Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 11810–11839.
DOI:10.3934/math.2026486
Received: 12 March 2026
Revised: 07 April 2026
Accepted: 15 April 2026
Published: 28 April 2026

Research article

Functional analysis framework for two-dimensional fractional dynamical systems with applications to Lorenz and Euler systems

Faten H. Damag[1,2], **Mohammed Alsharafi**[3,4,*] **and Abeer Hamdan Alblowy**[1]

[1] Department of Mathematics, Faculty of Sciences, University of Ha'il, Ha'il 2440, Saudi Arabia
[2] Department of Computer Science, Faculty of Applied Sciences, Taiz University, Taiz 6803, Yemen
[3] Department of Mathematics, Faculty of Arts and Science, Yildiz Technical University, Istanbul, Turkey
[4] Department of Mathematics, Sana'a University, Sana'a, Yemen

* **Correspondence:** Email: alsharafi205010@gmail.com.

Abstract: This paper develops a functional analysis framework for a class of two-dimensional Erdélyi–Kober (EK) fractional dynamical systems defined in locally compact Hausdorff spaces under the compact-open topology. The considered model incorporates delay and nonlinear interactions through EK fractional operators, which allow the description of dynamical systems with memory effects. By constructing appropriate operators in a Banach space setting, we analyze their continuity, boundedness, and the Lipschitz properties. Using classical results from functional analysis and operator theory, sufficient conditions are derived to establish the solvability and Hyers–Ulam stability of the proposed fractional system. The theoretical results demonstrate that the considered model remains stable under small perturbations of the system parameters. To illustrate the applicability of the developed framework, the results are applied to two significant fractional models: the EK fractional Lorenz system describing chaotic dynamics, and the two-dimensional fractional Euler system arising in fluid mechanics. These applications confirm the effectiveness of the proposed functional analytic approach for studying nonlinear fractional dynamical systems with memory and delay effects.

Keywords: fractional calculus; EK operator; locally compact Hausdorff space; Lorenz system; two-dimensional fractional Euler system
Mathematics Subject Classification: 26A33, 34A08, 35Q31, 47H10, 54H11, 76N15

1. Introduction

Fractional calculus is a well-established area of mathematical analysis in which the classical operators of differentiation and integration are generalized to orders that may take arbitrary real or complex values. This extension offers a versatile mathematical framework that is particularly effective in representing processes characterized by memory and hereditary effects. Because of these features, fractional calculus has become a valuable tool for modeling a wide range of physical and engineering phenomena observed in real applications. The idea of differentiation of non-integer order can be traced back to 1695, when Leibniz posed a question regarding the interpretation of a derivative of order $\frac{1}{2}$ in a letter to L'Hôpital. Later developments in the nineteenth century by mathematicians such as Riemann, Liouville, and Grünwald provided the rigorous theoretical groundwork for what is now known as fractional calculus [1, 2]. Their contributions introduced several fractional operators that continue to play a fundamental role in modern fractional analysis. In recent years, fractional calculus has attracted growing interest since fractional-order models are often capable of describing complex dynamical systems with greater accuracy than classical integer-order models. A key feature of fractional derivatives is their nonlocal nature, which allows the current state of a system to depend not only on its instantaneous behavior, but also on its previous evolution [3]. Liu et al. [4] introduced a spectral element method scheme for efficiently solving the fractional telegraph problem with improved accuracy and stability. Liu and Ding [5] proposed a radial basis function neural network approach for efficiently approximating solutions of diffusion partial differential equations. This property makes fractional models particularly suitable for studying systems that exhibit memory effects. This nonlocal property makes fractional models especially useful for describing phenomena with memory effects. Various types of fractional derivatives have been introduced in the literature, including Caputo–Fabrizio, Hilfer, Katugampola, and EK derivatives. These operators differ mainly in the kernel functions they employ, which determine how memory effects are incorporated into the model. Fractional calculus has become a rapidly growing research area that combines theoretical developments with numerous practical applications [6]. For example, in viscoelastic materials, fractional models provide more accurate descriptions of stress strain relationships. In heat transfer and diffusion processes, fractional operators capture anomalous diffusion observed in complex materials and biological systems. In control theory, fractional controllers often exhibit improved stability and flexibility compared with classical proportional integral derivative controllers. Furthermore, fractional models have found applications in finance for describing long-term market dependence and in epidemiology for modeling delayed disease transmission [7]. These applications demonstrate the versatility of fractional calculus in representing systems with memory and hereditary effects. Another important mathematical framework for describing physical phenomena is provided by partial differential equations (PDEs). Classical PDEs are widely used to model processes such as heat conduction, wave propagation, elasticity, and diffusion. However, classical models are typically local in nature, meaning that the evolution of the system at a given point depends only on nearby values. This locality assumption is often insufficient for complex systems characterized by long range interactions, memory effects, or multiscale dynamics. To address these limitations, fractional partial differential equations (FPDEs) have been introduced by replacing integer-order derivatives with fractional ones [8]. FPDEs incorporate nonlocal behavior and memory effects, allowing a more realistic description of various complex phenomena. Fractional differential equations and systems have therefore become a central topic of investigation in both theoretical and applied mathematics. They have been used to model anomalous diffusion, viscoelastic flows, quantum mechanics, turbulence, and stochastic transport. Applications also arise in plasma physics, geophysics, biological systems, and financial mathematics [9]. As a consequence, considerable efforts have been devoted to studying qualitative properties of fractional systems, including solvability, stability, and long-term dynamics. In this context, fixed-point methods have proved to be powerful tools for analyzing fractional differential equations. For instance, Wang et al. [1] established new contraction principles and applied them to fractional models arising in economic dynamics. Balegh and GhezaL [10] developed a theoretical framework for delay fractional differential equations using fixed-point techniques to investigate solvability results. Cona and Kocağ [11] studied systems of fractional differential equations through fixed-point methods, while Belhadj et al. [12] generalized Darbo's fixed-point theorem and applied it to nonlinear fractional differential equations. Liu et al. [13] developed a pseudospectral method based on Legendre cardinal functions to solve the fractional Klein–Gordon equation with high precision. Liu and Xue [14] presented a convergent multi-step iterative method for solving systems of nonlinear equations with enhanced convergence performance. Motivated by: these developments, we consider the following general two-dimensional EK fractional dynamical system

$$\begin{cases} {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_1(\xi,\rho) = \varsigma_1\zeta_1(\xi,\rho) + \mathcal{V}\left\{\xi,\rho,\zeta_1(\xi,\rho-\vartheta),\zeta_1\left[\xi,\rho-\omega(\zeta_3(\xi,\rho)\zeta_2(\xi,\rho))\right]\right\}, \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_2(\xi,\rho) = \varsigma_2\zeta_2(\xi,\rho) + \mathcal{V}'\left\{\xi,\rho,\zeta_2(\xi,\rho-\vartheta),\zeta_2\left[\xi,\rho-\omega(\zeta_3(\xi,\rho)\zeta_1(\xi,\rho))\right]\right\}, \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_3(\xi,\rho) = \varsigma_3\zeta_3(\xi,\rho) + \mathcal{V}''\left\{\xi,\rho,\zeta_3(\xi,\rho-\vartheta),\zeta_3\left[\xi,\rho-\omega(\zeta_1(\xi,\rho)\zeta_2(\xi,\rho))\right]\right\}, \\ \zeta_1(\xi,\rho) = \theta_1(\xi,\rho),\ \zeta_2(\xi,\rho) = \theta_2(\xi,\rho),\ \zeta_3(\xi,\rho) = \theta_3(\xi,\rho), \quad 0 \le \rho \le \omega_0, \\ \zeta_1(\xi,0) = l_1(\xi), \quad \zeta_2(\xi,0) = l_2(\xi), \quad \zeta_3(\xi,0) = l_3(\xi), \end{cases} \tag{1.1}$$

for all $(\xi,\rho) \in \Delta_{\rho_0} := \Delta \times [0,\rho_0]$, where $\Delta \subset \mathbb{R}$ is a bounded locally compact subset of $\mathbb{R}$. Here, ${}^{EK}\mathcal{D}^{\alpha}_{0,\rho}$ denotes the EK fractional derivative of order $0 < \alpha \le 1$, $\omega_0 = \max\{\vartheta, \sup_t \omega(t)\}$ with $\omega_0 < \rho_0$, $\vartheta > 0$, $\varsigma_{1,2,3}$ are constants, ω is a continuous function mapping into $(0,\omega_0]$, $\theta_{1,2,3}$ are continuous bounded functions, and $\mathcal{V}_{1,2,3}$ are continuous operators.

To illustrate the applicability of the proposed theoretical framework, we consider two important models: the fractional Lorenz system and the two-dimensional fractional Euler system. Fractional Lorenz type systems have been widely studied due to their ability to describe complex chaotic dynamics. Ahmad et al. [15] analyzed the fractional Lorenz system and established solvability results using contraction principles. Awais et al. [16] investigated stochastic and hyperchaotic behaviors in variable order fractional Lorenz systems. In addition, Meera and Selvaganesan [17] proposed a generalized complex fractional Lorenz system and demonstrated its potential applications in security frameworks. On the other hand, the two-dimensional Euler equations play a fundamental role in fluid dynamics. Their analytical and dynamical properties have been extensively studied in the literature. Hui et al. [18] proposed a unified coordinate system that improves numerical treatments of two-dimensional Euler equations. Chen [19] investigated the stability of transonic shock fronts in Euler systems, while Wang and Yang [20] studied planar diffusion waves in damped two-dimensional Euler equations. Moreover, Dullin et al. [21] analyzed the instability of equilibria in two-dimensional Euler flows on the torus, revealing complex dynamical structures in fluid models.

Fractional differential systems have become an essential tool for describing many real-world processes that exhibit memory and hereditary characteristics. Despite the large body of literature in this area, many studies concentrate mainly on numerical experiments or specific models, often without establishing a comprehensive theoretical framework within suitable topological structures. Motivated by this limitation, the present work examines a general two-dimensional EK fractional dynamical system defined in locally compact Hausdorff spaces endowed with the compact-open topology.

In recent years, various control strategies have been developed for complex dynamical systems, including sliding mode control with variable convergence rates, asynchronously switched control methods, and H^{∞} control techniques. These approaches have been widely used to analyze stability and robustness in nonlinear systems. Although these methods provide effective tools for control and stabilization, they are mainly focused on control design, while the present work emphasizes the theoretical analysis of fractional dynamical systems within a functional analytic framework.

The main results of this paper are summarized as follows. First, we prove the continuity, boundedness, and Lipschitz properties of the corresponding operators in an appropriate Banach space setting. Second, by employing the Banach and Schaefer fixed point theorems, we establish conditions guaranteeing the existence and uniqueness of solutions for the considered system. Third, the Hyers–Ulam stability of the system is investigated, demonstrating that the obtained solutions remain stable under small perturbations. To illustrate the applicability of the theoretical results, we study two representative examples, namely the fractional Lorenz system and the two-dimensional fractional Euler system, which highlight the effectiveness and flexibility of the proposed framework.

The remainder of the paper is organized as follows. Section 2 introduces the required preliminaries and basic results. Section 3 focuses on the continuity and compactness analysis of the associated operators. In Section 4, we prove the solvability, uniqueness and Hyers–Ulam stability results. Finally, Section 5 applies the developed theory to the fractional Lorenz system and the two-dimensional fractional Euler system.

The novelty of this work is the use of the EK operator within a functional analysis framework in locally compact Hausdorff spaces. This allows us to study existence, uniqueness, and stability of solutions in a more general setting compared to previous works.

Several recent studies have investigated fractional-order systems with exponential kernels as well as two-dimensional time-delay fractional systems, mainly focusing on numerical simulations or specific models. While these approaches provide useful insights into the behavior of such systems, they often lack a unified theoretical framework within an appropriate topological setting. In contrast, the present work develops a functional analysis framework for two-dimensional Erdélyi–Kober fractional dynamical systems in locally compact Hausdorff spaces. This approach allows us to establish general results concerning existence, uniqueness, and Hyers–Ulam stability, which are not fully addressed in the aforementioned studies. Therefore, the main contribution of this work is to fill this gap by providing a rigorous and general theoretical foundation for a broader class of fractional systems.

2. Preliminaries

The definitions and main results related to EK fractional operators are initially formulated in the single variable setting [22–25]. In this work, we extend these notions to the two variable case. For brevity, we refer to the existing proofs in the single-variable framework as the extension to two variables follows by a natural and direct generalization.

Definition 2.1. [24] Let $\Delta \subset \mathbb{R}$ and $\zeta_1 : \Delta \times [0, \rho_0] \to \mathbb{R}$ be a given mapping. Let $0 < \beta \leq 1$ and $\sigma \in \mathbb{R}$. The EK fractional integral of order α is given by

$$\mathcal{I}_{\rho}^{\alpha,\beta,\sigma}\zeta_{1\xi}(\rho) = \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)} \int_0^{\rho} (\rho^{\beta} - \varrho^{\beta})^{\alpha-1} \varrho^{\beta\sigma} \zeta_1(\xi, \varrho) \, d\varrho, \tag{2.1}$$

where $\zeta_{1\xi}(\rho) := \zeta_1(\xi, \rho)$.

Define the differential operator

$$A_\rho := \frac{1}{\beta\rho^{\beta-1}}\frac{\partial}{\partial\rho},$$

and define its iterates recursively by $A_\rho^n f := A_\rho(A_\rho^{n-1} f),\ n \in \mathbb{N}$.

The extension of the EK operators from the single-variable case to the two-variable setting is performed by treating the spatial variable ξ as a parameter. For each fixed $\xi \in \Delta$, the operators act on the variable ρ as in the classical one-dimensional case. Consequently, the main properties of the operators are obtained by applying the corresponding single-variable results pointwise with respect to ξ, ensuring the validity of the extension.

Definition 2.2. [24] Let $\Delta \subset \mathbb{R}$ and let ζ_1 be a mapping defined on $\Delta \times [0, \rho_0]$. Assume that $0 < \beta \leq 1$, $n = \lfloor \alpha \rfloor$, and $\sigma \in \mathbb{R}$. Then, the EK fractional operator of order α is defined by

$$\mathcal{D}_\rho^{\alpha,\beta,\sigma}\zeta_{1\xi}(\rho) = \begin{cases} \rho^{-\beta\sigma} A_\rho^n \left(\rho^{\beta(n+\sigma)} \mathcal{I}_\rho^{n-\alpha,\beta,\sigma}\zeta_1(\xi,\rho)\right), & n-1 < \alpha < n; \\ \zeta_{1\rho^n}^{(n)}(\xi,\rho), & \alpha := n \in \mathbb{N}. \end{cases} \tag{2.2}$$

The EK fractional integral and derivative are defined for $\rho > 0$. Although the factor $\rho^{-\beta(\alpha+\sigma)}$ appears in 2.1, the integral term vanishes sufficiently fast as $\rho \to 0^+$ under the assumptions $0 < \alpha \leq 1$, $0 < \beta \leq 1$, and $\zeta_1 \in C(\Delta \times [0, \rho_0])$. In particular, the kernel $(\rho^\beta - \varrho^\beta)^{\alpha-1}\varrho^{\beta\sigma}$ remains integrable on $[0, \rho]$, ensuring that the limit as $\rho \to 0^+$ exists and the expression remains finite. Therefore, no pole occurs at $\rho = 0$, and the operators are well-defined together with the prescribed initial conditions. Recalling [25], we have

$$\mathcal{D}_\rho^{\alpha,\beta,\sigma}\mathcal{I}_\rho^{\alpha,\beta,\sigma}(\zeta_1(\xi,\rho)) = \zeta_1(\xi,\rho).$$

In this paper, $AC([0,\rho_0])$ denotes the space of all absolutely continuous functions defined on the interval $[0,\rho_0]$, that is,

$$AC([0,\rho_0]) := \{h : [0,\rho_0] \to \mathbb{R} : h \text{ is absolutely continuous on } [0,\rho_0]\}.$$

Moreover, for $n \in \mathbb{N}$, we define

$$AC^n([0,\rho_0]) := \left\{h : [0,\rho_0] \to \mathbb{R}\ :\ A_y^k(y^{\beta\sigma}h(y)) \in AC([0,\rho_0]),\ \forall\, k = 1, 2, \ldots, n\right\}.$$

Theorem 2.3. [23] Let $\zeta_1(\xi, \cdot) \in AC^n([0,\rho_0])$ for all $\xi \in \Delta$ and $n - 1 \leq \alpha < n$. Then,

$$\begin{aligned} \mathcal{I}_\rho^{\alpha,\beta,\sigma}\mathcal{D}_\rho^{\alpha,\beta,\sigma}(\zeta_1(\xi,\rho)) &= \zeta_1(\xi,\rho) \\ &\quad - \sum_{k=0}^{n-1} \frac{\rho^{-\beta(\sigma+k)}}{k!} A_\rho^k\!\left(\rho^{\beta\sigma}\zeta_1(\xi,\rho)\right)\Big|_{\rho=0}, \end{aligned} \tag{2.3}$$

where $n \in \mathbb{N}$.

Theorem 2.4. [26] Let $(S, \|\cdot\|)$ be a Banach space and suppose that $\mathcal{P} : S \to S$ has a contraction property. Then, there is a unique $s \in X$ such that $\mathcal{P}(s) = s$. Moreover, for every $x_0 \in X$, the iterative process $s_{n+1} = \mathcal{P}(s_n)$ converges to s.

Theorem 2.5. [26] Let S be a Banach space and $\mathcal{P} : X \to X$ be a compact and continuous operator. Assume that

$$\mathcal{M} = \{s \in S : s = \lambda\mathcal{P}(s),\ \lambda \in [0,1]\}$$

is bounded in S. Then, $\mathcal{P}$ has at least one fixed point in S.

Theorem 2.6. [27] Let K be a compact metric space with metric d, and let $\mathcal{E} \subset C(K)$ be a collection of real-valued continuous functions on K. Then, $\mathcal{E}$ is relatively compact in $(C(K), \|\cdot\|_\infty)$ if and only if $\mathcal{E}$ is uniformly bounded and equicontinuous. In particular, every sequence in $\mathcal{E}$ has uniformly convergent subsequence.

3. Properties of the nonlinear operators

In this section, we analyze the analytical characteristics of the operators related to the EK fractional system. The hypothesis that Δ is a bounded locally compact Hausdorff space guarantees the compactness of Δ_{ρ_0}. This property ensures that the compact-open topology coincides with the topology induced by the supremum norm, which allows the effective use of fixed point techniques. A subset S of a space X is a locally compact if for every $x \in S$ there is an open set O of x such that the closure $\overline{O}$ has a compact property in X. A space X is called a Hausdorff space whenever any two distinct points can be separated by disjoint open neighborhoods. Let Δ be a bounded locally compact subset of $\mathbb{R}$. Denote by $\mathbb{R}^{\Delta_{\rho_0}}$ the set of all real-valued functions defined on Δ_{ρ_0}, and define $\mathcal{B} := C_{\Delta_0} \times C_{\Delta_0} \times C_{\Delta_0}$ as a Banach space under the norm

$$\|(\zeta_1, \zeta_2, \zeta_3)\|_{\mathcal{B}} = \max\left\{|\zeta_1(\xi,\rho)|, |\zeta_2(\xi,\rho)|, |\zeta_3(\xi,\rho)| : (\xi,\rho) \in \Delta_{\rho_0}\right\}.$$

Here $C_{\Delta_0} := C(\Delta \times [0,\rho_0]) \subseteq \mathbb{R}^{\Delta_{\rho_0}}$ denotes the family of all mappings that have continuity property on Δ_{ρ_0} into $\mathbb{R}$. The family of open balls $B(\xi_0, r)$ in $\mathbb{R}$, equipped with the metric $d(\xi,\rho) = |\xi - \rho|$, forms a subbase for the topology on $\mathbb{R}$, known as the usual topology. Since Δ_{ρ_0} is a Hausdorff subspace of the product space $\mathbb{R}^2$ endowed with the usual topology, the family

$$\{\langle C, B(\xi_0, r)\rangle : C \text{ is compact set in } \Delta_{\rho_0},\ \xi_0 \in \mathbb{R},\ r > 0\}$$

generates a subbase for a topology on C_{Δ_0}, called the compact-open topology, where

$$\langle C, B(\xi_0, r)\rangle = \{\zeta_1 \in C_{\Delta_0} : \zeta_1(C) \subset B(\xi_0, r)\}.$$

Because $\Delta \subset \mathbb{R}$ is assumed to be bounded and locally compact, it follows that Δ_{ρ_0} is compact. Therefore, on the space C_{Δ_0}, the compact-open topology coincides with the topology that is generated by the supremum norm

$$\|\zeta_1\|_\infty = \sup\left\{|\zeta_1(\xi,\rho)| : (\xi,\rho) \in \Delta_{\rho_0}\right\}.$$

Hence, the notions of continuity, compactness and convergence proved under the compact-open topology coincide with those derived from the supremum norm topology. This equivalence validates the application of fixed point Theorems 2.4 and 2.5 in Banach spaces equipped with the supremum norm throughout this study.

So, throughout this paper the space $\mathbb{R}^{\Delta_{\rho_0}}$ is considered with the compact-open topology. Define the operators $\mathbb{J}_{\zeta_1}, \mathbb{J}_{\zeta_2}, \mathbb{J}_{\zeta_3} : \Delta_{\rho_0} \times \mathbb{R}^{\Delta_{\rho_0}} \times \mathbb{R}^{\Delta_{\rho_0}} \times \mathbb{R}^{\Delta_{\rho_0}} \to \mathbb{R}$ by

$$\mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3) = \varsigma_1\zeta_1(\xi,\rho) + \mathcal{V}\{\xi,\rho,\zeta_1(\xi,\rho-\vartheta),\zeta_1[\xi,\rho-\omega(\zeta_3(\xi,\rho)\zeta_2(\xi,\rho))]\}, \tag{3.1}$$

$$\mathbb{J}_{\zeta_2}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3) = \varsigma_2\zeta_2(\xi,\rho) + \mathcal{V}'\{\xi,\rho,\zeta_2(\xi,\rho-\vartheta),\zeta_2[\xi,\rho-\omega(\zeta_3(\xi,\rho)\zeta_1(\xi,\rho))]\}, \tag{3.2}$$

and

$$\mathbb{J}_{\zeta_3}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3) = \varsigma_3\zeta_3(\xi,\rho) + \mathcal{V}''\{\xi,\rho,\zeta_3(\xi,\rho-\vartheta),\zeta_3[\xi,\rho-\omega(\zeta_1(\xi,\rho)\zeta_2(\xi,\rho))]\}. \tag{3.3}$$

Apply the EK integral operator of (1.1), and use Theorem 2.3 to get that

$$\zeta_1(\xi,\rho) = l_1(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho, \tag{3.4}$$

$$\zeta_2(\xi,\rho) = l_2(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_2}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho, \tag{3.5}$$

and

$$\zeta_3(\xi,\rho) = l_3(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_3}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho. \tag{3.6}$$

Define the operators $\mathcal{A}_{\zeta_1}, \mathcal{A}_{\zeta_2} : \mathbb{R}^{\Delta_{\rho_0}} \times \mathbb{R}^{\Delta_{\rho_0}} \times \mathbb{R}^{\Delta_{\rho_0}} \to \mathbb{R}^{\Delta_{\rho_0}}$ by

$$\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho) = l_1(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho, \tag{3.7}$$

$$\mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho) = l_2(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_2}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho, \tag{3.8}$$

and

$$\mathcal{A}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho) = l_3(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_3}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho. \tag{3.9}$$

The operator $\mathbb{J}_{\zeta_1}$ is said to have the Lipschitz condition if there is $\lambda_{\zeta_1} > 0$ where

$$\left|\mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3) - \mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_{11},\zeta_{21})\right| \leq \lambda_{\zeta_1}\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|_{\mathcal{B}}$$

for all $(\xi,\rho) \in \Delta_{\rho_0}$.

Theorem 3.1. If $\mathbb{J}_{\zeta_1}$ has the Lipschitz condition, then $\mathcal{A}_{\zeta_1}$ has the Lipschitz property.

Proof. Since $\mathbb{J}_{\zeta_1}$ has the Lipschitz property, then there is $\lambda_{\zeta_1} > 0$ where

$$\left|\mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3) - \mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_{11},\zeta_{21},\zeta_{31})\right| \leq \lambda_{\zeta_1}\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|_{\mathcal{B}}.$$

Hence,

$$
\begin{aligned}
&\left|\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)-\mathcal{A}_{\zeta_1}(\zeta_{11},\zeta_{21},\zeta_{31})(\xi,\rho)\right| \leq \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\\
&\times\left|\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)d\varrho-\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_{11},\zeta_{21},\zeta_{31})d\varrho\right|\\
&\leq \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}\left|\rho^{\beta}-\varrho^{\beta}\right|^{\alpha-1}\varrho^{\beta\sigma}\ \left|\mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3)-\mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_{11},\zeta_{21},\zeta_{31})\right|d\varrho\\
&\leq \frac{\lambda_{\zeta_1}\ \rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|_{\mathcal{B}}\,.
\end{aligned}
$$

Then, we have

$$
\left\|\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)-\mathcal{A}_{\zeta_1}(\zeta_{11},\zeta_{21},\zeta_{31})\right\| \leq \frac{\lambda_{\zeta_1}\ \rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|_{\mathcal{B}}\,, \tag{3.10}
$$

where $\|\cdot\|$ denotes the supremum norm in the Banach space $\mathcal{B}$. Note that for $0<\beta<1$ and $\rho\in[0,\rho_0]$, the term $\rho^{1-\beta}$ is non-negative and bounded. Moreover, the Gamma function $\Gamma(\cdot)$ is well-defined and positive for the considered parameters. Hence, $\frac{\lambda_{\zeta_1}\ \rho_0^{1-\beta}\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}>0$, that is, $\mathcal{A}_{\zeta_1}$ has the Lipschitz property. □

Remark 3.2. Similarly, for Theorem 3.1 and its proof, if $\mathbb{J}_{\zeta_2}$ and $\mathbb{J}_{\zeta_3}$ have the Lipschitz property, then $\mathcal{A}_{\zeta_2}$ and $\mathcal{A}_{\zeta_3}$ have the Lipschitz property, respectively. That is, there are two constants $\lambda_{\zeta_2}>0$ and $\lambda_{\zeta_3}>0$ such that

$$
\left\|\mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)-\mathcal{A}_{\zeta_2}(\zeta_{11},\zeta_{21},\zeta_{31})\right\| \leq \frac{\lambda_{\zeta_2}\ \rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|_{\mathcal{B}}\,, \tag{3.11}
$$

and

$$
\left\|\mathcal{A}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)-\mathcal{A}_{\zeta_3}(\zeta_{11},\zeta_{21},\zeta_{31})\right\| \leq \frac{\lambda_{\zeta_3}\ \rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|_{\mathcal{B}}\,. \tag{3.12}
$$

Define the operators $\mathcal{G}_{\zeta_1},\mathcal{G}_{\zeta_2},\mathcal{G}_{\zeta_3}:\mathcal{B}\to\mathbb{R}^{\Delta_{\rho_0}}$ as

$$
\mathcal{G}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)=\varsigma_1\zeta_1+\mathcal{V}\left[\pi_1,\pi_2,\zeta_1\circ(\pi_1\times\eta(\vartheta)),\zeta_1\circ(\pi_1\times\eta(\omega\circ(\zeta_2\zeta_3)))\right], \tag{3.13}
$$

$$
\mathcal{G}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)=\varsigma_2\zeta_2+\mathcal{V}'\left[\pi_1,\pi_2,\zeta_2\circ(\pi_1\times\eta(\vartheta)),\zeta_2\circ(\pi_1\times\eta(\omega\circ(\zeta_1\zeta_3)))\right], \tag{3.14}
$$

and

$$
\mathcal{G}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)=\varsigma_3\zeta_3+\mathcal{V}''\left[\pi_1,\pi_2,\zeta_3\circ(\pi_1\times\eta(\vartheta)),\zeta_3\circ(\pi_1\times\eta(\omega\circ(\zeta_1\zeta_2)))\right], \tag{3.15}
$$

where $\pi_{1,2}$ denotes the standard projection mappings and $\eta(\vartheta)(\xi,\rho)=\rho-\vartheta$. To establish the Lipschitz condition of $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$, and $\mathcal{G}_{\zeta_3}$, observe that if $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$ satisfy the Lipschitz property, then we obtain

$$|\langle \mathcal{G}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)-\mathcal{G}_{\zeta_1}(\zeta_{11},\zeta_{21},\zeta_{31}),(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\rangle| \le \lambda_{\zeta_1}\,\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|^2_{\mathcal{B}},$$

$$|\langle \mathcal{G}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)-\mathcal{G}_{\zeta_2}(\zeta_{11},\zeta_{21},\zeta_{31}),(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\rangle| \le \lambda_{\zeta_2}\,\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|^2_{\mathcal{B}},$$

and

$$|\langle \mathcal{G}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)-\mathcal{G}_{\zeta_3}(\zeta_{11},\zeta_{21},\zeta_{31}),(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\rangle| \le \lambda_{\zeta_3}\,\|(\zeta_1,\zeta_2,\zeta_3)-(\zeta_{11},\zeta_{21},\zeta_{31})\|^2_{\mathcal{B}}.$$

Theorem 3.3. The continuity of $\mathbb{J}_{\zeta_1}$ implies to the continuity of $\mathcal{G}_{\zeta_1}$.

Proof. Assume that $\mathbb{J}_{\zeta_1}$ is a continuous operator. To establish the continuity of $\mathcal{G}_{\zeta_1}$, it suffices to show that $\mathcal{G}^{-1}_{\zeta_1}(\langle Q, B(\xi_0,r)\rangle)$ is open in $\mathcal{B}$ for every open $\langle Q, B(\xi_0,r)\rangle$ in $\mathbb{R}^{\Delta_{\rho_0}}$. Observe that

$$\begin{aligned}\mathcal{G}^{-1}_{\zeta_1}(\langle Q, B(\xi_0,r)\rangle) &= \{\,(\varsigma,\varsigma',\varsigma'')\in\mathcal{B}:\ \mathcal{G}_{\zeta_1}(\varsigma,\varsigma',\varsigma'')(Q)\subset B(\xi_0,r)\,\}\\ &= \{\,(\varsigma,\varsigma',\varsigma'')\in\mathcal{B}:\ \mathbb{J}_{\zeta_1}(\{(\varsigma,\varsigma',\varsigma'')\}\times Q)\subset B(\xi_0,r)\,\}.\end{aligned}$$

Fix $(\varsigma_0,\varsigma_0',\varsigma_0'')\in\mathcal{G}^{-1}_{\zeta_1}(\langle Q,B(\xi_0,r)\rangle)$. Then, $\mathbb{J}_{\zeta_1}(\varsigma_0,\varsigma_0',\varsigma_0'',h)\in B(\xi_0,r)$ for all $h\in Q$, i.e. $\{(\varsigma_0,\varsigma_0',\varsigma_0'')\}\times Q\subset\mathbb{J}^{-1}_{\zeta_1}(B(\xi_0,r))$. Since $\mathbb{J}^{-1}_{\zeta_1}(B(\xi_0,r))\subset\Delta_{\rho_0}\times\mathcal{B}$ is open and $\mathbb{J}_{\zeta_1}$ is continuous, then for any point $d\in Q$ there is an open set $E_d\subset\mathcal{B}$ of $(\varsigma_0,\varsigma_0',\varsigma_0'')$ and $F_d\subset\Delta_{\rho_0}$ of h such that $F_d\times E_d\subset\mathbb{J}^{-1}_{\zeta_1}(B(\xi_0,r))$. The class $\{F_d\}_{d\in Q}$ is an open covering of the compact set Q, so choose $h_1,\ldots,h_n\in Q$ with $Q\subset\bigcup_{i=1}^n F_{h_i}$, and set $E:=\cap_{i=1}^n E_{h_i}$. Then, E is an open neighborhood of $(\varsigma_0,\varsigma_0',\varsigma_0'')$ in $\mathcal{B}$, and hence for any $(\varsigma,\varsigma',\varsigma'')\in E$ and any $k\in Q$, there is i with $k\in F_{h_i}$ such that

$$(h,k)\in F_{h_i}\times E_{h_i}\subset\mathbb{J}^{-1}_{\zeta_1}(B(\xi_0,r)).$$

Then, $\mathbb{J}_{\zeta_1}(Q,\varsigma,\varsigma',\varsigma'')\subset B(\xi_0,r)$. Hence, $\mathcal{G}_{\zeta_1}(\varsigma,\varsigma',\varsigma'')\in\langle Q,B(\xi_0,r)\rangle$. Consequently, $E\subset\mathcal{G}^{-1}_{\zeta_1}(\langle Q,B(\xi_0,r)\rangle)$, which means that $\mathcal{G}^{-1}_{\zeta_1}(\langle Q,B(\xi_0,r)\rangle)$ is an open set. Hence, $\mathcal{G}_{\zeta_1}$ is continuous. □

Theorem 3.4. If $\mathcal{G}_{\zeta_1}$ has the continuity and Δ has a locally compact property in $\mathbb{R}$, then the map of $\mathbb{J}_{\zeta_1}$ to $\Delta_{\rho_0}\times\mathcal{B}$ is continuous.

Proof. Assume that $\mathcal{G}_{\zeta_1}$ has the continuity and Δ has a locally compact property in $\mathbb{R}$. We show that $\mathbb{J}^{-1}_{\zeta_1}(B(\xi,r))$ is open in $\Delta_{\rho_0}\times\mathcal{B}$ for any ball $B(\xi,r)\subset\mathbb{R}$. Let $B(\xi_0,r)$ be any ball in $\mathbb{R}$ and $(\xi_0',\rho_0',\varsigma_0,\varsigma_0',\varsigma_0'')$ be any point in $\mathbb{J}^{-1}_{\zeta_1}(B(\xi_0,r))$. Then,

$$\mathcal{G}_{\zeta_1}(\varsigma_0,\varsigma_0',\varsigma_0'')(\xi_0',\rho_0')\in B(\xi_0,r).$$

Since $\mathbb{R}$ is Hausdorff and Δ has the locally compact property in $\mathbb{R}$, then Δ_{ρ_0} has the locally compact property and is a Hausdorff set. Thus, there is an open set F containing (ξ_0',ρ_0') such that its closure $\overline{F}$ is compact and contains (ξ_0',ρ_0'). Let $Q:=\overline{F}$. Then, Q is compact and $(\xi_0',\rho_0')\in F\subset Q$. Because $\mathcal{G}_{\zeta_1}(\varsigma_0,\varsigma_0',\varsigma_0'')(\xi_0',\rho_0')\in B(\xi_0,r)$, we obtain that $\mathcal{G}_{\zeta_1}(\varsigma_0,\varsigma_0',\varsigma_0'')(Q)$ intersects $B(\xi_0,r)$ in (ξ_0',ρ_0'). Define

$$\langle Q,B(\xi_0,r)\rangle=\{(\varsigma,\varsigma')\in\mathbb{R}^{\Delta_{\rho_0}}\times\mathbb{R}^{\Delta_{\rho_0}}:\mathcal{G}_{\zeta_1}(\varsigma,\varsigma')(Q)\subset B(\xi_0,r)\}.$$

Since $\langle Q, B(\xi_0, r)\rangle$ is open in $\mathbb{R}^{\Delta_{\rho_0}}$ and $\mathcal{G}_{\zeta_1}$ is continuous, there is an open set $N \subset \mathcal{B}$ containing $(\varsigma_0, \varsigma_0', \varsigma_0'')$ such that

$$\mathcal{G}_{\zeta_1}(N) \subset \langle Q, B(\xi_0, r)\rangle.$$

Hence, for any $x \in N$, $\mathcal{G}_{\zeta_1}(x)(Q) \subset B(\xi_0, r)$. Consider

$$F \times N \subset \Delta_{\rho_0} \times \mathcal{B}.$$

If $(\xi, \rho, \varsigma, \varsigma', \varsigma'') \in F \times N$, then $(\xi, \rho) \in F \subset Q$ and $\mathcal{G}_{\zeta_1}(\varsigma, \varsigma', \varsigma'')(Q) \subset B(\xi_0, r)$. Therefore,

$$\mathbb{J}_{\zeta_1}(\xi, \rho, \varsigma, \varsigma', \varsigma'') = \mathcal{G}_{\zeta_1}(\varsigma, \varsigma', \varsigma'')(\xi, \rho) \in B(\xi_0, r).$$

Thus,

$$F \times N \subset \mathbb{J}_{\zeta_1}^{-1}(B(\xi_0, r)).$$

Since every point of $\mathbb{J}_{\zeta_1}^{-1}(B(\xi_0, r))$ admits such an open neighborhood, the set $\mathbb{J}_{\zeta_1}^{-1}(B(\xi_0, r))$ is open. As $B(\xi_0, r)$ is arbitrary in $\mathbb{R}$, we conclude that $\mathbb{J}_{\zeta_1}$ is continuous. □

Remark 3.5. Similarly, conclusions hold for Theorems 3.3 and 3.4 together with their proofs:

1) If $\mathbb{J}_{\zeta_2}$ and $\mathbb{J}_{\zeta_3}$ are continuous, then $\mathcal{G}_{\zeta_2}$ and $\mathcal{G}_{\zeta_3}$ are also continuous, respectively.
2) If the operators $\mathcal{G}_{\zeta_2}$ and $\mathcal{G}_{\zeta_3}$ are continuous and Δ has the locally compact property in $\mathbb{R}$, then the restrictions $\mathbb{J}_{\zeta_2}$ and $\mathbb{J}_{\zeta_3}$ on $\Delta_{\rho_0} \times \mathcal{B}$ are continuous, respectively.

4. Existence and stability analysis

In this section, we establish the solvability and uniqueness of the considered fractional system by employing Banach and Schaefer fixed point Theorems 2.4 and 2.5. The Hyers–Ulam stability of the corresponding solutions is investigated. Define the operator $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ as

$$\mathcal{A}(\zeta_1, \zeta_2, \zeta_3)(\xi, \rho) = (\mathcal{A}_{\zeta_1}(\zeta_1, \zeta_2, \zeta_3)(\xi, \rho),\ \mathcal{A}_{\zeta_2}(\zeta_1, \zeta_2, \zeta_3)(\xi, \rho),\ \mathcal{A}_{\zeta_3}(\zeta_1, \zeta_2, \zeta_3)(\xi, \rho))$$

for all $(\xi, \rho) \in \Delta_{\rho_0}$. Consider the following assumptions:

R1: Assume Δ is a bounded and locally compact subset of $\mathbb{R}$, $\zeta_1, \zeta_2, \zeta_3 \in C_{\Delta_0}$, and consider the space C_{Δ_0} under the compact-open topology.

R2: There exist positive real numbers ϑ_{ζ_1}, ϑ_{ζ_2}, and ϑ_{ζ_3} such that

$$\left|\mathbb{J}_{\zeta_1}(\xi, \rho, \zeta_1, \zeta_2, \zeta_3)\right| \le \vartheta_{\zeta_1},\ \left|\mathbb{J}_{\zeta_2}(\xi, \rho, \zeta_1, \zeta_2, \zeta_3)\right| \le \vartheta_{\zeta_2},\ \left|\mathbb{J}_{\zeta_3}(\xi, \rho, \zeta_1, \zeta_2, \zeta_3)\right| \le \vartheta_{\zeta_3}.$$

R3: The operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$ have the Lipschitz property, with corresponding the Lipschitz constants $\lambda_{\zeta_1} > 0$, $\lambda_{\zeta_2} > 0$, and $\lambda_{\zeta_3} > 0$, respectively.

Lemma 4.1. If the condition R1 holds, then the operators $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$, and $\mathcal{G}_{\zeta_3}$ are continuous.

Proof. From R1, we have that $\zeta_1, \zeta_2, \zeta_3$ are continuous. Since $\mathcal{V}_{1,2,3}$ are continuous, then so too are the operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$. By Theorem 3.3 and the part (1) in Remark 3.5, $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$, and $\mathcal{G}_{\zeta_3}$ are continuous. □

Theorem 4.2. If the conditions R1 and R2 hold, then (1.1) has at least one solution.

Proof. To solve system (1.1), we apply the Schaefer Theorem 2.5. For any $\epsilon > 0$, let

$$M_\epsilon = \{(\zeta_1, \zeta_2, \zeta_3) \in \mathcal{B} : \|(\zeta_1, \zeta_2, \zeta_3)\|_{\mathcal{B}} \le \epsilon\}.$$

First, for the continuity of $\mathcal{A}$, we will prove that $\|\mathcal{A}(\zeta_{1n}, \zeta_{2n}, \zeta_{3n}) - \mathcal{A}(\zeta_1, \zeta_2, \zeta_3)\|_{\mathcal{B}} \to 0$ when the sequence $(\zeta_{1n}, \zeta_{2n}, \zeta_{3n})$ converges to a point $(\zeta_1, \zeta_2, \zeta_3)$ in $\mathcal{B}$. Let $(\zeta_{1n}, \zeta_{2n}, \zeta_{3n}) \to (\zeta_1, \zeta_2, \zeta_3)$. By the continuity of ω, ζ_1, ζ_2, and ζ_3, we get the following pointwise convergences:

$$\zeta_{1n}(\xi,\rho) \to \zeta_1(\xi,\rho),\ \ \zeta_{2n}(\xi,\rho) \to \zeta_2(\xi,\rho),\ \ \zeta_{3n}(\xi,\rho) \to \zeta_3(\xi,\rho),$$

$$\zeta_{1n}(\xi,\rho-\vartheta) \to \zeta_1(\xi,\rho-\vartheta),\ \ \omega(\zeta_{2n}(\xi,\rho)\zeta_{3n}(\xi,\rho)) \to \omega(\zeta_2(\xi,\rho)\zeta_3(\xi,\rho)),$$

and

$$\zeta_{1n}(\xi, \omega(\zeta_{2n}(\xi,\rho)\zeta_{3n}(\xi,\rho))) \to \zeta_1(\xi, \omega(\zeta_2(\xi,\rho)\zeta_3(\xi,\rho))).$$

By the continuity of $\mathcal{V}$, we have

$$\begin{aligned}&\mathcal{V}(\xi,\rho,\zeta_{1n}(\xi,\rho-\vartheta),\zeta_{1n}(\xi,\rho-\omega(\zeta_{2n}(\xi,\rho)\zeta_{3n}(\xi,\rho))))\\ &\to \mathcal{V}(\xi,\rho,\zeta_1(\xi,\rho-\vartheta),\zeta_1(\xi,\rho-\omega(\zeta_2(\xi,\rho)\zeta_3(\xi,\rho)))).\end{aligned}$$

For any $(\xi,\rho) \in \Delta_{\rho_0}$, we have

$$\begin{aligned}&\left|\mathcal{A}_{\zeta_1}(\zeta_{1n},\zeta_{2n},\zeta_{3n})(\xi,\rho) - \mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right|\\ &\le \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\left|\int_0^\rho (\rho^\beta - \varrho^\beta)^{\alpha-1}\varrho^{\beta\sigma}\left[\mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_{1n},\zeta_{2n},\zeta_{3n}) - \mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right]d\varrho\right|\\ &\le \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^\rho \left|\rho^\beta - \varrho^\beta\right|^{\alpha-1}\varrho^{\beta\sigma}\left|\mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_{1n},\zeta_{2n},\zeta_{3n}) - \mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|d\varrho.\end{aligned} \tag{4.1}$$

By Lemma 4.1, the operator $\mathcal{G}_{\zeta_1}$ is continuous. Since Δ has the Hausdorff locally compact property, it follows from Theorem 3.4 that $\mathbb{J}_{\zeta_1}$ is also continuous. Consequently, for every $(\xi,\rho) \in \Delta_{\rho_0}$, we obtain

$$\left|\mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_{1n},\zeta_{2n},\zeta_{3n}) - \mathbb{J}_{\zeta_1}(\xi,\rho,\zeta_1,\zeta_2,\zeta_3)\right| \to 0.$$

Because Δ_{ρ_0} is compact and the convergence occurs with respect to the supremum norm, the convergence is uniform. In particular,

$$\zeta_{1n} \to \zeta_1$$

uniformly. Then, for any $(\xi,\rho) \in \Delta_{\rho_0}$ and $n \in \mathbb{N}$, we have

$$|\zeta_{1n}| < C_{\zeta_1} \quad \text{and} \quad |\zeta_1| < C_{\zeta_1}$$

for some constant C_{ζ_1}. Therefore, by the uniform convergence on the compact interval, we get

$$\sup_{(\xi,\rho)\in\Delta_{\rho_0}} \left\|\mathcal{A}_{\zeta_1}(\zeta_{1n},\zeta_{2n},\zeta_{3n})(\xi,\rho) - \mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right\| \to 0.$$

Similarly, we have

$$\sup_{(\xi,\rho)\in\Delta_{\rho_0}} \left\|\mathcal{A}_{\zeta_2}(\zeta_{1n},\zeta_{2n},\zeta_{3n})(\xi,\rho) - \mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right\| \to 0,$$

and

$$\sup_{(\xi,\rho)\in\Delta_{\rho_0}} \left\|\mathcal{A}_{\zeta_3}(\zeta_{1n},\zeta_{2n},\zeta_{3n})(\xi,\rho)-\mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right\| \to 0.$$

Hence, $\|\mathcal{A}(\zeta_{1n},\zeta_{2n},\zeta_{3n})-\mathcal{A}(\zeta_1,\zeta_2,\zeta_3)\|_{\mathcal{B}} \to 0$. That is, $\mathcal{A}$ is continuous.

Next, we show that the boundedness of the set space remains invariant as a topological property under the action of the continuous operator $\mathcal{A}$. Let $(\zeta_1,\zeta_2,\zeta_3)\in M_\epsilon$. Then,

$$\begin{aligned}
&\left|\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right| \\
&\le |l_1(\xi)| + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^\rho \left|\rho^\beta-\varrho^\beta\right|^{\alpha-1}\varrho^{\beta\sigma}\,\left|\mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|d\varrho \\
&\le |l_1(\xi)| + \frac{\vartheta_{\zeta_1}\,\rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\left|\mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right| \\
&\le |l_2(\xi)| + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^\rho \left|\rho^\beta-\varrho^\beta\right|^{\alpha-1}\varrho^{\beta\sigma}\,\left|\mathbb{J}_{\zeta_2}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|d\varrho \\
&\le |l_2(\xi)| + \frac{\vartheta_{\zeta_2}\,\rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)},
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\left|\mathcal{A}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)\right| \\
&\le |l_3(\xi)| + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^\rho \left|\rho^\beta-\varrho^\beta\right|^{\alpha-1}\varrho^{\beta\sigma}\,\left|\mathbb{J}_{\zeta_3}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|d\varrho \\
&\le |l_3(\xi)| + \frac{\vartheta_{\zeta_3}\,\rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}.
\end{aligned} \tag{4.4}$$

Hence, $\mathcal{A}(M_\epsilon)\subseteq M_{\epsilon'}$, where

$$\begin{aligned}
\epsilon' = \max\Bigg\{ & |l_1(\xi)| + \frac{\vartheta_{\zeta_1}\,\rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}, |l_2(\xi)| + \frac{\vartheta_{\zeta_2}\,\rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}, \\
& |l_3(\xi)| + \frac{\vartheta_{\zeta_3}\,\rho_0^{1-\beta}\,\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\Bigg\}.
\end{aligned}$$

Since ϵ' does not depend on $(\zeta_1,\zeta_2,\zeta_3)$, it follows that $\mathcal{A}(M_\epsilon)$ forms a uniformly bounded set.

Third, to verify the equicontinuity of the operator $\mathcal{A}$, let $(\xi_1,\rho_1),(\xi,\rho)\in\Delta_{\rho_0}$. be arbitrary points.

Note that

$$\begin{aligned}
&\left|\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)-\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi_1,\rho_1)\right| \le \frac{\beta\,(\rho-\rho_1)^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\\
&\times\int_0^{\rho}\left[(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}-(\rho_1^{\beta}-\varrho^{\beta})^{\alpha-1}\right]\varrho^{\beta\sigma}\ \left|\mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)-\mathbb{J}_{\zeta_1}(\xi_1,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|d\varrho\\
&\le\frac{\beta\,(\rho-\rho_1)^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\\
&\times\int_0^{\rho}\left|\rho^{\beta}-\varrho^{\beta})^{\alpha-1}-(\rho_1^{\beta}-\varrho^{\beta})^{\alpha-1}\right|\varrho^{\beta\sigma}\ \left[\left|\mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|+\left|\mathbb{J}_{\zeta_1}(\xi_1,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|\right]d\varrho\\
&\le\frac{2\epsilon'\,\beta\,(\rho-\rho_1)^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}\left|(\rho-\varrho)^{\alpha-1}-(\rho_1-\varrho)^{\alpha-1}\right|\varrho^{\beta\sigma}d\varrho.
\end{aligned}\tag{4.5}$$

Since the kernels are integrable, it follows that

$$\left|\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)-\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi_1,\rho_1)\right|\to 0\quad\text{as}\quad(\xi,\rho)\to(\xi_1,\rho_1).$$

Similarly, we have

$$\left|\mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)-\mathcal{A}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3)(\xi_1,\rho_1)\right|\to 0\quad\text{as}\quad(\xi,\rho)\to(\xi_1,\rho_1),$$

and

$$\left|\mathcal{A}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)-\mathcal{A}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3)(\xi_1,\rho_1)\right|\to 0\quad\text{as}\quad(\xi,\rho)\to(\xi_1,\rho_1).$$

Hence,

$$\|\mathcal{A}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)-\mathcal{A}(\zeta_1,\zeta_2,\zeta_3)(\xi_1,\rho_1)\|_{\mathcal{B}}\to 0\quad\text{as}\quad(\xi,\rho)\to(\xi_1,\rho_1).$$

Consequently, $\mathcal{A}$ fulfills the equicontinuity requirement. Together with the previously established boundedness properties, the Arzela–Ascoli Theorem 2.6 ensures that $\mathcal{A}$ is a compact operator.

Finally, we show that $\Theta=\{(\zeta_1,\zeta_2,\zeta_3)\in\mathcal{B}:(\zeta_1,\zeta_2,\zeta_3)=t\mathcal{A}(\zeta_1,\zeta_2,\zeta_3),\ t\in[0,1]\}$ is bounded in $\mathcal{B}$. Assume that $(\zeta_1,\zeta_2,\zeta_3)=t\mathcal{A}(\zeta_1,\zeta_2,\zeta_3)$ for some $t\in[0,1]$. Then,

$$\begin{aligned}
|\zeta_1(\xi,\rho)|&=|t\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)|=t|\mathcal{A}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3)(\xi,\rho)|\\
&\le t\left[|l_1(\xi)|+\frac{\vartheta_{\zeta_1}\rho_0^{1-\beta}\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\right].
\end{aligned}\tag{4.6}$$

Similarly,

$$|\zeta_2(\xi,\rho)|\le t\left[|l_2(\xi)|+\frac{\vartheta_{\zeta_2}\rho_0^{1-\beta}\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\right],\tag{4.7}$$

and

$$|\zeta_3(\xi,\rho)|\le t\left[|l_3(\xi)|+\frac{\vartheta_{\zeta_3}\rho_0^{1-\beta}\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}\right].\tag{4.8}$$

Therefore, Θ has boundedness in $\mathcal{B}$. By using the Schaefer Theorem 2.5, a fixed point of $\mathcal{A}$ in M_ϵ is a solution of (1.1). □

Theorem 4.3. If R1-R3 hold and

$$\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3} < \frac{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)}{\rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)},$$

then system (1.1) has a unique solution.

Proof. From the condition R3 and Theorem 3.1, we get that for $(\zeta_1, \zeta_2, \zeta_3), (\zeta_{11}, \zeta_{21}, \zeta_{31}) \in \mathcal{B}$,

$$\left\|\mathcal{A}_{\zeta_1}(\zeta_1, \zeta_2, \zeta_3) - \mathcal{A}_{\zeta_1}(\zeta_{11}, \zeta_{21}, \zeta_{31})\right\| \leq \frac{\lambda_{\zeta_1}\, \rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}},$$

$$\left\|\mathcal{A}_{\zeta_2}(\zeta_1, \zeta_2, \zeta_3) - \mathcal{A}_{\zeta_2}(\zeta_{11}, \zeta_{21}, \zeta_{31})\right\| \leq \frac{\lambda_{\zeta_2}\, \rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}},$$

and

$$\left\|\mathcal{A}_{\zeta_3}(\zeta_1, \zeta_2, \zeta_3) - \mathcal{A}_{\zeta_3}(\zeta_{11}, \zeta_{21}, \zeta_{31})\right\| \leq \frac{\lambda_{\zeta_3}\, \rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}}.$$

Hence,

$$\begin{aligned}
\|\mathcal{A}(\zeta_1, \zeta_2, \zeta_3) - \mathcal{A}(\zeta_{11}, \zeta_{21}, \zeta_{31})\| \leq{}& \frac{\lambda_{\zeta_1}\, \rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}} \\
&+ \frac{\lambda_{\zeta_2}\, \rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}} \\
&+ \frac{\lambda_{\zeta_3}\, \rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}} \\
&= \frac{\rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} (\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3}) \|(\zeta_1, \zeta_2, \zeta_3) - (\zeta_{11}, \zeta_{21}, \zeta_{31})\|_{\mathcal{B}}.
\end{aligned}$$

Hence, whenever the condition

$$\frac{\rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)} (\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3}) < 1, \quad \text{i.e.,} \quad \lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3} < \frac{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)}{\rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}.$$

is satisfied, then the operator $\mathcal{A}$ becomes a contraction mapping. An application of the Banach fixed point Theorem 2.4, then guarantees the existence and uniqueness of the solution to the system. □

Now, we analyze the stability of solutions to the fractional nonlinear system (1.1). Specifically, we investigate the Hyers–Ulam stability, which guarantees that small perturbations in the initial data or system parameters lead only to small deviations in the corresponding solutions. This property highlights the robustness and reliability of the proposed fractional model. The results obtained provide a solid theoretical foundation for the stability of the system within the adopted topological and functional analytic framework. Let $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3)$ be any approximate solution of (1.1) with

$$\begin{cases} \left|{}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\hat{\zeta}_1(\xi,\rho) - \mathbb{J}_{\zeta_1}(\xi,\rho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)\right| \leq \Omega_1, \\ \left|{}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\hat{\zeta}_2(\xi,\rho) - \mathbb{J}_{\zeta_2}(\xi,\rho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)\right| \leq \Omega_2, \\ \left|{}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\hat{\zeta}_3(\xi,\rho) - \mathbb{J}_{\zeta_3}(\xi,\rho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)\right| \leq \Omega_3, \end{cases} \tag{4.9}$$

for all $(\xi,\rho) \in \Delta_{\rho_0}$, where $\Omega_1, \Omega_2, \Omega_3 > 0$. By [28], (1.1) has Hyers–Ulam stability if we have a unique solution $(\zeta_1, \zeta_2, \zeta_3)$ of (1.1) where

$$\left\|\hat{\zeta}_1 - \zeta_1\right\| \leq \Omega_1'\Omega_1, \quad \left\|\hat{\zeta}_2 - \zeta_2\right\| \leq \Omega_2'\Omega_2 \quad \text{and} \quad \left\|\hat{\zeta}_3 - \zeta_3\right\| \leq \Omega_3'\Omega_3,$$

where $\Omega_1', \Omega_2', \Omega_3' > 0$ are constants independent of $\Omega_1, \Omega_2, \Omega_3$, and $\|\cdot\|$ represents the supremum norm on C_{Δ_0}.

Theorem 4.4. If R1–R3 hold and

$$\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3} < \frac{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)}{\rho_0^{1-\beta}\Gamma\left(\sigma + \frac{1}{\beta}\right)},$$

then (1.1) has Hyers–Ulam stability.

Proof. Let $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3)$ be an approximate solution, and let $(\zeta_1, \zeta_2, \zeta_3)$ be the solution of (1.1). By (4.9), there exist three continuous maps g, g', and g'' defined on Δ_{ρ_0} such that

$$|g(\xi,\rho)| \leq \varepsilon_1, \quad |g'(\xi,\rho)| \leq \varepsilon_2, \quad |g''(\xi,\rho)| \leq \varepsilon_3.$$

$${}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\hat{\zeta}_1(\xi,\rho) = \mathbb{J}_{\zeta_1}(\xi,\rho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3) + g(\xi,\rho),$$

$${}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\hat{\zeta}_2(\xi,\rho) = \mathbb{J}_{\zeta_2}(\xi,\rho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3) + g'(\xi,\rho),$$

and

$${}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\hat{\zeta}_3(\xi,\rho) = \mathbb{J}_{\zeta_3}(\xi,\rho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3) + g''(\xi,\rho),$$

for every $(\xi,\rho) \in \Delta_{\rho_0}$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. Accordingly, the approximate solution $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3)$ can be expressed as

$$\begin{aligned} \hat{\zeta}_1(\xi,\rho) = l_1(\xi) &+ \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta} - \varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_1}(\xi,\varrho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)d\varrho \\ &+ \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta} - \varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ g(\xi,\varrho)d\varrho, \end{aligned}$$

$$\begin{aligned} \hat{\zeta}_2(\xi,\rho) = l_2(\xi) &+ \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta} - \varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_2}(\xi,\varrho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)d\varrho \\ &\frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta} - \varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ g'(\xi,\varrho)d\varrho, \end{aligned}$$

and

$$\begin{aligned}\hat{\zeta}_3(\xi,\rho) &= l_3(\xi) + \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ \mathbb{J}_{\zeta_3}(\xi,\varrho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)d\varrho\\ &+ \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}(\rho^{\beta}-\varrho^{\beta})^{\alpha-1}\varrho^{\beta\sigma}\ g''(\xi,\varrho)d\varrho\end{aligned}$$

for all $(\xi,\rho)\in\Delta_{\rho_0}$. Then, by Theorem (4.3) for all $(\xi,\rho)\in\Delta_{\rho_0}$, we have

$$\begin{aligned}&\left|\hat{\zeta}_1(\xi,\rho)-\zeta_1(\xi,\rho)\right| \le \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\\ &\times\int_0^{\rho}\left|\rho^{\beta}-\varrho^{\beta}\right|^{\alpha-1}\varrho^{\beta\sigma}\ \left[\left|\mathbb{J}_{\zeta_1}(\xi,\varrho,\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-\mathbb{J}_{\zeta_1}(\xi,\varrho,\zeta_1,\zeta_2,\zeta_3)\right|+|g(\xi,\varrho)|\right]d\varrho\\ &\le \frac{\beta\rho^{-\beta(\alpha+\sigma)}}{\Gamma(\alpha)}\int_0^{\rho}\left|\rho^{\beta}-\varrho^{\beta}\right|^{\alpha-1}\varrho^{\beta\sigma}\ \left[\lambda_{\zeta_1}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_1\right]d\varrho\\ &\le \frac{\rho_0^{1-\beta}\,\Gamma\!\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\!\left(\alpha+\sigma+\frac{1}{\beta}\right)}\left[\lambda_{\zeta_1}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_1\right].\end{aligned}$$

That is,

$$\left\|\hat{\zeta}_1-\zeta_1\right\|\le\mathbb{F}^{\alpha}\left[\lambda_{\zeta_1}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_1\right],$$

where $\mathbb{F}^{\alpha}:=\frac{\rho_0^{1-\beta}\Gamma\left(\sigma+\frac{1}{\beta}\right)}{\Gamma\left(\alpha+\sigma+\frac{1}{\beta}\right)}$. Similarly, we have

$$\left\|\hat{\zeta}_2-\zeta_2\right\|\le\mathbb{F}^{\alpha}\left[\lambda_{\zeta_2}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_2\right],$$

and

$$\left\|\hat{\zeta}_3-\zeta_3\right\|\le\mathbb{F}^{\alpha}\left[\lambda_{\zeta_3}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_3\right].$$

Hence,

$$\begin{aligned}\left\|\hat{\zeta}_1-\zeta_1\right\| &\le\mathbb{F}^{\alpha}\left[\lambda_{\zeta_1}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_1\right]\\ &\le\mathbb{F}^{\alpha}\left[\frac{\mathbb{F}^{\alpha}\lambda_{\zeta_1}}{1-\left(\mathbb{F}^{\alpha}(\lambda_{\zeta_1}+\lambda_{\zeta_2}+\lambda_{\zeta_3})\right)}\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}\right]+\mathbb{F}^{\alpha}\varepsilon_1,\end{aligned}$$

$$\begin{aligned}\left\|\hat{\zeta}_2-\zeta_2\right\| &\le\mathbb{F}^{\alpha}\left[\lambda_{\zeta_2}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_2\right]\\ &\le\mathbb{F}^{\alpha}\left[\frac{\mathbb{F}^{\alpha}\lambda_{\zeta_2}}{1-\left(\mathbb{F}^{\alpha}(\lambda_{\zeta_1}+\lambda_{\zeta_2}+\lambda_{\zeta_3})\right)}\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}\right]+\mathbb{F}^{\alpha}\varepsilon_2,\end{aligned}$$

and

$$\begin{aligned}\left\|\hat{\zeta}_3-\zeta_3\right\| &\le\mathbb{F}^{\alpha}\left[\lambda_{\zeta_3}\left\|(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3)-(\zeta_1,\zeta_2,\zeta_3)\right\|_{\mathcal{B}}+\varepsilon_3\right]\\ &\le\mathbb{F}^{\alpha}\left[\frac{\mathbb{F}^{\alpha}\lambda_{\zeta_3}}{1-\left(\mathbb{F}^{\alpha}(\lambda_{\zeta_1}+\lambda_{\zeta_2}+\lambda_{\zeta_3})\right)}\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}\right]+\mathbb{F}^{\alpha}\varepsilon_3.\end{aligned}$$

Set $\varepsilon := \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Then

$$\left\|\hat{\zeta}_1 - \zeta_1\right\| \leq \varepsilon\, \mathbb{F}^\alpha \left[\frac{\mathbb{F}^\alpha \lambda_{\zeta_1}}{1 - \left(\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3})\right)} + 1\right],$$

$$\left\|\hat{\zeta}_2 - \zeta_2\right\| \leq \varepsilon\, \mathbb{F}^\alpha \left[\frac{\mathbb{F}^\alpha \lambda_{\zeta_2}}{1 - \left(\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3})\right)} + 1\right],$$

and

$$\left\|\hat{\zeta}_3 - \zeta_3\right\| \leq \varepsilon\, \mathbb{F}^\alpha \left[\frac{\mathbb{F}^\alpha \lambda_{\zeta_3}}{1 - \left(\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3})\right)} + 1\right].$$

Take

$$\varepsilon_1' := \mathbb{F}^\alpha \left[\frac{\mathbb{F}^\alpha \lambda_{\zeta_1}}{1 - \left(\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3})\right)} + 1\right], \quad \varepsilon_2' := \mathbb{F}^\alpha \left[\frac{\mathbb{F}^\alpha \lambda_{\zeta_2}}{1 - \left(\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3})\right)} + 1\right],$$

and

$$\varepsilon_3' := \mathbb{F}^\alpha \left[\frac{\mathbb{F}^\alpha \lambda_{\zeta_3}}{1 - \left(\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3})\right)} + 1\right].$$

The Hyers-Ulam stability parameters ε_1', ε_2', and ε_3' indicate the proximity between the approximate solution and the exact one. These quantities depend on the denominator

$$1 - \mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3}),$$

which reflects the contraction behavior of the operator. When the values of $\mathbb{F}^\alpha$, λ_{ζ_1}, λ_{ζ_2} and λ_{ζ_3} decrease, the denominator becomes larger and consequently the constants ε_{123}' become smaller. This results in a stronger stability property. In order to guarantee such stability, it is necessary to impose

$$\mathbb{F}^\alpha(\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3}) \leq 1, \quad \text{i.e.,} \quad \lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3} < \frac{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)}{\rho_0^{1-\beta}\,\Gamma\left(\sigma + \frac{1}{\beta}\right)}.$$

Hence, (1.1) has Hyers–Ulam stability. □

5. Applications to fractional models

In this section, the obtained theoretical results are applied to the fractional Lorenz system and the two-dimensional fractional Euler system. We verify the required conditions and establish the existence of unique and stable solutions for these models.

First, we apply Theorems 4.2 and 4.3 to the EK fractional Lorenz system (5.1). The system is reformulated in an abstract operator framework suitable for the application of fixed-point techniques by introducing the unknown vector $(\zeta_1, \zeta_2, \zeta_3)$ together with the associated nonlinear operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$. We work in the Banach space $C([0, \rho_0], \mathbb{R})$ and consider a closed bounded subset with uniformly bounded norms. The required assumptions R1, R2, and R3 are verified, and the corresponding constants $\vartheta_{\zeta_1}, \vartheta_{\zeta_2}, \vartheta_{\zeta_3}$ and $\lambda_{\zeta_1}, \lambda_{\zeta_2}, \lambda_{\zeta_3}$ are explicitly determined. The considered model represents a fractional extension of the classical Lorenz system, incorporating memory effects through the EK fractional derivative and describing spatiotemporal chaotic dynamics. The EK fractional Lorenz system is given by

$$\begin{cases} {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_1(\rho) = \epsilon(\zeta_2(\rho) - \zeta_1(\rho)), \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_2(\rho) = \zeta_1(\rho)(\omega - \zeta_3(\rho)) - \zeta_2(\rho), \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_3(\rho) = \zeta_1(\rho)\zeta_2(\rho) - \mu\zeta_3(\rho), \\ \zeta_1(0) = l_1, \quad \zeta_2(0) = l_2, \quad \zeta_3(0) = l_3, \\ \zeta_1, \zeta_2, \zeta_3 \in C([0,\rho_0], \mathbb{R}), \end{cases} \tag{5.1}$$

where ζ_1 represents the intensity of convective motion, ζ_2 represents the horizontal temperature variation, ζ_3 represents the vertical temperature variation, ρ is a time variable, ϵ is the Prandtl number, ω is the Rayleigh number, and μ is a geometric factor.

From system 1.1, we observe that $\varsigma_1 = -\epsilon$, $\varsigma_2 = -1$, and $\varsigma_3 = -\mu$. Then, the nonlinear operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$ are given by

$$\begin{aligned} \mathbb{J}_{\zeta_1}(\zeta_1, \zeta_2, \zeta_3)(\rho) &:= \epsilon(\zeta_2(\rho) - \zeta_1(\rho)), \\ \mathbb{J}_{\zeta_2}(\zeta_1, \zeta_2, \zeta_3)(\rho) &:= \zeta_1(\rho)(\omega - \zeta_3(\rho)) - \zeta_2(\rho), \\ \mathbb{J}_{\zeta_3}(\zeta_1, \zeta_2, \zeta_3)(\rho) &:= \zeta_1(\rho)\zeta_2(\rho) - \mu\zeta_3(\rho). \end{aligned}$$

Step 1: Satisfying the continuity property for the operators $\mathbb{J}_{\omega}$, $\mathbb{J}_{\zeta_1}$, and $\mathbb{J}_E$: Since $\zeta_1, \zeta_2, \zeta_3 \in C([0,\rho_0], \mathbb{R})$, then the operators $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$, and $\mathcal{G}_{\zeta_3}$ are continuous. Since $[0,\rho_0]$ is locally compact, then by Theorem 3.4 and Remark 3.5, $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$ are continuous.

Step 2: Satisfying the boundedness property for the operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, and $\mathbb{J}_{\zeta_3}$ to satisfy condition R2 on 5.1: Let

$$B := C([0,\rho_0], \mathbb{R}) \times C([0,\rho_0], \mathbb{R}) \times C([0,\rho_0], \mathbb{R})$$

with the norm

$$\|(\zeta_1, \zeta_2, \zeta_3)\|_B := \max\{\|\zeta_1\|_\infty, \|\zeta_2\|_\infty, \|\zeta_3\|_\infty\},$$

where $\|f\|_\infty := \sup_{\rho\in[0,\rho_0]} |f(\rho)|$. Since the functions $\zeta_1, \zeta_2, \zeta_3 \in BC^1(\Delta_0)$ in the system 5.1, then there are positive constants $\delta_{\zeta_1}, \delta_{\zeta_2}, \delta_{\zeta_3} > 0$ such that $\|\zeta_1\|_{BC^1} \leq \delta_{\zeta_1}$, $\|\zeta_2\|_{BC^1} \leq \delta_{\zeta_2}$, and $\|\zeta_3\|_{BC^1} \leq \delta_{\zeta_3}$. Fix $\delta > 0$ and define the closed bounded set

$$M := \{(\zeta_1, \zeta_2, \zeta_3) \in B : \ \|(\zeta_1, \zeta_2, \zeta_3)\|_B \leq \delta\}.$$

We observe that $\mathcal{M}$ is nonempty, closed, bounded, and convex in $\mathcal{B}$.

(1) For $(\zeta_1, \zeta_2, \zeta_3) \in M$, we have $\|\zeta_1\|_\infty, \|\zeta_2\|_\infty, \|\zeta_3\|_\infty \leq \delta$ and

$$|\mathbb{J}_{\zeta_1}(\zeta_1, \zeta_2, \zeta_3)(\rho)| = |\epsilon||\zeta_2(\rho) - \zeta_1(\rho)| \leq |\epsilon|(|\zeta_2(\rho)| + |\zeta_1(\rho)|) \leq 2|\epsilon|\delta.$$

Take $\vartheta_{\zeta_1} := 2|\epsilon|\delta > 0$.

(2) For $(\zeta_1, \zeta_2, \zeta_3) \in M$, we have

$$|\mathbb{J}_{\zeta_2}(\zeta_1, \zeta_2, \zeta_3)(\rho)| = |\zeta_1(\rho)(\omega - \zeta_3(\rho)) - \zeta_2(\rho)| \leq \delta(|\omega| + \delta) + \delta.$$

Take $\vartheta_{\zeta_2} := \delta(|\omega| + \delta) + \delta > 0$.

(3) For $(\zeta_1, \zeta_2, \zeta_3) \in M$, we have

$$|\mathbb{J}_{\zeta_3}(\zeta_1, \zeta_2, \zeta_3)(\rho)| = |\zeta_1(\rho)\zeta_2(\rho) - \mu\zeta_3(\rho)| \leq \delta^2 + |\mu|\delta.$$

Take $\vartheta_{\zeta_3} := \delta^2 + |\mu|\delta > 0$. Hence, R2 holds on M with the positive constants $\vartheta_{\zeta_1}, \vartheta_{\zeta_2}, \vartheta_{\zeta_3}$.

We have also

$$\lambda_{\zeta_1} + \lambda_{\zeta_2} + \lambda_{\zeta_3} = 2|\varepsilon| + (|\omega| + 2\delta + 1) + (2\delta + |\mu|). \tag{5.2}$$

Hence, Theorem 4.3 guarantees the existence and uniqueness of the solution of system (5.1) provided that

$$2|\varepsilon| + |\omega| + |\mu| + 4\delta + 1 < \frac{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)}{\rho_0^{1-\beta}\Gamma\left(\sigma + \frac{1}{\beta}\right)}. \tag{5.3}$$

Therefore, Theorems 4.2 and 4.3 hold for system 5.1. That is, the system 5.1 has a unique solution.

Next, we apply Theorems 4.2 and 4.3 to the two-dimensional fractional Euler system (5.4). The system is reformulated in an abstract operator framework suitable for the application of fixed-point methods by introducing the unknown vector $(\zeta_1, \zeta_2, \zeta_3, D)$ and defining the associated nonlinear operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, $\mathbb{J}_{\zeta_3}$, and $\mathbb{J}_D$. We will work in the bounded space $BC^1(\Delta \times \Delta \times [0, \rho_0], \mathbb{R})$, and consider a closed bounded subset in which the density satisfies the uniform positivity condition $\zeta_1 \geq \tau_{\min} > 1$. Within this framework, we verify conditions R1, R2, and R3. The corresponding constants $\vartheta_{\zeta_1}, \vartheta_{\zeta_2}, \vartheta_{\zeta_3}, \vartheta_D$, and $\lambda_{\zeta_1}, \lambda_{\zeta_2}, \lambda_{\zeta_3}, \lambda_D$ are explicitly determined. The two-dimensional Euler system describes the motion of a compressible inviscid fluid and is derived from the fundamental conservation laws of mass, momentum in two spatial directions, and energy. This model plays a fundamental role in gas dynamics, aerodynamics, and the study of multidimensional shock waves. The EK fractional two-dimensional Euler system is given by

$$\begin{cases} {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_1 = -\frac{\partial}{\partial x}(\zeta_1\zeta_2) - \frac{\partial}{\partial y}(\zeta_1\zeta_3) \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_2 = \dfrac{1}{\zeta_1}\left[\zeta_2\left[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)\right] - \frac{\partial}{\partial x}(\zeta_1\zeta_2^2 + p) - \frac{\partial}{\partial y}(\zeta_1\zeta_2\zeta_3)\right] \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}\zeta_3 = \dfrac{1}{\zeta_1}\left[\zeta_3\left[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)\right] - \frac{\partial}{\partial x}(\zeta_1\zeta_2\zeta_3) - \frac{\partial}{\partial y}(\zeta_1\zeta_3^2 + p)\right] \\ {}^{EK}\mathcal{D}^{\alpha}_{0,\rho}D = -\frac{\partial}{\partial x}(\zeta_2(D + p)) - \frac{\partial}{\partial y}(\zeta_3(D + p)) \\ \zeta_1(x, y, 0) = l_1(x, y), \quad \zeta_2(x, y, 0) = l_2(x, y), \\ \zeta_3(x, y, 0) = l_3(x, y), \quad e(x, y, 0) = g(x, y) \\ \zeta_1, \zeta_2, \zeta_3 \in C^1(\Delta_0) := C^1(\Delta \times \Delta \times [0, \rho_0], \mathbb{R}). \end{cases} \tag{5.4}$$

Here, $\Delta \subset \mathbb{R}$ is a bounded and locally compact subset, and $\rho_0 > 0$. The unknown $\zeta_1(x, y, \rho) > 1$ denotes the mass density, $\zeta_2(x, y, \rho)$ and $\zeta_3(x, y, \rho)$ denote the velocity components in the x- and y-directions, respectively, and $D(x, y, \rho)$ is the total energy density defined by $D = \zeta_1 e + \frac{1}{2}\zeta_1\left(\zeta_2^2 + \zeta_3^2\right)$, where $e(x, y, \rho)$ is the internal energy per unit mass which is bounded continuous function. The variables x and y are spatial coordinates and ρ is the time variable. To close the system, an equation of state is required. For an ideal gas, the pressure $p(x, y, \rho)$ is given by $p = (\lambda - 1)\zeta_1 e$, where $\lambda > 1$ is the adiabatic index. By $C^1(\Delta_0)$, we mean the space of all bounded, continuously differentiable real-valued functions on $\Delta \times \Delta \times [0, \rho_0]$ whose first-order derivatives are bounded and continuous.

According to the system 1.1, we observe that $\varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4 = 0$. Then, the nonlinear operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, $\mathbb{J}_{\zeta_3}$, and $\mathbb{J}_D$ are given by

$$\begin{aligned}
\mathbb{J}_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3,D) &:= -\frac{\partial}{\partial x}(\zeta_1\zeta_2) - \frac{\partial}{\partial y}(\zeta_1\zeta_3),\\
\mathbb{J}_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3,D) &:= \frac{1}{\zeta_1}\Big(\zeta_2[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)] - \frac{\partial}{\partial x}(\zeta_1\zeta_2^2 + p) - \frac{\partial}{\partial y}(\zeta_1\zeta_2\zeta_3)\Big),\\
\mathbb{J}_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3,D) &:= \frac{1}{\zeta_1}\Big(\zeta_3[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)] - \frac{\partial}{\partial x}(\zeta_1\zeta_2\zeta_3) - \frac{\partial}{\partial y}(\zeta_1\zeta_3^2 + p)\Big),\\
\mathbb{J}_D(\zeta_1,\zeta_2,\zeta_3,D) &:= -\frac{\partial}{\partial x}(\zeta_2(D+p)) - \frac{\partial}{\partial y}(\zeta_3(D+p)).
\end{aligned}$$

Note that

$$l_4(x) = D(x,0) = l_1(x)g(x) + \frac{1}{2}l_1(x)[(l_2(x))^2 + (l_3(x))^2].$$

Step 1: Satisfying the continuity property for the operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, $\mathbb{J}_{\zeta_3}$, and $\mathbb{J}_D$: Since $\zeta_1,\zeta_2,\zeta_3 \in BC^1(\Delta_0)$ and e is continuous, then the operators $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$, $\mathcal{G}_{\zeta_3}$, and $\mathcal{G}_D$ are continuous. Since Δ is a locally compact, then by Theorem 3.4 and Remark 3.5, $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, $\mathbb{J}_{\zeta_3}$, and $\mathbb{J}_D$ are continuous, respectively.

Step 2: Satisfying the boundedness property for the operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, $\mathbb{J}_{\zeta_3}$, and $\mathbb{J}_D$ to satisfy condition R2 on 5.4. Let

$$\mathcal{B} := BC^1(\Delta_0)\times BC^1(\Delta_0)\times BC^1(\Delta_0)\times BC^1(\Delta_0),$$

with norm

$$\|(\zeta_1,\zeta_2,\zeta_3,D)\|_{\mathcal{B}} := \max\{\|\zeta_1\|_{BC^1}, \|\zeta_2\|_{BC^1}, \|\zeta_3\|_{BC^1}, \|D\|_{BC^1}\},$$

where

$$\|f\|_{BC^1} := \|f\|_\infty + \|\frac{\partial}{\partial x}f\|_\infty + \|\frac{\partial}{\partial y}f\|_\infty.$$

Since the functions $\zeta_1,\zeta_2,\zeta_3 \in BC^1(\Delta_0)$ and e is bounded in the system 5.4, then there are positive constants $\delta_{\zeta_1},\delta_{\zeta_2},\delta_{\zeta_3}\delta_e > 0$ such that $\|\zeta_1\|_{BC^1} \le \delta_{\zeta_1}$, $\|\zeta_2\|_{BC^1} \le \delta_{\zeta_2}$, $\|\zeta_3\|_{BC^1} \le \delta_{\zeta_3}$, and $\|e\|_{BC^1} \le \delta_e$. Fix $\delta > 0$ and $\tau_{\min} > 1$ to be constant. Define

$$\mathcal{M} := \Big\{(\zeta_1,\zeta_2,\zeta_3,D) \in \mathcal{B} : \|\zeta_1\|_{BC^1}, \|\zeta_2\|_{BC^1}, \|\zeta_3\|_{BC^1}, \|D\|_{BC^1} \le \delta,\ \zeta_1 \ge \tau_{\min}\Big\}.$$

We observe that $\mathcal{M}$ is nonempty. For example if $l_1 \ge \tau_{\min}$ and l_2, l_3 are bounded, pick a constant in time extension at the level of the fixed-point setup. It is easy to see that $\mathcal{M}$ is closed, bounded, and convex in $\mathcal{B}$.

(1) For $(\zeta_1,\zeta_2,D) \in \mathcal{M}$, we have

$$|\mathbb{J}_{\zeta_1}| \le 2\delta^2 + 2\delta^2 = 4\delta^2.$$

Take $\vartheta_{\zeta_1} := 4\delta^2 > 0$.

(2) For the bound of p and its first derivatives, on $\mathcal{M}$, we have

$$|p| \le (\lambda-1)\Big(|D| + \frac{1}{2}|\zeta_1|(\zeta_2^2 + \zeta_3^2)\Big) \le (\lambda-1)\Big(\delta + \frac{1}{2}\delta(2\delta^2)\Big) = (\lambda-1)(\delta + \delta^3).$$

Also,

$$\frac{\partial}{\partial x}p = (\lambda - 1)\Big(\frac{\partial}{\partial x}D - \frac{1}{2}\frac{\partial}{\partial x}(\zeta_1(\zeta_2^2 + \zeta_3^2))\Big).$$

Now,

$$\begin{aligned}\frac{\partial}{\partial x}(\zeta_1(\zeta_2^2 + \zeta_3^2)) &= (\frac{\partial}{\partial x}\zeta_1)(\zeta_2^2 + \zeta_3^2) + \zeta_1\frac{\partial}{\partial x}(\zeta_2^2 + \zeta_3^2)\\ &= (\frac{\partial}{\partial x}\zeta_1)(\zeta_2^2 + \zeta_3^2) + 2\zeta_1(\zeta_2\frac{\partial}{\partial x}\zeta_2 + \zeta_3\frac{\partial}{\partial x}\zeta_3),\end{aligned}$$

so,

$$|\frac{\partial}{\partial x}(\zeta_1(\zeta_2^2 + \zeta_3^2))| \leq \delta(2\delta^2) + 2\delta(\delta\cdot\delta + \delta\cdot\delta) = 2\delta^3 + 4\delta^3 = 6\delta^3.$$

Hence,

$$|\frac{\partial}{\partial x}p| \leq (\lambda - 1)\Big(\delta + \frac{1}{2}\cdot 6\delta^3\Big) = (\lambda - 1)(\delta + 3\delta^3).$$

Similarly, $|\frac{\partial}{\partial y}p| \leq (\lambda - 1)(\delta + 3\delta^3)$.

(3) For the operator $\mathbb{J}_{\zeta_2}$, write $\mathbb{J}_{\zeta_2} = \dfrac{1}{\zeta_1}\mathcal{N}_{\zeta_2}$ where

$$\mathcal{N}_{\zeta_2} := \zeta_2[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)] - \frac{\partial}{\partial x}(\zeta_1\zeta_2^2 + p) - \frac{\partial}{\partial y}(\zeta_1\zeta_2\zeta_3).$$

Since $\zeta_1 \geq \tau_{\min}$ on $\mathcal{M}$, we have $|1/\zeta_1| \leq 1/\tau_{\min}$. Since

$$\Big|\zeta_2[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)]\Big| \leq |\zeta_2|(|\frac{\partial}{\partial x}(\zeta_1\zeta_2)| + |\frac{\partial}{\partial y}(\zeta_1\zeta_3)|) \leq \delta(2\delta^2 + 2\delta^2) = 4\delta^3,$$

and

$$|\frac{\partial}{\partial x}(\zeta_1\zeta_2^2)| = |(\frac{\partial}{\partial x}\zeta_1)\zeta_2^2 + 2\zeta_1\zeta_2(\frac{\partial}{\partial x}\zeta_2)| \leq \delta\cdot\delta^2 + 2\delta\cdot\delta\cdot\delta = 3\delta^3,$$

then,

$$|\frac{\partial}{\partial x}(\zeta_1\zeta_2^2 + p)| \leq 3\delta^3 + (\lambda - 1)(\delta + 3\delta^3).$$

Hence,

$$|\frac{\partial}{\partial y}(\zeta_1\zeta_2\zeta_3)| = |(\frac{\partial}{\partial y}\zeta_1)\zeta_2\zeta_3 + \zeta_1(\frac{\partial}{\partial y}\zeta_2)\zeta_3 + \zeta_1\zeta_2(\frac{\partial}{\partial y}\zeta_3)| \leq \delta^3 + \delta^3 + \delta^3 = 3\delta^3.$$

Combining

$$|\mathcal{N}_{\zeta_2}| \leq 4\delta^3 + (3\delta^3 + (\lambda - 1)(\delta + 3\delta^3)) + 3\delta^3 = 10\delta^3 + (\lambda - 1)(\delta + 3\delta^3),$$

then, $|\mathbb{J}_{\zeta_2}| \leq \frac{1}{\tau_{\min}}\Big(10\delta^3 + (\lambda - 1)(\delta + 3\delta^3)\Big)$. Take

$$\vartheta_{\zeta_2} := \frac{1}{\tau_{\min}}\Big(10\delta^3 + (\lambda - 1)(\delta + 3\delta^3)\Big) > 0.$$

(4) For the operator $\mathbb{J}_{\zeta_3}$, similarly, writing $\mathbb{J}_{\zeta_3} = \dfrac{1}{\zeta_1}\mathcal{N}_{\zeta_3}$ with

$$\mathcal{N}_{\zeta_3} := \zeta_3[\frac{\partial}{\partial x}(\zeta_1\zeta_2) + \frac{\partial}{\partial y}(\zeta_1\zeta_3)] - \frac{\partial}{\partial x}(\zeta_1\zeta_2\zeta_3) - \frac{\partial}{\partial y}(\zeta_1\zeta_3^2 + p),$$

we have

$$|\zeta_3[\frac{\partial}{\partial x}(\zeta_1\zeta_2)+\frac{\partial}{\partial y}(\zeta_1\zeta_3)]|\le\delta(2\delta^2+2\delta^2)=4\delta^3,$$

$$|\frac{\partial}{\partial x}(\zeta_1\zeta_2\zeta_3)|\le 3\delta^3,\qquad |\frac{\partial}{\partial y}(\zeta_1\zeta_3^2+p)|\le 3\delta^3+(\lambda-1)(\delta+3\delta^3),$$

and hence, $|\mathcal{N}_{\zeta_3}|\le 10\delta^3+(\lambda-1)(\delta+3\delta^3)$. Hence,

$$|\mathbb{J}_{\zeta_3}|\le\frac{1}{\tau_{\min}}\Big(10\delta^3+(\lambda-1)(\delta+3\delta^3)\Big).$$

Take

$$\vartheta_{\zeta_3}:=\frac{1}{\tau_{\min}}\Big(10\delta^3+(\lambda-1)(\delta+3\delta^3)\Big)>0.$$

(5) For the operator $\mathbb{J}_D$, we have

$$\mathbb{J}_D=-\frac{\partial}{\partial x}(\zeta_2(D+p))-\frac{\partial}{\partial y}(\zeta_3(D+p)).$$

Since

$$|D+p|\le|D|+|p|\le\delta+(\lambda-1)(\delta+\delta^3),$$

$$|\frac{\partial}{\partial x}(D+p)|\le|\frac{\partial}{\partial x}D|+|\frac{\partial}{\partial x}p|\le\delta+(\lambda-1)(\delta+3\delta^3),$$

and

$$|\frac{\partial}{\partial y}(D+p)|\le\delta+(\lambda-1)(\delta+3\delta^3)$$

Then, by using the product rule, we get

$$|\frac{\partial}{\partial x}(\zeta_2(D+p))|\le|\frac{\partial}{\partial x}\zeta_2|\,|D+p|+|\zeta_2|\,|\frac{\partial}{\partial x}(D+p)|\le\delta(\delta+(\lambda-1)(\delta+\delta^3))+\delta(\delta+(\lambda-1)(\delta+3\delta^3)),$$

and similarly,

$$|\frac{\partial}{\partial y}(\zeta_3(D+p))|\le\delta(\delta+(\lambda-1)(\delta+\delta^3))+\delta(\delta+(\lambda-1)(\delta+3\delta^3)).$$

Hence,

$$|\mathbb{J}_D|\le 2\Big[\delta(\delta+(\lambda-1)(\delta+\delta^3))+\delta(\delta+(\lambda-1)(\delta+3\delta^3))\Big].$$

Take

$$\vartheta_D:=2\Big[\delta(\delta+(\lambda-1)(\delta+\delta^3))+\delta(\delta+(\lambda-1)(\delta+3\delta^3))\Big]>0.$$

Step 3: Satisfying the Lipschitz property for the operators $\mathbb{J}_{\zeta_1}$, $\mathbb{J}_{\zeta_2}$, $\mathbb{J}_{\zeta_3}$, and $\mathbb{J}_D$ to satisfy condition R3 on 5.4: For all $m=(\zeta_1,\zeta_2,\zeta_3,D)$ and $\tilde{m}=(\tilde{\zeta}_1,\tilde{\zeta}_2,\tilde{\zeta}_3,\tilde{D})$, lrt $\Omega\zeta_1=\zeta_1-\tilde{\zeta}_1$, $\Omega\zeta_2=\zeta_2-\tilde{\zeta}_2$, $\Omega\zeta_3=\zeta_3-\tilde{\zeta}_3$, $\Omega D=D-\tilde{D}$. Then,

$$\|\Omega\zeta_1\|_\infty,\|\frac{\partial}{\partial x}\Omega\zeta_1\|_\infty,\|\frac{\partial}{\partial y}\Omega\zeta_1\|_\infty\le\|m-\tilde{m}\|_{\mathscr{B}},$$

and similarly for $\Omega\zeta_2,\Omega\zeta_3,\Omega D$.

(1) The Lipschitz constant for $\mathbb{J}_{\zeta_1}$: We have

$$\mathbb{J}_{\zeta_1}(m) - \mathbb{J}_{\zeta_1}(\tilde{m}) = -\frac{\partial}{\partial x}(\zeta_1\zeta_2 - \tilde{\zeta}_1\tilde{\zeta}_2) - \frac{\partial}{\partial y}(\zeta_1\zeta_3 - \tilde{\zeta}_1\tilde{\zeta}_3)$$

$$= -\frac{\partial}{\partial x}(\Omega\zeta_1\,\zeta_2 + \tilde{\zeta}_1\,\Omega\zeta_2) - \frac{\partial}{\partial y}(\Omega\zeta_1\,\zeta_3 + \tilde{\zeta}_1\,\Omega\zeta_3).$$

By using the product rule and $\|\zeta_2\|_{BC^1}, \|\zeta_3\|_{BC^1}, \|\tilde{\zeta}_1\|_{BC^1} \le \delta$, we have

$$\|\frac{\partial}{\partial x}(\Omega\zeta_1\,\zeta_2)\|_\infty \le 2\delta\|m - \tilde{m}\|_{\mathcal{B}},\ \ \|\frac{\partial}{\partial x}(\tilde{\zeta}_1\,\Omega\zeta_2)\|_\infty \le 2\delta\|m - \tilde{m}\|_{\mathcal{B}},$$

and similarly,

$$\|\frac{\partial}{\partial y}(\Omega\zeta_1\,\zeta_3)\|_\infty \le 2\delta\|m - \tilde{m}\|_{\mathcal{B}},\ \ \|\frac{\partial}{\partial y}(\tilde{\zeta}_1\,\Omega\zeta_3)\|_\infty \le 2\delta\|m - \tilde{m}\|_{\mathcal{B}}.$$

Hence,

$$\|\mathbb{J}_{\zeta_1}(m) - \mathbb{J}_{\zeta_1}(\tilde{m})\|_\infty \le 8\delta\|m - \tilde{m}\|_{\mathcal{B}}.$$

Take $\lambda_{\zeta_1} := 8\delta > 0$.

(2) The Lipschitz constants for $\mathbb{J}_{\zeta_2}$ and $\mathbb{J}_{\zeta_3}$: Write

$$\mathbb{J}_{\zeta_2}(m) = \frac{1}{\zeta_1}\mathcal{N}_{\zeta_2}(m) \ \text{ and } \mathbb{J}_{\zeta_2}(\tilde{m}) = \frac{1}{\tilde{\zeta}_1}\mathcal{N}_{\zeta_2}(\tilde{m}).$$

Then,

$$\mathbb{J}_{\zeta_2}(m) - \mathbb{J}_{\zeta_2}(\tilde{m}) = \frac{1}{\zeta_1}(\mathcal{N}_{\zeta_2}(m) - \mathcal{N}_{\zeta_2}(\tilde{m})) + \mathcal{N}_{\zeta_2}(\tilde{m})\left(\frac{1}{\zeta_1} - \frac{1}{\tilde{\zeta}_1}\right).$$

Since $\zeta_1, \tilde{\zeta}_1 \ge \tau_{\min}$ on $\mathcal{M}$, then $\left\|\frac{1}{\zeta_1}\right\|_\infty \le \frac{1}{\tau_{\min}}$, and hence,

$$\left\|\frac{1}{\zeta_1} - \frac{1}{\tilde{\zeta}_1}\right\|_\infty = \left\|\frac{\tilde{\zeta}_1 - \zeta_1}{\zeta_1\tilde{\zeta}_1}\right\|_\infty \le \frac{1}{\tau_{\min}^2}\|\Omega\zeta_1\|_\infty \le \frac{1}{\tau_{\min}^2}\|m - \tilde{m}\|_{\mathcal{B}}.$$

Since $\mathcal{N}_{\zeta_2}$ is uniformly bounded on $\mathcal{M}$, that is, $\sup_{m\in\mathcal{M}}\|\mathcal{N}_{\zeta_2}(m)\|_\infty \le C_{N,\zeta_2}(\delta,\lambda)$, and by product-rule estimates together with $\|p(m) - p(\tilde{m})\|_{BC^1} \le C_p\|m - \tilde{m}\|_{\mathcal{B}}$, then there exists $L_{N,\zeta_2}(\delta,\lambda) > 0$ such that

$$\|\mathcal{N}_{\zeta_2}(m) - \mathcal{N}_{\zeta_2}(\tilde{m})\|_\infty \le L_{N,\zeta_2}(\delta,\lambda)\|m - \tilde{m}\|_{\mathcal{B}}.$$

Hence,

$$\|\mathbb{J}_{\zeta_2}(m) - \mathbb{J}_{\zeta_2}(\tilde{m})\|_\infty \le \left(\frac{1}{\tau_{\min}}L_{N,\zeta_2}(\delta,\lambda) + \frac{1}{\tau_{\min}^2}C_{N,\zeta_2}(\delta,\lambda)\right)\|m - \tilde{m}\|_{\mathcal{B}}.$$

Take

$$\lambda_{\zeta_2} := \frac{1}{\tau_{\min}}L_{N,\zeta_2}(\delta,\lambda) + \frac{1}{\tau_{\min}^2}C_{N,\zeta_2}(\delta,\lambda) > 0.$$

The same argument applies to $\mathbb{J}_{\zeta_3}$ yielding constants $C_{N,\zeta_3}(\delta,\lambda)$ and $L_{N,\zeta_3}(\delta,\lambda)$, take

$$\lambda_{\zeta_3} := \frac{1}{\tau_{\min}}L_{N,\zeta_3}(\delta,\lambda) + \frac{1}{\tau_{\min}^2}C_{N,\zeta_3}(\delta,\lambda) > 0.$$

(3) The Lipschitz bounds for the pressure map: Define $p(m) = (\lambda - 1)(D - \frac{1}{2}\zeta_1(\zeta_2^2 + \zeta_3^2))$ and $p(\tilde{m})$. On $\mathcal{M}$, we have $\|\zeta_1\|_\infty, \|\zeta_2\|_\infty, \|\zeta_3\|_\infty, \|D\|_\infty \le \delta$. Then,

$$|\zeta_2^2 - \tilde{\zeta}_2^{\,2}| \le (|\zeta_2| + |\tilde{\zeta}_2|)|\zeta_2 - \tilde{\zeta}_2| \le 2\delta|\Omega\zeta_2|,$$

and $|\zeta_3^2 - \tilde{\zeta}_3^{\,2}| \le 2\delta|\Omega\zeta_3|$ gives a constant $C_p = C_p(\delta, \lambda) > 0$ such that

$$\|p(m) - p(\tilde{m})\|_{BC^1} \le C_p\|m - \tilde{m}\|_{\mathcal{B}}.$$

(4) The Lipschitz constant for $\mathbb{J}_D$, we have

$$\mathbb{J}_D(m) - \mathbb{J}_D(\tilde{m}) = -\frac{\partial}{\partial x}(\zeta_2(D + p(m)) - \tilde{\zeta}_2(\tilde{D} + p(\tilde{m}))) - \frac{\partial}{\partial y}(\zeta_3(D + p(m)) - \tilde{\zeta}_3(\tilde{D} + p(\tilde{m}))),$$

and

$$\zeta_2(D + p(m)) - \tilde{\zeta}_2(\tilde{D} + p(\tilde{m})) = \Omega\zeta_2\,(D + p(m)) + \tilde{\zeta}_2(\Omega D + (p(m) - p(\tilde{m}))).$$

By using the product rule and the bound $\|p(m) - p(\tilde{m})\|_{BC^1} \le C_p\|m - \tilde{m}\|_{\mathcal{B}}$, there is a constant $\lambda_D = \lambda_D(\delta, \lambda) > 0$ such that

$$\|\mathbb{J}_D(m) - \mathbb{J}_D(\tilde{m})\|_\infty \le \lambda_D\|m - \tilde{m}\|_{\mathcal{B}}.$$

If

$$\begin{aligned} &8\delta + \frac{1}{\tau_{\min}}L_{N,\zeta_2}(\delta, \lambda) + \frac{1}{\tau_{\min}^2}C_{N,\zeta_2}(\delta, \lambda) + \frac{1}{\tau_{\min}}L_{N,\zeta_3}(\delta, \lambda) \\ &+ \frac{1}{\tau_{\min}^2}C_{N,\zeta_3}(\delta, \lambda) + \lambda_D(\delta, \lambda) < \frac{\Gamma\left(\alpha + \sigma + \frac{1}{\beta}\right)}{\rho_0^{1-\beta}\Gamma\left(\sigma + \frac{1}{\beta}\right)}, \end{aligned}$$

then, Theorems 4.2 and 4.3 hold for the system 5.4. That is, the system 5.4 has a unique solution.

For example, if Δ is any closed interval in the system 5.1 with $\epsilon = 10$, $\omega = 28$, $\mu = \frac{8}{3}$, $\beta = \sigma = 1$, and with initial conditions $l_1 = l_2 = l_3 = 2$, then by using the expansion new iterative method [2], the solution is

$$\zeta_1(\rho) \approx 2 + 500\frac{\rho^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{16420}{3}\frac{\rho^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots, \tag{5.5}$$

$$\zeta_2(\rho) \approx 2 + 50\frac{\rho^{\alpha}}{\Gamma(\alpha + 1)} - \frac{142}{3}\frac{\rho^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{115562}{9}\frac{\rho^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots, \tag{5.6}$$

$$\zeta_3(\rho) \approx 2 - \frac{4}{3}\frac{\rho^{\alpha}}{\Gamma(\alpha + 1)} + \frac{932}{9}\frac{\rho^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{16988}{27}\frac{\rho^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots. \tag{5.7}$$

This solution is graphically presented in four cases, $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$; see Figure 1. Δ is any closed interval.

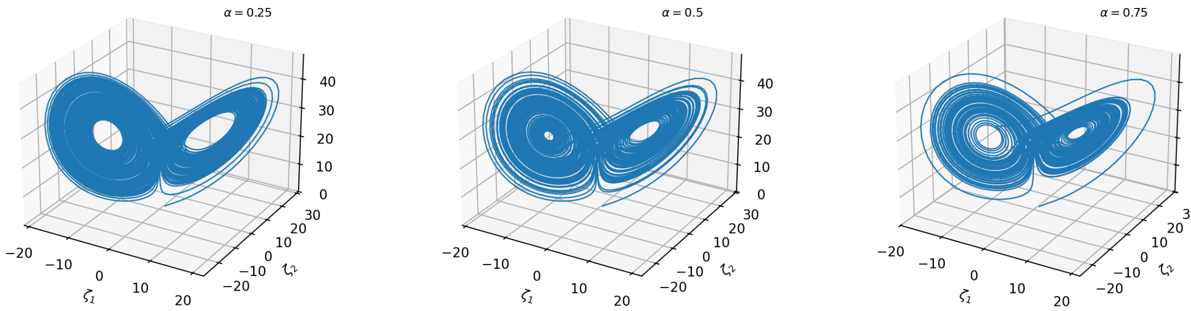

Figure 1. Solution of 5.1 with $\epsilon = 10$, $\omega = 28$, $\mu = \frac{8}{3}$, $\beta = \sigma = 1$, and with initial conditions $l_1 = l_2 = l_3 = 2$.

In system 5.1, if $\Delta = [1, 2]$ is a closed interval with $\beta = \sigma = 1$, $\lambda = 2$, and initial conditions $\zeta_1(x, y, 0) = x$, $\zeta_2(x, y, 0) = y$, $\zeta_3(x, y, 0) = e(x, y, 0) = xy$, then by using the expansion new iterative method, [**?**], the solution is

$$\zeta_1(x, y, \rho) \approx x - (y + x^2)\frac{\rho^\alpha}{\Gamma(\alpha + 1)} + \cdots \tag{5.8}$$

$$\zeta_2(x, y, \rho) \approx y - y(x + 2)\frac{\rho^\alpha}{\Gamma(\alpha + 1)} + \cdots \tag{5.9}$$

$$\zeta_3(x, y, \rho) \approx xy - (y^2 + x^2 y + x)\frac{\rho^\alpha}{\Gamma(\alpha + 1)} + \cdots \tag{5.10}$$

$$\begin{aligned} D(x, y, \rho) \approx{}& x^2 y + \frac{1}{2}xy^2 + \frac{1}{2}x^3 y^2 \\ &- \left(4xy^2 + \frac{1}{2}y^3 + \frac{3}{2}x^2 y^3 + 4x^3 y + \frac{3}{2}x^2 y^2 + \frac{3}{2}x^4 y^2\right)\frac{\rho^\alpha}{\Gamma(\alpha + 1)} + \cdots \end{aligned} \tag{5.11}$$

This solution ζ_1, ζ_2, ζ_3, and D of 5.4 (with $\beta = \sigma = 1$, $\lambda = 2$, $\rho = 3$) is graphically presented in two cases, $\alpha = 0.25$, and $\alpha = 0.75$; see Figures 2–5.

In the special case $\alpha = 1$, the fractional model reduces to the classical Lorenz system. This confirms that the proposed formulation is consistent with the well-known integer-order model and can be viewed as its natural generalization. The results obtained for the fractional Euler system are in agreement with the classical case. The use of fractional operators provides a more general framework that captures memory effects without altering the fundamental structure of the model.

Figure 1 illustrates the behavior of the solution of the fractional system. It can be observed that the solution remains bounded and stable over the considered interval, which confirms the theoretical results obtained in Section 4. Figure 2 shows the evolution of the system dynamics. It is clear that the solution follows a smooth pattern, and the fractional parameter has a noticeable effect on the system behavior. Figure 3 presents the behavior of the solution under the given conditions. The results indicate consistency with the theoretical analysis and demonstrate stable dynamics. Figure 4 illustrates the influence of the fractional operator on the system. It can be seen that memory effects play an important role in shaping the solution behavior. Figure 5 shows the overall behavior of the system. The obtained results are in agreement with the theoretical findings and confirm the validity of the proposed model.

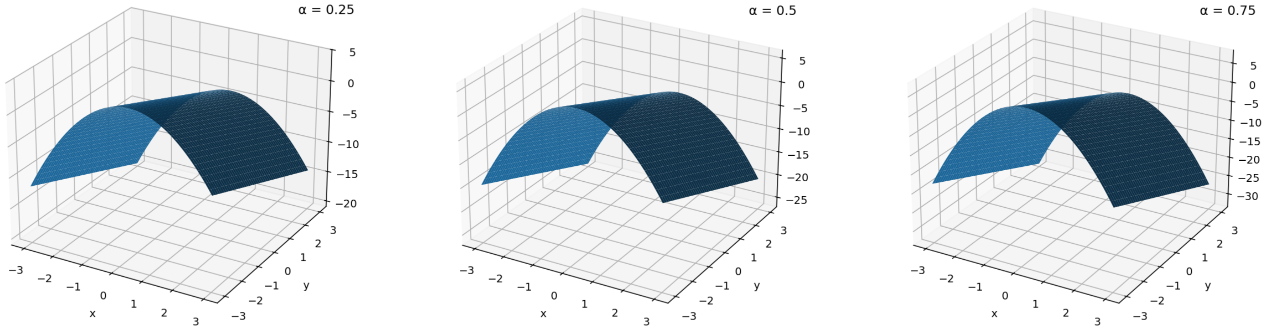

Figure 2. Solution ζ_1 of 5.4 with $\beta = \sigma = 1$, $\lambda = 2$, $\rho = 3$, and with initial conditions $\zeta_1(x, y, 0) = x$, $\zeta_2(x, y, 0) = y$, $\zeta_3(x, y, 0) = e(x, y, 0) = xy$.

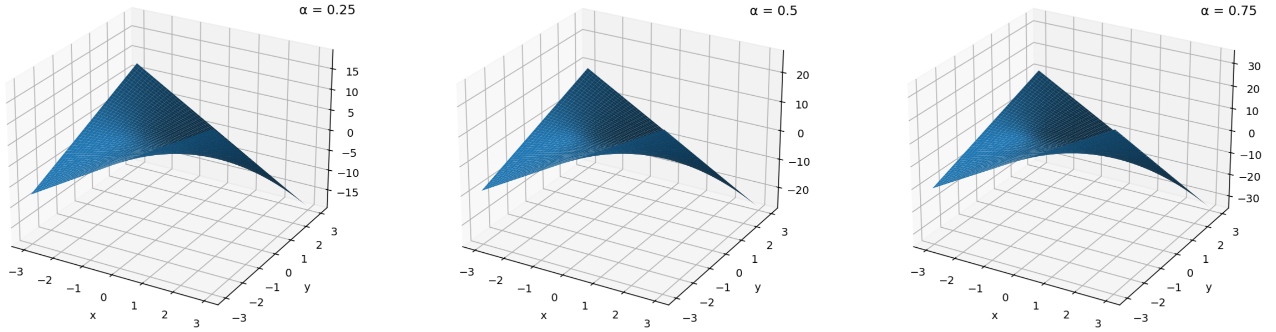

Figure 3. Solution ζ_2 of 5.4 with $\beta = \sigma = 1$, $\lambda = 2$, $\rho = 3$, and with initial conditions $\zeta_1(x, y, 0) = x$, $\zeta_2(x, y, 0) = y$, $\zeta_3(x, y, 0) = e(x, y, 0) = xy$.

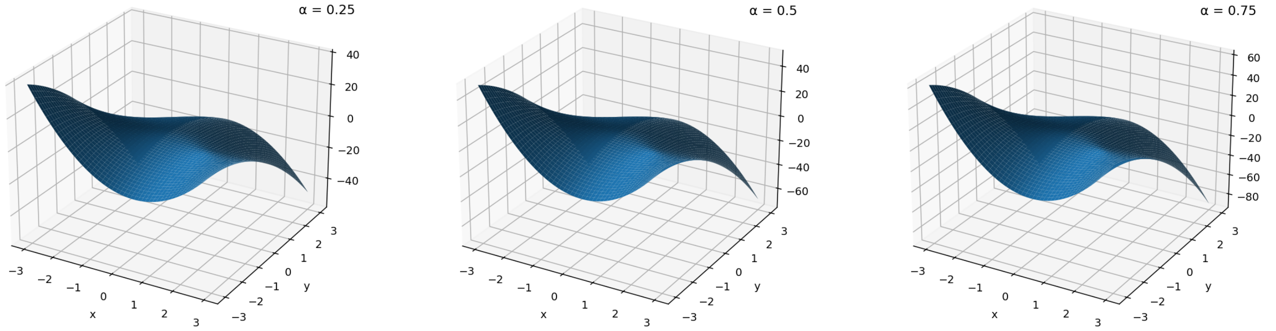

Figure 4. Solution ζ_3 of 5.4 with $\beta = \sigma = 1$, $\lambda = 2$, $\rho = 3$, and with initial conditions $\zeta_1(x, y, 0) = x$, $\zeta_2(x, y, 0) = y$, $\zeta_3(x, y, 0) = e(x, y, 0) = xy$.

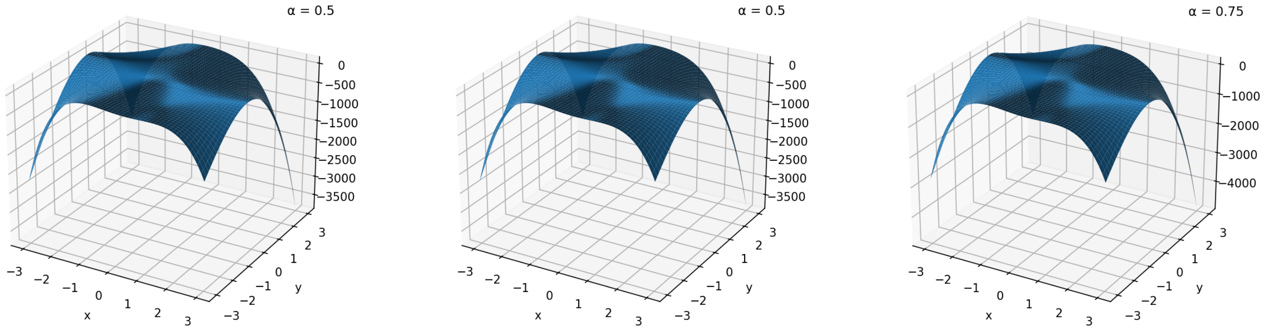

Figure 5. Solution D of 5.4 with $\beta = \sigma = 1$, $\lambda = 2$, $\rho = 3$, and with initial conditions $\zeta_1(x, y, 0) = x$, $\zeta_2(x, y, 0) = y$, $\zeta_3(x, y, 0) = e(x, y, 0) = xy$.

Although the proposed framework provides a rigorous theoretical analysis for a class of fractional dynamical systems, it is subject to certain limitations. The results rely on specific assumptions, including the Lipschitz conditions and boundedness of the operators, which may restrict their applicability to more general systems. Moreover, the present study focuses on theoretical analysis without extensive numerical validation. Future work may address these limitations by considering more general models, relaxing the assumptions, and developing efficient numerical methods.

6. Conclusions

In this work, we examined a class of two-dimensional EK fractional dynamical systems using a functional analytic approach within locally compact Hausdorff spaces. By investigating the corresponding operators in a suitable Banach space setting, their fundamental analytical features are derived, and the Banach and Schaefer fixed point theorems are applied to establish the existence as well as the uniqueness of solutions. Moreover, the Hyers–Ulam stability of the considered system is analyzed, indicating that the solutions preserve stability when subjected to small perturbations. To illustrate the effectiveness of the theoretical findings, two representative models are investigated, namely the fractional Lorenz system and the two-dimensional fractional Euler system. For each model, the imposed hypotheses are verified, and the existence of unique stable solutions is confirmed. Possible future investigations may involve extending the proposed framework to broader classes of fractional operators, studying higher-dimensional dynamical systems and stochastic fractional models, and developing numerical techniques for approximating the obtained solutions.

Author contributions

Faten H. Damag: Conceptualization, Methodology, Validation, Investigation, Resources, Writing–review and editing; Mohammed Alsharafi: Conceptualization, Methodology, Validation, Investigation, Resources, Writing–original draft; Abeer Hamdan Alblowy: Methodology, Validation, Investigation, Resources, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Wang, M. Din, M. Zhou, Some fixed point results for novel contractions with applications in fractional differential equations for market equilibrium and economic growth, *Fractal Fract.*, **9** (2025), 324. https://doi.org/10.3390/fractalfract9050324
2. F. H. Damag, and A. Saif, On solving modified time Caputo fractional Kawahara equations in the framework of Hilbert algebras using the Laplace residual power series method, *Fractal and Fractional*, **9** (2025), 301. https://doi.org/10.3390/fractalfract9050301
3. R. Ashraf, R. Nawaz, O. Alabdali, N. Fewster-Young, A. H. Ali, F. Ghanim, et al., A new hybrid optimal auxiliary function method for approximate solutions of non-linear fractional partial differential equations, *Fractal Fract.*, **7** (2023), 673. https://doi.org/10.3390/fractalfract7090673
4. T. Liu, R. Q. Xue, B. L. Ding, D. A. Juraev, B. N. Saray, F. Soleymani, A novel and effective scheme for solving the fractional telegraph problem via the spectral element method, *Fractal Fract.*, **8** (2024), 711. https://doi.org/10.3390/fractalfract8120711
5. T. Liu, B. L. Ding, A radial basis function neural network approach for solving a diffusion partial differential equation efficiently, *Appl. Math. Comput.*, **509** (2026), 129651. https://doi.org/10.1016/j.amc.2025.129651
6. A. E. Hamza, O. Osman, A. Ali, A. Alsulami, K. Aldwoah, A. Mustafa, et al., Fractal-fractional modeling of liver fibrosis disease and its mathematical results with subinterval transitions, *Fractal Fract.*, **8** (2024), 638. https://doi.org/10.3390/fractalfract8110638
7. M. Adel, K. Aldwoah, F. Alahmadi, M. S. Osman, The asymptotic behavior for a binary alloy in energy and material science: The unified method and its applications, *J. Ocean Eng. Sci.*, **9** (2024), 373–378. https://doi.org/10.1016/j.joes.2022.03.006
8. Y. L. Li, F. W. Liu, I. W. Turner, T. Li, Time-fractional diffusion equation for signal smoothing, *Appl. Math. Comput.*, **326** (2018), 108–116. https://doi.org/10.1016/j.amc.2018.01.036
9. K. A. Aldwoah, M. A. Almalahi, K. Shah, M. Awadalla, R. H. Egami, Dynamics analysis of dengue fever model with harmonic mean type under fractal-fractional derivative, *AIMS Mathematics*, **9** (2024), 13894–13926. https://doi.org/10.3934/math.2024676
10. M. Balegh, A. GhezaL, Theoretical analysis and applications of fixed-point theorems in delay fractional differential equations, *J. Appl. Math. Comput.*, **71** (2025), 867–881. https://doi.org/10.1007/s12190-025-02513-0
11. L. Cona, A. H. Kocağ, Fixed point approach for fractional order differential equation systems, *Journal of New Results in Science*, **14** (2025), 124–137.

12. M. Belhadj, J. R. Roshan, S. S. Radenović, Generalizations of Darbo's fixed point theorem and its application to the solvability of a nonlinear fractional differential equation, *Nonlinear Anal.-Model.*, **30** (2025), 1081–1102. https://doi.org/10.15388/namc.2025.30.43745
13. T. Liu, B. Ding, B. N. Saray, D. A. Juraev, E. E. Elsayed, On the pseudospectral method for solving the fractional Klein–Gordon equation using Legendre cardinal functions, *Fractal Fract.*, **9** (2025), 177. https://doi.org/10.3390/fractalfract9030177
14. T. Liu, R. Q. Xue, A convergent multi-step efficient iteration method to solve nonlinear equation systems, *J. Appl. Math. Comput.*, **71** (2025), 2571–2588. https://doi.org/10.1007/s12190-024-02324-9
15. H. Ahmad, F. U. Din, M. Younis, Fractional order Lorenz dynamics: Investigating existence and uniqueness via basic contraction, *J. Appl. Math. Comput.*, **71** (2025), 6827–6858. https://doi.org/10.1007/s12190-025-02550-9
16. M. Awais, M. A. Khan, Z. Bashir, Exploring the stochastic patterns of hyperchaotic Lorenz systems with variable fractional order and radial basis function networks, *Cluster Comput.*, **27** (2024), 9031–9064. https://doi.org/10.1007/s10586-024-04431-5
17. K. Meera, N. Selvaganesan, Generalized fractional order complex Lorenz system based dual layer security enhancement approach, *IEEE Access*, **13** (2025), 112162–112175. https://doi.org/10.1109/ACCESS.2025.3583894
18. W. H. Hui, P. Y. Li, Z. W. Li, A unified coordinate system for solving the two-dimensional Euler equations, *J. Comput. Phys.*, **153** (1999), 596–637. https://doi.org/10.1006/jcph.1999.6295
19. S. X. Chen, Stability of transonic shock fronts in two-dimensional Euler systems, *T. A. Math. Soc.*, **357** (2005), 287–308.
20. W. K. Wang, T. Yang, Existence and stability of planar diffusion waves for 2-D Euler equations with damping, *J. Differ. Equations*, **242** (2007), 40–71. https://doi.org/10.1016/j.jde.2007.07.002
21. H. R. Dullin, R. Marangell, J. Worthington, Instability of equilibria for the two-dimensional Euler equations on the torus, *SIAM J. Appl. Math.*, **76** (2016), 1446–1470. https://doi.org/10.1137/15M1043054
22. W. Alfwzan, S.-W. Yao, F. M. Allehiany, S. Ahmad, S. Saifullah, M. Inc, Analysis of fractional non-linear tsunami shallow-water mathematical model with singular and non singular kernels, *Results Phys.*, **52** (2023), 106707. https://doi.org/10.1016/j.rinp.2023.106707
23. S. G. Samko, A. A. Kilbas, *Fractional integrals and derivatives: theory and applications*, New York: CRC Press, 1993.
24. T. Asifa, R. Srivastava, R. Alharbi, R. Md Kasmani, S. Qureshi, New inequalities using multiple Erdelyi-Kober fractional integral operators, *Fractal Fract.*, **8** (2024), 180. https://doi.org/10.3390/fractalfract8040180
25. V. Kiryakova, J. Paneva-Konovska, Generalized Erdélyi-Kober fractional integrals and images of special functions, *Fractal Fract.*, **9** (2025), 567. https://doi.org/10.3390/fractalfract9090567
26. E. Zeidler, *Nonlinear functional analysis and its applications I: fixed-point theorems*, New York: Springer, 1986.
27. W. Rudin, *Principles of mathematical analysis*, 3 Eds., New York: McGraw-Hill, 1976.

28. I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.*, **26** (2010), 103–107.