be obtained, including several results of Wang and Chen [10, Section 2] as special cases. For example, we have

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)} H_n^{(l+1)}}{2^n} = 2^r \left\{ \sum_{i=1}^{r} 2^{-i} \Omega_i(1, l+1; \tfrac{1}{2}) + \ln(2)\zeta(l+1) + \zeta(\bar{1}, \bar{2}, \{1\}_{l-1}) \right\},$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\zeta_n^\star(\{1\}_l)}{2^n n \cdot n!} = \sum_{n=k}^{\infty} \frac{\zeta_n(\{1\}_k)\zeta_n^\star(\{1\}_l)}{2^n n}$$

$$= \frac{2\ln^k(2)\lambda(l+1)}{k!} + \sum_{j=1}^{k} \frac{\ln^{k-j}(2)}{(k-j)!} \left\{ \zeta(l+1, \bar{1}, \{1\}_{j-1}) - \zeta(\overline{l+1}, \{1\}_j) \right\}.$$

It is interesting to note that the first series involves the products of harmonic numbers and hyperharmonic numbers, while the second series is similar to Hoffman's series (1.5). Moreover, by specifying the parameters, some representative examples are presented.

2. Calculation of infinite series $\varOmega_r(k, l; q)$ and integral $\mathcal{I}(r, k, l; q)$

Define the series $\varOmega_r(k, l; q)$ involving r-Stirling numbers of the first kind by

$$\varOmega_r(k, l; q) := \sum_{n=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{q^n}{n!(n+1)^l},$$

for integers $k \geq 0$, $l > r \geq 0$, and real number $q \in [-1, 1]$. It is clear that $\varOmega_0(0, l; q) = 0$ because $\begin{bmatrix} n \\ 0 \end{bmatrix}_0 = 0$ for $n \geq 1$. Moreover, as shown by Chen et al. [23, Section 2], the following two lemmas on the series $\varOmega_r(k, l; q)$ can be obtained.

Lemma 2.1. *For any integers $k \geq 2$, $l \geq 1$, and real number $q \in [-1, 0) \cup (0, 1]$, the series $\varOmega_0(k, l; q)$ satisfy the following recurrence relation:*

$$\varOmega_0(k, l; q) = (1 - q^{-1}) \operatorname{Li}_{\{1\}_k}(q) - q^{-1} \sum_{i=2}^{l} \operatorname{Li}_{i,\{1\}_{k-1}}(q) + \sum_{i=1}^{l} \varOmega_0(k-1, i; q),$$

with the initial case

$$\varOmega_0(1, l; q) = \sum_{n=1}^{\infty} \frac{q^n}{n(n+1)^l} = (1 - q^{-1}) \ln \frac{1}{1-q} - q^{-1} \sum_{i=2}^{l} \operatorname{Li}_i(q) + l. \tag{2.1}$$

By convention, when $q = 1$, the first term on the right-hand side of each of the above two equations vanishes.

Lemma 2.2. *For any integers $k \geq 1$, $l > r \geq 1$, and real number $q \in [-1, 0) \cup (0, 1]$, the series $\varOmega_r(k, l; q)$ satisfy the following recurrence relation:*

$$(r-1)\varOmega_r(k, l; q) = \varOmega_{r-1}(k, l-1; q) + (r-2)\varOmega_{r-1}(k, l; q) + \varOmega_{r-1}(k-1, l; q) - \varOmega_r(k-1, l; q),$$

with the initial cases

$$\varOmega_1(k, l; q) = \sum_{n=1}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{q^n}{n!(n+1)^l} = q^{-1} \operatorname{Li}_{l,\{1\}_k}(q), \tag{2.2}$$

$$\varOmega_r(0, l; q) = \sum_{n=1}^{\infty} \binom{r+n-1}{n} \frac{q^n}{(n+1)^l} = \frac{q^{-1}}{(r-1)!} \sum_{i=0}^{r-1} \begin{bmatrix} r-1 \\ i \end{bmatrix} \operatorname{Li}_{l-i}(q) - 1. \tag{2.3}$$

According to Lemmas 2.1 and 2.2, all the r-Stirling series $\varOmega_r(k, l; q)$ are reducible to $\mathbb{Q}$-linear combinations of polylogarithms $\operatorname{Li}_i(q)$ and generalized polylogarithms $\operatorname{Li}_{i,\{1\}_p}(q)$. In particular, we can use the Mathematica package developed in [23] to compute some special series of the form $\varOmega_r(k, l; q)$.

In the following lemma, we present an integral expression for the series $\varOmega_r(k, l; q)$.

Lemma 2.3. *For any integers $k \geq 0$ and $l \geq r \geq 0$, we have*

$$\mathcal{I}(r, k, l; q) := \int_0^1 \frac{\ln^k(1-qt) \ln^l(t)}{(1-qt)^r} \mathrm{d}t = (-1)^{k+l} k! l! \varOmega_r(k, l+1; q) + (-1)^l l! \delta_{k,0}, \tag{2.4}$$

where $\delta_{k,0}$ is the Kronecker delta.

Proof. Using the generating function (1.6) and the integral $\int_0^1 t^{n-1}\ln^l(t)\mathrm{d}t = \frac{(-1)^l l!}{n^{l+1}}$ yields

$$\mathcal{I}(r,k,l;q) = (-1)^k k! \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{q^n}{n!} \int_0^1 t^n \ln^l(t)\mathrm{d}t = (-1)^{k+l} k! l! \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{q^n}{n!(n+1)^{l+1}}.$$

According to the definition of the r-Stirling numbers of the first kind, for $k \geq 1$, the summation on the right-hand side is just $\Omega_r(k, l+1; q)$, while for $k = 0$, it equals $\Omega_r(0, l+1; q) + 1$ because $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_r = 1$. Then, Eq (2.4) follows. □

3. r-Stirling series related to integral $\mathcal{I}(r,k,l;1/2)$

Lemma 2.3 shows that the integral $\mathcal{I}(r,k,l;q)$ can be evaluated via the r-Stirling series $\Omega_r(k,l;q)$. In this section, we present a r-Stirling series identity related to the integral $\mathcal{I}(r,k,l;\frac{1}{2})$.

Theorem 3.1. *For any integers $k, r \geq 0$ and $l \geq 1$, with $l \geq r$, the following series identity holds:*

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{H_n^{(l+1)}}{2^n n!} = \frac{(-1)^{k+l} 2^r}{k! l!} \left\{ \sum_{i=1}^{r} 2^{-i} \mathcal{I}(i,k,l;\tfrac{1}{2}) - \sum_{j=1}^{k} \binom{k}{j} (-\ln(2))^{k-j} \mathcal{J}(j,l) \right\}, \tag{3.1}$$

where

$$\begin{aligned} \mathcal{J}(j,l) :&= \int_0^1 \frac{\ln^j(1+x)\ln^l(1-x)}{x}\mathrm{d}x \\ &= (-1)^l l! \left\{ j! \sum_{i=0}^{j-1} \frac{\ln^i(2)}{i!} \zeta(\bar{1}, \{1\}_{j-i-1}, \bar{2}, \{1\}_{l-1}) + \zeta(l+1)\ln^j(2) \right\}. \end{aligned} \tag{3.2}$$

Therefore, the r-Stirling series on the left-hand side of (3.1) *can be expressed in terms of AMZVs.*

Proof. Consider the integral

$$\mathcal{H}_1(r,k,l) := \int_0^1 \frac{\ln^k(1-\frac{x}{2})\ln^l(x)}{(1-\frac{x}{2})^r(1-x)}\mathrm{d}x.$$

On the one hand, by using the generating function (1.6) and the integral

$$\int_0^1 \frac{x^n \ln^l(x)}{1-x}\mathrm{d}x = \sum_{i=0}^{\infty} \int_0^1 x^{n+i}\ln^l(x)\mathrm{d}x = (-1)^l l! \{\zeta(l+1) - H_n^{(l+1)}\}, \tag{3.3}$$

we rewrite $\mathcal{H}_1(r,k,l)$ as

$$\begin{aligned} \mathcal{H}_1(r,k,l) &= (-1)^k k! \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{2^n n!} \int_0^1 \frac{x^n \ln^l(x)}{1-x}\mathrm{d}x \\ &= (-1)^{k+l} k! l! \left\{ \frac{2^r \ln^k(2)\zeta(l+1)}{k!} - \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{H_n^{(l+1)}}{2^n n!} \right\}. \end{aligned} \tag{3.4}$$

On the other hand, substituting the partial fraction decomposition

$$\frac{1}{(1-\frac{x}{2})^r(1-x)} = \frac{2^r}{1-x} - \sum_{i=1}^{r} \frac{2^{r-i}}{(1-\frac{x}{2})^i}$$

into the integral $\mathcal{H}_1(r,k,l)$ yields

$$\begin{aligned}\mathcal{H}_1(r,k,l) &= 2^r \int_0^1 \frac{\ln^k(1-\frac{x}{2})\ln^l(x)}{1-x}\mathrm{d}x - \sum_{i=1}^{r} 2^{r-i} \int_0^1 \frac{\ln^k(1-\frac{x}{2})\ln^l(x)}{(1-\frac{x}{2})^i}\mathrm{d}x \\ &= 2^r \sum_{j=0}^{k} \binom{k}{j} (-\ln(2))^{k-j} \mathcal{J}(j,l) - \sum_{i=1}^{r} 2^{r-i} \mathcal{I}(i,k,l;\tfrac{1}{2})\,. \end{aligned} \tag{3.5}$$

As shown in [10, Lemma 2.4], for $j, l \geq 1$, the integral $\mathcal{J}(j,l)$ satisfies the AMZV expression (3.2), which is also valid for $j = 0$ because $\int_0^1 \frac{\ln^l(1-x)}{x}\mathrm{d}x = (-1)^l l!\zeta(l+1)$. Next, combining with (3.4) and (3.5), we obtain

$$\begin{aligned}\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{H_n^{(l+1)}}{2^n n!} = \frac{(-1)^{k+l}2^r}{k!l!} &\left\{ \sum_{i=1}^{r} 2^{-i} \mathcal{I}(i,k,l;\tfrac{1}{2}) - \sum_{j=0}^{k} \binom{k}{j} (-\ln(2))^{k-j} \mathcal{J}(j,l) \right\} \\ &+ \frac{2^r \ln^k(2)\zeta(l+1)}{k!}\,. \end{aligned} \tag{3.6}$$

Then Eq (3.1) is derived from the cancellation of the $j = 0$ term of the second summation with the last term on the right-hand side of (3.6). Finally, based on Section 2, the integral $\mathcal{I}(r,k,l;\frac{1}{2})$ is reducible to a $\mathbb{Q}$-linear combination of $\mathrm{Li}_i(\frac{1}{2})$ and $\mathrm{Li}_{i,\{1\}_p}(\frac{1}{2})$, and ultimately to a combination of unit-exponent AMZVs via

$$\mathrm{Li}_{i,\{1\}_p}(\tfrac{1}{2}) = (-1)^{p+1}\zeta(\bar{1},\{1\}_p,\bar{1},\{1\}_{i-2})\,,\ i \geq 2\,,\ p \geq 0 \tag{3.7}$$

(see also [23, Eq (1.9)]). This concludes the proof. □

Example 3.1. *By combining Theorem 3.1 with the lemmas in Section 2, we obtain the evaluations of the following r-Stirling series:*

$$\sum_{n=3}^{\infty} \frac{\begin{bmatrix} n \\ 3 \end{bmatrix}_1 H_n^{(2)}}{2^n n!} = -\tfrac{1}{60}\ln^5(2) - \tfrac{1}{240}\ln(2)\pi^4 + \tfrac{5}{8}\ln^2(2)\zeta(3) - \tfrac{7}{48}\pi^2\zeta(3) + \tfrac{7}{4}\zeta(5)\,,$$

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{\begin{bmatrix} n \\ 2 \end{bmatrix}_2 H_n^{(3)}}{2^n n!} = &-\ln^2(2) + \tfrac{1}{6}\pi^2 + \tfrac{1}{6}\ln(2)\pi^2 - 2\zeta(3) - \tfrac{1}{12}\ln^4(2) + \tfrac{1}{12}\ln^2(2)\pi^2 \\ &+ \tfrac{1}{40}\pi^4 - 2\ln(2)\zeta(3) - 2\,\mathrm{Li}_4(\tfrac{1}{2}) + \tfrac{1}{10}\ln^5(2) - \tfrac{1}{9}\ln^3(2)\pi^2 \\ &- \tfrac{7}{120}\ln(2)\pi^4 + \tfrac{7}{2}\ln^2(2)\zeta(3) - \tfrac{1}{3}\pi^2\zeta(3) + 11\zeta(5) - 8\,\mathrm{Li}_5(\tfrac{1}{2})\,. \end{aligned}$$

Moreover, by specializing the parameter r in Theorem 3.1, we obtain Corollary 3.2.

Corollary 3.2. *For any integers $k, l \geq 1$, the following two parametric Stirling series are expressible in terms of AMZVs:*

$$\sum_{n=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{H_n^{(l+1)}}{2^n n!} = (-1)^{k-1} \zeta(\bar{1}, \{1\}_{k-1}, \bar{2}, \{1\}_{l-1}) + \frac{\ln^k(2)\zeta(l+1)}{k!}\,, \tag{3.8}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{H_n^{(l+1)}}{2^n n!} &= \sum_{n=1}^{\infty} \frac{H_n^{(l+1)} \zeta_n(\{1\}_k)}{2^n} \\ &= 2\left\{ \mathrm{Li}_{l+1,\{1\}_k}(\tfrac{1}{2}) + (-1)^{k-1}\zeta(\bar{1}, \{1\}_{k-1}, \bar{2}, \{1\}_{l-1}) + \frac{\ln^k(2)\zeta(l+1)}{k!} \right\}, \end{aligned} \tag{3.9}$$

where Eq (3.8) *is in fact Wang and Chen's [10, Theorem 2.5].*

Proof. Setting $r = 0$ in Theorem 3.1 and using Eq (3.2), we have

$$\begin{aligned} \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{H_n^{(l+1)}}{2^n n!} &= \frac{(-1)^{k+l+1}}{k!l!} \sum_{j=1}^{k} \binom{k}{j} (-\ln(2))^{k-j} \mathcal{J}(j, l) \\ &= \sum_{j=1}^{k} \sum_{i=0}^{j-1} (-1)^{j-1} \frac{\ln^{k-j+i}(2)}{(k-j)!i!} \zeta(\bar{1}, \{1\}_{j-i-1}, \bar{2}, \{1\}_{l-1}) - \frac{\ln^k(2)\zeta(l+1)}{k!} \sum_{j=1}^{k} \binom{k}{j} (-1)^j, \end{aligned}$$

which gives Eq (3.8) after simplification for $k, l \geq 1$ (see also the proof of [10, Theorem 2.5] for details). Next, setting $r = 1$ in Theorem 3.1 and using Eq (1.3) yields

$$\begin{aligned} \sum_{n=1}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{H_n^{(l+1)}}{2^n n!} &= \sum_{n=1}^{\infty} \frac{H_n^{(l+1)} \zeta_n(\{1\}_k)}{2^n} \\ &= \frac{(-1)^{k+l}}{k!l!} \left\{ \mathcal{I}(1, k, l; \tfrac{1}{2}) - 2 \sum_{j=1}^{k} \binom{k}{j} (-\ln(2))^{k-j} \mathcal{J}(j, l) \right\}, \end{aligned}$$

which, together with Eqs (2.2), (2.4) and (3.8), leads us to Eq (3.9). □

Remark 3.3. *According to [25, Section 2.1], the following recurrence relation holds for the r-Stirling numbers of the first kind:*

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_r = \langle r \rangle_n\,, \quad \begin{bmatrix} n \\ k \end{bmatrix}_r = \frac{1}{k} \sum_{j=0}^{k-1} (-1)^{k-j-1} (H_{n+r-1}^{(k-j)} - H_{r-1}^{(k-j)}) \begin{bmatrix} n \\ j \end{bmatrix}_r, \quad \textit{for } k, r \in \mathbb{N}\,.$$

Here $\langle x \rangle_n$ are the rising factorials defined by $\langle x \rangle_0 := 1$ and $\langle x \rangle_n := x(x+1)\cdots(x+n-1)$ for $n \geq 1$. This recurrence relation shows that the r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_r$ can be expressed in terms of harmonic numbers. For instance,

$$\begin{aligned} &\begin{bmatrix} n \\ 0 \end{bmatrix}_r = \langle r \rangle_n = n! \binom{r+n-1}{n}, \quad \begin{bmatrix} n \\ 1 \end{bmatrix}_r = \langle r \rangle_n (H_{n+r-1} - H_{r-1}) = n! h_n^{(r)}\,, \\ &\begin{bmatrix} n \\ 2 \end{bmatrix}_r = \frac{\langle r \rangle_n}{2} \{ (H_{n+r-1} - H_{r-1})^2 - (H_{n+r-1}^{(2)} - H_{r-1}^{(2)}) \}\,. \end{aligned}$$

In particular, by combining (1.3)*, the MHSs* $\zeta_n(\{1\}_k)$ *can also be expressed in terms of harmonic numbers by the following recurrence relation:*

$$\zeta_n(\{1\}_0) = 1\,, \quad \zeta_n(\{1\}_k) = \frac{1}{k}\sum_{j=0}^{k-1}(-1)^{k-j-1}H_n^{(k-j)}\zeta_n(\{1\}_j)\,, \quad \textit{for } k \in \mathbb{N}\,.$$

Therefore, we can obtain some infinite series involving harmonic numbers from Theorem 3.1 and Corollary 3.2.

Example 3.2. *Setting* $(k, l) = (2, 1)$ *in* (3.8) *and substituting* $H_{n+1}^{(2)} = H_n^{(2)} + \frac{1}{(n+1)^2}$ *yields*

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{2^n(n+1)} = -2\zeta(\bar{1}, 1, \bar{2}) + \ln^2(2)\zeta(2) - 2\,\mathrm{Li}_{3,1}(\tfrac{1}{2}) = -\tfrac{1}{12}\ln^4(2) - \tfrac{1}{144}\pi^4 + \tfrac{3}{2}\ln(2)\zeta(3)\,.$$

Additionally, from (3.9)*, we have*

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{2^n} &= -\tfrac{1}{3}\ln^3(2) + \tfrac{3}{2}\zeta(3)\,,\\
\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{2^n} &= \tfrac{1}{6}\ln^4(2) - \tfrac{1}{12}\ln^2(2)\pi^2 - \tfrac{1}{240}\pi^4 + \tfrac{3}{2}\ln(2)\zeta(3) + 2\,\mathrm{Li}_4(\tfrac{1}{2})\,,\\
\sum_{n=1}^{\infty}\frac{(H_n^2 - H_n^{(2)})H_n^{(2)}}{2^n} &= -\tfrac{1}{3}\ln^4(2) + \tfrac{1}{6}\ln^2(2)\pi^2 + \tfrac{13}{360}\pi^4 - \ln(2)\zeta(3) - 4\,\mathrm{Li}_4(\tfrac{1}{2})\,.
\end{aligned}$$

Note that Mező [26] gave the values of $\sum_{n=1}^{\infty}\frac{H_n^3}{2^n}$ *and* $\sum_{n=1}^{\infty}\frac{H_n^2}{2^n}$*, which are analogous to our series.*

Next, by specializing the parameter k in Theorem 3.1, we obtain Corollary 3.4.

Corollary 3.4. *For any integers* $l \geq r \geq 0$ *with* $l \geq 1$*, the following two series involving harmonic numbers and hyperharmonic numbers satisfy:*

$$\sum_{n=0}^{\infty}\frac{\binom{r+n-1}{n}H_n^{(l+1)}}{2^n} = \sum_{i=0}^{r-1}\sum_{j=0}^{i}\frac{2^{r-i}}{i!}\begin{bmatrix} i \\ j \end{bmatrix}\mathrm{Li}_{l+1-j}(\tfrac{1}{2})\,, \tag{3.10}$$

$$\sum_{n=1}^{\infty}\frac{h_n^{(r)}H_n^{(l+1)}}{2^n} = 2^r\left\{\sum_{i=1}^{r}2^{-i}\varOmega_i(1, l+1; \tfrac{1}{2}) + \ln(2)\zeta(l+1) + \zeta(\bar{1}, \bar{2}, \{1\}_{l-1})\right\}. \tag{3.11}$$

Therefore, they are reducible to combinations of AMZVs.

Proof. Setting $k = 0$ in Theorem 3.1 and using Eqs (2.3) and (2.4), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\binom{r+n-1}{n}H_n^{(l+1)}}{2^n} &= \frac{(-1)^l}{l!}\sum_{i=1}^{r}2^{r-i}\mathcal{I}(i, 0, l; \tfrac{1}{2}) = \sum_{i=1}^{r}2^{r-i}\left\{\varOmega_i(0, l+1; \tfrac{1}{2}) + 1\right\}\\
&= \sum_{i=1}^{r}\frac{2^{r-i+1}}{(i-1)!}\sum_{j=0}^{i-1}\begin{bmatrix} i-1 \\ j \end{bmatrix}\mathrm{Li}_{l+1-j}(\tfrac{1}{2})\,.
\end{aligned}$$

Then replacing i by $i + 1$ yields Eq (3.10). Equation (3.11) is obtained by setting $k = 1$ in Theorem 3.1 and applying Eq (2.4). □

Example 3.3. *Here are some special cases of Corollary 3.4:*

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(n+1)H_n^{(3)}}{2^n} &= -\ln^2(2)+\tfrac{1}{6}\pi^2+\tfrac{2}{3}\ln^3(2)-\tfrac{1}{3}\ln(2)\pi^2+\tfrac{7}{2}\zeta(3)\,,\\
\sum_{n=0}^{\infty}\frac{\binom{n+2}{2}H_n^{(4)}}{2^n} &= -\tfrac{1}{2}\ln^2(2)+\tfrac{1}{12}\pi^2+\tfrac{5}{6}\ln^3(2)-\tfrac{5}{12}\ln(2)\pi^2+\tfrac{35}{8}\zeta(3)+8\operatorname{Li}_4(\tfrac{1}{2})\,,\\
\sum_{n=1}^{\infty}\frac{h_n^{(2)}H_n^{(3)}}{2^n} &= \ln^2(2)-\tfrac{1}{6}\pi^2-\tfrac{1}{6}\ln(2)\pi^2+2\zeta(3)+\tfrac{1}{3}\ln^4(2)-\tfrac{1}{6}\ln^2(2)\pi^2-\tfrac{1}{120}\pi^4\\
&\quad+3\ln(2)\zeta(3)+4\operatorname{Li}_4(\tfrac{1}{2})\,,\\
\sum_{n=1}^{\infty}\frac{h_n^{(3)}H_n^{(4)}}{2^n} &= \tfrac{3}{4}\ln^2(2)-\tfrac{1}{8}\pi^2-\tfrac{3}{4}\ln^3(2)+\tfrac{7}{24}\ln(2)\pi^2-\tfrac{47}{16}\zeta(3)+\tfrac{5}{24}\ln^4(2)+\tfrac{1}{144}\pi^4\\
&\quad-\tfrac{5}{8}\ln(2)\zeta(3)+5\operatorname{Li}_4(\tfrac{1}{2})-\tfrac{1}{15}\ln^5(2)-\tfrac{1}{90}\ln(2)\pi^4-\tfrac{1}{12}\pi^2\zeta(3)\\
&\quad+7\zeta(5)+\tfrac{1}{2}\ln^2(2)\zeta(3)\,.
\end{aligned}
$$

4. r-Stirling series related to integral $\mathcal{I}(r,k,l;-1)$

In this section, we present two r-Stirling series identities related to the integral $\mathcal{I}(r,k,l;-1)$.

Theorem 4.1. *For any integers $k,l,r\geq 0$, with $l\geq r$, the following series identity holds:*

$$
\sum_{n=k}^{\infty}\begin{bmatrix} n\\ k\end{bmatrix}_r\frac{\zeta_{n+1}^{\star}(\{1\}_l)}{2^n(n+1)!}=\frac{2^r}{k!l!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j+l}\ln^{k-j}(2)\mathcal{I}(r,j,l;-1)\,. \tag{4.1}
$$

Therefore, the r-Stirling series on the left-hand side of (4.1) *can be expressed in terms of AMZVs.*

Proof. Let us consider the integral

$$
\mathcal{H}_2(r,k,l):=\int_0^1\frac{\ln^k(1-\frac{x}{2})\ln^l(1-x)}{(1-\frac{x}{2})^r}\mathrm{d}x\,.
$$

On the one hand, by applying the generating function (1.6) and the integral

$$
\int_0^1 x^n\ln^l(1-x)\mathrm{d}x=(-1)^l l!\frac{\zeta_{n+1}^{\star}(\{1\}_l)}{n+1} \tag{4.2}
$$

(see [27, Eq (2.5)]), we obtain

$$
\mathcal{H}_2(r,k,l)=(-1)^{k+l}k!l!\sum_{n=k}^{\infty}\begin{bmatrix} n\\ k\end{bmatrix}_r\frac{\zeta_{n+1}^{\star}(\{1\}_l)}{2^n(n+1)!}\,.
$$

On the other hand, the substitution $x\to 1-x$ in $\mathcal{H}_2(r,k,l)$ gives

$$
\mathcal{H}_2(r,k,l)=2^r\sum_{j=0}^{k}\binom{k}{j}(-\ln(2))^{k-j}\mathcal{I}(r,j,l;-1)\,.
$$

Combining the above two expressions gives (4.1) immediately, and the final assertion follows from Lemma 2.3, along with the definition of AMZVs. □

Remark 4.2. *Note that by Eq* (1.4)*, the following recurrence relation holds for* $\zeta_n^\star(\{1\}_l)$*:*

$$\zeta_n^\star(\{1\}_0) = 1\,, \quad \zeta_n^\star(\{1\}_l) = \frac{1}{l}\sum_{j=0}^{l-1} H_n^{(l-j)}\zeta_n^\star(\{1\}_j)\,, \quad \textit{for } l \in \mathbb{N}\,,$$

which shows that $\zeta_n^\star(\{1\}_l)$ *can be expressed in terms of harmonic numbers (see also [24, Section 2.1] and [28, Section 2.4]). In particular, we have*

$$\begin{aligned}&\zeta_n^\star(1) = H_n\,, \quad \zeta_n^\star(1,1) = \tfrac{1}{2}(H_n^2 + H_n^{(2)})\,, \quad \zeta_n^\star(\{1\}_3) = \tfrac{1}{6}(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)})\,,\\ &\zeta_n^\star(\{1\}_4) = \tfrac{1}{24}(H_n^4 + 6H_n^2H_n^{(2)} + 3(H_n^{(2)})^2 + 8H_nH_n^{(3)} + 6H_n^{(4)})\,.\end{aligned}$$

Thus, by using Remark 4.2 and specializing the parameters r, k, l in Theorem 4.1, we obtain the evaluations of some special r-Stirling series.

Example 4.1. *We have*

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{\left[{n \atop 3}\right]_1 (H_{n+1}^2 + H_{n+1}^{(2)})}{2^n(n+1)!} &= -\tfrac{1}{90}\ln^6(2) + \tfrac{1}{36}\ln^4(2)\pi^2 + \tfrac{1}{24}\ln^2(2)\pi^4 + \tfrac{53}{7560}\pi^6\\ &- \tfrac{2}{3}\ln^3(2)\zeta(3) - \tfrac{33}{8}\ln(2)\zeta(5) - \tfrac{1}{3}\ln(2)\pi^2\zeta(3) + \zeta(3)^2 - 2\zeta(\bar{5},1) - 8\,\mathrm{Li}_6(\tfrac{1}{2})\,,\\ \sum_{n=2}^{\infty} \frac{\left[{n \atop 2}\right]_2 \zeta_{n+1}^\star(\{1\}_3)}{2^n(n+1)!} &= 3\zeta(3) + \tfrac{1}{3}\ln^4(2) - \tfrac{1}{3}\ln^2(2)\pi^2 - \tfrac{11}{90}\pi^4 + 4\ln(2)\zeta(3) + 8\,\mathrm{Li}_4(\tfrac{1}{2})\\ &- \tfrac{1}{15}\ln^5(2) + \tfrac{1}{9}\ln^3(2)\pi^2 + \tfrac{11}{90}\ln(2)\pi^4 - 2\ln^2(2)\zeta(3) - \tfrac{2}{3}\pi^2\zeta(3) - \tfrac{1}{2}\zeta(5) + 8\,\mathrm{Li}_5(\tfrac{1}{2})\,.\end{aligned}$$

Moreover, the expressions for some parametric series can be established from Theorem 4.1, as shown in Corollaries 4.3 and 4.4.

Corollary 4.3. *For any integers* $k, l \geq 0$*, the following parametric Stirling series is reducible to AMZVs:*

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{\zeta_{n+1}^\star(\{1\}_l)}{2^n(n+1)!} = \sum_{n=k}^{\infty} \frac{\zeta_n(\{1\}_k)\zeta_{n+1}^\star(\{1\}_l)}{2^n(n+1)} = -2\sum_{j=0}^{k} \frac{\ln^j(2)}{j!}\zeta(\overline{l+1}, \{1\}_{k-j})\,, \tag{4.3}$$

which is equivalent to [10, Theorem 2.12, Eq (38)].

Proof. Setting $r = 1$ in Theorem 4.1, and combining Eqs (1.3) and (2.2)–(2.4), we obtain Eq (4.3) for $l \geq 1$. Note that (4.3) also holds for $l = 0$:

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{1}{2^n(n+1)!} = \frac{2\ln^{k+1}(2)}{(k+1)!} = -2\sum_{j=0}^{k} \frac{\ln^j(2)}{j!}\zeta(\bar{1}, \{1\}_{k-j})\,,$$

according to the generating function (1.6) and the relation $\zeta(\bar{1}, \{1\}_j) = (-1)^{j+1}\frac{\ln^{j+1}(2)}{(j+1)!}$ (see, e.g., [10, Eq (37)]). Thus, the proof is complete. □

Example 4.2. *Combining Remark 4.2 and Corollary 4.3, the following special series can be evaluated:*

$$\sum_{n=1}^{\infty}\frac{H_n\zeta_{n+1}(\{1\}_3)}{2^n(n+1)}=\tfrac{7}{360}\pi^4\ln(2)-\tfrac{1}{6}\pi^2\zeta(3)+\tfrac{29}{16}\zeta(5)\,,$$

$$\sum_{n=2}^{\infty}\frac{\zeta_n(1,1)\zeta^{\star}_{n+1}(1,1)}{2^n(n+1)}=\sum_{n=2}^{\infty}\frac{(H_n^2-H_n^{(2)})(H_{n+1}^2+H_{n+1}^{(2)})}{2^{n+2}(n+1)}$$
$$=-\tfrac{1}{30}\ln^5(2)+\tfrac{1}{24}\pi^4\ln(2)+\tfrac{1}{18}\pi^2\ln^3(2)-\ln^2(2)\zeta(3)-\tfrac{1}{6}\pi^2\zeta(3)-\tfrac{33}{16}\zeta(5)+4\,\mathrm{Li}_5(\tfrac{1}{2})\,.$$

Besides the special cases given in Corollary 4.3, the next corollary related to MHSSs $\zeta_n^{\star}(\{1\}_l)$ and hyperharmonic numbers can be established.

Corollary 4.4. *For any integers $l\ge r\ge 1$, the following two parametric series are reducible to AMZVs:*

$$\sum_{n=0}^{\infty}\frac{\binom{r+n-1}{n}\zeta^{\star}_{n+1}(\{1\}_l)}{2^n(n+1)}=\frac{2^r}{(r-1)!}\sum_{i=0}^{r-1}\begin{bmatrix}r-1\\ i\end{bmatrix}\eta(l+1-i)\,,\tag{4.4}$$

$$\sum_{n=1}^{\infty}\frac{h_n^{(r)}\zeta^{\star}_{n+1}(\{1\}_l)}{2^n(n+1)}=2^r\left\{\varOmega_r(1,l+1;-1)+\frac{\ln(2)}{(r-1)!}\sum_{i=0}^{r-1}\begin{bmatrix}r-1\\ i\end{bmatrix}\eta(l+1-i)\right\}\,,\tag{4.5}$$

and for any integer $l\ge 0$, the following parametric series is reducible to zeta values:

$$\sum_{n=1}^{\infty}\frac{\zeta^{\star}_{n+1}(\{1\}_l)}{2^n n(n+1)}=-\sum_{i=1}^{l+1}\eta(i)+l+1\,.\tag{4.6}$$

Proof. For $r\ge 1$, setting $k=0$ in Theorem 4.1 and using Eqs (2.3) and (2.4), we have

$$\sum_{n=0}^{\infty}\frac{\binom{r+n-1}{n}\zeta^{\star}_{n+1}(\{1\}_l)}{2^n(n+1)}=\frac{(-1)^l2^r}{l!}\mathcal{I}(r,0,l;-1)=2^r\left\{\varOmega_r(0,l+1;-1)+1\right\}$$
$$=-\frac{2^r}{(r-1)!}\sum_{i=0}^{r-1}\begin{bmatrix}r-1\\ i\end{bmatrix}\mathrm{Li}_{l+1-i}(-1)\,.\tag{4.7}$$

Then, Eq (4.4) follows from the fact that $\mathrm{Li}_i(-1)=-\eta(i)$. Next, setting $k=1$ in Theorem 4.1 and applying Eq (2.4), we have

$$\sum_{n=1}^{\infty}\frac{h_n^{(r)}\zeta^{\star}_{n+1}(\{1\}_l)}{2^n(n+1)}=\frac{(-1)^l2^r}{l!}\left\{\ln(2)\mathcal{I}(r,0,l;-1)-\mathcal{I}(r,1,l;-1)\right\}$$
$$=2^r\left\{\ln(2)(\varOmega_r(0,l+1;-1)+1)+\varOmega_r(1,l+1;-1)\right\}\,,$$

which, together with Eqs (2.1) and (4.7), yields the expression (4.5) for $r\ge 1$, and the expression (4.6) for $r=0$. □

Example 4.3. *Equation* (4.4) *gives the following three series:*

$$\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{2^n n}=\tfrac{3}{2}\zeta(3)\,,\quad \sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{2^n}=\tfrac{1}{3}\pi^2\,,\quad \sum_{n=1}^{\infty}\frac{(n+1)\zeta_n^{\star}(\{1\}_3)}{2^n}=\tfrac{1}{3}\pi^2+3\zeta(3)\,,$$

and Eqs (4.5) *and* (4.6) *give the next three:*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta^{\star}_{n+1}(\{1\}_3)}{2^n n(n+1)} &= 4-\ln(2)-\tfrac{1}{12}\pi^2-\tfrac{3}{4}\zeta(3)-\tfrac{7}{720}\pi^4\,,\\ \sum_{n=1}^{\infty}\frac{h_n^{(2)}(H_{n+1}^2+H_{n+1}^{(2)})}{2^n(n+1)} &= -\tfrac{2}{3}\pi^2+\tfrac{2}{3}\ln(2)\pi^2+5\zeta(3)\,,\\ \sum_{n=1}^{\infty}\frac{h_n^{(3)}\zeta^{\star}_{n+1}(\{1\}_3)}{2^n(n+1)} &= -\tfrac{1}{2}\pi^2+\tfrac{1}{3}\ln(2)\pi^2+\zeta(3)-\tfrac{1}{3}\ln^4(2)+\tfrac{1}{3}\ln^2(2)\pi^2\\ &\quad+\tfrac{11}{90}\pi^4-4\ln(2)\zeta(3)-8\,\mathrm{Li}_4(\tfrac{1}{2})\,.\end{aligned}$$

Similarly to Theorem 4.1, the next result related to the integral $\mathcal{I}(r,k,l;-1)$ can be established.

Theorem 4.5. *For any integers $k,l\geq 1$ and $r\geq 0$, with $l\geq r$, the following series identity holds:*

$$\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_r\frac{\zeta^{\star}_n(\{1\}_l)}{2^n n\cdot n!}=\frac{(-1)^{k+l}}{k!l!}\sum_{j=0}^{k}\binom{k}{j}(-\ln(2))^{k-j}\left\{\mathcal{K}(j,l)+\sum_{i=1}^{r}2^{i-1}\mathcal{I}(i,j,l;-1)\right\}\,, \tag{4.8}$$

where

$$\mathcal{K}(j,l):=\int_0^1\frac{\ln^j(1+x)\ln^l(x)}{1-x}\mathrm{d}x=\begin{cases}(-1)^{j+l}j!l!\zeta(l+1,\bar{1},\{1\}_{j-1}) & j,l\geq 1\,,\\ (-1)^l l!\zeta(l+1) & j=0,l\geq 1\,.\end{cases} \tag{4.9}$$

Therefore, the r-Stirling series on the left-hand side of (4.8) *can be expressed in terms of AMZVs.*

Proof. For $k\geq 1$, using (1.6) and (4.2), we expand the following integral:

$$\mathcal{H}_3(r,k,l):=\int_0^1\frac{\ln^k(1-\frac{x}{2})\ln^l(1-x)}{(1-\frac{x}{2})^r x}\mathrm{d}x$$

in two different ways and obtain

$$\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_r\frac{\zeta^{\star}_n(\{1\}_l)}{2^n n\cdot n!}=\frac{(-1)^{k+l}2^r}{k!l!}\sum_{j=0}^{k}\binom{k}{j}(-\ln(2))^{k-j}\int_0^1\frac{\ln^j(1+x)\ln^l(x)}{(1+x)^r(1-x)}\mathrm{d}x\,. \tag{4.10}$$

Applying partial fraction decomposition to the inner integral in (4.10) gives

$$\begin{aligned}\int_0^1\frac{\ln^j(1+x)\ln^l(x)}{(1+x)^r(1-x)}\mathrm{d}x &= \frac{1}{2^r}\int_0^1\frac{\ln^j(1+x)\ln^l(x)}{1-x}\mathrm{d}x+\sum_{i=1}^{r}\frac{1}{2^{r-i+1}}\int_0^1\frac{\ln^j(1+x)\ln^l(x)}{(1+x)^i}\mathrm{d}x\\ &= \frac{1}{2^r}\mathcal{K}(j,l)+\sum_{i=1}^{r}\frac{1}{2^{r-i+1}}\mathcal{I}(i,j,l;-1)\,.\end{aligned}$$

Then, we obtain (4.8). Note that the integral $\mathcal{K}(j,l)$ satisfies the AMZV expression (4.9) (see [10, Theorem 2.9] for details). Hence, the final assertion follows. □

Example 4.4. *Here are two special r-Stirling series obtained from Theorem 4.5:*

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{\begin{bmatrix}n\\3\end{bmatrix}_1(H_n^2+H_n^{(2)})}{2^n n\cdot n!}&=-\tfrac{1}{18}\ln^6(2)+\tfrac{1}{48}\ln^4(2)\pi^2+\tfrac{1}{48}\ln^2(2)\pi^4+\tfrac{247}{30240}\pi^6-\tfrac{7}{12}\ln^3(2)\zeta(3)\\&\quad-\tfrac{7}{16}\ln(2)\pi^2\zeta(3)+\tfrac{75}{64}\zeta(3)^2-4\zeta(\bar{5},1)-2\ln^2(2)\operatorname{Li}_4(\tfrac{1}{2})\\&\quad-\tfrac{1}{2}\pi^2\operatorname{Li}_4(\tfrac{1}{2})-4\ln(2)\operatorname{Li}_5(\tfrac{1}{2})-4\operatorname{Li}_6(\tfrac{1}{2}),\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\begin{bmatrix}n\\2\end{bmatrix}_2\zeta_n^\star(\{1\}_3)}{2^n n\cdot n!}&=\tfrac{3}{2}\zeta(3)+\tfrac{1}{6}\ln^4(2)-\tfrac{1}{6}\ln^2(2)\pi^2-\tfrac{11}{180}\pi^4+2\ln(2)\zeta(3)+4\operatorname{Li}_4(\tfrac{1}{2})\\&\quad-\tfrac{1}{30}\ln^5(2)+\tfrac{1}{18}\ln^3(2)\pi^2+\tfrac{11}{180}\ln(2)\pi^4-\ln^2(2)\zeta(3)-\tfrac{1}{3}\pi^2\zeta(3)\\&\quad-\tfrac{1}{4}\zeta(5)+4\operatorname{Li}_5(\tfrac{1}{2})-\tfrac{1}{144}\ln^4(2)\pi^2+\tfrac{1}{144}\ln^2(2)\pi^4+\tfrac{67}{30240}\pi^6\\&\quad-\tfrac{7}{48}\ln(2)\pi^2\zeta(3)+\tfrac{1}{16}\zeta(3)^2-\tfrac{5}{2}\zeta(\bar{5},1)-\tfrac{1}{6}\pi^2\operatorname{Li}_4(\tfrac{1}{2}).\end{aligned}$$

By specializing the parameters r,k in Theorem 4.5, we obtain the expressions for three parametric series.

Corollary 4.6. *For any integers $k,l\geq 1$, the following Stirling series are reducible to AMZVs:*

$$\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}\frac{\zeta_n^\star(\{1\}_l)}{2^n n\cdot n!}=\sum_{j=1}^{k}\frac{\ln^{k-j}(2)}{(k-j)!}\zeta(l+1,\bar{1},\{1\}_{j-1})+\frac{\ln^k(2)\zeta(l+1)}{k!},\tag{4.11}$$

$$\begin{aligned}\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}\frac{\zeta_n^\star(\{1\}_l)}{2^n n\cdot n!}&=\sum_{n=k}^{\infty}\frac{\zeta_n(\{1\}_k)\zeta_n^\star(\{1\}_l)}{2^n n}\\&=\frac{2\ln^k(2)\lambda(l+1)}{k!}+\sum_{j=1}^{k}\frac{\ln^{k-j}(2)}{(k-j)!}\left\{\zeta(l+1,\bar{1},\{1\}_{j-1})-\zeta(\overline{l+1},\{1\}_j)\right\},\end{aligned}\tag{4.12}$$

and for any integers $l\geq r\geq 0$ with $l\geq 1$, the following series are reducible to AMZVs:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{h_n^{(r)}\zeta_n^\star(\{1\}_l)}{2^n n}&=\sum_{i=1}^{r}2^{i-1}\varOmega_i(1,l+1;-1)+\ln(2)\sum_{i=0}^{r-1}\sum_{j=0}^{i}\frac{2^i}{i!}\begin{bmatrix}i\\j\end{bmatrix}\eta(l+1-j)\\&\quad+\zeta(l+1,\bar{1})+\ln(2)\zeta(l+1).\end{aligned}\tag{4.13}$$

Here, Eq (4.11) *can also be found in [10, Theorem 2.9, Eq (34)], and $\lambda(s)$ is the Dirichlet lambda function defined by $\lambda(s):=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}$.*

Proof. In Theorem 4.5, setting $r=0$ gives Eq (4.11) immediately, while setting $r=1$ yields

$$\begin{aligned}\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}\frac{\zeta_n^\star(\{1\}_l)}{2^n n\cdot n!}&=\frac{(-1)^{k+l}}{k!l!}\sum_{j=1}^{k}\binom{k}{j}(-\ln(2))^{k-j}\left\{\mathcal{K}(j,l)+\mathcal{I}(1,j,l;-1)\right\}\\&\quad+\frac{(-1)^l\ln^k(2)}{k!l!}\left\{\mathcal{K}(0,l)+\mathcal{I}(1,0,l;-1)\right\},\end{aligned}$$

which, together with Eqs (2.2)–(2.4) and (4.9), leads us to Eq (4.12). Additionally, substituting $k=1$ into Theorem 4.5 yields Eq (4.13). □

Example 4.5. *The following four series are obtained from Eq* (4.12)*:*

$$\sum_{n=1}^{\infty}\frac{H_n^2}{2^n n}=\tfrac{7}{8}\zeta(3)\,,\quad \sum_{n=1}^{\infty}\frac{H_n\zeta_n^\star(\{1\}_3)}{2^n n}=-\tfrac{7}{48}\pi^2\zeta(3)+\tfrac{93}{32}\zeta(5)\,,$$

$$\sum_{n=2}^{\infty}\frac{(H_n^2-H_n^{(2)})H_n}{2^n n}=-\tfrac{1}{4}\ln^4(2)+\tfrac{1}{4}\ln^2(2)\pi^2+\tfrac{17}{240}\pi^4-\tfrac{21}{4}\ln(2)\zeta(3)-6\,\mathrm{Li}_4(\tfrac{1}{2})\,,$$

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\zeta_n(1,1)\zeta_n^\star(1,1)}{2^n n}=\frac{1}{4}\sum_{n=2}^{\infty}\frac{H_n^4-(H_n^{(2)})^2}{2^n n}&=-\tfrac{1}{15}\ln^5(2)+\tfrac{1}{18}\pi^2\ln^3(2)-\tfrac{7}{8}\ln^2(2)\zeta(3)\\&\quad-\tfrac{7}{32}\pi^2\zeta(3)+\tfrac{155}{32}\zeta(5)-2\ln(2)\,\mathrm{Li}_4(\tfrac{1}{2})-2\,\mathrm{Li}_5(\tfrac{1}{2})\,,\end{aligned}$$

and the next two are derived from Eq (4.13)*:*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{h_n^{(2)}(H_n^2+H_n^{(2)})}{2^n n}&=-\tfrac{1}{3}\pi^2+\tfrac{1}{3}\ln(2)\pi^2+\tfrac{5}{2}\zeta(3)-\tfrac{1}{6}\ln^4(2)+\tfrac{1}{6}\ln^2(2)\pi^2+\tfrac{49}{720}\pi^4\\&\quad-\tfrac{7}{2}\ln(2)\zeta(3)-4\,\mathrm{Li}_4(\tfrac{1}{2})\,,\\ \sum_{n=1}^{\infty}\frac{h_n^{(3)}\zeta_n^\star(\{1\}_3)}{2^n n}&=-\tfrac{1}{4}\pi^2+\tfrac{1}{6}\ln(2)\pi^2-\zeta(3)-\tfrac{1}{3}\ln^4(2)+\tfrac{1}{3}\ln^2(2)\pi^2+\tfrac{11}{90}\pi^4\\&\quad-4\ln(2)\zeta(3)-8\,\mathrm{Li}_4(\tfrac{1}{2})-\tfrac{7}{48}\pi^2\zeta(3)+\tfrac{93}{32}\zeta(5)\,.\end{aligned}$$

5. Conclusions

In this paper, by using the generating function of the r-Stirling numbers of the first kind $\left[{n \atop k}\right]_r$ and special integrals, we obtain three general identities for the following r-Stirling series:

$$\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_r\frac{H_n^{(l+1)}}{2^n n!}\,,\qquad \sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_r\frac{\zeta_{n+1}^\star(\{1\}_l)}{2^n(n+1)!}\,,\qquad \sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_r\frac{\zeta_n^\star(\{1\}_l)}{2^n n\cdot n!}\,,$$

and show that they are all expressible in terms of AMZVs via the r-Stirling series $\Omega_r(k,l;q) := \sum_{n=1}^{\infty}\left[{n \atop k}\right]_r\frac{q^n}{n!(n+1)^l}$. Furthermore, we derive AMZV expressions for several parametric series involving binomial coefficients, Stirling numbers, harmonic numbers, and hyperharmonic numbers.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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