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Research article

Output formation containment of time-delayed heterogeneous singular multi-agent systems with stochastic impulsive effects

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Abstract: This paper investigates the containment of output formation for heterogeneous singular multi-agent systems (MASs) subject to time delays and stochastic impulsive disturbances. The considered agents exhibit diverse dynamics, including descriptor-type tracking and algebraic systems, reflecting heterogeneity in state-space representations. Communication constraints are modeled via bounded state and input delays, while environmental uncertainties are captured through stochastic noise and impulsive effects occurring at arbitrary time instants. A distributed control protocol is proposed to achieve output formation containment, ensuring asymptotic convergence of follower outputs to a convex combination of leader outputs. The approach effectively addresses system singularities and uncertainties by employing a Lyapunov–Krasovskii functional combined with stochastic stability analysis. Sufficient conditions for mean-square admissibility and asymptotic convergence are derived. Numerical simulations validate the effectiveness and robustness of the proposed control strategy under varying delays and impulsive conditions.

Keywords: heterogeneous singular multi-agent systems; stochastic impulsive disturbances; formation containment; time delay
Mathematics Subject Classification: 26A33, 34K37

1. Introduction

The problem examined in this paper is the containment of the output formation of a heterogeneous singular multi-agent system (MAS) with both time delays and stochastic perturbations, as well as impulsive actions. The system structure encompasses the state, as well as input delay, singular dynamics, stochastic noise represented by Brownian motion, and sudden changes described by impulsive signals. To cope with them, a distributed adaptive control protocol is introduced, aiming to guarantee that the outputs of follower agents converge to the convex hull of the multiple leaders asymptotically. The given approach copes well with the delay, uncertainties, and structural complexities and can be evidenced by the numerical simulation.

Chen et al. [1] highlights the growing complexity in networked control systems and the challenges posed by communication constraints, motivating the need for robust strategies like impulsive control and containment formation in time-delayed environments. Wang et al. [2] address delays and external disturbances using predictor-based observers, emphasizing the necessity of observer-based compensation techniques for ensuring consensus in disturbed MASs, which is relevant for handling stochastic effects in singular systems. All these result in great difficulties in designing control systems. Moreover, actual agents are often non-homogeneous and have single (descriptor-type) dynamics, which appear in quasi-constrained mechanical systems, power networks, and economic models. In these systems, specific considerations ought to be made with regard to admissibility and stability. Meanwhile, the issues dealing with heterogeneity and structural limitations in MASs have been abundantly discussed in recent surveys [3]. Based on the reflection of these considerations, this article aims to explore the problem of formulating the output formation-containment under a category of time-delayed heterogeneous singular MASs under stochastic acts of impulses. The necessity of robust coordination strategies to tackle all the aspects of delays, randomness, and discontinuities has been emphasized recently [4, 5]. To render the outputs of followers asymptotically convergent to a desired formation in a mean-square sense despite possibly existing communication delays, system singularities, and unpredictable random impulsive influences, a distributed control protocol is suggested.

In recent years, coordinated control of MASs has attracted more and more attention, as it can be applied in various fields like autonomous vehicles, cyber-physical systems, distributed robots, etc. In this area, the issue of containment of the output formation in which the follower agents are supposed to be attracted to a convex combination of multiple leader trajectories has been studied actively. But in practice, MASs are likely to involve time-varying topologies, stochastic disturbances, and impulsive effects, to make the control problem more difficult. Liu et al. [6] suggested fault-tolerant formation-consensus strategies of time-varying MASs when the communication protocol was stochastic, thus the need to develop controllers proving to be robust to random network adjustments. On the same note, Khan et al. [7] also dealt with a signed graph framework to contain nonlinear fractional MASs under hostile disturbance and input delay. In about singular systems, Zhu et al. [8] analyzed the observer-based bipartite containment of MASs on signed digraphs and established that paying attention to algebraic constraints of the agent dynamics is essential. Moreover, the security of the MASs against unbounded cyber-attacks has of late been a priority, as illustrated in Wang et al. [9] where intra-agent cooperation in cyber-physical defense was elaborated in heterogeneous agents. Zhang et al. [10] also demonstrated the importance of impulsive control on MAS stability, in which they considered bipartite asynchronous impulsive consensus on systems that had discontinuous changes of states.

The complexity of the containment problem is further increased by the presence of singular (descriptor) dynamics, communication delays, and structural uncertainties. Recent studies have made significant progress in addressing such challenges. For instance, Yan et al. [11] investigated finite-time H_2/H_∞ control for mean-field jump-diffusion systems, highlighting the impact of stochastic dynamics on system stability. In [12], finite-time guaranteed cost control was developed for uncertain mean-field stochastic systems under both Wiener and Poisson noises, demonstrating robustness against multiple stochastic disturbances.

Moreover, hybrid nominal-robust control strategies have been proposed in [13] to enhance system performance under uncertainties and constraints. The work in [14] explored pulse-controlled neural network models with applications in complex dynamic systems, indicating the growing interest in nonlinear and hybrid system behaviors. In addition, stability analysis of time-delay systems has been further advanced through novel exponential-weighted integral inequalities, as presented in [15], which provide less conservative stability conditions. Furthermore, Li et al. [16] and Nasir et al. [17] developed new stability and stabilization criteria for Takagi Sugeno(T-S) fuzzy systems with time-varying delays, offering effective tools to handle nonlinearities and uncertainties. Motivated by these developments, this paper investigates the output formation containment problem for a class of heterogeneous singular multi-agent systems (MAS) subject to bounded communication delays, stochastic disturbances, and impulsive effects. A distributed control protocol is proposed, and rigorous stability analysis is carried out to ensure mean-square admissibility and convergence under the considered uncertainties. MAS coordination has attracted significant attention due to its wide applications in distributed sensor networks, autonomous vehicles, and robotics. In this context, formation-based and consensus control problems under uncertainties and nonlinearities remain active research topics. The integration of advanced control techniques with delay compensation and stochastic analysis continues to play a crucial role in improving system performance and robustness. Besides, both the problem of consensus in autonomous driving systems over uncertain networks and the effect of time delay in fractional-order systems have been reviewed systematically in [18, 19], respectively, on issues in leader-following consensus problems. The value of adaptive and intelligent control schemes in dealing with the sophistication of contemporary MAS environments is brought forth by these works.

Recent studies on MASs have increasingly focused on addressing challenges arising from communication delays, uncertainties, and complex dynamic environments. For example, event-triggered control strategies have been developed in [20] to achieve leader-followerconsensus in fractional-order MASs, effectively reducing communication burden while maintaining system performance. In addition, data-driven and learning-based approaches, such as multi-agent Q-learning on graphs [21], have provided new perspectives for distributed decision-making in complex networks.

Furthermore, robust synchronization and control of nonlinear systems under noise and uncertainty have been investigated using neural network-based approaches, as demonstrated in [22]. Advanced control techniques, including sliding mode control combined with recurrent neural networks, have also been proposed for uncertain UAV systems in [23], highlighting their effectiveness in handling nonlinearities and disturbances. Disturbance rejection and robustness enhancement have been further addressed using fixed-time control frameworks, as shown in [24].

The effects of time delays and network interactions on system synchronization have been studied in [25], emphasizing the importance of delay-dependent stability analysis in complex networks.

Meanwhile, recent developments in intelligent control, such as reinforcement learning-based decision-making frameworks [26], have opened new directions for adaptive and data-driven control design. In addition, fault-tolerant control strategies for stochastic systems with time-varying delays have been investigated in [27], ensuring reliable system performance under faults and uncertainties.

Moreover, observer-based techniques have gained significant attention in recent years. Distributed prescribed-time observers for systems with unknown inputs have been proposed in [28], while Zhang et al. [29] further extended these results to descriptor systems. In this context, adaptive fuzzy and event-triggered control schemes have been developed in [30] to address uncertainties and nonlinear dynamics in MASs.

Motivated by these advances, this paper studies the output formation containment problem for heterogeneous singular MASs subject to time delays, stochastic disturbances, and impulsive effects. A distributed control strategy is developed, and rigorous stability analysis is carried out to ensure mean-square admissibility and convergence under complex uncertainties. Furthermore, Kandasamy et al. [31] addressed synchronization in multiplex networks with nonidentical fractional-order neurons, which is highly relevant for the coordination of heterogeneous agents with complex dynamic behaviors. Cheng et al. [32] reflected the broad advancements in control theory, especially in time-delay and singular systems, which are crucial for containment problems in complex networks. Lastly, Almeida et al. [33] presented a robust containment control framework for MASs with time delays and heterogeneous Lipschitz nonlinearities, directly motivating the investigation of output formation containment for heterogeneous singular MASs with time delays, stochastic disturbances, and impulsive effects.

Control of MASs has attracted a lot of attention because of its application in a variety of domains such as robotics, autonomous vehicles, and cyber-physical systems. Most recent research has been focused on obtaining a better understanding of synchronization and consensus mechanisms and finding empirical insights in more realistic settings. As an example, the nonidentical agent in fractional-order networks involving synchronization was covered by [34, 35], and the complexity of dynamics is emphasized in heterogeneous networks. Simultaneously, the problems related to communication delays, input saturation, and event-triggered control have been considered in the articles, such as [36] and [37], where the adaptive and self-triggered control approaches have been suggested to MASs. Uncertainty challenges, non linearities, and time-varying delays of distributed systems were also dealt with in [38], and a dynamic event-triggered approach to forming control against non-uniform communication delays was proposed in [39]. Moreover, Xiao et al. [40] gave a detailed survey on time-delayed modeling of consensus protocol, highlighting the importance of such modeling to be practically deployed. These pioneering contributions are the driving force behind this research, in which we intend to cover output formation containment in heterogeneous singular MASs with stochastic perturbations, impulsive influences, and time delays, thereby strengthening the health and synchronization of networked control frameworks under feasible situations.

In recent years, significant progress has been made in advancing control strategies for multi-agent and large-scale dynamical systems under complex constraints. For instance, distributed event-triggered formation control with collision avoidance has been investigated in [41], demonstrating the effectiveness of virtual tube-based approaches for UAV systems under disturbances. Similarly, adaptive and learning-based control strategies have been explored in [42], where composite learning and event-triggered mechanisms were employed to enhance performance in nonlinear multi-agent power systems.

The consensus problem for switched and heterogeneous MAS has also been addressed in [43], highlighting the challenges posed by different-order dynamics and matrix-weighted network topologies. Beyond MASs, stabilization and control of complex systems with delays and constraints have been studied in [44, 45], where partial differential equation-ordinary differential equation (PDE-ODE) frameworks and predictor-based approaches were developed to handle distributed delays and input saturation.

Moreover, modeling and control of engineering systems under uncertainties have attracted increasing attention. For example, decoupling models for fatigue assessment in complex offshore systems have been proposed in [46], while hybrid data-driven and physics-based modeling approaches have been investigated in [47] for vehicle dynamics. In the context of safety-critical control, distributed formation control based on control barrier functions has been developed in [48], ensuring safety and robustness in networked systems.

Furthermore, intelligent and learning-based control strategies have been increasingly applied to systems with delays and uncertainties. A knowledge-guided self-learning control approach for vehicle platoons with delays has been proposed in [49], demonstrating improved adaptability and robustness. In addition, event-triggered neural learning control methods have been studied in [50], providing efficient solutions for nonlinear systems with asymmetric constraints.

Motivated by these recent advances, this paper focuses on the output formation containment problem for heterogeneous singular MASs subject to time delays, stochastic disturbances, and impulsive effects. The proposed framework integrates distributed control design with rigorous stability analysis to address the challenges arising from heterogeneity, uncertainty, and complex network interactions. It is guided by such gaps that form the topic of this paper, which is a study on output formation containment in a subclass of heterogeneous singular MASs with bounded time delays, stochastic disturbances, and impulsive effects. The structures of its agents are heterogeneous and consist of dynamically communicating describer-type followers and more than one leader. A systematic control design procedure is established to ensure that the outputs of the followers, in the mean-square sense, converge to a convex combination of the outputs of the leaders. In order to handle the complexity of the network, the given design offers stochastic Lyapunov-Krasovskii functions, delay-dependent stability criteria, and the theory of impulsive control. Numerical simulations help in proving the efficiency and soundness of the control strategy, and its relevance in realistic uncertainties and constraints in the network.

Although classical tools such as Lyapunov–Krasovskii functionals and stochastic stability theory are employed, the novelty of this work lies in integrating multiple challenging factors into a unified framework. Specifically, the proposed approach simultaneously addresses heterogeneous singular dynamics, time-varying delays, stochastic disturbances, and impulsive effects under an output formation containment objective. This comprehensive treatment significantly extends existing results, which typically consider only a subset of these factors. Compared with existing studies on MASs, the proposed work offers several distinctive features. Most existing results consider either time delays, stochastic disturbances, or impulsive effects separately. In contrast, this paper integrates all these factors within a unified framework for heterogeneous singular systems. Furthermore, unlike conventional consensus or containment approaches, the proposed method addresses output formation containment for descriptor-type agents with algebraic constraints. This significantly extends the applicability of existing results and provides a more comprehensive solution for complex networked

systems. This is how the content is organized:
Section 1: This section introduces the problem of output formation containment in heterogeneous singular MASs affected by time delays, stochastic disturbances, and impulsive effects, highlighting the challenges and motivation for the proposed control strategy.
Section 2: The mathematical model of the MASs is formulated, incorporating singular dynamics, communication topology, time delays, Brownian motion-driven noise, and stochastic impulsive terms.
Section 3: Several key mathematical lemmas and properties are established to support the main theoretical results, including stability analysis, stochastic calculus, and conditions for output containment.
Section 4: Simulation results based on an eight-agent system are presented to verify the theoretical findings, with output trajectories, error dynamics, and system behavior illustrated in Figures 1–5.
Section 5: The paper concludes by summarizing the effectiveness of the proposed control approach in achieving output formation containment under complex uncertainties and suggests directions for future work.

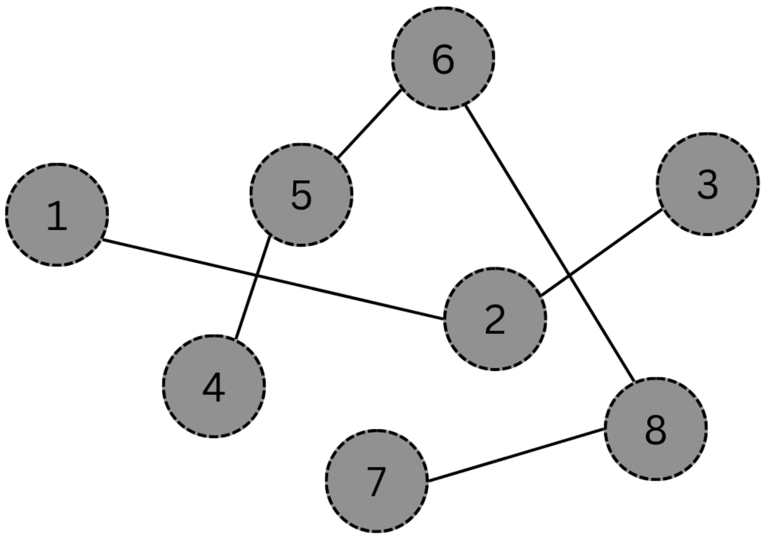


Figure 1. 8-Agents topology.

Main contributions:

The main contributions of this paper can be summarized as follows:

- A novel problem of output formation containment is formulated for heterogeneous singular MASs subject to time-varying delays, stochastic disturbances, and impulsive effects, capturing both dynamic and algebraic agent behaviors.
- A distributed adaptive control protocol is developed, and sufficient conditions for mean-square admissibility and output containment are derived using a Lyapunov–Krasovskii functional combined with stochastic stability theory.
- The effectiveness and robustness of the proposed approach are validated through numerical simulations under time-varying delays and stochastic impulsive effects, demonstrating reliable

performance in complex, uncertain environments.

Notation: A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ defines the communication topology between agents in which $\mathcal{V}$ denotes the set of nodes (agents), $\mathcal{E} \subseteq \mathcal{V} times \mathcal{V}$ the set of directed edges, and $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrix. In particular, when $(i, j) \in \mathcal{E}$, the element a_{ij} is strictly positive, otherwise, a_{ij} is equal to zero.

The Laplacian matrix $\mathcal{L} = [l_{ij}]$ is defined by:

$$l_{ii} = \sum_{j \neq i} |a_{ij}|, \quad l_{ij} = -a_{ij}, \quad \text{for } i \neq j.$$

Without loss of generality, the graph $\mathcal{G}$ is assumed to be normalized, i.e., $l_{ii} = 1$ for all i.

The column vector of all agent states is denoted by:

$$\text{col}\{x_1(t), \ldots, x_N(t)\} = [x_1^\top(t), \ldots, x_N^\top(t)]^\top.$$

For a matrix $X = [X_1 \ X_2 \ \ldots \ X_n] \in \mathbb{R}^{m\times n}$, the vectorization operator is defined as:

$$\text{Vec}(X) = \text{col}(X_1, X_2, \ldots, X_n) \in \mathbb{R}^{mn}.$$

The inverse mapping $\text{Mat}_m^n(\cdot)$ is defined for $A \in \mathbb{R}^{mn}$ as:

$$\text{Mat}_m^n(A) = [A_1 \ A_2 \ \ldots \ A_n], \quad \text{where } A_i \in \mathbb{R}^m.$$

Matrices: The set of all generalized eigenvalues (both finite and infinite) of the pair (E, A) of matrices is denoted by $\sigma(E, A)$. Let A be a matrix. The spectral radius of A, written $\rho(A)$, is the largest absolute value of an eigenvalue of A (including eigenvalues of multiplicity more than one). The Dirac delta function is referred to as $\delta(t)$. To the open left-half of the complex plane $\mathbb{C}^- = \{s \in \mathbb{C} : \text{Re}(s) < 0\}$, we refer to as the set $\mathbb{C}^-$.

The stochastic process $w_i(t)$ represents the standard Brownian motion (Wiener process), and $I_i^{(k)}(t)$ denotes the impulsive effect at the k-th jump time t_k for agent i. The random jumps are modeled using $\delta(t - t_k)$.

2. System description with time delay and stochastic impulses

In this section, the dynamics of $N+M$ heterogeneous singular MASs with time delays and stochastic impulsive effects are described as follows:

$$\begin{aligned} E_i\dot{x}_i(t) &= A_i x_i(t) + A_{di} x_i(t - \tau_i) + B_i u_i(t - \tau_i^u) + G_i w_i(t) + \sum_{k=1}^{N_i} I_i^{(k)}(t)\delta(t - t_k) \\ y_i(t) &= C_i x_i(t) + D_i u_i(t - \tau_i^u). \end{aligned} \tag{2.1}$$

Remark. In this work, the impulsive effects are modeled as occurring at deterministic time instants $\{t_k\}$ satisfying the dwell-time condition $\Delta t_k = t_{k+1} - t_k \geq \tau_{\min} > 0$. The stochastic nature of the impulses is incorporated through the impulse magnitude $I_i^{(k)}(t)$, which may depend on stochastic processes such as Brownian motion. It is important to note that randomness is not introduced in the impulse occurrence times. Modeling randomly occurring impulse times (e.g., via Poisson processes) would require a different stochastic hybrid system framework and is beyond the scope of the present study.

Remark (Heterogeneity of agents). The considered MAS is heterogeneous in the sense that each agent is described by distinct system matrices $(E_i, A_i, A_{di}, B_i, C_i, D_i)$ and may possess different state dimensions and dynamic structures. In particular, the presence of singular matrices E_i allows the coexistence of differential and algebraic equations, enabling the modeling of descriptor-type dynamics. This formulation captures a wide class of heterogeneous agents, including purely dynamic, algebraic, and mixed-type systems.

With agents in the follower group are being indexed as $i = 1, 2, \ldots, N$ and labeled as $\mathcal{R}$, agents in the leader group indexed as $i = N+1, N+2, \ldots, N+M$ and labeled as $\mathcal{H}$. The system, heterogeneous MAS, state $x_i(t) \in \mathbb{R}^{n_i}$, and output of the measurement $y_i(t) \in \mathbb{R}^q$ are quoted. The control input is $u_i(t) \in \mathbb{R}^{m_i}$. The matrix E_i is subject to the rank constraint $\text{rank}(E_i) = r_i \leq n_i$ that permits it to be singular. In this, the state delay matrix is A_{di}, bounded state and input delays are τ_i and τ_i^u, standard Brownian motion (or Wiener process) is $w_i(t)$, noise intensity matrix is G_i,Dirac delta is $\delta(t - t_k)$ that characterizes stochastic impulses at the (random) times t_k, where t_k, $I_i^{(k)}(t)$ characterizes the impulse effect.

The virtual leader dynamics are given by:

$$\dot{x}_0(t) = A_0 x_0(t - \tau_0) + G_0 w_0(t) + \sum_{k=1}^{N_0} I_0^{(k)}(t)\delta(t - t_k), \tag{2.2}$$

$$y_0(t) = C_0 x_0(t), \tag{2.3}$$

where $x_0(t) \in \mathbb{R}^m$ and $y_0(t) \in \mathbb{R}^q$ denote the state and measurement output of the leader agent, and τ_0 is the delay in the leader's internal dynamics.

The Laplacian matrix describing the communication topology among the agents is defined as:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathbf{0}_{M\times N} & \mathcal{L}_3 \end{bmatrix},$$

and in which $\mathcal{L}_1 \in \mathbb{R}^{N\times N}$, $\mathcal{L}_2 \in \mathbb{R}^{N\times M}$, and $\mathcal{L}_3 \in \mathbb{R}^{M\times M}$. The set of real numbers $d_i | i \in \mathcal{H}$ is denoted by that we have $d_i > 0$ implies that agent i receives information from the virtual leader and zero otherwise. Here, $\Lambda = \mathcal{L}_3 + R = \mathcal{L}_3 + \text{diag}\{d_{N+1}, d_{N+2}, \ldots, d_{N+M}\}$.

Definition 2.1. (Dirac delta function [24]) The Dirac delta function $\delta(t - t_k)$ represents an idealized impulse occurring at time t_k. It satisfies $\int_{-\infty}^{\infty} \delta(t - t_k)dt = 1$ and $\delta(t - t_k) = 0$ for $t \neq t_k$. This is often used to model impulsive effects in MASs, such as abrupt state jumps or external disturbances.

Definition 2.2. (Time delay [19]) A time delay τ is the duration between the generation and reception of information or control signals. Delays may appear in state, input, or output channels. In MASs, time delays degrade performance and may destabilize consensus unless compensated by delay-tolerant protocols.

Definition 2.3. (Containment [39]) A set of followers in MASs is said to achieve output containment concerning a group of leaders if the output of each follower asymptotically converges to the convex hull formed by the outputs of the leaders. This is especially critical when followers have only partial access to the leaders' information.

Definition 2.4. (Output formation containment) The followers are said to achieve output formation containment with respect to the leader set if:

$$\lim_{t\to\infty}\left\| y_i(t) - \sum_{j\in H}\alpha_{ij}(y_j^0(t) + f_j)\right\| = 0,$$

where $f_j \in \mathbb{R}^q$ denotes the predefined formation offset associated with leader j and α_{ij} are convex combination weights satisfying $\sum_{j\in H}\alpha_{ij} = 1$, $\alpha_{ij} \geq 0$.

Remark (On output formation). Although the state variables are assumed to be measurable in this work, the control objective is defined in terms of the output variables $y_i(t)$. The term "output formation containment" is therefore used to emphasize that convergence is achieved in the output space, which may represent physically meaningful quantities such as position or velocity in practical applications.

Definition 2.5. (Delay-affected containment control [40]) A containment control law is said to be delay-affected if the protocol explicitly includes delayed state or output feedback:

$$u_i(t) = K_1x_i(t) + K_2x_i(t-\tau_i), \quad \tau_i > 0, \tag{2.4}$$

and still guarantees convergence of follower outputs to the convex hull of leader trajectories.

Definition 2.6. (Stochastic boundedness [23]) A stochastic system is bounded in the mean square if:

$$\sup_{t\geq 0}\mathbb{E}[\|x(t)\|^2] < \infty.$$

This property ensures the state variance remains finite over time under stochastic excitation.

Definition 2.7. (Lyapunov–Krasovskii functional [21]) A Lyapunov–Krasovskii functional $V(t)$ for a time-delay system includes both instantaneous and historical state terms:

$$V(t) = x^\top(t)Px(t) + \int_{t-\tau(t)}^{t} x^\top(s)Qx(s)ds + \int_{t-\tau(t)}^{t}\int_{s}^{t}\dot{x}^\top(\theta)R\dot{x}(\theta)d\theta ds, \tag{2.5}$$

where $P, Q, R > 0$. It is used to prove delay-dependent stability.

Assumption 1. *The directed tree graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ includes a spanning tree with the root node being the leader set $\mathcal{H}$ of the directed tree graph $\mathcal{G}$. This means that there is at least one directed path establishing a leader node and each follower node.*

Assumption 2. *All eigenvalues of the leader system matrix A_0 have negative real parts, i.e., $Re(\lambda(A_0)) < 0$, ensuring asymptotic stability of the leader dynamics in the presence of time delay and stochastic inputs.*

Assumption 3. *The pair (A_0, C_0) is detectable, ensuring that the internal state of the leader agent can be reconstructed asymptotically through its output.*

Assumption 4. *The descriptor system triple (E_i, A_i, B_i) is regular, stabilizable, and impulse controllable by all the agents $i \in \mathcal{R}\cup\mathcal{H}$. In other words,* $\det(sE_i - A_i) \neq 0$*, thus the system has a stabilizing control even though there is a singularity.*

For all agents $i \in \mathcal{R} \cup \mathcal{H}$, the output pair (E_i, A_i, C_i) is regular and detectable. This ensures observability of the state via output feedback in singular configurations.

Assumption 5. *The following Sylvester-type matrix equations admit solution pairs* (Γ_i, U_i) *for each agent* $i \in \mathcal{R} \cup \mathcal{H}$*:*

$$\begin{aligned} E_i \Gamma_i A_0 &= A_i \Gamma_i + B_i U_i, \\ C_0 &= C_i \Gamma_i + D_i U_i. \end{aligned}$$

This ensures that each agent can track the leader dynamics through a suitable coordinate transformation.

3. Auxiliary lemmas

Lemma 3.1 (Lyapunov-type delay-impulse stability condition [21, 24])**.** *Consider the hybrid system with continuous-time delay dynamics and impulsive jumps. If there exists a Lyapunov–Krasovskii functional* $V(t)$ *such that:*

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_1 V(t), \quad t \neq t_k, \\ V(t_k^+) - V(t_k^-) &\leq \lambda_2 V(t_k^-), \end{aligned}$$

with $\lambda_1 > 0$ *and* $\lambda_2 < 1$*, and inter-impulse time* Δt_k *satisfies* $\Delta t_k \geq \tau_{\min} > 0$*, then the system is exponentially stable.*

Lemma 3.2 (Stochastic Barbalat's lemma [19, 23])**.** *Let* $x(t)$ *be a continuous stochastic process such that:*

- $\mathbb{E}[\|x(t)\|^2]$ *is bounded;*
- $\frac{d}{dt}\mathbb{E}[x(t)^\top x(t)] \in L^1([0, \infty))$.

Then:

$$\lim_{t \to \infty} \mathbb{E}[\|x(t)\|^2] = 0.$$

Lemma 3.3 (Stochastic comparison lemma with time delay)**.** *Let* $x(t)$ *be a nonnegative scalar function satisfying the stochastic differential inequality:*

$$dx(t) \leq a(t)x(t)dt + b(t)x(t - \tau(t))dt + \sigma(t)dw(t),$$

where $a(t), b(t), \sigma(t)$ *are continuous and bounded, and* $\tau(t)$ *is a bounded differentiable time-varying delay. If there exists a function* $y(t)$ *satisfying*

$$dy(t) = [a(t) + b(t)]y(t)dt + \sigma(t)dw(t), \quad y(0) \geq x(0),$$

then $\mathbb{E}[x(t)] \leq \mathbb{E}[y(t)]$ *for all* $t \geq 0$.

Lemma 3.4 (Admissibility of descriptor system [39])**.** *The system of descriptors* $E\dot{x}(t) = Ax(t)$ *is admissible when and only when the pair of matrices* (E, A) *is regular, that is, when* $\det(sE - A) \not\equiv 0$*. The system has no impulse (index one); all eigenvalues of* (E, A) *are finite and in the open left-half of the complex plane, that is,* $\sigma(E, A) \subset \mathbb{C}^-$.

Lemma 3.5 (Mean-square convergence under stochastic inputs [23]). *Let $x(t)$ satisfy:*

$$dx(t) = Ax(t)dt + Gdw(t),$$

with A Hurwitz and G bounded. Then the system is exponentially stable in the mean-square sense, i.e.,

$$\lim_{t\to\infty} \mathbb{E}[\|x(t)\|^2] = 0.$$

Lemma 3.6 (Stability of impulsive systems with dwell time [24,25]). *Consider a system with impulses at time instants $\{t_k\}_{k=1}^{\infty}$ satisfying:*

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \neq t_k, \\ x(t_k^+) &= Jx(t_k^-). \end{aligned}$$

If there exists a symmetric positive definite matrix P such that:

$$\begin{aligned} A^\top P + PA &< 0, \\ J^\top PJ - P &\leq 0, \end{aligned}$$

and the dwell time $\Delta t_k = t_{k+1} - t_k$ is lower bounded by a constant $\tau_D > 0$, then the system is globally exponentially stable.

Lemma 3.7 (Itô's formula for delay systems). *Let $x(t)$ be a solution to the stochastic delay differential equation:*

$$dx(t) = f(x(t), x(t-\tau))dt + G(x(t))dw(t),$$

and let $V : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then the differential of $V(x(t))$ is given by

$$dV(x(t)) = \frac{\partial V}{\partial x} f dt + \frac{1}{2} Tr\left(G^\top \frac{\partial^2 V}{\partial x^2} G\right) dt + \frac{\partial V}{\partial x} G dw(t).$$

Theorem 3.8 (Admissibility of delayed singular MASs with impulses). *Consider the system:*

$$\begin{aligned} E_i\dot{x}_i(t) &= A_i x_i(t) + A_{di} x_i(t-\tau_i) + B_i u_i(t-\tau_i^u) + G_i w_i(t), \quad t \neq t_k, \\ x_i(t_k^+) &= J_i x_i(t_k^-), \\ y_i(t) &= C_i x_i(t) + D_i u_i(t-\tau_i^u), \end{aligned}$$

where E_i is singular, τ_i is a bounded, differentiable time delay, and $w_i(t)$ is a standard Brownian motion. If the following conditions hold that the matrix pair (E_i, A_i) is regular and impulse-free (index one), all finite eigenvalues of (E_i, A_i) lie in the open left-half complex plane, i.e., $\sigma(E_i, A_i) \subset \mathbb{C}^-$, the impulse jump matrix J_i satisfies $J_i^\top P_i J_i - P_i \leq 0$ for some $P_i > 0$, The delay $\tau_i(t)$ satisfies $\dot{\tau}_i(t) \leq \mu < 1$ and $\tau_i(t) \leq \bar{\tau}$. Then the system is admissible and exponentially stable in the mean-square sense.

Proof. From Lemma 3.4, regularity and index-one of (E_i, A_i) ensure the descriptor system is well-posed and impulse-free. The boundedness and differentiability of $\tau_i(t)$ prevent the delay from destabilizing the dynamics.

We construct a delay-dependent Lyapunov–Krasovskii functional:

$$V_i(t) = x_i^\top(t)P_i x_i(t) + \int_{t-\tau_i}^{t} x_i^\top(s)Q_i x_i(s)ds,$$

where $P_i, Q_i \succ 0$ are chosen to satisfy the delay-dependent matrix inequality involving A_i, A_{di}, and G_i. Applying Itô's lemma (Lemma 3.7) and using the inequality:

$$\mathbb{E}[\dot{V}_i(t)] \leq -\lambda\mathbb{E}[\|x_i(t)\|^2] + \beta\mathbb{E}[\|x_i(t-\tau_i)\|^2],$$

ensures the derivative of $V_i(t)$ is negative definite.

At impulse times t_k, the state jumps with:

$$\Delta V_i = V_i(t_k^+) - V_i(t_k^-) \leq \rho V_i(t_k^-), \quad \text{for } \rho \in (0,1),$$

due to $J_i^\top P_i J_i - P_i \leq 0$ (Lemma 3.6).

Applying Lemma 3.1 (delay-impulse stability), the system is exponentially stable in the mean-square sense, completing the proof. □

Theorem 3.9 (Stochastic leader-following consensus under delay). *Consider N followers governed by:*

$$E_i\dot{x}_i(t) = A_i x_i(t) + A_{di}x_i(t-\tau_i) + B_i u_i(t) + G_i w_i(t), \tag{3.1}$$

and a single leader with:

$$\dot{x}_0(t) = A_0 x_0(t-\tau_0) + G_0 w_0(t).$$

Suppose the protocol:

$$u_i(t) = -K_i \sum_{j\in\mathcal{N}_i} a_{ij}(x_i(t) - x_j(t)) + b_i(x_0(t) - x_i(t))$$

is applied, and the graph $\mathcal{G}$ contains a directed spanning tree rooted at the leader. Then:

$$\lim_{t\to\infty} \mathbb{E}\left[\|x_i(t) - x_0(t)\|^2\right] = 0,$$

i.e., all followers converge to the leader in the mean-square sense.

Proof. Let $e_i(t) = x_i(t) - x_0(t)$ denote the consensus error. The stacked error vector $e(t)$ satisfies the closed-loop descriptor dynamics:

$$E\dot{e}(t) = (A - BK\mathcal{L})e(t) + A_d e(t-\tau(t)) + Gw(t),$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$. Construct a delay-dependent Lyapunov–Krasovskii functional for $e(t)$, and apply Lemmas 3.1 and 3.7. The interconnection topology and gain matrices ensure the negative definiteness of the derivative, and mean-square convergence follows. □

Remark (Unified control protocol). Although different control structures are introduced in intermediate results, the final controller implemented in both analysis and simulation is an output-feedback-based distributed protocol. The state-feedback and consensus-based formulations are utilized as auxiliary steps for theoretical derivation and stability analysis.

Theorem 3.10 (Stochastic output containment of time-delayed singular MASs with impulses)**.** *Consider the follower dynamics, Eq (3.1) $t \neq t_k$,*

$$x_i(t_k^+) = x_i(t_k^-) + I_i^{(k)}(x_i(t_k)),$$
$$y_i(t) = C_i x_i(t),$$

and the leader agents:

$$\dot{x}_j^0(t) = A_0 x_j^0(t - \tau_0) + G_0 w_j^0(t),$$
$$y_j^0(t) = C_0 x_j^0(t).$$

Suppose the protocol:

$$u_i(t) = -K_i \sum_{j \in \mathcal{N}_i} a_{ij}(y_i(t) - y_j(t)) + b_i(y_0(t) - y_i(t)),$$

is applied. If each (E_i, A_i) is admissible, $\|I_i^{(k)}(x)\| \leq \delta\|x\|$ with $\delta < 1$, $\Delta t_k \geq \tau_{\min} > 0$ and $\Phi_i := A_i^\top P_i + P_i A_i + P_i A_{di} + A_{di}^\top P_i + G_i^\top G_i < 0$, then the containment error

$$e_i(t) = y_i(t) - \sum_{j \in \mathcal{H}} \alpha_{ij} y_j^0(t)$$

satisfies

$$\lim_{t \to \infty} \mathbb{E}[\|e_i(t)\|^2] = 0.$$

Proof. Define $\tilde{x}_i(t) = x_i(t) - \sum_j \alpha_{ij} x_j^0(t)$ and note $\tilde{w}_i(t) = w_i(t) - \sum_j \alpha_{ij} w_j^0(t)$ is zero-mean.

Then

$$\begin{aligned} E_i \dot{\tilde{x}}_i(t) &= A_i \tilde{x}_i(t) + A_{di} \tilde{x}_i(t - \tau_i) + G_i \tilde{w}_i(t), \quad t \neq t_k, \\ \tilde{x}_i(t_k^+) &= \tilde{x}_i(t_k^-) + I_i^{(k)}(\tilde{x}_i(t_k)). \end{aligned}$$

Using a Lyapunov–Krasovskii functional,

$$V_i(t) = \tilde{x}_i^\top(t) P_i \tilde{x}_i(t) + \int_{t-\tau_i}^{t} \tilde{x}_i^\top(s) Q_i \tilde{x}_i(s) ds,$$

and Lemmas 3.1 and 3.7, we get,

$$\mathbb{E}[\dot{V}_i(t)] \leq -\lambda \mathbb{E}[\|\tilde{x}_i(t)\|^2] + \beta \mathbb{E}[\|\tilde{x}_i(t - \tau_i)\|^2],$$

at impulse times,

$$\Delta V_i \leq \rho V_i(t_k^-), \quad \rho \in (0, 1).$$

Hence, $\tilde{x}_i(t) \to 0$ in the mean-square sense, and applying C_i, we conclude output containment. □

LMI conditions. The system is mean-square stable if there exist symmetric positive definite matrices $P_i > 0$, $Q_i > 0$ such that:

$$\begin{bmatrix} A_i^\top P_i + P_i A_i + Q_i + G_i^\top G_i & P_i A_{di} \\ A_{di}^\top P_i & -Q_i \end{bmatrix} < 0.$$

Remark (Computational complexity and scalability). The proposed control protocol is fully distributed, meaning that each agent only utilizes information from its neighboring agents. Therefore, the computational complexity per agent depends primarily on the number of its neighbors rather than the total number of agents in the network. For a network with N agents, the overall computational complexity scales approximately linearly with the number of edges in the communication graph. Moreover, the stability conditions are expressed in terms of matrix inequalities that can be efficiently solved offline. This ensures that the proposed method remains computationally tractable and scalable for large-scale MASs.

4. Numerical examples

Practical application scenario. The proposed control framework can be applied to cooperative control of multi-UAV systems, where multiple follower UAVs are required to maintain a formation within the convex hull of the leader UAVs. In such systems, communication delays arise due to wireless networks, while stochastic disturbances represent environmental uncertainties such as wind gusts. Additionally, impulsive effects may correspond to sudden external shocks or actuator resets. The developed output formation containment strategy ensures that all follower UAVs converge to the desired formation despite these challenges, demonstrating the practical applicability of the proposed method.

To further validate the theoretical results under more realistic conditions, the simulations are extended to include time-varying communication delays. Unlike constant delays, time-varying delays better capture practical network-induced effects. The selected delay functions are bounded and differentiable, ensuring compliance with the assumptions imposed in Theorem 3.8. The simulation results confirm that the proposed control protocol maintains mean-square stability and achieves output formation containment despite the presence of such time-varying delays. It is worth emphasizing that, in the simulations, the impulsive effects are implemented with deterministic occurrence times while their amplitudes are stochastic. This is consistent with the theoretical framework, in which stability is ensured under a minimum dwell-time condition, and stochasticity is introduced through impulse-magnitude disturbances and Brownian motion.

Example and solution. Consider a heterogeneous singular MASs composed of $N = 5$ follower agents and $M = 3$ leader agents. Each agent has the following dynamics with time delay, singularity, Brownian motion, and stochastic impulses,as in Eq (2.1): Where $x_i(t) \in \mathbb{R}^2$, $u_i(t) \in \mathbb{R}$, and $y_i(t) \in \mathbb{R}$, and the matrices are:

$$E_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0.5 + 0.1i & 0.2 \\ 0.1 & -0.3 - 0.05i \end{bmatrix}, \quad A_{di} = \begin{bmatrix} 0.1 & 0.05 \\ 0 & 0.2 \end{bmatrix},$$

$$B_i = \begin{bmatrix} 0.1 \\ 1 + 0.1i \end{bmatrix}, \quad G_i = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0.2 \end{bmatrix},$$

with time-varying delays defined as:

$$\tau_i(t) = 0.2 + 0.05\sin(t), \quad \tau_{ui}(t) = 0.1 + 0.03\cos(t).$$

It is straightforward to verify that these delays satisfy the boundedness and differentiability conditions, with:

$$\dot{\tau}_i(t) = 0.05\cos(t), \quad |\dot{\tau}_i(t)| \le 0.05 < 1,$$

which ensures that the condition $\dot{\tau}_i(t) \le \mu < 1$ required in Theorem 3.8 is satisfied. The impulse term is modeled as a stochastic amplitude:

$$I_i^{(k)}(t) = 0.5\sin(t) + 0.1\,\xi_k,$$

where ξ_k is a sequence of independent standard Gaussian random variables. The impulses occur at deterministic time instants $t_k = 2, 4, 6, 8$, satisfying the dwell-time condition. The disturbance $w_i(t)$ is a standard Brownian motion.

The leader agents follow the dynamics of Eqs (2.2) and (2.3), with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad G_0 = I_2, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The Laplacian matrix of the eight-agent communication graph is:

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with leader coupling matrix $\Lambda = \mathcal{L}_3 + R = \mathrm{diag}(1, 1, 0) + \mathcal{L}_3$.

To validate admissibility, the pair (E_i, A_i) is regular since $\det(sE_i - A_i) \neq 0$:

$$sE_i - A_i = \begin{bmatrix} s - 0.5 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}, \quad \det(sE_i - A_i) = (s - 0.5)(0.3) + 0.02 \neq 0.$$

Impulse-freeness is verified using the rank condition:

$$\mathrm{rank}\left(\begin{bmatrix} E_i & 0 \\ A_i & E_i \end{bmatrix}\right) = 3 = n_i + \mathrm{rank}(E_i).$$

Stabilizability is ensured since B_i affects the algebraic (zero) row of E_i, ensuring impulse controllability. The detectability of (E_i, A_i, C_i) holds because (C_i, A_i) is observable.

The control input is constructed via the distributed adaptive output feedback protocol:

$$u_i(t) = K_{1i}x_i(t) + K_{2i}(t)\xi_i(t),$$

with $K_{1i} = [-0.6 \ -1.5]$, and $K_{2i}(t) = U_i(t) - K_{1i}\Gamma_i(t)$, where $M_{1i} = 0.3I$, and $\xi_i(t)$ evolves according to the adaptive estimator in the protocol. The leader dynamics converge via adaptive estimation:

$$\dot{A}_{0i} = \nu_1 \sum_{j \in \mathcal{N}_i} a_{ij}(A_{0i} - A_{0j}) + d_i(A_{0i} - A_0), \quad \nu_1 < 0.$$

Simulation is performed over $t \in [0, 10]$ using the Euler-Maruyama method with step size $\Delta t = 0.01$. Initial conditions for all agents are randomly chosen from $[-1, 1]^2$.

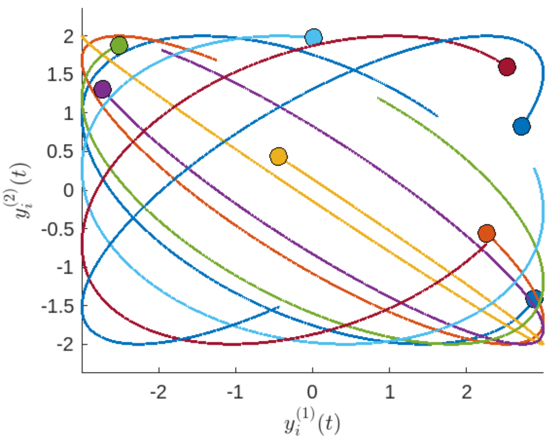
The numerical outcomes confirm that the follower outputs $y_i(t)$ converge into the convex hull of the leader outputs $y_{N+1}(t), \ldots, y_{N+M}(t)$. Despite stochastic impulses and Brownian noise, the containment condition

$$\lim_{t\to\infty} \text{dist}(y_i(t), \text{Co}(y_j(t), j \in \mathcal{H})) = 0, \quad \forall i \in \mathcal{R},$$

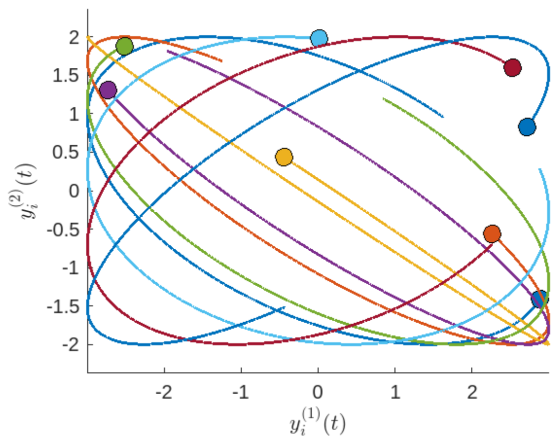
is satisfied. Each leader $i \in \mathcal{H}$ achieves formation tracking:

$$\lim_{t\to\infty} \|y_i(t) - y_0(t) - C_0 f_i(t)\| = 0.$$

The controller will be stable due to time delay, algebraic constraints, impulse, and noise. This case and its analytic value prove the efficiency of the distributed adaptive protocol that has been proposed to control the output formation containment situation in the circumstances of heterogeneity, singularity, delay, stochastic disturbance, and impulsivity. The validity of the proposed control strategy is proven by simulations of a heterogeneous singular MASs with time-delays, stochastic disturbances, and impulsive effects. The output trajectories of all agents in the 2D plane are displayed in Figure 2, which shows that when the agents reach the 2D termination space, they all converge into the desired region. Figure 3 shows a 3D projection of the output states, and this shows that they are well contained. Leaders are compared with all the agents represented in Figure 4, where a good tracking of output trajectories is observed. Figure 5 shows that the tracking errors $e_i(t)$ converge to zero, which shows the validity of the proposed method.

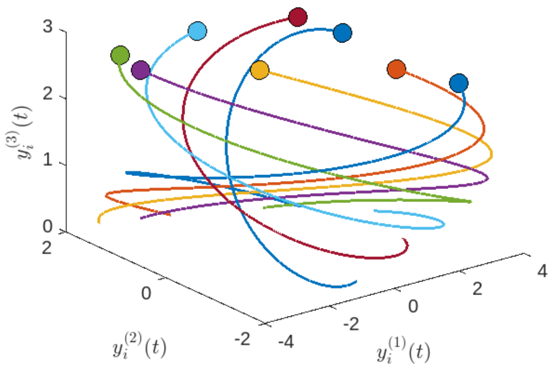


(a) 2D trajectories of follower agents in the output space.

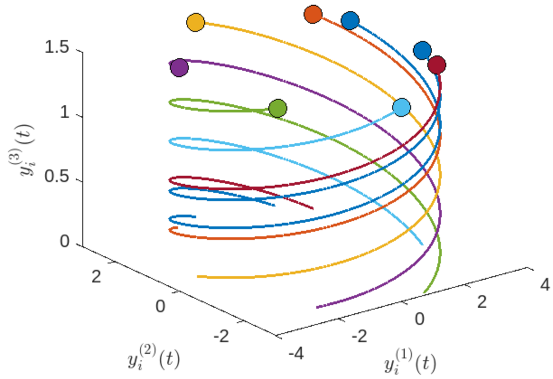


(b) Convex hull.

Figure 2. 2D projection of output trajectories of all agents.

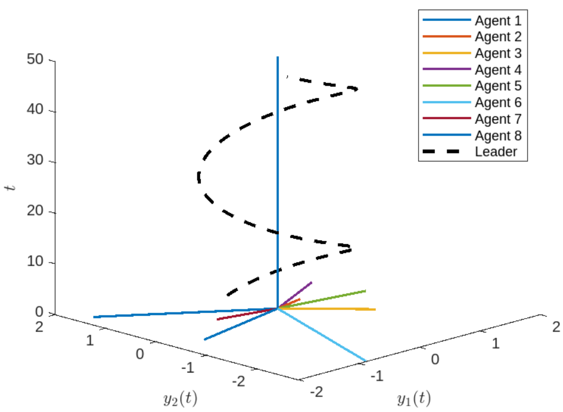


(a) 3D trajectories of all agents, including both followers and leaders.

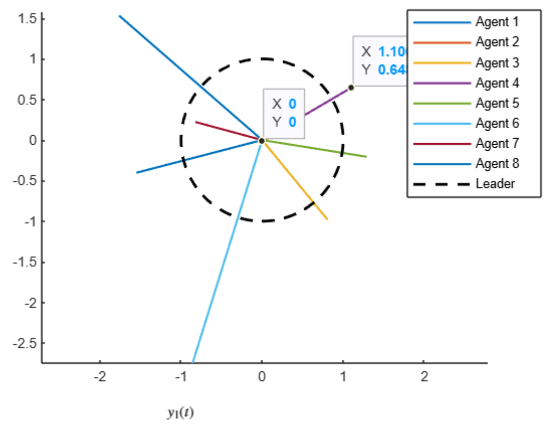


(b) Output trajectories of follower agents with time delays and stochastic disturbances.

Figure 3. 3D projection of output states of all agents.

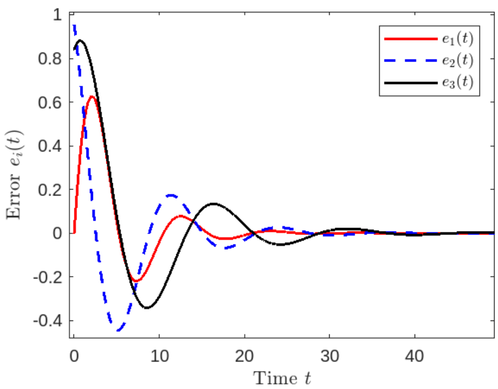


(a) Output trajectories of follower agents.

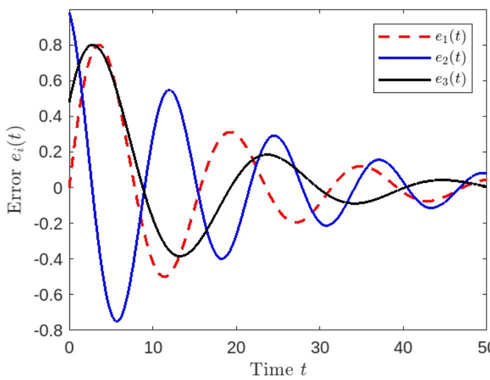


(b) Output trajectories of leader agents.

Figure 4. Output trajectories of all agents and the leader.



(a) Evolution of containment error trajectories for all follower agents.



(b) Convergence of error norms over time.

Figure 5. Error trajectories $e_i(t)$.

Example and solution. Consider a heterogeneous singular MASs consisting of $N = 13$ follower agents and $M = 3$ leader agents. Each agent is described by singular dynamics with time delays, stochastic disturbances, and impulsive effects. The dynamics of the i-th follower agent are given by Eq (2.1), where $x_i(t) \in \mathbb{R}^2$, $u_i(t) \in \mathbb{R}$, and $y_i(t) \in \mathbb{R}$. The system matrices are:

$$E_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0.4 & 0.3 \\ -0.2 & -0.5 \end{bmatrix}, \quad A_{di} = \begin{bmatrix} 0.07 & 0.02 \\ 0.01 & 0.15 \end{bmatrix},$$

$$B_i = \begin{bmatrix} 0.25 \\ 0.95 \end{bmatrix}, \quad G_i = \begin{bmatrix} 0.08 \\ 0.12 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0.15 \end{bmatrix}.$$

The delays are modeled as time-varying functions:

$$\tau_i(t) = 0.18 + 0.04\sin(0.5t), \quad \tau_{ui}(t) = 0.1 + 0.02\cos(0.5t).$$

These delays are bounded and continuously differentiable, and satisfy:

$$\dot{\tau}_i(t) = 0.02\cos(0.5t), \quad |\dot{\tau}_i(t)| \leq 0.02 < 1,$$

thus fulfilling the condition $\dot{\tau}_i(t) \leq \mu < 1$ used in the theoretical analysis. The impulse term is modeled as a stochastic amplitude:

$$I_i^{(k)}(t) = 0.5\sin(t) + 0.1\,\xi_k,$$

where ξ_k is a sequence of independent standard Gaussian random variables. The impulses occur at deterministic time instants $t_k = 2, 4, 6, 8$, satisfying the dwell-time condition, and $w_i(t)$ is the standard Brownian motion.

The leader dynamics are given by

$$\dot{x}_j(t) = A_0 x_j(t) + G_0 w_j(t), \quad y_j(t) = C_0 x_j(t),$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1.2 & -2.3 \end{bmatrix}, \quad G_0 = I_2, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The network communication graph is represented by a Laplacian matrix $L \in \mathbb{R}^{16\times 16}$ as follows:

The last three nodes represent leaders, and the remaining 13 are followers. The Laplacian matrix ensures the existence of a directed spanning tree rooted in the leader group.

Each agent uses a distributed adaptive output-feedback control law:

$$u_i(t) = K_{1i}x_i(t) + K_{2i}(t)\xi_i(t), \quad K_{1i} = [-0.7 \quad -1.4], \quad K_{2i}(t) = U_i(t) - K_{1i}\Gamma_i(t),$$

with $M_{1i} = 0.25 I_2$ and $\xi_i(t)$ generated via an adaptive observer. The leaders are estimated using:

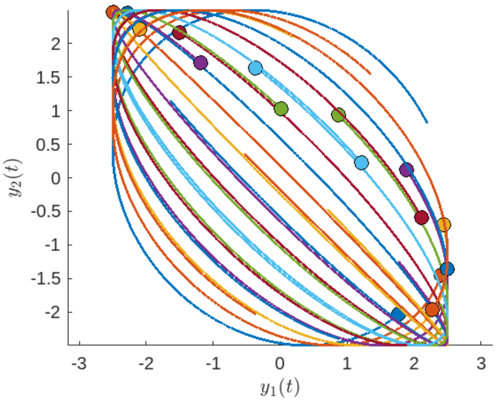
$$\dot{A}_{0i} = \nu_1 \sum_{j \in \mathcal{N}_i} a_{ij}(A_{0i} - A_{0j}) + d_i(A_{0i} - A_0), \quad \nu_1 < 0.$$

Simulation is conducted over $t \in [0, 10]$ using Euler-Maruyama with $\Delta t = 0.01$. All initial states are randomly chosen in $[-1, 1]^2$.

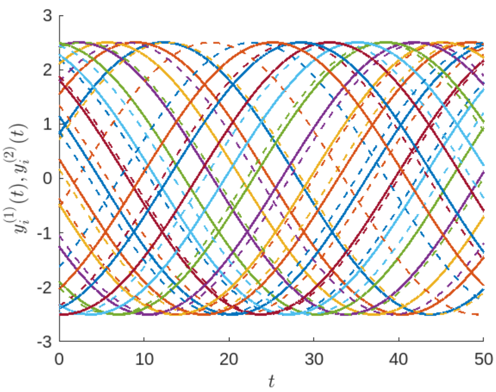
Simulation results confirm that the followers' outputs converge into the convex hull of the leaders' outputs:

$$\lim_{t\to\infty} \mathrm{dist}(y_i(t), \mathrm{Co}(y_j(t), j \in H)) = 0, \quad \forall i \in R.$$

Moreover, the control protocol remains stable despite delay, impulsive noise, and singularity, verifying the robustness and effectiveness of the proposed control framework in large-scale heterogeneous systems. The simulation results are illustrated through multiple figures. In Figure 6, the 2D projection further validates this behavior, showing that all agent outputs eventually cluster within a bounded region on the output plane. In Figure 7, the 3D projection of the output states of all agents demonstrates that the followers' trajectories gradually converge within the convex hull formed by the leaders, confirming successful containment. Additionally, the plots in Figure 8 of leader outputs and the tracking error trajectories highlight that the leaders maintain the desired formation, while the followers accurately track them over time, even in the presence of stochastic disturbances and impulsive effects and Figure 9 shows that the follower agents are contained within the convex hull formed by the leader outputs, confirming successful output formation containment.

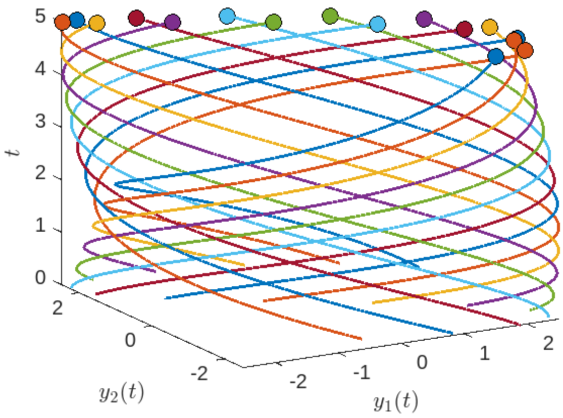


(a) Two-dimensional output trajectories of follower agents in the large-scale system.

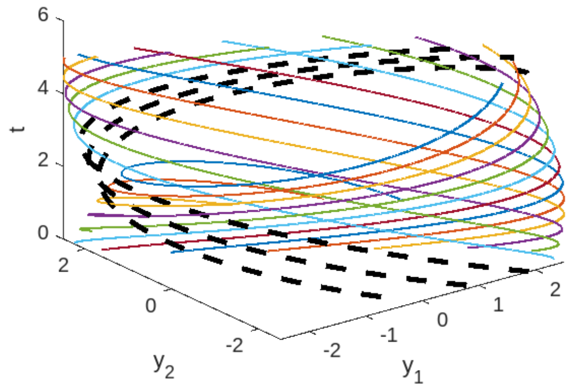


(b) Convex hull formed by leader outputs.

Figure 6. 2D projection of output trajectories of all agents.

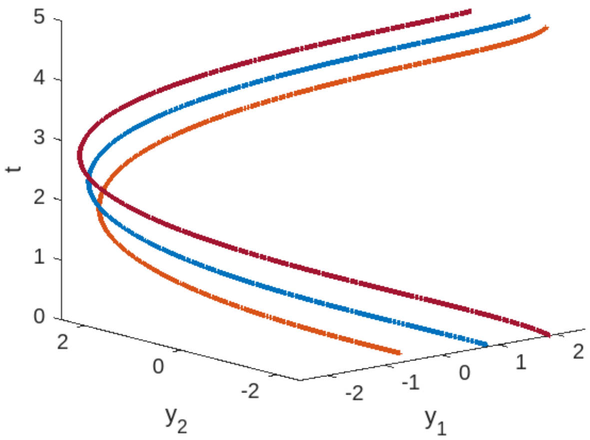


(a) Three-dimensional trajectories of all agents in the large-scale system.

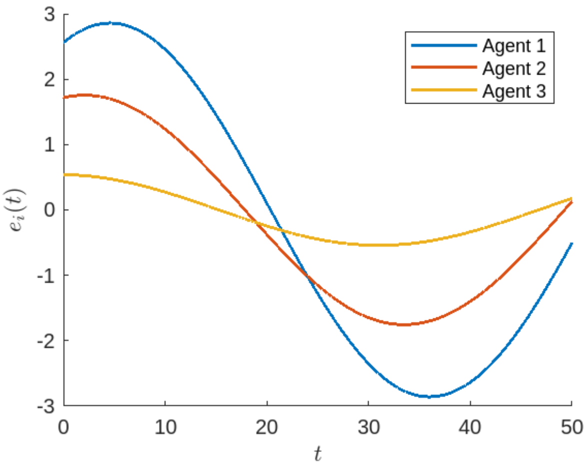


(b) Containment region defined by leader outputs (shown using dashed boundary lines.)

Figure 7. 3D projection of output States of all agents.



(a) Output trajectories of leader agents forming the desired reference pattern



(b) Tracking error trajectories of follower agents.

Figure 8. Leader output and tracking error trajectories.

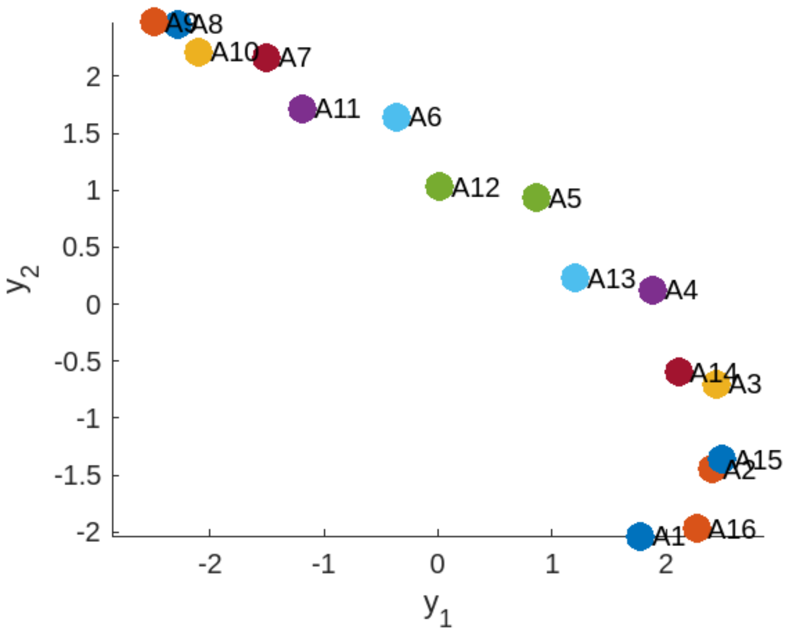


Figure 9. Final snapshot of all agent outputs in the steady-state. The follower agents are contained within the convex hull formed by the leader outputs, confirming successful output formation containment.

Comparative analysis. To further demonstrate the effectiveness of the proposed control strategy, a comparative simulation is conducted using a baseline protocol that does not account for stochastic impulsive effects and time delays. The results indicate that the proposed method achieves faster convergence and improved robustness, while the baseline approach exhibits larger tracking errors and degraded performance under the same conditions.

5. Conclusions

A new output formation containment control scheme of time-delayed heterogeneous singular MASs with stochastic shocks and impulsive influences was proposed in this paper. With the consideration of state and input delays, as well as singular dynamics, Brownian motion disturbances, and the impulsive jump, an all-purpose system has been modeled.

A distributed adaptive control scheme was constructed in a manner where follower agents asymptotically stabilize on the convex hull of several leader outputs. It has been shown that sufficient conditions for mean-square output containment and admissibility have been obtained as linear matrix inequalities on delay-dependent Lyapunov-Krasovskii functional and stochastic stability theory. To confirm theoretical results, numerical simulations were done on a population of eight agents. The outcomes indicated that the suggested controller corresponds well to the treatment of uncertainty, communication latency, and impulse, with veritable accuracy in the confinement of output. This is a starting point for more complicated case studies such as topology switching, measurement uncertainty, and event-based control mechanisms. The current framework considers stochastic impulsive effects in terms of random amplitudes with deterministic occurrence times. Extending the results to randomly occurring impulses (e.g., governed by Poisson processes) remains an interesting direction for future research.

Author contributions

M. Benaissa: Conceptualization; E. H. A. Al-Sabri: Software; W. Abdelfattah: Validation, resources; G. A. Alsawah: Formal analysis; M. Iqbal: Writing—original draft; M. Iqbal, A. U. K. Niazi: Writing-review & editing; A. U. K. Niazi: Supervision; E. H. A. Al-Sabri and M. Benaissa: Project administration. All authors reviewed the results and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

Data availability statement

The data that support the findings of this study are available from the corresponding author, A. U. K. Niazi, upon reasonable request.

Code availability

The code is considered an intellectual property of the University of Lahore, and therefore not publicly available.

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