Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 11760–11775.
DOI: 10.3934/math.2026484
Received: 10 February 2026
Revised: 13 April 2026
Accepted: 24 April 2026
Published: 28 April 2026

Research article

On a sequence related to Mersenne numbers and its connections to the Horadam sequence

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Abstract: Mersenne numbers, introduced by Marin Mersenne in the early 17th century and known since antiquity for their deep links to prime number theory, have long fascinated mathematicians. In this study, we introduce a second-order sequence derived from consecutive Mersenne numbers, which we refer to as the Trisenne sequence. Our objective is to investigate its structural properties and to establish relationships between the Trisenne numbers and the general Horadam (or generalized Fibonacci) sequence through their generating functions. The connection between these families is analyzed using both ordinary and exponential generating functions, with particular emphasis on classical cases such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences. We also discuss the historical context and mathematical lineage of Horadam sequences, tracing their origin to the work of Edouard Lucas and Alwyn Horadam. Theoretical results are illustrated with explicit examples, and several related identities are presented.

Keywords: Mersenne numbers; Horadam sequence; generating function; recurrence relation; binomial transform
Mathematics Subject Classification: 11B37, 11B39, 11B83, 05A15

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1. Introduction and motivation

Mersenne numbers, defined by $M_n = 2^n - 1$ for non-negative integers n, have been studied since ancient Greek mathematics and were later systematized by the French philosopher and mathematician Marin Mersenne in the 17th century. Owing to their fundamental role in number theory, particularly in the search for large prime numbers, Mersenne numbers have become important objects in modern mathematics, computer science, and cryptography.

One of the central aspects of Mersenne numbers is their connection with prime numbers. The Mersenne sequence contains prime numbers, known as Mersenne primes, of the form $2^n - 1$. A straightforward observation shows that if M_n is a prime number, then n must also be prime; however, not all such numbers are prime. As of early 2026, 52 Mersenne primes are known; the largest being $2^{136279841} - 1$.

Over the past decades, numerous generalizations of the classical Mersenne numbers have been proposed and analyzed. These include generalized and k-Mersenne numbers [1–3], Mersenne polynomials and matrices [4], as well as Mersenne–Lucas and hybrid structures [5, 6]. The algebraic and combinatorial properties of such sequences, encompassing recurrence relations, identities, and generating functions, have been thoroughly investigated; see, for example, [7–9]. In particular, new families of generalized k-Mersenne and Gaussian Mersenne numbers, along with their polynomial analogues, were introduced in [1]. Further extensions involving matrix methods and generalizations of classical identities of Catalan and Cassini type were developed in [2]. A different approach was taken in [3], where numbers of the form $a^n - (a-1)^n$, termed global generalized Mersenne numbers with base a, were introduced and their congruence properties for prime indices were studied. Matrix-theoretic aspects of Mersenne numbers were also examined in [9], where determinant formulas for certain Toeplitz–Hessenberg matrices with Mersenne entries were derived, leading to multi-sum identities involving products of Mersenne numbers and multinomial coefficients. Additional background information may be found in the On-Line Encyclopedia of Integer Sequences [10] under entry A000225.

Another active line of research concerns the interaction between Mersenne-type sequences and generalized second-order linear recurrence sequences. In particular, links with Horadam sequences and related families have been established using generating-function techniques and algebraic methods [11]. The general theory of Horadam sequences, including their structural properties, recurrence relations, generating functions, and applications, has been extensively developed; see, for example, [12, 13].

Finite sums of consecutive terms constitute a standard topic in the study of second-order linear recurrence sequences. For sequences such as Fibonacci, Lucas, and Horadam sequences, partial sums often satisfy linear recurrence relations and admit explicit closed forms or generating functions; see, for example, [14–16]. These results demonstrate that summation processes may yield sequences with nontrivial recurrence behavior and rich algebraic structure.

Despite the extensive literature on Mersenne numbers and their generalizations, finite sums of consecutive Mersenne numbers have not been systematically investigated. In particular, constructions that combine such sums with low-order linear recurrence relations remain largely unexplored. Although certain Mersenne-based sequences satisfy higher-order recurrences or arise as special cases of generalized Horadam sequences, a systematic and explicit investigation of sums of consecutive Mersenne numbers leading to second-order linear recurrences does not appear to have been carried out. This naturally raises the question of when finite sums of Mersenne numbers lead to low-order linear recurrences and how such sequences fit into the Horadam framework.

Motivated by this observation, we consider the integer sequence defined as the sum of three consecutive Mersenne numbers,

$$T_n = M_n + M_{n+1} + M_{n+2}, \quad n \geq 0. \tag{1.1}$$

We refer to this sequence as the *Trisenne sequence*. It is recorded in [10] as A156127; however, while finite sums of consecutive terms have been explored in various contexts, a dedicated and systematic study of its recurrence structure and algebraic properties has not been extensively carried out.

Although an explicit formula for T_n can be derived directly, the main contribution of this paper lies in the systematic analysis of the Trisenne sequence within the class of second-order linear recurrences. In particular, despite being defined as a sum of three consecutive Mersenne numbers, the sequence satisfies a second-order recurrence relation. This feature places it naturally within the framework of classical recurrence theory and enables a detailed analysis using standard tools.

The aim of this paper is therefore twofold: to provide a systematic and explicit study of the Trisenne sequence, including its algebraic properties, recurrence relations, closed forms, and generating functions, and to explore its connections with standard number sequences and with the general Horadam framework. In doing so, we place the Trisenne sequence within the broader context of Mersenne-based constructions and second-order linear recurrences.

2. Basic properties of the Trisenne sequence

In this section, we describe recurrence relations, closed forms, identities, and generating functions for the Trisenne sequence.

It follows directly from the definition that the Trisenne numbers satisfy the second-order recurrence relation

$$T_n = 3T_{n-1} - 2T_{n-2}, \quad n \geq 2, \tag{2.1}$$

with initial conditions $T_0 = 4$ and $T_1 = 11$. As a consequence of (2.1), the sequence also satisfies

$$T_n = 2T_{n-1} + 3, \quad n \geq 1. \tag{2.2}$$

From (1.1), we derive the following Binet-type formula of the sequence $(T_n)_{n\geq 0}$:

$$T_n = 7{\cdot}2^n - 3, \quad n \geq 0.$$

The Trisenne sequence can be extended to negative indices by means of the relation

$$T_{-n} = \frac{1}{2}\left(3T_{-n+1} - T_{-n+2}\right), \quad n \geq 1,$$

which allows a natural extension of the sequence to all integer indices.

We further derive several classical identities associated with second-order recurrence sequences, adapted here to the Trisenne numbers, namely the Catalan, Cassini, d'Ocagne, Honsberger, and Vajda identities. Their proofs follow standard techniques for second-order linear recurrence sequences. For integers n, m, r, and s, the following formulas hold:

- Catalan identity: $T_{n-r}T_{n+r} - T_n^2 = -21 \cdot 2^{n-r}(2^r - 1)^2$;
- Cassini identity: $T_{n-1}T_{n+1} - T_n^2 = -21 \cdot 2^{n-1}$;
- d'Ocagne identity: $T_{m+s}T_{n+r} - T_{m+r}T_{n+s} = -21(2^s - 2^r)(2^m - 2^n)$;
- Honsberger identity: $T_{m-1}T_n + T_mT_{n+1} = 245 \cdot 2^{m+n-1} - 63(2^{m-1} + 2^n) + 18$;
- Vajda identity: $T_{n+r}T_{n+s} + T_nT_{n+r+s} = 98 \cdot 2^{2n+r+s} - 21 \cdot 2^n(2^r + 1)(2^s + 1) + 18$.

The Catalan identities play a central role in combinatorics, particularly in the enumeration of Catalan-type structures. Cassini's identity, a special case of the Catalan identity, has applications in logic, including Curry's paradox, which was introduced by P. Curry in 1953. The Honsberger identity frequently appears in combinatorial applications of the Fibonacci sequence and in number theory. The d'Ocagne identity is useful for deriving further identities and studying properties of Fibonacci-type sequences, while Vajda's identity generalizes the d'Ocagne identity (see [12, 17, 18] for more details).

We conclude this section by presenting several generating functions that will be instrumental in subsequent proofs. Using standard techniques (see, for instance, [15]), we derive the ordinary (non-exponential) generating functions for the sequences $(T_n)_{n\geq 0}$, $(T_{2n+1})_{n\geq 0}$, and $(T_{2n})_{n\geq 0}$ as follows:

$$t(x) = \sum_{n=0}^{\infty} T_n x^n = \frac{4-x}{1-3x+2x^2}, \tag{2.3}$$

$$t_1(x) = \sum_{n=0}^{\infty} T_{2n+1} x^n = \frac{11-2x}{1-5x+4x^2}, \tag{2.4}$$

$$t_2(x) = \sum_{n=0}^{\infty} T_{2n} x^n = \frac{4+5x}{1-5x+4x^2}. \tag{2.5}$$

In addition, the exponential generating functions of these sequences are given by

$$\tau(x) = \sum_{n=0}^{\infty} T_n \frac{x^n}{n!} = 7e^{2x} - 3e^x, \tag{2.6}$$

$$\tau_1(x) = \sum_{n=0}^{\infty} T_{2n+1} \frac{x^n}{n!} = 14e^{4x} - 3e^x, \tag{2.7}$$

$$\tau_2(x) = \sum_{n=0}^{\infty} T_{2n} \frac{x^n}{n!} = 7e^{4x} - 3e^x. \tag{2.8}$$

These results lay the groundwork for studying the connection between the Trisenne sequence and the Horadam framework and for deriving further identities and representations in the subsequent sections.

Remark 1. It is worth noting the behavior of other finite sums of consecutive Mersenne numbers. Let $S_n^{(k)} = \sum_{i=0}^{k-1} M_{n+i}$. For $k = 2$, the sum of two consecutive Mersenne numbers yields $S_n^{(2)} = 3\cdot 2^n - 2$. For $k = 4$ and $k = 5$, we obtain $S_n^{(4)} = 15\cdot 2^n - 4$ and $S_n^{(5)} = 31\cdot 2^n - 5$, respectively. Interestingly, regardless of the number of terms k, the resulting sequence always satisfies the same homogeneous second-order recurrence relation $S_n^{(k)} = 3S_{n-1}^{(k)} - 2S_{n-2}^{(k)}$ for $n \geq 2$, mirroring the recurrence structure of the Trisenne sequence and thereby fitting seamlessly into the broader Horadam framework.

3. Trisenne–Horadam identities using ordinary generating functions

A generalized Fibonacci sequence, also known as a Horadam sequence, $(w_n)_{n\geq 0} = \big(w_n(a, b; p, q)\big)_{n\geq 0}$ is defined by a second-order homogeneous linear recurrence relation

$$w_n = pw_{n-1} + qw_{n-2}, \quad n \geq 2, \tag{3.1}$$

with initial conditions $w_0 = a$ and $w_1 = b$, where a, b, p, and q are integers with $p > 0$ and $q \neq 0$.

This general class of sequences was systematically studied by Alwyn Horadam in the 1960s (see [12, 13]) and is closely related to the sequences investigated by Édouard Lucas in the 1870s. These generalizations played an important role in clarifying the relationships between fundamental sequences introduced earlier by Lucas [19] and stimulated further research on second- and higher-order recurrence sequences, as well as the development of various analytical techniques [20]. For a concise overview of Horadam numbers and related research, the reader is referred to the survey paper [21].

This sequence can be extended to negative indices using

$$w_{-n} = -\frac{pw_{-n+1} - w_{-n+2}}{q}, \quad n \geq 1.$$

For any integer n, the Binet formula for w_n, in the non-degenerate case $p^2 + 4q > 0$, is given by

$$w_n = Ar_1^n + Br_2^n, \quad n \geq 0,$$

where

$$r_1 = \frac{p + \sqrt{p^2 + 4q}}{2} \text{ and } r_2 = \frac{p - \sqrt{p^2 + 4q}}{2}$$

are the distinct zeros of the characteristic polynomial $x^2 - px - q$, and

$$A = \frac{a}{2} + \frac{2b - ap}{2\sqrt{p^2 + 4q}}, \quad B = \frac{a}{2} - \frac{2b - ap}{2\sqrt{p^2 + 4q}}.$$

Throughout this work, we frequently utilize the generating function of the sequence $(w_n)_{n\geq 0}$. As noted in [15], this sequence has the ordinary generating function

$$w(x) = \sum_{n=0}^{\infty} w_n x^n = \frac{a + (b - ap)x}{1 - px - qx^2}, \tag{3.2}$$

where $|x| < \frac{1}{\max(|r_1|,|r_2|)}$. By decomposing $w(x)$ into its even and odd parts, we obtain the generating functions for the subsequences $(w_{2n})_{n\geq 0}$ and $(w_{2n+1})_{n\geq 0}$. More precisely, we use the identities

$$w_1(x) = \frac{w(\sqrt{x}) - w(-\sqrt{x})}{2\sqrt{x}} \quad \text{and} \quad w_2(x) = \frac{w(\sqrt{x}) + w(-\sqrt{x})}{2}.$$

A straightforward computation then yields

$$w_1(x) = \sum_{n=0}^{\infty} w_{2n+1} x^n = \frac{b + (apq - bq)x}{1 - (p^2 + 2q)x + q^2x^2}, \tag{3.3}$$

$$w_2(x) = \sum_{n=0}^{\infty} w_{2n} x^n = \frac{a + (bp - aq - ap^2)x}{1 - (p^2 + 2q)x + q^2x^2}. \tag{3.4}$$

The Horadam sequence generalizes various other numerical and polynomial sequences, such as the Fibonacci sequence $F_n = w_n(0, 1; 1, 1)$, the Lucas sequence $L_n = w_n(2, 1; 1, 1)$, the Jacobsthal sequence $J_n = w_n(0, 1; 1, 2)$, the Jacobsthal–Lucas sequence $j_n = w_n(2, 1; 1, 2)$, the Pell sequence

$P_n = w_n(0,1;2,1)$, the Pell–Lucas sequence $Q_n = w_n(2,1;2,1)$, and the Mersenne sequence $M_n = w_n(0,1;3,-2)$. Moreover, it follows directly from (2.1) that the Trisenne numbers T_n also belong to the Horadam sequence family, specifically $T_n = w_n(4,11;3,-2)$.

In this section, we derive several relationships between the Trisenne and Horadam sequences. Our first result establishes an explicit connection between the Trisenne and Horadam numbers through their ordinary generating functions. The proof method follows the same approach as in [11, 14].

Theorem 2. *For $n \geq 1$, the following formula holds:*

$$\begin{aligned}&(2a+b-ap)(p+q-1)\,T_n\\=&(p+4q+5)\,w_{n+1}-(p^2+4pq-3p-11q+2)\,w_n\\&+\ 9ap+3bp-9b-6a-3a(p^2+q)-(2p+q-4)(p+q-1)\sum_{k=0}^{n-1}T_{n-k-1}w_k.\end{aligned} \tag{3.5}$$

Proof. By (2.3) and (3.2), we obtain

$$\frac{a+(b-ap)x}{w(x)} = 1-3x+2x^2-(p-3)x-(q+2)x^2 = \frac{4-x}{t(x)}-(p-3)x-(q+2)x^2,$$

and thus

$$(a+(b-ap)x)\,t(x) = (4-x)w(x)-((p-3)x+(q+2)x^2)\,t(x)w(x).$$

Expanding both sides of the last equation as a power series in x yields

$$(a+(b-ap)x)\sum_{n=1}^{\infty}T_nx^n = (4-x)\sum_{n=1}^{\infty}w_nx^n-((p-3)x+(q+2)x^2)\sum_{n=1}^{\infty}T_nx^n\sum_{n=1}^{\infty}w_nx^n.$$

Using the Cauchy product rule for the multiplication of two power series,

$$\sum_{n=1}^{\infty}a_nx^n\sum_{n=1}^{\infty}b_nx^n = \sum_{n=1}^{\infty}\sum_{k=0}^{n}a_kb_{n-k}x^n, \tag{3.6}$$

we obtain

$$\begin{aligned}&aT_0+a\sum_{n=1}^{\infty}T_{n+1}x^{n+1}+(b-ap)\sum_{n=1}^{\infty}T_nx^{n+1}\\=&4w_0+4\sum_{n=1}^{\infty}w_{n+1}x^{n+1}-\sum_{n=1}^{\infty}w_nx^{n+1}-\ (p-3)\sum_{n=1}^{\infty}\sum_{k=0}^{n}T_{n-k}w_kx^{n+1}\\&-(q+2)\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}T_{n-k-1}w_kx^{n+1}.\end{aligned}$$

Comparing coefficients of x^{n+1} on both sides, we have

$$aT_{n+1}+(b-ap)T_n = 4w_{n+1}-w_n-(p-3)\sum_{k=0}^{n}T_{n-k}w_k-(q+2)\sum_{k=0}^{n-1}T_{n-k-1}w_k$$

and then, by applying (2.2), we obtain

$$(2a+b-ap)T_n = 4w_{n+1} - (4p-11)w_n - 3a - (2p+q-4)\sum_{k=0}^{n-1} T_{n-k-1}w_k - 3(p-3)\sum_{k=0}^{n-1} w_k. \quad (3.7)$$

Finally, by the summation formula for Horadam numbers,

$$\sum_{k=0}^{n} w_k = \frac{w_{n+1} + qw_n - b + (p-1)\,a}{p+q-1},$$

we obtain (3.5) from (3.7) after some algebraic manipulations. □

By selecting $(a,b,p,q) = (0,1,1,1),\ (2,1,1,1),\ (0,1,2,1),\ (0,1,1,2)$, the following identities are obtained from (3.5), respectively.

Example 3. *For any integer n,*

$$T_n = 10F_{n+1} + 7F_n - 6 + \sum_{k=1}^{n-1} T_{n-k-1}F_k, \qquad T_n = \frac{10L_{n+1} + 7L_n}{3} - 4 + \frac{1}{3}\sum_{k=0}^{n-1} T_{n-k-1}L_k,$$

$$T_n = \frac{11P_{n+1} + 3P_n}{2} - \frac{3}{2} - \sum_{k=1}^{n-1} T_{n-k-1}P_k, \qquad T_n = 7\,(J_{n+1} + J_n) - 3.$$

Using the generating functions (3.3), (2.4) and (3.4), (2.5), we can similarly derive two additional relations between the Trisenne numbers and odd/even indexed Horadam numbers. These relationships are presented in the following theorem.

Theorem 4. *For $n \geq 1$, the following formulas hold:*

$$\begin{aligned}(a(p^2+q) - 4a - bp)(p^2-(q-1)^2)T_{2n} &= (4p^2-(q-4)^2)(p^2-(q-1)^2)\sum_{k=0}^{n-1} T_{2(n-k-1)}w_{2k} \\ &\quad - (16p^2 + 5p^2q - 9q - 15q^2 + 4q^3 + 20)w_{2n} + pq(5p^2 + 4q^2 + 10q - 41)\,w_{2n-1} \\ &\quad + 9(ap^4 - bp^3 + 3ap^2q - 5ap^2 + aq^2 - 5aq - 2bpq + 5bp + 4a), \end{aligned} \quad (3.8)$$

and

$$\begin{aligned}(bq - apq - 4b)(p^2-(q-1)^2)T_{2n+1} &= (4p^2-(q-4)^2)(p^2-(q-1)^2)\sum_{k=0}^{n-1} T_{2(n-k)-1}w_{2k+1} \\ &\quad - p(44p^2 - 53q^2 + 88q - 8)w_{2n} - q(44p^2 - 2p^2q - 57q^2 + 11q^3 + 54q - 8)\,w_{2n-1} \\ &\quad - 9(ap^3q - bp^2q + 2apq^2 - bq^2 + 5bq - 5apq - 4b). \end{aligned} \quad (3.9)$$

Proof. Proceeding as in the proof of Theorem 2, and using (2.5) and (3.4), we deduce that

$$(a + (bp - aq - ap^2)x)\,t_2(x) = (4+5x)w_2(x) - (p^2 + 2q - 5 - (q^2-4)x)\,xt_2(x)w_2(x).$$

Expanding both sides of the above equation as a power series in x, and applying the Cauchy product rule (3.6), we obtain

$$\begin{aligned}&aT_{2n+2}+(bp-aq-ap^2)T_{2n}\\&=4w_{2n+2}+5w_{2n}-(p^2+2q-5)\sum_{k=0}^{n}T_{2(n-k)}w_{2k}+(q^2-4)\sum_{k=0}^{n-1}T_{2(n-k-1)}w_{2k}.\end{aligned}$$

Using (2.1) and (3.1), and after some algebraic manipulations, this leads to

$$\begin{aligned}&(4a+bp-a(p^2+q))T_{2n}\\&=4pw_{2n+1}-(4(p^2+q)-25)w_{2n}-9a\\&-(4p^2-q^2+8q-16)\sum_{k=0}^{n-1}T_{2(n-k-1)}w_{2k}-9(p^2+2q-5)\sum_{k=0}^{n-1}w_{2k}.\end{aligned}$$

Finally, applying the summation formula

$$\sum_{k=0}^{n}w_{2k}=\frac{pw_{2n+1}+(q-q^2)w_{2n}-a-bp+a(p^2+q)}{(p-q+1)(p+q-1)},$$

yields (3.8).

To prove (3.9), we use the relation

$$(b+(ap-b)qx)\,t_1(x)=(11-2x)w_1(x)-((p^2+2q-5)x-(q^2-4)x^2)\,t_1(x)w_1(x)$$

and the known formula

$$\sum_{k=0}^{n}w_{2k+1}=\frac{pw_{2n+2}+(q-q^2)w_{2n+1}-apq+bq-b}{(p-q+1)(p+q-1)}.$$

□

By substituting specific values $(a,b,p,q)=(0,1,1,1),\ (0,1,2,1),\ (0,1,1,2)$ into (3.8) and (3.9), we obtain the following identities.

Example 5. *For any integer n,*

$$T_{2n}=21F_{2n}+22F_{2n-1}-18+5\sum_{k=0}^{n-1}T_{2(n-k-1)}F_{2k},$$

$$T_{2n}=\frac{42P_{2n}+7P_{2n-1}}{4}+\frac{9}{4}-\frac{7}{2}\sum_{k=0}^{n-1}T_{2(n-k-1)}P_{2k},$$

$$T_{2n}=4J_{2n+1}+13J_{2n},\qquad T_{2n+1}=\frac{31}{2}J_{2n+1}+11J_{2n}-\frac{9}{2},$$

$$T_{2n+1}=14P_{2n+1}+\frac{35}{6}P_{2n}-3-\frac{7}{3}\sum_{k=0}^{n-1}T_{2(n-k)-1}P_{2k+1}.$$

4. Trisenne–Horadam identities via exponential generating functions

In this section, we utilize the structure of exponential generating functions to establish our next results. Consider the Lucas sequence of the first kind $(u_n)_{n\geq 0} = (w_n(0, b; p, q))_{n\geq 0}$. We will derive several formulas that establish connections between u_n and Trisenne numbers T_n involving binomial coefficients.

Let $u(x)$, $u_1(x)$, and $u_2(x)$ represent the exponential generating function for the sequences $(u_n)_{n\geq 0}$, $(u_{2n+1})_{n\geq 0}$, and $(u_{2n})_{n\geq 0}$, respectively. By employing classical techniques [15, Section 3], we obtain the following expressions:

$$u(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!} = \frac{2b}{\Delta} e^{\frac{px}{2}} \sinh\left(\frac{\Delta x}{2}\right), \tag{4.1}$$

$$u_1(x) = \sum_{n=0}^{\infty} u_{2n+1} \frac{x^n}{n!} = \frac{b}{\Delta} e^{\frac{p^2+2q}{2}x} \left(p \sinh\left(\frac{p\Delta x}{2}\right) + \Delta \cosh\left(\frac{p\Delta x}{2}\right)\right), \tag{4.2}$$

$$u_2(x) = \sum_{n=0}^{\infty} u_{2n} \frac{x^n}{n!} = \frac{2b}{\Delta} e^{\frac{p^2+2q}{2}x} \sinh\left(\frac{p\Delta x}{2}\right), \tag{4.3}$$

where $\Delta = \sqrt{p^2 + 4q}$.

Next, we present a relationship between Trisenne and Horadam numbers involving binomial coefficients.

Theorem 6. *For any integer n, the following identity holds:*

$$T_n = 4 + \frac{7}{b\Delta^{n-1}} \sum_{k=1}^{n} \binom{n}{k} \left(\frac{3\Delta - p}{2}\right)^{n-k} u_k. \tag{4.4}$$

Proof. From (2.6) and (4.1), we have

$$b\tau(\Delta x) - 4be^{\Delta x} = 7\Delta u(x) e^{\frac{3\Delta - p}{2}x}.$$

Expanding both sides of the last equation as a power series in x, we get

$$b \sum_{n=0}^{\infty} T_n \frac{(\Delta x)^n}{n!} - 4b \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!} = 7\Delta \sum_{n=0}^{\infty} u_n \frac{x^n}{n!} \sum_{n=0}^{\infty} \left(\frac{3\Delta - p}{2}\right)^n \frac{x^n}{n!}.$$

Equivalently, this can be expressed as

$$b \sum_{n=0}^{\infty} T_n \Delta^n \frac{x^n}{n!} - 4b \sum_{n=0}^{\infty} \Delta^n \frac{x^n}{n!} = 7\Delta \sum_{n=0}^{\infty} \sum_{k=0}^{n} u_k \frac{x^k}{k!} \left(\frac{3\Delta - p}{2}\right)^{n-k} \frac{x^{n-k}}{(n-k)!}.$$

By comparing the coefficients on both sides, we obtain the desired result. □

From (4.4), the following corollaries can be immediately derived.

Example 7. *For any integer n,*

$$T_n = 4 + \frac{7}{(\sqrt{5})^{n-1}} \sum_{k=1}^{n} \binom{n}{k} \left(\frac{3\sqrt{5}-1}{2} \right)^{n-k} F_k, \tag{4.5}$$

$$T_n = 4 + 28 \left(\frac{4}{3} \right)^{n-1} \sum_{k=1}^{n} \binom{n}{k} \frac{J_k}{4^k},$$

$$T_n = 4 + 7 \left(\frac{\sqrt{2}}{4} \right)^{n-1} \sum_{k=1}^{n} \binom{n}{k} (3\sqrt{2}-1)^{n-k} P_k.$$

From (4.5), using $F_k = \frac{1}{\sqrt{5}}(\alpha^k - (-1/\alpha)^k)$, where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio, we observe that the nth Trisenne number can be expressed in terms of α as follows:

$$T_n = 4 + 7 \left(\frac{8-\alpha}{5} \right)^{n} \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha^{2k} - (-1)^k}{(3+\alpha)^k}.$$

Similarly, we can use the generating functions (4.2), (2.7) and (4.3), (2.8) to derive two additional relations between the Trisenne numbers and the numbers u_n. These relations are stated in the following theorem.

Theorem 8. *For any integer n, the following identities hold:*

$$T_{2n+1} = 11 - \frac{14\Delta(4^n+1)}{p} + \frac{28\Delta}{bp} \left(\frac{3}{p\Delta} \right)^{n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{5p\Delta - 3p^2 - 6q}{6} \right)^{n-k} u_{2k+1},$$

$$T_{2n} = 4 + \frac{7\Delta}{b} \left(\frac{3}{p\Delta} \right)^{n} \sum_{k=1}^{n} \binom{n}{k} \left(\frac{5p\Delta - 3p^2 - 6q}{6} \right)^{n-k} u_{2k}.$$

Proof. The proof is similar to the proof of Theorem 6. The first and second formulas follow, respectively, from the functional relations

$$\begin{aligned} 14p\Delta u_1(x) + 7b(p^2+2q-p\Delta)e^{\frac{p^2+2q+p\Delta}{2}x} - 7b(p^2+2q+p\Delta)e^{\frac{p^2+2q-p\Delta}{2}x} \\ = b(p^2+q)\left(\tau_1\left(\frac{p\Delta}{3}x \right) e^{\left(\frac{p^2+q}{2} - \frac{5p\Delta}{6}\right)x} - 11e^{\frac{p^2+q-p\Delta}{2}x} \right) \end{aligned}$$

and

$$b\tau_2\left(\frac{p\Delta}{3}x \right) - 4be^{\frac{p\Delta}{3}x} = 7\Delta u_2(x) e^{\frac{5p\Delta - 3p^2 - 6q}{6}x},$$

writing in terms of power series and collecting terms. The details are omitted. □

By selecting appropriate values for a, b, p, and q, the following identities can be derived from Theorem 8.

Example 9. *For any integer n,*

$$T_{2n+1} = 11 - 14\sqrt{5}(4^n+1) + 28\sqrt{5} \left(\frac{3\sqrt{5}}{5} \right)^{n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{5\sqrt{5}-9}{6} \right)^{n-k} F_{2k+1},$$

$$T_{2n+1} = 11 - 14\sqrt{2}\,(4^n + 1) + 28\sqrt{2}\left(\frac{3\sqrt{2}}{8}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{5\sqrt{8} - 9}{3}\right)^{n-k} P_{2k+1},$$

$$T_{2n} = 4 + 7\sqrt{5}\left(\frac{3}{\sqrt{5}}\right)^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{5\sqrt{5} - 9}{6}\right)^{n-k} F_{2k}, \quad T_{2n} = 4 + 21 J_{2n},$$

$$T_{2n} = 4 + 14\sqrt{2}\left(\frac{3\sqrt{2}}{8}\right)^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{10\sqrt{2} - 9}{3}\right)^{n-k} P_{2k}.$$

Next, we explore connections between the Lucas sequence of the second kind $(v_n)_{n\geq 0} = (w_n(2, p; p, q))_{n\geq 0}$ and Trisenne numbers. It is known that the exponential generating functions for the sequences $(v_n)_{n\geq 0}$, $(v_{2n})_{n\geq 0}$, and $(v_{2n+1})_{n\geq 0}$ are given by

$$v(x) = \sum_{n=0}^{\infty} v_n \frac{x^n}{n!} = 2e^{\frac{px}{2}} \cosh\left(\frac{\Delta x}{2}\right), \tag{4.6}$$

$$v_1(x) = \sum_{n=0}^{\infty} v_{2n+1} \frac{x^n}{n!} = e^{\frac{p^2+2q}{2}x} \left(p \cosh\left(\frac{p\Delta x}{2}\right) + \Delta \sinh\left(\frac{p\Delta x}{2}\right)\right),$$

$$v_2(x) = \sum_{n=0}^{\infty} v_{2n} \frac{x^n}{n!} = 2e^{\frac{p^2+2q}{2}x} \cosh\left(\frac{p\Delta x}{2}\right),$$

where $\Delta = \sqrt{p^2 + 4q}$.

Theorem 10. *For $n \geq 0$, the following identities hold:*

$$T_n = -10 + \frac{7}{\Delta^n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{3\Delta - p}{2}\right)^{n-k} v_k, \tag{4.7}$$

$$T_{2n+1} = 11 - \frac{14p(4^n + 1)}{\Delta} + \frac{28}{\Delta}\left(\frac{3}{p\Delta}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{5p\Delta - 3p^2 - 6q}{6}\right)^{n-k} v_{2k+1},$$

$$T_{2n} = -10 + 7\left(\frac{3}{p\Delta}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{5p\Delta - 3p^2 - 6q}{6}\right)^{n-k} v_{2k}.$$

Proof. We will prove only the first formula. The proofs of others are similar, so we omit them. Using (2.6) and (4.6), we obtain

$$7v(x)e^{\frac{3\Delta - p}{2}x} = \tau(\Delta x) + 10e^{\Delta x}.$$

From the formula above, we now deduce that

$$7\sum_{n=0}^{\infty} v_n \frac{x^n}{x!} \sum_{n=0}^{\infty} \left(\frac{3\Delta - p}{2}\right)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} T_n \Delta^n \frac{x^n}{n!} + 10\sum_{n=0}^{\infty} \Delta^n \frac{x^n}{n!},$$

and, by applying the Cauchy product rule (3.6) and simplifying, we obtain Eq (4.7). □

Some particular cases of Theorem 10 are stated in the next example.

Example 11. *For any integer n,*

$$T_n = -10 + \frac{7}{(\sqrt{5})^n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{3\sqrt{5}-1}{2}\right)^{n-k}L_k, \quad T_n = -10 + 7\left(\frac{4}{3}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{j_k}{4^k},$$

$$T_{2n+1} = 11 + \frac{14(4^n+1)}{\sqrt{5}} + \frac{28}{\sqrt{5}}\left(\frac{3}{\sqrt{5}}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{5\sqrt{5}-9}{6}\right)^{n-k}L_{2k+1},$$

$$T_{2n+1} = \frac{19}{3} - \frac{14}{3}4^n + \frac{28}{3}j_{2n+1}, \quad T_{2n} = -10 + 7j_{2n},$$

$$T_{2n} = -10 + 7\left(\frac{3}{\sqrt{5}}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{5\sqrt{5}-9}{6}\right)^{n-k}L_{2k}.$$

5. Horadam–Trisenne identities using binomial transform

Theorem 3 highlights an important issue. Consider the sequence a_k defined by $a_k = \left(\frac{3\Delta-p}{2}\right)^{-k}u_k$ and the sequence b_n given by $b_n = \frac{b\Delta^{n-1}}{7}\left(\frac{2}{3\Delta-p}\right)^n(T_n-4)$. It can be shown (see, for example, [16]) that b_n is the binomial transform of a_n, and its inverse is defined by

$$b_n = \sum_{k=0}^{n}\binom{n}{k}a_k \quad \Longleftrightarrow \quad a_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}b_k. \tag{5.1}$$

We can apply similar considerations to the formulas in Theorem 8.

From Theorems 6 and 8, the inverse relation given in (5.1) immediately yields the following results.

Theorem 12. *For any integer n, we have*

$$u_n = \frac{b\Delta^{n-1}}{7}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{p-3\Delta}{2\Delta}\right)^{n-k}(T_k-4), \tag{5.2}$$

$$u_{2n+1} = \frac{b}{2}\left(\frac{p\Delta}{3}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{3p^2+6q-5p\Delta}{2p\Delta}\right)^{n-k}\left(\frac{p}{14\Delta}(T_{2k+1}-11)+4^k+1\right),$$

$$u_{2n} = \frac{b}{7\Delta}\left(\frac{p\Delta}{3}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{3p^2+6q-5p\Delta}{2p\Delta}\right)^{n-k}(T_{2k}-4).$$

Proof. We prove only (5.2); the proofs of the remaining identities can be obtained similarly.

From (4.4), we have

$$\left(\frac{2}{3\Delta-p}\right)^n\frac{b\Delta^{n-1}}{7}(T_n-4) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{3\Delta-p}\right)^k u_k.$$

Denoting

$$b_n = \left(\frac{2}{3\Delta-p}\right)^n\frac{b\Delta^{n-1}}{7}(T_n-4), \qquad a_k = \left(\frac{2}{3\Delta-p}\right)^k u_k,$$

and using binomial transform together with its inverse (5.1), we obtain

$$\left(\frac{2}{3\Delta - p}\right)^n u_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(\frac{2}{3\Delta - p}\right)^k \frac{b\Delta^{k-1}}{7}(T_k - 4),$$

from which (5.2) follows. □

By taking specific values for the parameters a, b, p, and q in Theorem 12, we obtain the following identities.

Example 13. *For any integer n,*

$$F_n = \frac{(\sqrt{5})^{n-1}}{7} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sqrt{5} - 15}{10}\right)^{n-k} T_k - (-1)^n \frac{4\sqrt{5}}{35\alpha^n},$$

$$F_{2n+1} = \frac{3}{140} \left(\frac{\sqrt{5}}{3}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{9\sqrt{5} - 25}{10}\right)^{n-k} T_{2k+1} + \frac{L_{2n}}{2} - \frac{11\sqrt{5}}{140\alpha^{2n}},$$

$$F_{2n} = \frac{3}{35} \left(\frac{\sqrt{5}}{3}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{9\sqrt{5} - 25}{10}\right)^{n-k} T_{2k} - \frac{4\sqrt{5}}{35\alpha^{2n}},$$

$$J_n = \frac{3^n}{21} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{4}{3}\right)^{n-k} T_k - (-1)^n \frac{4}{21}, \quad J_{2n+1} = \frac{1}{84} T_{2n+1} + \frac{31}{84} + 2^{2n-1}.$$

Similarly, by applying the binomial transform (5.1) to Theorem 10, we obtain expressions that represent the sequence $(v_n)_{n\geq 0}$ in terms of the Trisenne numbers.

Theorem 14. *For $n \geq 0$, we have*

$$v_n = \frac{\Delta^n}{7} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{p - 3\Delta}{2\Delta}\right)^{n-k} (T_k + 10), \tag{5.3}$$

$$v_{2n} = \frac{1}{7} \left(\frac{p\Delta}{3}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{3p^2 + 6q - 5p\Delta}{2p\Delta}\right)^{n-k} (T_{2k} + 10),$$

$$v_{2n+1} = \frac{\Delta}{28} \left(\frac{p\Delta}{3}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{3p^2 + 6q - 5p\Delta}{2p\Delta}\right)^{n-k} (\Delta(T_{2k+1} - 11) + 14p(4^k + 1)),$$

where $\Delta = \sqrt{p^2 + 4q}$.

Proof. The proof of formula (5.3) follows from the binomial transform (5.1) by defining

$$b_n = \frac{\Delta^n}{7} \left(\frac{2}{3\Delta - p}\right)^n (T_n + 10), \qquad a_k = \left(\frac{2}{3\Delta - p}\right)^k v_k.$$

The proofs of the remaining formulas are similar and are therefore omitted. □

Example 15. *It follows from Theorem 14 that for any integer n,*

$$
\begin{aligned}
L_n &= \frac{10}{7(-\alpha)^n} + \frac{\sqrt{5}^n}{7}\sum_{k=0}^{n}\left(\frac{\sqrt{5}-15}{10}\right)^{n-k} T_k,\\
Q_n &= \frac{10(1-\sqrt{2})^n}{7} + \frac{(2\sqrt{2})^n}{7}\sum_{k=0}^{n}\left(\frac{\sqrt{2}-6}{4}\right)^{n-k} T_k,\\
j_{2n+1} &= \frac{3T_{2n+1}-19}{28} + 2^{2n-1}, \qquad j_{2n} = \frac{T_{2n+1}+10}{7},\\
L_{2n} &= \frac{10}{7\alpha^{2n}} + \frac{1}{7}\left(\frac{\sqrt{5}}{3}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{9\sqrt{5}-25}{10}\right)^{n-k} T_{2k},
\end{aligned}
$$

where $\alpha = (1+\sqrt{5})/2$.

6. Polynomial extensions

Using the generating function approach, we can generalize our results to polynomial sequences, offering avenues for further research.

In [4], the bivariate Mersenne polynomials, denoted by $M_n(x,y)$, were introduced and defined by the recurrence relation

$$M_n(x,y) = 3yM_{n-1}(x,y) - 2xM_{n-2}(x,y), \quad n \geq 2,$$

with initial conditions $M_0(x,y) = 0$ and $M_1(x,y) = 1$.

By analogy with Trisenne numbers, we now introduce the *Trisenne polynomials*, denoted by $P_n(x,y)$, and define them as follows:

$$P_n(x,y) = M_n(x,y) + M_{n+1}(x,y) + M_{n+2}(x,y), \quad n \geq 0.$$

As an illustration, we present the formula in Theorem 2 for the Trisenne polynomials and the Chebyshev polynomials of the first and second kinds, $T_n(x) = w_n(1,x;2x,-1)$ and $U_n(x) = w_n(1,2x;2x,-1)$, respectively:

$$
\begin{aligned}
P_n(x,y) - \frac{3y}{2}P_{n-1}(x,y) = (1+3y)\left(\frac{3y}{2x}\right)^n T_n(x) + (1-2x)\left(\frac{3y}{2x}\right)^{n-1} T_{n-1}(x)\\
+\left(1-\frac{8x^3}{9y^2}\right)\sum_{k=0}^{n-2}\left(\frac{3y}{2x}\right)^{n-k} P_k(x,y)\,T_{n-2-k}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
P_n(x,y) =&(1+3y)\left(\frac{3y}{2x}\right)^n U_n(x) + (1-2x)\left(\frac{3y}{2x}\right)^{n-1} U_{n-1}(x)\\
&+\left(1-\frac{8x^3}{9y^2}\right)\sum_{k=0}^{n-2}\left(\frac{3y}{2x}\right)^{n-k} P_k(x,y)\,U_{n-2-k}(x).
\end{aligned}
$$

7. Conclusions

This paper establishes a connection between Mersenne-based constructions and Horadam-type recurrences, using the Trisenne sequence as a representative example. Although the proposed sequence itself does not introduce fundamentally new recurrence behavior, it demonstrates that the Trisenne sequence can be naturally embedded into the class of second-order linear recurrences and analyzed within the Horadam framework using generating functions. In particular, this approach allows us to establish explicit connections with classical sequences such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences, and to derive corresponding identities and representations. The broader contribution of this study lies in demonstrating how generating function techniques can be used to systematically relate such special cases to standard Horadam families.

Author contributions

Engin Eser: Conceptualization, methodology, validation, formal analysis, investigation, original draft preparation, funding acquisition, review and editing; Taras Goy: Conceptualization, methodology, validation, formal analysis, investigation, original draft preparation, supervision, review and editing; Engin Özkan: Conceptualization, methodology, validation, formal analysis, investigation, supervision, Project administration, review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous reviewers for their careful reading of the manuscript and for their valuable comments and suggestions, which helped to improve the quality and clarity of the paper.

Conflict of interest

The authors declare no conflicts of interest.

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