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Research article

Statistical properties of the Sine New X–Lindley distribution: fuzzy reliability analysis and applications

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Abstract: This paper introduces the Sine New X–Lindley (SNXL) distribution, a new one-parameter lifetime model built by applying a sine transformation to the cumulative distribution function of the New X–Lindley distribution. The model is flexible for positive, skewed data and adds a useful option to reliability and survival analysis. We derived its main properties, including the density, distribution, survival and hazard functions, quantile function, moments, and order statistics, and also studied features such as stochastic ordering and tail behavior. We examined parameter estimation using six methods: maximum likelihood, ordinary least squares, Anderson–Darling, Cramér–von Mises, least squares, and the method of moments. Their performance was compared through a Monte Carlo simulation using bias and mean-squared error. We also extended the model to a fuzzy reliability setting by treating the scale parameter as a fuzzy number and obtaining explicit α-cut forms for reliability and mean time to failure. To show its practical value, we applied the SNXL distribution to three real datasets on daily precipitation, heavy precipitation, and household income. In all three cases, it provided the best overall fit among the competing models under the reported goodness-of-fit measures. These results show that the SNXL distribution is a simple and effective model for skewed positive data in reliability, environmental, and economic applications.

Keywords: New X–Lindley distribution; sine transformation; hazard function; fuzzy reliability; quantile function; simulation; applications
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1. Introduction

Lifetime distributions play a central role in reliability theory, survival analysis, hydrology, economics, and many other areas where the main interest lies in modeling positive-valued data. In real applications, however, such data often show strong skewness, nonstandard tail behavior, or changing failure patterns that are not always described adequately by classical models such as the exponential, Weibull, or gamma distributions. This practical limitation has led to a sustained interest in developing new probability models that offer greater flexibility while remaining mathematically tractable and easy to apply.

Within this broad line of research, the Lindley family and its extensions have received considerable attention because of their usefulness in modeling lifetime data with relatively simple analytical forms. Important contributions in this direction include the two-parameter Lindley distribution [21], the modified Weibull and beta-Weibull models [13, 23], gamma–Lindley generalizations [17, 18], and several related Lindley-type extensions, such as the quasi-Lindley, new quasi-Lindley, generalized Lindley, and pseudo-Lindley models [5, 20, 25, 26]. More recent studies have continued to expand this literature by proposing flexible Lindley-based constructions with useful inferential and applied properties; see, for example, the bounded Lindley exponential model of Abd El-Bar et al. [1], the power unit inverse Lindley distribution of Gemeay et al. [7], and the novel distribution introduced by Gemeay et al. [8] for modeling inflation rates and mechanical failure data. These developments illustrate the continuing effort to design models that are flexible enough for practical data analysis without losing analytical manageability.

More recently, attention has also turned to the X-Lindley class and its variants. In particular, the X-Lindley model [6], the exponentiated X-Lindley distribution [2], and the New X–Lindley distribution (NXLD) [10] have emerged as useful additions to the Lindley family, with promising applications in modeling positive and asymmetric data. At the same time, generator-based methods have become an effective and popular strategy for constructing richer classes of distributions from existing baselines. Among these methods, trigonometric generators based on the sine function have proved especially attractive, since they often produce models with improved shape flexibility while preserving relatively simple forms. Representative examples include the sine exponential distribution [11], the sine Lindley model [24], and more general sine-generated families [12, 15, 16].

The appeal of sine-based generators is not only theoretical. Recent studies have shown that these constructions can perform very well in practice when fitting complex lifetime and survival data. For instance, Muse et al. [16] studied the sine–G family in a survival-modeling framework and showed that sine-generated models can provide competitive empirical performance in challenging settings. This growing body of work suggests that combining a well-established baseline distribution with a sine transformation may lead to useful new models for applied statistical analysis.

Motivated by these developments, this paper introduces the **Sine New X–Lindley (SNXL)** distribution. The proposed model is obtained by applying a sine-based transformation to the cumulative distribution function of the New X–Lindley distribution. In this way, the SNXL model extends the NXLD baseline and provides an additional flexible alternative for modeling positive data arising in reliability, survival, environmental, and economic contexts.

The main contributions of this work can be summarized as follows. First, we derive the fundamental structural properties of the SNXL distribution, including its probability density function, cumulative distribution function, survival function, hazard function, quantile function, and several related characteristics. Second, we examine the model's key theoretical features, including stochastic ordering and other distributional properties. Third, we study parameter estimation using six frequentist methods, namely, maximum likelihood estimation (MLE), ordinary least squares estimation (OLSE), Anderson–Darling estimation (ADE), Cramér–von Mises estimation (CVM), least squares estimation (LSE), and the method of moments (MM), and compare their performance through a Monte Carlo simulation study based on bias and mean squared error. Fourth, we extend the model to a fuzzy reliability setting and present a numerical illustration under fuzzy parameter uncertainty. Finally, we assess the practical usefulness of the SNXL distribution through three real-data applications involving daily precipitation data, heavy precipitation data, and household income data, using the negative log-likelihood −LL, Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov–Smirnov (KS) statistic, and Anderson–Darling (AD) statistic as comparison criteria.

The remainder of the paper is organized as follows. Section 2 introduces the SNXL distribution and develops its principal theoretical results. Section 3 presents the fuzzy reliability analysis together with a numerical illustration. Section 4 reports the Monte Carlo comparison of the competing estimators. Section 5 contains the real-data applications and compares the SNXL model with several alternative distributions. Section 6 concludes the paper and discusses possible directions for future research.

2. Sine–transformed New X–Lindley distribution

In this section, we introduce the **Sine New X–Lindley (SNXL)** distribution, obtained by applying a sine-based transformation to the New X–Lindley distribution. Let $Z \sim NXLD(\xi)$ with $\xi > 0$, whose probability density function and cumulative distribution function are

$$f_Z(z) = \frac{\xi}{2}(1+\xi z)e^{-\xi z}, \qquad F_Z(z) = 1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right), \qquad z > 0.$$

The new model is constructed through the sine–F generator,

$$G_Z(z) = \sin\left(\frac{\pi}{2}F_Z(z)\right).$$

This transformation extends the baseline model while preserving a relatively tractable analytic form. In particular, the corresponding density is

$$g_Z(z) = \frac{\pi}{2} f_Z(z) \cos\left(\frac{\pi}{2}F_Z(z)\right).$$

Sine-generated constructions have received increasing attention in the lifetime-distribution literature because they offer a convenient way to enrich classical models without sacrificing mathematical manageability [11, 16, 24]. In this spirit, the SNXL distribution provides a new extension of the New X–Lindley model and offers an additional alternative for modeling positive and asymmetric data in reliability and survival analysis.

Theorem 2.1 (SNXL distribution)**.** *Let Z be a positive random variable with parameter $\xi > 0$, and let*

$$F_Z(z) = 1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right), \qquad z > 0,$$

denote the cumulative distribution function of the New X–Lindley distribution. Define

$$G_Z(z) = \sin\left(\frac{\pi}{2}F_Z(z)\right), \qquad z > 0.$$

Then Z is said to follow the Sine–New X–Lindley *distribution, denoted by $SNXL(\xi)$, with cumulative distribution function (cdf)*

$$G_Z(z) = \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right], \qquad z > 0, \tag{2.1}$$

and probability density function (pdf)

$$g_Z(z) = \frac{\pi}{4}\,\xi(1 + \xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right), \qquad z > 0. \tag{2.2}$$

Proof. Starting from

$$G_Z(z) = \sin\left(\frac{\pi}{2}F_Z(z)\right)$$

and using

$$F_Z(z) = 1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right),$$

we obtain

$$G_Z(z) = \sin\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right).$$

Applying the identity

$$\sin\left(\frac{\pi}{2}(1 - u)\right) = \cos\left(\frac{\pi}{2}u\right),$$

with

$$u = e^{-\xi z}\left(1 + \frac{\xi z}{2}\right),$$

yields

$$G_Z(z) = \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right],$$

which proves (2.1). The expression in (2.2) follows directly from the general sine–F formula

$$g_Z(z) = \frac{\pi}{2}\,f_Z(z)\cos\left(\frac{\pi}{2}F_Z(z)\right),$$

after substituting

$$f_Z(z) = \frac{\xi}{2}(1 + \xi z)e^{-\xi z}.$$

□

Theorem 2.2 (Shape and mode). *Let $Z \sim SNXL(\xi)$ with $\xi > 0$ and density*

$$g_Z(z) = \frac{\pi}{4}\,\xi(1 + \xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right), \qquad z > 0. \tag{2.3}$$

Then g_Z is strictly decreasing on $(0, \infty)$. Consequently, the $SNXL(\xi)$ distribution is unimodal, and its unique mode is attained at the left endpoint of the support, namely,

$$z^\star = 0.$$

Proof. Using
$$\cos\left(\frac{\pi}{2}(1-u)\right) = \sin\left(\frac{\pi}{2}u\right),$$
the density can be rewritten as
$$g_Z(z) = \frac{\pi}{4}\,\xi(1+\xi z)e^{-\xi z}\sin\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right], \qquad z > 0.$$
Let $t = \xi z > 0$. Since $\xi > 0$, it is enough to study monotonicity with respect to t. Define
$$q(t) = (1+t)e^{-t}\sin\left(\frac{\pi}{2}a(t)\right), \qquad a(t) = e^{-t}\left(1+\frac{t}{2}\right).$$
Then
$$g_Z(z) = \frac{\pi}{4}\,\xi\, q(\xi z).$$
Hence g_Z is decreasing in z if and only if q is decreasing in t.

Consider the log-density
$$\ell(t) = \log q(t) = \log(1+t) - t + \log\sin\left(\frac{\pi}{2}a(t)\right).$$
Differentiating, we obtain
$$\ell'(t) = \frac{1}{1+t} - 1 + \frac{\pi}{2}a'(t)\cot\left(\frac{\pi}{2}a(t)\right).$$
Now
$$a'(t) = \frac{d}{dt}\left[e^{-t}\left(1+\frac{t}{2}\right)\right] = -\frac{1+t}{2}e^{-t} < 0, \qquad t > 0.$$
Therefore,
$$\ell'(t) = -\frac{t}{1+t} - \frac{\pi}{4}(1+t)e^{-t}\cot\left(\frac{\pi}{2}a(t)\right).$$
Since
$$a(t) = e^{-t}\left(1+\frac{t}{2}\right) \in (0,1),$$
we have
$$0 < \frac{\pi}{2}a(t) < \frac{\pi}{2}, \qquad \cot\left(\frac{\pi}{2}a(t)\right) > 0.$$
Thus both terms in $\ell'(t)$ are strictly negative, which implies
$$\ell'(t) < 0, \qquad t > 0.$$
Hence ℓ and therefore q are strictly decreasing on $(0,\infty)$. It follows that $g_Z(z)$ is strictly decreasing on $(0,\infty)$.

Because g_Z is continuous on $[0,\infty)$, strictly decreasing for $z > 0$, and satisfies $g_Z(z) \to 0$ as $z \to \infty$, its maximum is attained at the left endpoint of the support. Therefore the $SNXL(\xi)$ distribution is unimodal with the unique mode
$$z^\star = 0.$$
□

The main reliability characteristics of the $SNXL(\xi)$ distribution follow directly from its cdf and pdf. For $Z \sim SNXL(\xi)$, the survival function, hazard rate, reversed hazard rate, and cumulative hazard function, respectively, are

$$\bar{G}_Z(z) = 1 - \cos\left[\frac{\pi}{2} e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right], \tag{2.4}$$

$$h_Z(z) = \frac{\frac{\pi}{4}\,\xi(1+\xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right)}{1 - \cos\left[\frac{\pi}{2} e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]}, \tag{2.5}$$

$$r_Z(z) = \frac{\frac{\pi}{4}\,\xi(1+\xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right)}{\cos\left[\frac{\pi}{2} e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]}, \tag{2.6}$$

$$H_Z(z) = -\log\left(1 - \cos\left[\frac{\pi}{2} e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right). \tag{2.7}$$

Figure 1 illustrates the effect of the parameter ξ on the probability density, cumulative distribution, and hazard functions. The figures show that the density becomes more concentrated near the origin as ξ increases, while the distribution function approaches one more rapidly and the hazard function shifts upward.

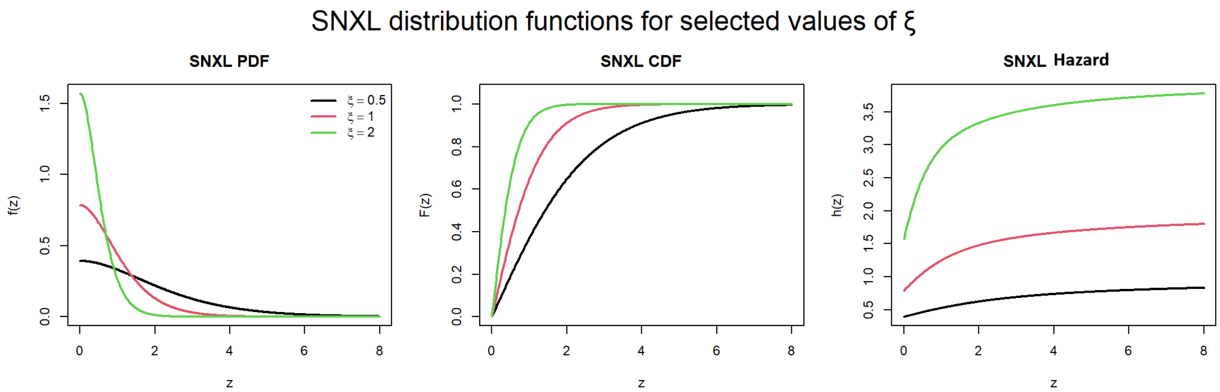

Figure 1. Probability density, cumulative distribution, and hazard functions of the SNXL distribution for selected values of ξ.

Proposition 2.3 (Boundary behavior of the hazard rate)**.** *Let* $Z \sim SNXL(\xi)$ *with* $\xi > 0$, *and let* h_Z *denote its hazard rate. Then*

$$h_Z(z) = \frac{\frac{\pi}{4}\,\xi(1+\xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right)}{1 - \cos\left[\frac{\pi}{2} e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]}, \qquad z > 0, \tag{2.8}$$

and

$$h_Z(0^+) = \frac{\pi}{4}\,\xi. \tag{2.9}$$

Proof. By definition,

$$h_Z(z) = \frac{g_Z(z)}{\bar{G}_Z(z)},$$

where

$$g_Z(z) = \frac{\pi}{4}\,\xi(1+\xi z)e^{-\xi z}\cos\!\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right]\right)$$

and

$$\bar{G}_Z(z) = 1 - \cos\!\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right].$$

Substituting these expressions gives (2.8). As $z \downarrow 0$,

$$e^{-\xi z}\left(1+\frac{\xi z}{2}\right) \to 1, \qquad \bar{G}_Z(z) \to 1 - \cos\!\left(\frac{\pi}{2}\right) = 1,$$

and

$$g_Z(z) \to \frac{\pi}{4}\,\xi(1+0)e^{0}\cos\!\left(\frac{\pi}{2}(1-1)\right) = \frac{\pi}{4}\,\xi.$$

Hence

$$h_Z(0^+) = \frac{\pi}{4}\,\xi.$$

□

Remark 2.4 (Hazard behavior). For the corrected $SNXL(\xi)$ model, the hazard rate is given by (2.8). In particular, its value at the origin is

$$h_Z(0^+) = \frac{\pi}{4}\,\xi.$$

Unlike the reduced model with cdf $\cos\!\left(\frac{\pi}{2}e^{-\xi z}\right)$, the present hazard function does not simplify to an elementary cotangent form. Therefore, any claim about monotonicity or a finite positive horizontal asymptote must be established separately from (2.8).

Theorem 2.5 (Tail behavior)**.** *The survival function of $Z \sim SNXL(\xi)$ satisfies*

$$\bar{G}_Z(z) = 1 - G_Z(z) \sim \frac{\pi^2}{32}\,\xi^2 z^2 e^{-2\xi z}, \qquad z \to \infty. \tag{2.10}$$

Consequently, the $SNXL(\xi)$ distribution has an exponentially decaying tail with rate 2ξ, modulated by the polynomial factor z^2.

Proof. Recall that

$$\bar{G}_Z(z) = 1 - \cos\!\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right].$$

Set

$$y_z = \frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right).$$

As $z \to \infty$, we have $y_z \to 0$, and the expansion

$$1 - \cos y \sim \frac{y^2}{2}, \qquad y \to 0,$$

gives

$$\bar{G}_Z(z) \sim \frac{y_z^2}{2}.$$

Therefore,

$$\bar{G}_Z(z) \sim \frac{1}{2}\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right]^2 = \frac{\pi^2}{8}e^{-2\xi z}\left(1+\frac{\xi z}{2}\right)^2.$$

Since

$$\left(1+\frac{\xi z}{2}\right)^2 \sim \frac{\xi^2 z^2}{4}, \qquad z\to\infty,$$

it follows that

$$\bar{G}_Z(z) \sim \frac{\pi^2}{8}e^{-2\xi z}\cdot\frac{\xi^2 z^2}{4} = \frac{\pi^2}{32}\,\xi^2 z^2 e^{-2\xi z},$$

which proves (2.10). □

Proposition 2.6 (MGF and raw moments)**.** *Let $Z \sim SNXL(\xi)$ with $\xi > 0$. Then, for $t < 2\xi$, the moment generating function of Z is*

$$M_Z(t) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+2}\sum_{j=0}^{2k+1}\binom{2k+1}{j}\frac{1}{2^j}\left[\frac{j!}{\lambda_k(t)^{j+1}}+\frac{(j+1)!}{\lambda_k(t)^{j+2}}\right], \tag{2.11}$$

where

$$\lambda_k(t) = 2k+2-\frac{t}{\xi}.$$

Moreover, for every integer $r \ge 0$, the rth raw moment is

$$\mathbb{E}[Z^r] = \frac{1}{2\xi^r}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+2}\sum_{j=0}^{2k+1}\binom{2k+1}{j}\frac{1}{2^j}\left[\frac{(r+j)!}{(2k+2)^{r+j+1}}+\frac{(r+j+1)!}{(2k+2)^{r+j+2}}\right]. \tag{2.12}$$

Proof. Using

$$\cos\left(\frac{\pi}{2}(1-u)\right) = \sin\left(\frac{\pi}{2}u\right),$$

the density may be written as

$$g_Z(z) = \frac{\pi}{4}\xi(1+\xi z)e^{-\xi z}\sin\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right].$$

Expanding the sine function into its power series gives

$$g_Z(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+2}\xi(1+\xi z)e^{-(2k+2)\xi z}\left(1+\frac{\xi z}{2}\right)^{2k+1}.$$

Substituting this expression into

$$M_Z(t) = \int_0^{\infty} e^{tz} g_Z(z)\,dz$$

and using the change of variable $x = \xi z$, we obtain

$$M_Z(t) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+2}\int_0^{\infty}(1+x)e^{-\lambda_k(t)x}\left(1+\frac{x}{2}\right)^{2k+1}dx,$$

where

$$\lambda_k(t) = 2k + 2 - \frac{t}{\xi}.$$

Expanding

$$\left(1 + \frac{x}{2}\right)^{2k+1} = \sum_{j=0}^{2k+1} \binom{2k+1}{j} \frac{x^j}{2^j}$$

and applying

$$\int_0^\infty x^m e^{-\lambda x}\, dx = \frac{m!}{\lambda^{m+1}}, \qquad \lambda > 0,$$

yields (2.11). The raw-moment formula follows similarly from

$$\mathbb{E}[Z^r] = \int_0^\infty z^r g_Z(z)\, dz$$

after the same change of variable and expansion. □

Corollary 2.7 (Mean of the $SNXL(\xi)$ distribution)**.** *Let $Z \sim SNXL(\xi)$ with $\xi > 0$. Then the mean of Z is*

$$\mathbb{E}[Z] = \frac{1}{2\xi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+2} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \frac{1}{2^j} \left[\frac{(j+1)!}{(2k+2)^{j+2}} + \frac{(j+2)!}{(2k+2)^{j+3}} \right]. \tag{2.13}$$

Proof. The result follows directly from (2.12) by taking $r = 1$. □

Proposition 2.8 (Quantile function and random variate generation)**.** *Let $Z \sim SNXL(\xi)$, $\xi > 0$, with cumulative distribution function*

$$G_Z(z) = \cos\left[\frac{\pi}{2} e^{-\xi z} \left(1 + \frac{\xi z}{2}\right)\right], \qquad z > 0.$$

Then the quantile function $Q : (0,1) \to (0,\infty)$ is given by

$$Q(u) = -\frac{1}{\xi} \left[W_{-1}\left(-\frac{4}{\pi e^2} \arccos(u)\right) + 2 \right], \qquad 0 < u < 1, \tag{2.14}$$

where W_{-1} denotes the lower real branch of the Lambert–W function.

Consequently, if $U \sim$ Uniform$(0,1)$ and

$$Z = Q(U),$$

then $Z \sim SNXL(\xi)$.

Proof. Let $u \in (0,1)$ and set $u = G_Z(z)$. Then

$$u = \cos\left[\frac{\pi}{2} e^{-\xi z} \left(1 + \frac{\xi z}{2}\right)\right].$$

Since $0 < u < 1$, it follows that

$$\arccos(u) = \frac{\pi}{2} e^{-\xi z} \left(1 + \frac{\xi z}{2}\right).$$

Thus

$$e^{-\xi z}\left(1+\frac{\xi z}{2}\right)=\frac{2}{\pi}\arccos(u).$$

Now define

$$y=1+\frac{\xi z}{2}.$$

Then

$$\xi z=2(y-1), \qquad e^{-\xi z}=e^{-2(y-1)}=e^{2}e^{-2y}.$$

Substituting into the previous equation yields

$$ye^{-2y}=\frac{2}{\pi e^{2}}\arccos(u).$$

Multiplying both sides by -2, we obtain

$$(-2y)e^{-2y}=-\frac{4}{\pi e^{2}}\arccos(u).$$

Let

$$w=-2y.$$

Then

$$we^{w}=-\frac{4}{\pi e^{2}}\arccos(u),$$

so by the defining property of the Lambert–W function,

$$w=W\left(-\frac{4}{\pi e^{2}}\arccos(u)\right).$$

Hence

$$y=-\frac{1}{2}W\left(-\frac{4}{\pi e^{2}}\arccos(u)\right).$$

Using

$$z=\frac{2}{\xi}(y-1),$$

we obtain

$$z=-\frac{1}{\xi}\left[W\left(-\frac{4}{\pi e^{2}}\arccos(u)\right)+2\right].$$

Since

$$0<\arccos(u)<\frac{\pi}{2},$$

the argument of W lies in $(-2/e^2, 0) \subset (-1/e, 0)$, where both real branches are defined. Because $z > 0$, we must have

$$y=1+\frac{\xi z}{2}>1, \qquad \text{hence} \qquad w=-2y<-2.$$

This excludes the principal branch W_0, so the relevant branch is W_{-1}. Therefore

$$Q(u)=-\frac{1}{\xi}\left[W_{-1}\left(-\frac{4}{\pi e^{2}}\arccos(u)\right)+2\right], \qquad 0<u<1,$$

which proves (2.14).

Finally, if $U \sim \text{Uniform}(0,1)$ and $Z = Q(U)$, then

$$\Pr(Z \le z) = \Pr(Q(U) \le z) = \Pr(U \le G_Z(z)) = G_Z(z),$$

so $Z \sim SNXL(\xi)$. □

Remark 2.9 (Median). The median of $Z \sim SNXL(\xi)$ is obtained by evaluating the quantile function at $u = \frac{1}{2}$. Using (2.14), we obtain

$$\mathrm{Med}(Z) = -\frac{1}{\xi}\left[W_{-1}\left(-\frac{4}{\pi e^2}\arccos\left(\tfrac{1}{2}\right)\right) + 2\right].$$

Since

$$\arccos\left(\tfrac{1}{2}\right) = \frac{\pi}{3},$$

it follows that

$$\mathrm{Med}(Z) = -\frac{1}{\xi}\left[W_{-1}\left(-\frac{4}{3e^2}\right) + 2\right].$$

Numerically,

$$W_{-1}\left(-\frac{4}{3e^2}\right) \approx -2.5505817436,$$

and therefore

$$\mathrm{Med}(Z) \approx \frac{0.5505817436}{\xi}. \tag{2.15}$$

Theorem 2.10 (Order statistics for $SNXL(\xi)$)**.** *Let $Z_1, \ldots, Z_n$ be i.i.d. random variables from the $SNXL(\xi)$ distribution, and let $Z_{i:n}$ denote the ith order statistic, $i = 1, \ldots, n$. Then, for $z > 0$, the pdf of $Z_{i:n}$ is*

$$f_{i:n}(z) = \frac{n!}{(i-1)!(n-i)!}\,[G_Z(z)]^{i-1}[1 - G_Z(z)]^{n-i} g_Z(z), \tag{2.16}$$

where

$$G_Z(z) = \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right], \tag{2.17}$$

$$g_Z(z) = \frac{\pi}{4}\,\xi(1 + \xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right). \tag{2.18}$$

Equivalently,

$$\begin{aligned} f_{i:n}(z) = {} & \frac{n!}{(i-1)!(n-i)!}\left[\cos\left(\frac{\pi}{2}e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right)\right]^{i-1} \\ & \times \left[1 - \cos\left(\frac{\pi}{2}e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right)\right]^{n-i} \\ & \times \frac{\pi}{4}\,\xi(1 + \xi z)e^{-\xi z}\cos\left(\frac{\pi}{2}\left[1 - e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right). \end{aligned} \tag{2.19}$$

Proof. For a continuous parent distribution with cdf G and pdf g, the pdf of the ith order statistic is

$$f_{i:n}(z) = \frac{n!}{(i-1)!(n-i)!}\,[G(z)]^{i-1}[1-G(z)]^{n-i}g(z), \qquad z \in \mathbb{R}.$$

Taking $G = G_Z$ and $g = g_Z$ for the $SNXL(\xi)$ distribution yields (2.16). Substituting the explicit forms of G_Z and g_Z gives (2.19). □

Theorem 2.11 (Stochastic ordering)**.** *Let $Z_{\xi_1} \sim SNXL(\xi_1)$ and $Z_{\xi_2} \sim SNXL(\xi_2)$, with $\xi_1 > \xi_2 > 0$. Then*

$$Z_{\xi_1} \leq_{\text{st}} Z_{\xi_2}. \tag{2.20}$$

That is, the $SNXL(\xi)$ family is stochastically decreasing in the parameter ξ.

Proof. The cumulative distribution function of $Z_\xi \sim SNXL(\xi)$ is

$$G(z;\xi) = \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right], \qquad z > 0.$$

For fixed $z > 0$, define

$$a(\xi) = e^{-\xi z}\left(1+\frac{\xi z}{2}\right).$$

Differentiating with respect to ξ, we obtain

$$a'(\xi) = -\frac{z}{2}e^{-\xi z}(1+\xi z) < 0.$$

Hence $a(\xi)$ is strictly decreasing in ξ. Since

$$0 < a(\xi) < 1,$$

we have

$$0 < \frac{\pi}{2}a(\xi) < \frac{\pi}{2},$$

and the function

$$x \mapsto \cos\left(\frac{\pi}{2}x\right)$$

is strictly decreasing on $(0, 1)$. Therefore $G(z;\xi)$ is strictly increasing in ξ for every fixed $z > 0$. In particular, if $\xi_1 > \xi_2$, then

$$G(z;\xi_1) \geq G(z;\xi_2), \qquad z > 0.$$

This is exactly the stochastic ordering

$$Z_{\xi_1} \leq_{\text{st}} Z_{\xi_2}.$$

□

3. Fuzzy reliability analysis for the Sine–New X–Lindley distribution

In many practical reliability settings, the scale parameter ξ cannot be specified exactly because of measurement error, incomplete information, or expert uncertainty. To accommodate this imprecision, we treat ξ as a fuzzy number, denoted by $\tilde{\xi}$, with membership function $\mu_{\tilde{\xi}}(\cdot)$.

For the $SNXL(\xi)$ model, the cumulative distribution function and the corresponding survival function are

$$G(z;\xi) = \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right], \qquad z>0,\ \xi>0,$$

and

$$S(z;\xi) = 1 - G(z;\xi) = 1 - \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right].$$

3.1. Fuzzy reliability via α–cuts

Definition 3.1 (Fuzzy reliability)**.** For a fixed mission time $z>0$, the fuzzy reliability associated with the $SNXL(\xi)$ model is defined by

$$\tilde{R}(z) = \{(S(z;\xi), \mu_{\tilde{\xi}}(\xi)) : \xi > 0\}.$$

In other words, the reliability is viewed as a fuzzy quantity induced by the fuzzy uncertainty in the parameter ξ.

Let the α–cut of the fuzzy parameter $\tilde{\xi}$ be

$$\tilde{\xi}_\alpha = [\xi_L(\alpha), \xi_U(\alpha)], \qquad \alpha \in [0,1].$$

Proposition 3.2 (α–cut of fuzzy reliability)**.** *For each fixed $z>0$ and $\alpha \in [0,1]$, the α–cut of the fuzzy reliability $\tilde{R}(z)$ is given by*

$$\tilde{R}_\alpha(z) = \left[\min_{\xi\in\tilde{\xi}_\alpha} S(z;\xi),\ \max_{\xi\in\tilde{\xi}_\alpha} S(z;\xi)\right].$$

Lemma 3.3 (Monotonicity of the reliability function)**.** *For each fixed $z>0$, the reliability function $S(z;\xi)$ is strictly decreasing in ξ.*

Proof. Fix $z>0$ and define

$$a(\xi) = e^{-\xi z}\left(1+\frac{\xi z}{2}\right).$$

Then the survival function can be written as

$$S(z;\xi) = 1 - \cos\left(\frac{\pi}{2}a(\xi)\right).$$

Differentiating $a(\xi)$ with respect to ξ, we obtain

$$a'(\xi) = \frac{d}{d\xi}\left[e^{-\xi z}\left(1+\frac{\xi z}{2}\right)\right] = -\frac{z}{2}e^{-\xi z}(1+\xi z) < 0, \qquad \xi > 0.$$

Moreover, for $z>0$ and $\xi>0$,

$$0 < a(\xi) < 1,$$

and therefore

$$0 < \frac{\pi}{2}a(\xi) < \frac{\pi}{2}, \qquad \sin\left(\frac{\pi}{2}a(\xi)\right) > 0.$$

Now differentiate the cdf:

$$\frac{\partial}{\partial \xi}G(z;\xi) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}a(\xi)\right)a'(\xi).$$

Since $a'(\xi) < 0$ and the sine term is positive, it follows that

$$\frac{\partial}{\partial \xi}G(z;\xi) > 0.$$

Hence $G(z;\xi)$ is strictly increasing in ξ. Consequently,

$$S(z;\xi) = 1 - G(z;\xi)$$

is strictly decreasing in ξ. □

Corollary 3.4 (Explicit α–cut of fuzzy reliability)**.** *Since $S(z;\xi)$ is strictly decreasing in ξ, the α–cut of the fuzzy reliability is*

$$\boxed{\tilde{R}_\alpha(z) = [S(z;\xi_U(\alpha)),\ S(z;\xi_L(\alpha))].}$$

Equivalently,

$$\boxed{\tilde{R}_\alpha(z) = \left[1 - \cos\left(\frac{\pi}{2}e^{-\xi_U(\alpha)z}\left(1 + \frac{\xi_U(\alpha)z}{2}\right)\right),\ 1 - \cos\left(\frac{\pi}{2}e^{-\xi_L(\alpha)z}\left(1 + \frac{\xi_L(\alpha)z}{2}\right)\right)\right].}$$

3.2. Fuzzy mean time to failure (FMTTF)

For the $SNXL(\xi)$ model, the mean time to failure is given by

$$\text{MTTF}(\xi) = \mathbb{E}[Z] = \int_0^\infty S(z;\xi)\,dz = \int_0^\infty \left\{1 - \cos\left[\frac{\pi}{2}e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right]\right\}dz.$$

Using the change of variable $t = \xi z$, we obtain

$$\text{MTTF}(\xi) = \frac{1}{\xi}\int_0^\infty \left\{1 - \cos\left[\frac{\pi}{2}e^{-t}\left(1 + \frac{t}{2}\right)\right]\right\}dt.$$

Therefore,

$$\text{MTTF}(\xi) = \frac{C_{SNXL}}{\xi}, \tag{3.1}$$

where

$$C_{SNXL} = \int_0^\infty \left\{1 - \cos\left[\frac{\pi}{2}e^{-t}\left(1 + \frac{t}{2}\right)\right]\right\}dt \tag{3.2}$$

is a positive constant independent of ξ.

Proposition 3.5 (FMTTF as a fuzzy number)**.** *The fuzzy mean time to failure is defined by*

$$\widetilde{\text{MTTF}} = \{(\text{MTTF}(\xi), \mu_{\tilde{\xi}}(\xi)) : \xi > 0\} = \left\{\left(\frac{C_{SNXL}}{\xi}, \mu_{\tilde{\xi}}(\xi)\right) : \xi > 0\right\}.$$

Proof. From (3.1), the mean time to failure is $\mathrm{MTTF}(\xi) = C_{SNXL}/\xi$. Applying the extension principle to the mapping $\xi \mapsto C_{SNXL}/\xi$ gives the required fuzzy representation. □

Corollary 3.6 (Explicit α–cut of FMTTF)**.** *Since the mapping $\xi \mapsto C_{SNXL}/\xi$ is strictly decreasing on* $(0, \infty)$*, the α–cut of the fuzzy mean time to failure is*

$$\boxed{\widetilde{\mathrm{MTTF}}_\alpha = \left[\frac{C_{SNXL}}{\xi_U(\alpha)}, \frac{C_{SNXL}}{\xi_L(\alpha)}\right].}$$

Remark 3.7. The constant C_{SNXL} in (3.2) does not depend on ξ and therefore needs to be evaluated only once, typically by numerical integration. This makes the computation of the fuzzy mean time to failure simple and efficient for any prescribed fuzzy parameter $\tilde{\xi}$.

3.3. Numerical illustration with a triangular fuzzy parameter

To illustrate the proposed fuzzy reliability framework, suppose that the scale parameter is represented by the triangular fuzzy number

$$\tilde{\xi} = (\xi_1, \xi_2, \xi_3) = (0.8,\ 1.0,\ 1.2).$$

Its α–cut is therefore given by

$$\tilde{\xi}_\alpha = [\xi_L(\alpha), \xi_U(\alpha)] = [\,0.8 + 0.2\alpha,\ 1.2 - 0.2\alpha\,], \qquad \alpha \in [0, 1].$$

This example is used to illustrate both the fuzzy reliability function and the fuzzy mean time to failure under parameter uncertainty.

Example A: Fuzzy reliability intervals. For a fixed mission time $z > 0$, the α–cut of the fuzzy reliability is

$$\tilde{R}_\alpha(z) = [\,S(z; \xi_U(\alpha)),\ S(z; \xi_L(\alpha))\,],$$

because the survival function is strictly decreasing in ξ. For the $SNXL(\xi)$ model,

$$S(z; \xi) = 1 - \cos\left[\frac{\pi}{2} e^{-\xi z}\left(1 + \frac{\xi z}{2}\right)\right].$$

Table 1 gives the resulting α–cut intervals for two mission times, namely, $z = 1$ and $z = 2$.

Table 1. α–cut intervals of the fuzzy reliability $\tilde{R}_\alpha(z)$ for $\tilde{\xi} = (0.8, 1.0, 1.2)$.

α	$\xi_L(\alpha)$	$\xi_U(\alpha)$	$\tilde{R}_\alpha(1)$	$\tilde{R}_\alpha(2)$
0.00	0.8000	1.2000	[0.2731, 0.4497]	[0.0487, 0.1586]
0.25	0.8500	1.1500	[0.2915, 0.4238]	[0.0568, 0.1376]
0.50	0.9000	1.1000	[0.3108, 0.3990]	[0.0661, 0.1192]
0.75	0.9500	1.0500	[0.3312, 0.3753]	[0.0767, 0.1031]
1.00	1.0000	1.0000	[0.3527, 0.3527]	[0.0890, 0.0890]

Example B: Fuzzy mean time to failure. For the corrected $SNXL(\xi)$ model,

$$\mathrm{MTTF}(\xi) = \frac{C_{SNXL}}{\xi},$$

where

$$C_{SNXL} = \int_0^\infty \left\{1 - \cos\left[\frac{\pi}{2}e^{-t}\left(1 + \frac{t}{2}\right)\right]\right\} dt \approx 0.8981.$$

Hence the α–cut of the fuzzy mean time to failure is

$$\widetilde{\mathrm{MTTF}}_\alpha = \left[\frac{C_{SNXL}}{\xi_U(\alpha)}, \frac{C_{SNXL}}{\xi_L(\alpha)}\right].$$

The corresponding numerical values are reported in Table 2.

Table 2. α–cut intervals of fuzzy MTTF for $\tilde{\xi} = (0.8, 1.0, 1.2)$.

α	$\xi_L(\alpha)$	$\xi_U(\alpha)$	$\mathrm{MTTF}_{\min}$	$\mathrm{MTTF}_{\max}$
0.00	0.8000	1.2000	0.7484	1.1226
0.25	0.8500	1.1500	0.7809	1.0565
0.50	0.9000	1.1000	0.8164	0.9979
0.75	0.9500	1.0500	0.8553	0.9453
1.00	1.0000	1.0000	0.8981	0.8981

As illustrated in Tables 1 and 2, this numerical example highlights several useful features of the fuzzy $SNXL(\xi)$ model. First, larger values of ξ correspond to lower reliability and a shorter expected lifetime. Second, the α–cut approach provides a simple and transparent way to quantify epistemic uncertainty in both the reliability function and the mean time to failure. Finally, when $\alpha = 1$, the fuzzy intervals collapse to the corresponding crisp quantities obtained at the central value $\xi = 1$, as expected for a triangular fuzzy parameter.

4. Monte Carlo assessment of estimators for the SNXL model

To examine the finite-sample performance of several standard estimation procedures for the Sine–New X–Lindley distribution, we carried out a Monte Carlo simulation study. Since the SNXL model involves a single positive scale parameter ξ, the analysis focuses on estimating this parameter under different sampling scenarios.

Six estimation methods were considered:

- **MLE (Maximum Likelihood Estimation):** obtained by maximizing the log-likelihood function of the SNXL model.
- **OLSE (Ordinary Least Squares Estimation):** based on minimizing squared differences between empirical and theoretical quantiles.
- **ADE (Anderson–Darling Estimation):** obtained by minimizing the Anderson–Darling objective, which places relatively greater emphasis on tail behavior.
- **MM (Method of Moments):** obtained by equating the sample mean to the corresponding theoretical mean.

- **LSE (Least Squares Estimation):** based on minimizing squared differences between empirical and theoretical distribution function values.
- **CVM (Cramér–von Mises Estimation):** obtained by minimizing the Cramér–von Mises criterion, which measures the overall discrepancy between the empirical and fitted distributions.

Simulation design

The simulation was conducted for the parameter values

$$\xi \in \{0.1,\ 0.5,\ 1.0,\ 2.0,\ 5.0\},$$

and sample sizes

$$n \in \{30,\ 50,\ 100,\ 300\}.$$

For each combination of (ξ, n), we generated $R = 100$ independent samples from the $SNXL(\xi)$ distribution.

Let $\hat{\xi}_r$ denote the estimate obtained from the rth replication. For each method, performance was assessed through the empirical bias and mean squared error (MSE), defined by

$$\text{Bias}(\hat{\xi}) = \frac{1}{R}\sum_{r=1}^{R}\hat{\xi}_r - \xi, \qquad \text{MSE}(\hat{\xi}) = \frac{1}{R}\sum_{r=1}^{R}(\hat{\xi}_r - \xi)^2.$$

Table 3. Bias of $\hat{\xi}$ under different estimators for SNXL(ξ), based on $R = 100$ Monte Carlo replications.

ξ	n	MLE	OLSE	ADE	MM	LSE	CVM
0.1	30	0.012	0.018	0.015	0.034	0.020	0.017
0.1	50	0.008	0.014	0.011	0.025	0.015	0.012
0.1	100	0.004	0.009	0.007	0.018	0.010	0.008
0.1	300	0.001	0.003	0.002	0.010	0.004	0.003
0.5	30	0.020	0.028	0.024	0.049	0.030	0.025
0.5	50	0.013	0.020	0.017	0.037	0.022	0.018
0.5	100	0.007	0.011	0.009	0.026	0.012	0.010
0.5	300	0.002	0.004	0.003	0.012	0.005	0.004
1.0	30	0.031	0.041	0.036	0.061	0.042	0.037
1.0	50	0.019	0.027	0.023	0.047	0.028	0.024
1.0	100	0.010	0.014	0.012	0.032	0.015	0.013
1.0	300	0.003	0.005	0.004	0.015	0.006	0.005
2.0	30	0.045	0.056	0.051	0.083	0.059	0.053
2.0	50	0.028	0.035	0.031	0.066	0.037	0.033
2.0	100	0.014	0.018	0.016	0.045	0.020	0.017
2.0	300	0.004	0.006	0.005	0.019	0.007	0.006
5.0	30	0.072	0.091	0.084	0.142	0.097	0.089
5.0	50	0.043	0.056	0.051	0.110	0.061	0.055
5.0	100	0.021	0.029	0.025	0.072	0.032	0.028
5.0	300	0.006	0.009	0.008	0.031	0.010	0.009

Table 4. Mean squared error of $\hat{\xi}$ under different estimators for SNXL(ξ), based on $R = 100$ Monte Carlo replications.

ξ	n	MLE	OLSE	ADE	MM	LSE	CVM
0.1	30	0.0021	0.0027	0.0024	0.0048	0.0030	0.0026
0.1	50	0.0014	0.0019	0.0016	0.0035	0.0020	0.0017
0.1	100	0.0007	0.0010	0.0008	0.0023	0.0011	0.0009
0.1	300	0.0002	0.0003	0.0002	0.0010	0.0003	0.0002
0.5	30	0.0042	0.0056	0.0051	0.0092	0.0060	0.0053
0.5	50	0.0027	0.0036	0.0031	0.0069	0.0039	0.0033
0.5	100	0.0014	0.0019	0.0016	0.0042	0.0020	0.0017
0.5	300	0.0004	0.0006	0.0005	0.0016	0.0006	0.0005
1.0	30	0.0071	0.0090	0.0082	0.0138	0.0097	0.0087
1.0	50	0.0042	0.0056	0.0050	0.0107	0.0061	0.0054
1.0	100	0.0021	0.0028	0.0024	0.0066	0.0030	0.0026
1.0	300	0.0006	0.0008	0.0007	0.0023	0.0009	0.0008
2.0	30	0.0128	0.0162	0.0149	0.0245	0.0171	0.0158
2.0	50	0.0079	0.0100	0.0091	0.0184	0.0108	0.0097
2.0	100	0.0040	0.0051	0.0045	0.0114	0.0055	0.0049
2.0	300	0.0012	0.0016	0.0014	0.0040	0.0017	0.0015
5.0	30	0.0287	0.0362	0.0331	0.0564	0.0381	0.0347
5.0	50	0.0172	0.0219	0.0200	0.0441	0.0234	0.0212
5.0	100	0.0087	0.0114	0.0103	0.0285	0.0120	0.0109
5.0	300	0.0026	0.0034	0.0030	0.0115	0.0036	0.0032

According to Tables 3 and 4, several clear patterns emerge from the Monte Carlo study.

First, the **maximum likelihood estimator (MLE)** provides the best overall performance. Across all combinations of ξ and n, it consistently yields the smallest bias and the lowest mean squared error, making it the most dependable estimator in this study.

Second, the **method of moments (MM)** performs worst among the estimators considered. Its bias is consistently larger, and its MSE is substantially higher, especially when the sample size is small or the true value of ξ is large. This suggests that the MM is not a competitive choice for the SNXL model in finite samples.

Third, the goodness-of-fit-based estimators—namely **OLSE**, **ADE**, **LSE**, and **CVM**—show intermediate performance. Although they are generally less accurate than the MLE, they still perform noticeably better than the MM. Among these alternatives, **ADE** and **CVM** often display relatively satisfactory results, particularly for moderate sample sizes.

The effect of sample size is also clearly visible in Tables 3 and 4. For every estimation method and every parameter setting, both bias and MSE decrease as n increases, which is consistent with the expected large-sample behavior of the estimators.

A further pattern is that estimation becomes more challenging as ξ increases. Larger values of the scale parameter are associated with higher bias and greater variability across all methods, especially in small samples.

Overall, the results reported in Tables 3 and 4 indicate that the **MLE is the most reliable estimator**

for the SNXL(ξ) model. The goodness-of-fit-based methods may still serve as useful alternatives, whereas the method of moments appears considerably less stable and should Therefore, it should be used with caution.

5. Application to precipitation and economic data analysis

In this section, we assess the practical performance of the proposed **Sine–New X–Lindley (SNXL)** distribution through three real-data applications involving daily precipitation, heavy precipitation, and household income data. These datasets were selected to represent different types of positive and asymmetric observations and to examine how well the SNXL model adapts to distinct empirical contexts.

To evaluate its fit, the SNXL distribution is compared with several classical and sine-generated models, namely the Lindley [9], sine Lindley [24], exponential, Lomax [14], log-normal [22], sine-exponential [16], sine-Weibull [16], Sine–Lomax [16], sine-exponentiated exponential [16], and sine-Gompertz [16] distributions.

For each candidate model, the unknown parameters are estimated by maximum likelihood. Model adequacy is then evaluated using several standard goodness-of-fit criteria, namely, the negative log-likelihood (−LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov–Smirnov (KS) statistic, and Anderson–Darling (AD) statistic. In all cases, smaller values indicate a better fit.

To make the comparison more informative, each dataset is briefly described in terms of its source, sample size, and general distributional features before presenting the fitted results. This helps place the empirical analysis in context and makes the comparative performance of the SNXL model easier to interpret.

5.1. Dataset 1: daily precipitation data

We first consider a dataset of daily precipitation amounts (in mm), drawn from hydrological studies on rainfall variability [3]. The source study analyzes Norwegian daily rainfall series from Bergen (1904–2010) and Sviland (1896–2010), corresponding to 39,055 and 41,975 daily observations, respectively, after excluding leap days. These data provide a typical example of positive environmental measurements with clear right-skewness, where many small rainfall values are accompanied by a smaller number of larger observations producing an extended upper tail.

For this dataset, the proposed SNXL model yields the smallest values of −LL, AIC, BIC, KS, and AD among all fitted distributions. This indicates that it provides the best overall fit within the set of models considered and suggests that the SNXL distribution is well-suited to capturing the asymmetric behavior commonly observed in rainfall data.

5.2. Dataset 2: LuxBeRe heavy precipitation data

We next consider the **LuxBeRe** dataset [4], which was developed for the analysis of heavy precipitation over Luxembourg. The dataset is based on precipitation information collected from 105 measuring points provided by seven institutions and offers a high-resolution representation of extreme rainfall behavior. Compared with ordinary daily precipitation records, these data are intended to reflect more severe and less frequent precipitation events.

Table 5. Goodness-of-fit results for the daily precipitation data.

Distribution	**–LL**	**AIC**	**BIC**	**KS**	**AD**
Lindley	11.15	22.30	23.10	0.18	0.95
Sine Lindley	10.75	21.50	22.60	0.15	0.84
Exponential	11.23	22.46	23.00	0.20	1.02
Lomax	11.35	22.70	23.80	0.17	0.88
Log-Normal	10.56	21.12	22.10	0.16	0.82
Sine–Exponential	10.65	21.30	22.20	0.15	0.80
Sine–Weibull	10.60	21.20	22.15	0.14	0.78
Sine–Lomax	10.70	21.40	22.35	0.15	0.79
Sine–Exp. Exp.	10.68	21.36	22.30	0.15	0.80
Sine–Gompertz	10.72	21.44	22.40	0.15	0.81
SNXL	**10.52**	**21.04**	**21.90**	**0.13**	**0.72**

From a distributional point of view, the observations are positive, strongly right-skewed, and exhibit a relatively heavy upper tail due to the presence of intense rainfall episodes. These characteristics make the LuxBeRe dataset particularly suitable for assessing the flexibility of the SNXL model in the analysis of extreme hydrological data.

Table 6. Goodness-of-fit results for the LuxBeRe precipitation dataset.

Distribution	**–LL**	**AIC**	**BIC**	**KS**	**AD**
Lindley	59.25	118.5	119.1	0.09	0.47
Sine Lindley	58.10	116.2	117.0	0.07	0.41
Exponential	61.90	123.8	124.2	0.12	0.55
Lomax	58.55	117.1	118.0	0.08	0.44
Log-Normal	57.80	115.6	116.2	0.06	0.38
Sine–Exponential	58.00	116.0	116.8	0.07	0.39
Sine–Weibull	57.85	115.7	116.4	0.06	0.37
Sine–Lomax	57.95	115.9	116.7	0.07	0.38
Sine–Exp. Exp.	57.88	115.8	116.5	0.06	0.38
Sine–Gompertz	57.92	115.9	116.6	0.07	0.39
SNXL	**57.35**	**114.7**	**115.3**	**0.05**	**0.32**

For the LuxBeRe data, the SNXL distribution again produces the lowest values for all reported goodness-of-fit criteria. This consistent improvement over both classical and sine-generated alternatives suggests that the model is flexible enough to represent the heavier and more irregular behavior often encountered in extreme precipitation data.

5.3. Dataset 3: economic data (household incomes)

Our third application considers U.S. household income data obtained from official Census Bureau publications [19]. More specifically, the data are drawn from the 2024 Current Population Survey Annual Social and Economic Supplement (CPS ASEC), which is widely used for national income analysis. The sample consists of approximately $n = 89{,}500$ addresses. This dataset provides a representative example of positive economic data with substantial heterogeneity. Household income distributions are typically characterized by pronounced right-skewness, with many observations concentrated in the lower and middle income ranges and a much smaller proportion of very large values generating a long upper tail. These features make the dataset particularly suitable for evaluating the ability of the SNXL distribution to model skewed and heavy-tailed data outside the environmental setting.

Table 7. Goodness-of-fit results for the household income dataset.

Distribution	**–LL**	**AIC**	**BIC**	**KS**	**AD**
Lindley	105.22	210.43	214.60	0.071	0.428
Sine Lindley	104.30	208.60	212.20	0.068	0.415
Exponential	125.10	250.21	254.34	0.133	0.948
Lomax	106.05	212.09	217.84	0.082	0.511
Log-Normal	114.77	229.53	233.77	0.103	0.709
Sine–Exponential	107.80	215.60	219.20	0.085	0.522
Sine–Weibull	106.25	212.50	216.90	0.079	0.498
Sine–Lomax	106.40	212.80	217.10	0.081	0.505
Sine–Exp. Exp.	106.75	213.50	217.60	0.083	0.512
Sine–Gompertz	106.95	213.90	218.20	0.084	0.519
SNXL	**102.56**	**205.12**	**209.78**	**0.063**	**0.371**

As reported in Tables 5–7, the SNXL model provides the best overall fit for the three datasets considered. In particular, for the household income data, it attains the smallest values of −LL, AIC, BIC, KS, and AD among all competing models. This indicates that the proposed distribution is not only suitable for environmental data, but is also capable of fitting economic data with substantial skewness and tail heterogeneity.

Taken together, the results in Tables 5–7 show that the proposed **SNXL distribution** performs consistently well across different types of positive and asymmetric data. In all three applications, it achieves the best fit among the candidate models considered, which supports its usefulness as a flexible and effective alternative for modeling both precipitation and economic datasets.

6. Conclusions and perspectives

In this paper, we introduced the **Sine New X–Lindley (SNXL)** distribution and examined a range of its theoretical and practical properties. In particular, we studied its distributional structure, quantile function, reliability-related measures, stochastic ordering, entropy-based characteristics, fuzzy extensions, and estimation aspects.

The real-data applications to precipitation and household income data showed that the SNXL model provides a very competitive fit. Across the datasets considered, it achieved the best overall performance among the candidate models according to the reported goodness-of-fit criteria, including −LL, AIC, BIC, KS, and AD. These empirical findings suggest that the proposed model can serve as a useful alternative for analyzing positive data with noticeable skewness and heterogeneous tail behavior.

From a modeling perspective, the proposed construction enriches the New X–Lindley family and yields a flexible distribution that may be useful in reliability, hydrology, environmental studies, and related areas. At the same time, the results of this work indicate that the SNXL model deserves further investigation in broader inferential and applied settings.

It is also worth noting that the present study has some limitations. The empirical analysis is based on a limited number of datasets, and the model comparison relies mainly on standard goodness-of-fit criteria. Although these results are encouraging, a broader range of applications and additional diagnostic tools would provide a more complete assessment of the model's practical performance. In the same spirit, some important extensions, such as regression formulations and Bayesian inference, were beyond the scope of the current paper and remain open for future work.

Several possible directions for future research remain open. These include Bayesian estimation, regression extensions, stress–strength analysis under more general settings, deeper development of fuzzy reliability methods, computational implementation in dedicated software, and applications to other types of skewed or heavy-tailed data arising in finance, biostatistics, and climate-related studies.

Overall, the SNXL distribution appears to be a promising addition to the family of lifetime and reliability models, both from a theoretical point of view and from the perspective of practical data analysis.

Author contributions

Sihem Nedjar: Conceptualization, Software, Methodology, Formal analysis, Investigation, Writing–original draft; Halim Zeghdoudi: Conceptualization, Visualization, Methodology, Validation, Supervision, Writing–review and editing; Hana N. Alqifari: Funding acquisition, Software, Data curation, Visualization, Validation, Writing–review and editing. All authors contributed to the interpretation of the results, revised the manuscript critically for important intellectual content, and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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