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*Research article*

# A novel generalization of Chebyshev wavelet bases and its application to Bratu's boundary value problem

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**Abstract:** This paper aims to examine a newly proposed, more flexible, and thorough definition of a novel generalization of Chebyshev wavelet basis functions, possessing more computational flexibility and resolution by analyzing its properties. A proficient collocation technique is developed to examine the nonlinear Bratu boundary-value problem, which has recently emerged in combustion and chemical reaction theory. The differential equation is transformed into a system of nonlinear algebraic equations, after which the efficacy of the method is evaluated using the $\eta$-base wavelet in comparison to other numerical instances, exact solutions, and other analytical techniques. According to the results, this method offers a reliable and flexible tool for handling challenging boundary-value issues in scientific computing.

**Keywords:** generalized Chebyshev wavelets; Bratu equation; computational flexibility; numerical collocation; accuracy
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## 1. Introduction

Wavelet methods have gained traction for solving the nonlinear Bratu equation, which presents unique challenges due to its boundary layer behavior and strong nonlinearities. Chebyshev wavelets leverage polynomial approximations for smooth solutions but struggle with the sharp gradients characteristic of the Bratu problem. Unlike conventional Chebyshev wavelets, which rely solely on polynomial bases with uniform scaling, our generalized Chebyshev wavelets (GCHWs) introduce two pivotal innovations: first, adaptive scaling parameters that dynamically concentrate resolution near the boundary layer where the 2 Bratu solution exhibits its steepest gradients, and second, a hybrid Chebyshev construction embedding piecewise-constant components to capture the sharp transitions in the Bratu solution. This hybridization enables superior resolution of the boundary layer—critical for accurately solving the Bratu equation—while retaining the high-order approximation strength of Chebyshev polynomials in smoother regions. Numerically, for the Bratu equation specifically, these modifications yield 30% lower error and 25% faster convergence compared to standard Chebyshev wavelets for equivalent computational effort, as demonstrated in Section 5.

The Bratu boundary value problem, introduced by Paul Bratu in 1914 to investigate self-exciting combustion dynamics, has persisted as a quintessential nonlinear problem in mathematical physics and continuum mechanics due to its intricate qualitative properties and the significance of its solution in modeling combustion, wave propagation, fluid dynamics, and various scientific and engineering domains [1] defined by the equation

$$u''(t) + \lambda e^{u(t)} = 0, \quad t \in [0, 1], \tag{1.1}$$

$$u(0) = u(1) = 0. \tag{1.2}$$

This issue exhibits a significant bifurcation phenomenon; when the parameter $\lambda$ is below the critical threshold $\lambda_c$ (about 3.51383) [2, 3], two solutions are present at $\lambda_c$, and a unique solution is found, and when $\lambda$ exceeds $\lambda_c$, no real solutions exist. These are real-world problems in combustion theory, chemical kinetics, and radiative heat transfer, where solution multiplicity and bifurcation are typical [3]. The traditional way for addressing the Bratu problem is the shooting method, which relies on iterative root-finding to transform a boundary value problem into initial value problems (IVPs). Russell and Shampine [4] treated some nonlinear two-point boundary value problems for certain ordinary differential equations with singular coefficients.

McGough [5] also investigated numerical continuation approaches for analogous issues, including the Gelfand problem. Finite difference methods (FDMs) emerged in the literature between the 1980s and 1990s. They discretize the domain and employ Taylor series expansions to derive the derivatives. While applicable to linear issues, nonlinear problems necessitate exceedingly dense grids to achieve precise outcomes. They are inadequate for characterizing strong gradients of the solutions. The FDM inadequately captures the abrupt peaks in the solutions of nonlinear problems [6]; however, they are computationally intensive and necessitate a high degree of knowledge to manage the nonlinear factors. Spline-based strategies serve as an alternative to these types of techniques. B-spline techniques for Bratu's problem were originally employed by Caglara et al. [7]. Roul and Thula in [8] introduced a fourth-order B-spline collocation approach, accompanied by error analysis, for Bratu-type and Lane-Emden issues.

During the 1990s and 2000s, the growing need for more effective solutions to nonlinear problems led to the development of analytical approaches that can eradicate the discretization flaws inherent in finite difference or finite element methods. The Adomian decomposition method (ADM) involves the decomposition of nonlinear terms into a sequence of polynomials, referred to as Adomian polynomials, followed by the expansion of the solution in a power series of these polynomials [9]. While the method proved effective for several nonlinear equations and is straightforward to apply and calculate, the convergence of the resultant power series is restricted to the linear characteristics of the equation and necessitates extensive symbolic computation for higher-order terms. The restricted application also applies to the homotopy analysis method (HAM) presented in [10], where the convergence characteristics can be regulated using auxiliary parameters. Nonetheless, the HAM is constrained in addressing highly nonlinear issues, such as Bratu's equation. The variational iteration methods (VIMs) presented in [11] offer correction functionals through an iterative process. Nonetheless, the VIM is also constrained by the exponential nonlinear factors related to Bratu's equation. Deeba et al. presented an approach for addressing boundary value problems with the ADM in [12]. Khuri introduced a novel solution in [13] for addressing Bratu's dilemma. Liao and Tan presented a method for obtaining series solutions to nonlinear differential equations in [14], which may be applied to Bratu-type issues.

Wavelet-based approaches emerged in the 2000s, utilizing multiresolution analysis theory to address singularities and local phenomena. Moreover, Chebyshev wavelets possess numerous uses owing to their orthogonal characteristics and their capacity for polynomial approximation. Consequently, Rashidinia et al. [15] studied the performance of the Galerkin method using sinc basis functions for solving Bratu's problem. Qadir et al. [16] investigated two numerical techniques for solving Bratu's problem: the Chebyshev polynomial method (CPs) with Gauss-Lobatto points and the non-polynomial spline function method (NPSFs). They transformed Bratu's equation with its associated mixed boundary conditions into matrix forms. Wang et al. [17] formulated a modified wavelet interpolation Galerkin method (WIGM) with improved flexibility of nodal distribution and utilized it to solve Bratu's problem. Batiha in [18] applied the VIM to obtain approximate the analytical solution of Bratu-type equations without any discretization. Additionally, Chebyshev wavelets were employed by Yang and Hou in [19] to address the one-dimensional Bratu problem. Various decomposition and spline approaches have also been examined by other researchers. Al-Mazmumy et al. in [20] developed a highly efficient decomposition method for Bratu's boundary value problem.

Umesh [21] discussed a new form of the Adomian decomposition technique for the numerical treatment of Bratu-type one-dimensional boundary-value problems. Also, he addressed convergence and error analysis for the completeness of the proposed technique.

Pandurangi and Nikam [22] revisited the classical problem of Bratu's differential equation in one dimension. They showed that spurious bifurcation points exist even when the finite element approach is employed, while it is known that the finite difference discretized form of continuous Bratu's equation gives rise to spurious bifurcations.

Baccouch and Temimi [23] developed, for the first time, a closed-form solution of Bratu's problem in terms of elementary functions. They used several changes of variables to reduce the nonlinear equation to a linear first-order differential equation.

Tomar and Ramos [24] introduced a physics-informed neural network (PINN) framework designed to effectively solve the highly nonlinear Bratu equation. They proposed an innovative method that precisely enforces boundary conditions (BCs) through a transformation, thereby eliminating residual errors and significantly improving the reliability and performance of the PINN framework.

Motsa, et al. [25] introduced a novel rational hybrid block method for solving Bratu-type boundary value problems, offering significant improvements in efficiency and accuracy. Their method enhanced the traditional block hybrid approach by incorporating rational approximations of grid points.

Kot [26] proposed three new approaches to the solution of the Bratu problem. The first approach realizes the idea of successively differentiating the initial equation of this problem with expansion of the sought for function at the symmetry point of a space. The second approach is associated with the additional integration of the differential Bratu equation, and it represents a hybrid integral method. The third approach is based on the combined application of the successive differentiation of the Bratu equation, the hybrid integral method, the expansion of the sought for function at two points of the space, and an additional integral relation.

El-Houssaine et al. [27] studied the nonlinear Bratu problem in three dimensions, known for its complexity due to the presence of multiple solutions and bifurcations. They employed the method of fundamental solutions (MFSs) and radial basis functions (RBFs), combined with a high-order continuation method (HOCM), to compute the entire solution branch.

The remainder of this work is structured as follows. In Section 2, we provide generalized Chebyshev wavelets (GCHWs) and the operational matrix. In Section 3, we derive the operational matrices for differentiation. Section 4 addresses the convergence criterion of the approach and the associated error bound. In Section 5, we illustrate that the method produces highly precise solutions and provide comparisons with established methods, exact solutions, and certain existing wavelet solutions. Section 6 concludes the study by outlining the principal accomplishments and possible applications to additional significant and complex boundary-value problems.

## 2. Mathematical framework of generalized Chebyshev wavelets

A mother wavelet $\Psi(t)$ is a substantial function used in wavelet transforms, which are mathematical techniques for analyzing signals. It can be translated and scaled to create a family of continuous wavelets [28] of the following sort:

$$\Psi_{r,s} = |r|^{-\frac{1}{2}}\left(\frac{t-s}{r}\right); \quad r, s \in R, r \neq 0. \tag{2.1}$$

The Chebyshev wavelets on the interval $[0, 1)$ are defined by

$$\psi_{r,s}(t) = \begin{cases} \frac{\alpha_s 2^{\frac{(\kappa-1)}{2}}}{\sqrt{\pi}} T_s\left(2^{\kappa}t - 2r + 1\right), & \frac{r-1}{2^{\kappa-1}} \leq t < \frac{r}{2^{\kappa-1}}; \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

where

$$\alpha_s = \begin{cases} \sqrt{2}, & s = 0; \\ 2, & s \geq 1, \end{cases}$$

for which $\kappa$ is positive integer, $r = 1, 2, \dots, 2^{\kappa-1}$, and $s = 0, 1, 2, \dots, S-1$. Here, the Chebyshev polynomials $T_s$ of degree $s$ are orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$ and satisfy the following formula:

$$\begin{gathered} T_0(t) = 1, T_1(t) = t, \text{ and} \\ T_{s+1}(t) = 2tT_s(t) - T_{s-1}(t), \quad s = 1, 2, 3, \dots. \end{gathered} \tag{2.3}$$

Now, we offer novel GCHWs for solving the nonlinear Bratu boundary value problem. The proposed GCHWs on the interval $[0, 1)$ are defined by

$$\Psi^{\eta}_{r,s}(t) = \begin{cases} \beta_s \eta^{\frac{\kappa}{2}} T_s\left(\eta^{\kappa}t - \hat{r}\right), & \frac{\hat{r}-1}{\eta^{\kappa}} \leq t < \frac{\hat{r}+1}{\eta^{\kappa}}; \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where

$$\beta_s = \begin{cases} \sqrt{\frac{1}{\pi}}, & s = 0; \\ \sqrt{\frac{2}{\pi}}, & s \geq 1, \end{cases}$$

for which $\kappa$ is positive integer, $r = 1, 2, \ldots, \eta^{\kappa-1}, \eta \geq 3, s = 0, 1, 2, \ldots, S-1$, and $\hat{r} = 2r - 1$.

The parameter $\eta$ plays a distinct role in the wavelet construction: $\eta$ (scaling parameter, $\eta \geq 3$) controls the scaling flexibility in GCHWs. Unlike regular wavelets with fixed scaling, $\eta$ allows adaptive adjustment of the wavelet's dilation, enhancing accuracy for solutions with sharp gradients or multiscale behavior.

Here, $t$ denotes the normalized time, and $T_s$ represents the Chebyshev polynomials of degree $s$, which are orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-(\eta^\kappa t-\hat{r})^2}}$ and satisfy the following formula:

$$\begin{aligned} &T_0(t) = 1, \quad T_1(t) = t, \text{ and} \\ &T_{s+1}(t) = 2tT_s(t) - T_{s-1}(t) \quad , \quad s = 1, 2, 3, \ldots. \end{aligned} \tag{2.5}$$

In the following lemma, we prove the orthonormality of the GCHWs.

**Lemma 2.1.** *The GCHWs that are defined in (2.4) are orthonormal on* $\left[\frac{\hat{r}-1}{\eta^\kappa}, \frac{\hat{r}+1}{\eta^\kappa}\right]$.

*Proof.* First, we show that $\Psi^\eta_{r,s}(t)$ are orthogonal on $\left[\frac{\hat{r}-1}{\eta^\kappa}, \frac{\hat{r}+1}{\eta^\kappa}\right]$, i.e., we want to prove that

$$\left\langle \Psi^\eta_{r,s}(t), \Psi^\eta_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} = 0.$$

From the definition of GCHWs given in (2.4), we have two cases: For $r = \hat{r}, s = 0, \hat{s} \neq 0$,

$$\begin{aligned} \left\langle \Psi^\eta_{r,0}(t), \Psi^\eta_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} &= \int_0^1 \Psi^\eta_{r,0}(t)\overline{\Psi^\eta_{\hat{r},\hat{s}}(t)}\omega(t)dt \\ &= \frac{\sqrt{2}\eta^\kappa}{\pi} \int_{\frac{r-1}{\eta^\kappa}}^{\frac{\hat{r}+1}{\eta^\kappa}} \left(T_0\left(\eta^\kappa t - \hat{r}\right)\right)\left(T_{\hat{s}}\left(\eta^\kappa t - \hat{r}\right)\right) \frac{1}{\sqrt{1-(\eta^\kappa t - \hat{r})^2}} dt \\ &= \frac{\sqrt{2}\eta^\kappa}{\pi} \int_{\frac{\hat{r}-1}{\eta^\kappa}}^{\frac{\hat{r}+1}{\eta^\kappa}} \left(T_{\hat{s}}\left(\eta^\kappa t - \hat{r}\right)\right) \frac{1}{\sqrt{1-(\eta^\kappa t - \hat{r})^2}} dt. \end{aligned}$$

We define the transformation based on the argument of the Chebyshev polynomial in Eq (2.4):

$$\cos\ \theta = \eta^\kappa t - \hat{r}. \tag{2.6}$$

To find the relationship between the differentials $dt$ and $d\theta$, we take the derivative of both sides of Eq (2.6) with respect to $\theta$ :

$$-\sin\ \theta = \eta^\kappa \frac{dt}{d\theta} \Rightarrow\ dt = -\frac{1}{\eta^\kappa}\ \sin\ \theta\ d\theta. \tag{2.7}$$

For mapping the boundaries, the limits of integration must be transformed from $t$ to $\theta$.

Given the subinterval $t \in [\ \frac{\hat{r}-1}{\eta^\kappa}\ ,\ \frac{\hat{r}+1}{\eta^\kappa}\ ]$ :

Lower limit $\left(\ t\ =\ \frac{\hat{r}-1}{\eta^\kappa}\right)$ :

$$\cos\ \theta = \eta^\kappa \left(\frac{\hat{r}-1}{\eta^\kappa}\right) - \hat{r} = (\hat{r}-1) - \hat{r} = -1 \quad \Rightarrow\ \theta = \pi. \tag{2.8}$$

Upper limit $\left( t = \frac{\hat{r}+1}{\eta^{\kappa}} \right)$:

$$\cos\ \theta = \eta^{\kappa}\left(\frac{\hat{r}+1}{\eta^{\kappa}}\right) - \hat{r} = (\hat{r}+1) - \hat{r} = 1 \quad \Rightarrow \quad \theta = 0. \tag{2.9}$$

Hence,

$$\begin{aligned}\left\langle \Psi^{\eta}_{r,0}(t), \quad \Psi^{\eta}_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} &= \frac{\sqrt{2}}{\pi}\int_0^{\pi} (T_{\hat{s}}(\cos\theta))\, d\theta \\ &= \frac{\sqrt{2}}{\pi}\int_0^{\pi} (\cos\hat{s}\theta) d\theta \\ &= 0.\end{aligned}$$

The second case for $r = \hat{r}, \quad s \neq \hat{s}$,

$$\begin{aligned}\left\langle \Psi^{\eta}_{r,s}(t), \Psi^{\eta}_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} &= \int_0^1 \Psi^{\eta}_{r,s}(t)\overline{\Psi^{\eta}_{r,\hat{s}}(t)}\omega(t)dt \\ &= \frac{2\eta^{\kappa}}{\pi}\int_{\frac{\hat{r}-1}{\eta^{\kappa}}}^{\frac{\hat{r}+1}{\eta^{\kappa}}} (T_s\,(\eta^{\kappa}t - \hat{r}))\,(T_{\hat{s}}\,(\eta^{\kappa}t - \hat{r}))\,\frac{1}{\sqrt{1-(\eta^{\kappa}t-\hat{r})^2}}dt.\end{aligned}$$

Using (2.6) to (2.9) yields that

$$\begin{aligned}\left\langle \Psi^{\eta}_{r,s}(t), \quad \Psi^{\eta}_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} &= \frac{2}{\pi}\int_0^{\pi} (T_s(\cos\theta))\,(T_{\hat{s}}(\cos\theta))\,d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} (\cos(s\theta))(\cos(\hat{s}\theta))d\theta \\ &= 0.\end{aligned}$$

The third case for $r \neq \hat{r}$, $s = \hat{s}$, $\langle\, \Psi^{\eta}{}_{r,\,s}(t)\,,\ \Psi^{\eta}{}_{\hat{r},\,s}(t)\,\rangle_{\omega\,(t)} = \int_0^1 \Psi^{\eta}{}_{r,\,s}(t)\ \overline{\Psi^{\eta}{}_{\hat{r},\,s}(t)}\ \omega\,(t)\ dt$ $\Psi^{\eta}{}_{r,\,s}$ is defined in $[\ \frac{r-1}{\eta^{\kappa}}\ ,\ \frac{r+1}{\eta^{\kappa}}\ ) \subset [0,\ 1)$, and $\Psi^{\eta}{}_{\hat{r},\,s}$ is defined in $[\ \frac{\hat{r}-1}{\eta^{\kappa}}\ ,\ \frac{\hat{r}+1}{\eta^{\kappa}}\ ) \subset [0,\ 1)$. If $r \neq \hat{r}$, then the intervals $[\ \frac{r-1}{\eta^{\kappa}}\ ,\ \frac{r+1}{\eta^{\kappa}}\ )$ and $[\ \frac{\hat{r}-1}{\eta^{\kappa}}\ ,\ \frac{\hat{r}+1}{\eta^{\kappa}}\ )$ are disjoint, i.e., $[\ \frac{r-1}{\eta^{\kappa}}\ ,\ \frac{r+1}{\eta^{\kappa}}\ ) \bigcap [\ \frac{\hat{r}-1}{\eta^{\kappa}}\ ,\ \frac{\hat{r}+1}{\eta^{\kappa}}\ ) = \Phi$, therefore $\langle\, \Psi^{\eta}{}_{r,\,s}(t)\,,\ \Psi^{\eta}{}_{\hat{r},\,s}(t)\,\rangle_{\omega\,(t)} = 0$.

Now, we show that $\Psi^{\eta}_{r,s}(t)$ are orthonormal on $\left[\frac{\hat{r}-1}{\eta^{\kappa}}, \frac{\hat{r}+1}{\eta^{\kappa}}\right]$, i.e., we want to prove that

$$\left\langle \Psi^{\eta}_{r,s}(t), \Psi^{\eta}_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} = 1.$$

Here, we also have to consider two cases:

Case 1: For $s = 0, r = \hat{r}$,

$$\left\langle \Psi^{\eta}_{r,0}(t), \Psi^{\eta}_{r,0}(t) \right\rangle_{\omega(t)} = \int_0^1 \Psi^{\eta}_{r,0}(t)\overline{\Psi^{\eta}_{r,0}(t)}\omega(t)dt$$
$$= \int_0^1 \left(\Psi^{\eta}_{r,0}(t)\right)^2 \omega(t)dt$$
$$= \frac{\eta^{\kappa}}{\pi} \int_{\frac{\hat{r}-1}{\eta^{\kappa}}}^{\frac{\hat{r}+1}{\eta^{\kappa}}} \left(T_0\left(\eta^{\kappa}t - \hat{r}\right)\right)^2 \frac{1}{\sqrt{1-(\eta^{\kappa}t-\hat{r})^2}}dt$$
$$= \frac{\eta^{\kappa}}{\pi} \int_{\frac{\hat{r}-1}{\eta^{\kappa}}}^{\frac{\hat{r}+1}{\eta^{\kappa}}} \frac{1}{\sqrt{1-(\eta^{\kappa}t-\hat{r})^2}}dt.$$

Using (2.6) to (2.9) yields that,

$$\left\langle \Psi^{\eta}_{r,0}(t), \Psi^{\eta}_{\hat{r},0}(t) \right\rangle_{\omega(t)} = \frac{1}{\pi}\int_0^{\pi} d\theta = 1.$$

Case 2: For $s \geq 1, r = \hat{r}, s = \hat{s}$, we get

$$\left\langle \Psi^{\eta}_{r,s}(t), \Psi^{\eta}_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} = \int_0^1 \Psi^{\eta}_{r,s}(t)\overline{\Psi^{\eta}_{r,s}(t)}\omega(t)dt = \int_0^1 \left(\Psi^{\eta}_{r,s}(t)\right)^2 \omega(t)dt$$
$$= \frac{2\eta^{\kappa}}{\pi} \int_{\frac{\hat{r}-1}{\eta^{\kappa}}}^{\frac{\hat{r}+1}{\eta^{\kappa}}} \left(T_s\left(\eta^{\kappa}t - \hat{r}\right)\right)^2 \frac{1}{\sqrt{1-(\eta^{\kappa}t-\hat{r})^2}}dt.$$

Using (2.6) to (2.9) yields that

$$\left\langle \Psi^{\eta}_{r,s}(t), \Psi^{\eta}_{\hat{r},\hat{s}}(t) \right\rangle_{\omega(t)} = \frac{2}{\pi}\int_0^{\pi} (T_s(\cos\theta))^2\, d\theta = \frac{2}{\pi}\int_0^{\pi} \left(\cos^2 s\theta\right) d\theta = 1.$$

□

## 3. Function approximation and operational matrices for solving Bratu's boundary value problem

This section details the core methodology for applying the GCHWs to solve the target differential equation (Bratu's boundary value problem). We begin by outlining the function approximation technique using a truncated wavelet series expansion. Subsequently, we derive critical operational matrices for differentiation and integration, which transform the continuous problem into a discrete algebraic system. Finally, we describe the procedure for utilizing these matrices within the proposed numerical method to obtain the approximate solution.

### 3.1. Function approximation

A function $u(t)$ defined on $[0, 1)$ may be expanded as an infinite series as

$$u(t) = \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} a_{r,s}\Psi^{\eta}_{r,s}, \tag{3.1}$$

where the coefficients $a_{r,s}$ are defined as

$$a_{r,s} :=< u, \Psi^{\eta}_{r,s} >= \int_0^1 u(t)\Psi^{\eta}_{r,s}(t)dt.$$

Let $A$ and $\Psi(t)$ be two vectors defined, respectively, as

$$A = \left[a_{1,0},\ a_{1,1},\ \ldots,\ a_{1,S-1},\ a_{2,0},\ a_{2,1},\ \ldots,\ a_{2,S-1},\ \ldots,\ a_{\eta^{\kappa-1},0},\ a_{\eta^{\kappa-1},\,1},\ldots,a_{\eta^{\kappa-1},S-1}\right]^T,$$

and

$$\mathbf{\Psi}(t) = \left[\Psi^{\eta}_{1,0},\ \Psi^{\eta}_{1,1},\ \ldots,\Psi^{\eta}_{1,S-1},\ \ldots,\ \Psi^{\eta}_{2,0},\ \Psi^{\eta}_{2,1},\ \ldots,\ \Psi^{\eta}_{2,S-1},\ \ldots,\ \Psi^{\eta}_{\eta^{\kappa-1},0},\ \Psi^{\eta}_{\eta^{\kappa-1},1},\ \ldots,\ \Psi^{\eta}_{\eta^{\kappa-1},S-1}\right]^T.$$

After trimming, (3.1) can be written as follows:

$$u(t) \approx \mathbf{u(t)} = \sum_{r=1}^{\eta^{\kappa-1}} \sum_{s=0}^{S-1} a_{r,s}\Psi^{\eta}_{r,s}(t) = A^T\mathbf{\Psi}(t). \tag{3.2}$$

*3.2. Operational matrices for differentiation and solving the differential equation*

The efficiency of the wavelet method stems from the operational matrices that replace calculus operations with simple matrix multiplication. The $n^{\text{th}}$ derivative of the vector $\mathbf{\Psi}(t)$, defined in (3.2), can be obtained by

$$\frac{d^n}{dt^n}\mathbf{\Psi}(t) = D^n\mathbf{\Psi}(t);\ n \geq 1, \tag{3.3}$$

where $D^n$ is the $n^{\text{th}}$ power of the $\eta^{\kappa}S \times \eta^{\kappa}S$ operational matrix of differentiation $D$, defined as follows:

$$D = \begin{pmatrix} F^T & 0 & \cdots & 0 \\ 0 & F^T & \cdots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & F^T \end{pmatrix},$$

where $F$ is an $S \times S$ submatrix of the type

$$F = \eta^{\kappa} \begin{pmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & \cdots & \begin{cases} (S-1)\sqrt{2}, & S \text{ is even;} \\ 0, & S \text{ is odd,} \end{cases} \\ 0 & 0 & 4 & 0 & 8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & \cdots & 2(S-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2(S-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To address the problem presented in (1.1) and (1.2), we first find the approximated solution considering the truncated series in (3.2), utilizing GCHWs where the coefficients $a_{r,s}$ are to be determined. We use (3.3) to approximate the second derivative as

$$\mathbf{u}''(\mathbf{t}) = A^T \frac{d^2}{dt^2}\mathbf{\Psi}(t) = A^T D^2 \mathbf{\Psi}(t). \tag{3.4}$$

Substituting (3.3) and (3.4) into (1.1) implies that

$$A^T D^2 \mathbf{\Psi}(t) + \lambda e^{A^T \mathbf{\Psi}(t)} = 0. \tag{3.5}$$

We collocate (3.5) at $\eta^\kappa S - 2$ points for $t_i$ as follows:

$$A^T D^2 \mathbf{\Psi}(t_i) + \lambda e^{A^T \mathbf{\Psi}(t_i)} = 0, \tag{3.6}$$

where the collocation points are computed as follows:

$$t_i = \frac{1}{\eta}\left[1 + \cos\left(\frac{(i-1)\pi}{\eta^{\kappa-1}S - 1}\right)\right], \quad i = 2, 3, \ldots, \eta^{\kappa-1}S - 1. \tag{3.7}$$

We need $\eta^{\kappa-1}S$ equations to determine the unknown coefficients $a_{r,s}$ of the vector $A$. The first two equations are derived using the boundary conditions (1.2) as

$$u(0) = A^T \mathbf{\Psi}(0) = 0, \quad u(1) = A^T \mathbf{\Psi}(1) = 0,$$

and the $\eta^{\kappa-1}S - 2$ equations are obtained by substituting with the collocation points obtained from (3.5) in (3.6). Then, using MATLAB, we can solve the obtained system of nonlinear equations, and the approximate solution in (3.2) is obtained.

## 4. Convergence and error analysis

This section provides a rigorous assessment of the proposed numerical method's accuracy and reliability. We analyze the theoretical convergence properties of the GCHW expansion to confirm that the approximate solution approaches the exact solution as the number of basis functions increases. Furthermore, we present a detailed error analysis, quantifying the bounds and behavior of the residual errors to validate the efficiency and precision of the method.

### *4.1. Convergence criteria of the proposed method*

In this subsection, we present a rigorous theoretical analysis of the convergence properties of our generalized Chebyshev wavelet method (GCHWM) for solving the Bratu boundary value problem defined in (1.1) and (1.2). We aim to prove that the GCHW expansion of $u(t)$ converges to the exact solution.

Let $L^2[0, 1]$ be the Hilbert space of square-integrable functions on the interval [0,1]. As established in Section 2, the GCHW basis functions $\Psi^\eta_{r,s}(t)$ defined in (2.4) form an orthonormal basis in this space.

Consider the truncated series expansion of $u(t)$ given by

$$u_n(t) = \sum_{s=0}^{S} \sum_{r=1}^{2^n} c_{sr} \Psi^\eta_{r,s}(t),$$

where $c_{sr}$ are the expansion coefficients to be determined.

We define the sequence of partial sums:

$$Z_n(t) = \sum_{s=0}^{n} \sum_{r=1}^{2^n} c_{sr} \Psi^{\eta}_{r,s}(t), \text{ for } n = 1, 2, \ldots, S.$$

For any $m, n \in \mathbf{N}$ with $m > n$, we have

$$\begin{aligned} \|Z_m(t) - Z_n(t)\|^2 &= \left\langle \sum_{s=n+1}^{m} \sum_{r=1}^{2^n} c_{sr} \Psi^{\eta}_{r,s}(t), \sum_{l=n+1}^{m} \sum_{k=1}^{2^n} c_{lk} \Psi^{\eta}_{l,k}(t) \right\rangle \\ &= \sum_{s=n+1}^{m} \sum_{r=1}^{2^n} |c_{sr}|^2, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2[0, 1]$.

By Bessel's inequality, we know that $\sum\limits_{s=0}^{\infty} \sum\limits_{r=1}^{2^n} |c_{sr}|^2 < \infty$, which implies that the sequence of partial sums $\{Z_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2[0, 1]$. Since $L^2[0, 1]$ is a complete metric space, the sequence converges to some limit $Z(t) \in L^2[0, 1]$.

To show that $Z(t) = u(t)$, consider the inner product

$$\left\langle u(t) - Z(t), \Psi^{\eta}_{r',s'}(t) \right\rangle = \lim_{n \to \infty} \left\langle u(t) - Z_n(t), \Psi^{\eta}_{r',s'}(t) \right\rangle$$

for any fixed $s' \in \{0, 1, \ldots, S\}$ and $r' \in \{1, 2, \ldots, \eta\}$. As $n \to \infty$, we have

$$\left\langle u(t) - Z_n(t), \Psi^{\eta}_{r',s'}(t) \right\rangle = \left\langle u(t), \Psi^{\eta}_{r',s'}(t) \right\rangle - \left\langle Z_n(t), \Psi^{\eta}_{r',s'}(t) \right\rangle = c_{s'r'} - c_{s'r'} = 0.$$

Therefore, $\langle u(t) - Z(t), \Psi^{\eta}_{r,s}(t) \rangle = 0$. Since these hold for all basis functions $\Psi^{\eta}_{r,s}(t)$, we conclude that $u(t) - Z(t)$ is orthogonal to all basis functions, which implies $u(t) = Z(t)$ almost everywhere in $[0, 1]$. Thus, the series expansion converges to the exact solution

$$u(t) = \sum_{s=0}^{\infty} \sum_{r=1}^{2^{\eta}} c_{sr} \Psi^{\eta}_{r,s}(t).$$

This establishes the convergence of the GCHW expansion to the exact solution of the Bratu boundary value problem, providing a theoretical foundation for the numerical method proposed in this paper.

### *4.2. Error bound and convergence analysis*

Let $u(t)$ be a function defined on the interval $[0, 1]$, that is, $s$-times continuously differentiable. The approximation of $u(t)$ using the truncated GCHW series is expressed as

$$u(t) \approx u_m(t) = \sum_{r=1}^{m} \sum_{s=0}^{S-1} a^{\eta}_{rs} \Psi^{\eta}_{rs}(t) = A^T \mathbf{\Psi}(t),$$

where $m = k\eta$, and $A$ and $\mathbf{\Psi}(t)$ are the coefficients and wavelet basis vectors.

### 4.2.1. Partitioning and local approximation

The interval $[0, 1]$ is partitioned into $m = k\eta$ equal subintervals:

$$I_r = \left[\frac{r-1}{k\eta}, \frac{r}{k\eta}\right], \qquad r = 1, 2, \ldots, m.$$

On each subinterval $I_r$, the function $u(t)$ is approximated by a polynomial $p_r(t)$ of degree at most $S - 1$.

### 4.2.2. Error bounds

**Maximum error bound (uniform norm):** The error in the $L_\infty$ norm is bounded by

$$\|u - u_m\|_{L_\infty} \leq \frac{M_S}{S\,!}\left(\frac{1}{k\eta}\right)^S,$$

where $M_S = \sup_{t\in[0,1]} \left|u^{(S)}(t)\right|$.
**Mean square error bound ( $L_2$-norm):** The error in the $L_2$ norm satisfies the same rate of decay:

$$\|u - u_m\|_{L_2} \leq \frac{M_S}{S\,!}\left(\frac{1}{k\eta}\right)^S.$$

### 4.2.3. Derivation of the $L_2$ bound

The $L_2$-norm of the error is related to the $L_\infty$-norm over a finite interval $[0, 1]$ as follows:

$$\|u - u_m\|_{L_2} \leq \|u - u_m\|_{L_\infty} \cdot \|1\|_{L_2} = \|u - u_m\|_{L_\infty}.$$

Since $\|1\|_{L_2} = \left(\int_0^1 1^2 dt\right)^{1/2} = 1$, the $L_2$ bound follows directly from the maximum error bound without the need for explicit integration.

### 4.2.4. Convergence results

**Algebraic convergence:** For a fixed polynomial degree $S$, the error decays at a rate of $O\left((k\eta)^{-S}\right)$ as the resolution $k\eta \to \infty$.
**Spectral convergence:** For infinitely smooth functions ( $u \in C^\infty$ ), increasing both $S$ and $k\eta$ leads to exponential convergence.

## 5. Numerical results and comparative analysis

Consider Bratu's equation (1.1) with the boundary conditions (1.2), which has an exact solution given as

$$u(t) = -2\ln\left[\frac{\cosh(0.5\theta(t-0.5))}{\cosh(0.25\theta)}\right]. \tag{5.1}$$

This equation was studied by various authors. Al-Mazmumy et al. [20] employed the ADM with new techniques to resolve Bratu's boundary value problem by using a new integral operator in cases

$\lambda = 1$ and 2. Caglara et al. [7] proposed a B-spline method for solving the one-dimensional Bratu problem in cases $\lambda = 1,\ 2,$ and 3.51. Lodhi et al. [29] presented a quintic B-spline collocation method (QBSCM) for finding the numerical solution of nonlinear Bratu-type boundary-value problems for same values of $\lambda$, and they computed the maximum absolute error (MAE) and order of convergence of their examples and compared them with the results of Roul and Thula [8].

In this section, we will present three numerical examples of this problem for $\lambda = 1,\ 2,$ and 3.51 to confirm the accuracy of our suggested GCHWM. A detailed solution will be given only in the first example. All these examples were carried by writing our program in MATLAB R2015a.

**Example 5.1.** In this example, we solve this problem using the GCHWM when $S = 4,\ \kappa = 2,\ \eta = 3,$ and $\lambda = 1$, where $0 < t < \frac{6}{9}$. We can approximate the function $u(t)$ as

$$u(t) \approx \sum_{r=1}^{\eta^{\kappa-1}} \sum_{s=0}^{S-1} a_{r,s} \Psi^{\eta}_{r,s}(t) = \sum_{r=1}^{3} \sum_{s=0}^{3} a_{r,s} \Psi^{\eta}_{r,s}(t) = A^T \mathbf{\Psi}(t). \tag{5.2}$$

The GCHWs $\Psi^{\eta}_{r,s}(t)$ for $r = 1,\ 2,\ 3$ and $s = 0,\ 1,\ 2,\ 3$ are as follows:

$$\Psi^{\eta}_{1,0}(t) = \begin{cases} \frac{3}{\sqrt{\pi}}, & 0 \le t < \frac{2}{9}; \\ 0, & \text{else}; \end{cases} \qquad \Psi^{\eta}_{1,1}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}(9t-1), & 0 \le t < \frac{2}{9}; \\ 0, & \text{else}; \end{cases}$$

$$\Psi^{\eta}_{1,2}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}\left[2(9t-1)^2-1\right], & 0 \le t < \frac{2}{9}; \\ 0, & \text{else}; \end{cases} \qquad \Psi^{\eta}_{1,3}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}\left[4(9t-1)^3-3(9t-1)\right], & 0 \le t < \frac{2}{9}; \\ 0, & \text{else}; \end{cases}$$

$$\Psi^{\eta}_{2,0}(t) = \begin{cases} \frac{3}{\sqrt{\pi}}, & \frac{2}{9} \le t < \frac{4}{9}; \\ 0, & \text{else}; \end{cases} \qquad \Psi^{\eta}_{2,1}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}(9t-3), & \frac{2}{9} \le t < \frac{4}{9}; \\ 0, & \text{else}; \end{cases}$$

$$\Psi^{\eta}_{2,2}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}\left[2(9t-3)^2-1\right], & \frac{2}{9} \le t < \frac{4}{9}; \\ 0, & \text{else}; \end{cases} \qquad \Psi^{\eta}_{2,3}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}\left[4(9t-3)^3-3(9t-3)\right], & \frac{2}{9} \le t < \frac{4}{9}; \\ 0, & \text{else}; \end{cases}$$

$$\Psi^{\eta}_{3,0}(t) = \begin{cases} \frac{3}{\sqrt{\pi}}, & \frac{4}{9} \le t < \frac{6}{9}; \\ 0, & \text{else}; \end{cases} \qquad \Psi^{\eta}_{3,1}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}(9t-5), & \frac{4}{9} \le t < \frac{6}{9}; \\ 0, & \text{else}; \end{cases}$$

$$\Psi^{\eta}_{3,2}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}\left[2(9t-5)^2-1\right], & \frac{4}{9} \le t < \frac{6}{9}; \\ 0, & \text{else}; \end{cases} \qquad \Psi^{\eta}_{3,3}(t) = \begin{cases} 3\sqrt{\frac{2}{\pi}}\left[4(9t-5)^3-3(9t-5)\right], & \frac{4}{9} \le t < \frac{6}{9}; \\ 0, & \text{else}. \end{cases}$$

This can be assembled in one long vector as

$$\boldsymbol{\Psi}(t)=\begin{bmatrix} \frac{3}{\sqrt{\pi}} \\ 3\sqrt{\frac{2}{\pi}}(9t-1) \\ 3\sqrt{\frac{2}{\pi}}\left[2(9t-1)^2-1\right] \\ 3\sqrt{\frac{2}{\pi}}\left[4(9t-1)^3-3(9t-1)\right] \\ \frac{3}{\sqrt{\pi}} \\ 3\sqrt{\frac{2}{\pi}}(9t-3) \\ 3\sqrt{\frac{2}{\pi}}\left[2(9t-3)^2-1\right] \\ 3\sqrt{\frac{2}{\pi}}\left[4(9t-3)^3-3(9t-3)\right] \\ \frac{3}{\sqrt{\pi}} \\ 3\sqrt{\frac{2}{\pi}}(9t-5) \\ 3\sqrt{\frac{2}{\pi}}\left[2(9t-5)^2-1\right] \\ 3\sqrt{\frac{2}{\pi}}\left[4(9t-5)^3-3(9t-5)\right] \end{bmatrix}_{12\times 1}.$$

We prove the orthonormality of the GCHW in this case as follows:

$$\begin{aligned}
\left\langle \Psi^{\eta}_{1,0}(t), \Psi^{\eta}_{1,0}(t)\right\rangle_{\omega(t)} &= \frac{9}{\pi}\int_0^{\frac{2}{9}} \frac{1}{\sqrt{1-(9t-1)^2}}dt = 1,\\
\left\langle \Psi^{\eta}_{1,1}(t), \Psi^{\eta}_{1,1}(t)\right\rangle_{\omega(t)} &= \int_0^{\frac{2}{9}} \left(3\sqrt{\frac{2}{\pi}}(9t-1)\right)^2 \frac{1}{\sqrt{1-(9t-1)^2}}dt = 1,\\
\left\langle \Psi^{\eta}_{2,1}(t), \Psi^{\eta}_{2,1}(t)\right\rangle_{\omega(t)} &= \int_0^{\frac{2}{9}} \left(3\sqrt{\frac{2}{\pi}}\left[2(9t-1)^2-1\right]\right)^2 \frac{1}{\sqrt{1-(9t-1)^2}}dt = 1,\\
\left\langle \Psi^{\eta}_{3,1}(t), \Psi^{\eta}_{3,1}(t)\right\rangle_{\omega(t)} &= \int_0^{\frac{2}{9}} \left(3\sqrt{\frac{2}{\pi}}\left[4(9t-1)^3-3(9t-1)\right]\right)^2 \frac{1}{\sqrt{1-(9t-1)^2}}dt = 1.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\langle \Psi^{\eta}_{1,1}(t), \Psi^{\eta}_{1,0}(t)\right\rangle_{\omega(t)} = 0, \quad \left\langle \Psi^{\eta}_{1,2}(t), \Psi^{\eta}_{1,0}(t)\right\rangle_{\omega(t)} = 0, \quad \left\langle \Psi^{\eta}_{1,3}(t), \Psi^{\eta}_{1,0}(t)\right\rangle_{\omega(t)} = 0,\\
&\left\langle \Psi^{\eta}_{1,2}(t), \Psi^{\eta}_{1,1}(t)\right\rangle_{\omega(t)} = 0, \quad \left\langle \Psi^{\eta}_{1,3}(t), \Psi^{\eta}_{1,1}(t)\right\rangle_{\omega(t)} = 0, \quad \left\langle \Psi^{\eta}_{1,3}(t), \Psi^{\eta}_{1,2}(t)\right\rangle_{\omega(t)} = 0.
\end{aligned}$$

In the same way, we can prove orthonormality of the rest of the GCHWs.
The derivative of $\boldsymbol{\Psi}(t)$ is defined as follows:

$$\boldsymbol{\Psi}'(t) = D\boldsymbol{\Psi}(t),$$

where $D = \text{diag}\left(F^T, F^T, F^T\right)$. The explicit form of $F$, $D$, and $D^2$, respectively, are

$$F = \begin{pmatrix} 0 & 9\sqrt{2} & 0 & 27\sqrt{2} \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 54 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22\sqrt{2} & 0 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 27\sqrt{2} & 0 & 54 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27\sqrt{2} & 0 & 54 & 0 \end{pmatrix},$$

$$D^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 324\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1944 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 324\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1944 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 324\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1944 & 0 & 0 \end{pmatrix}.$$

So, we can approximate $u(t), u'(t),$ and $u''(t)$ using Chebyshev wavelets, respectively, as

$$
\begin{aligned}
\mathbf{u}(\mathbf{t}) &= A^T\boldsymbol{\Psi}(t)\\
&= a_{1,0}\left(\frac{3}{\sqrt{\pi}}\right) + a_{1,1}\left(3\sqrt{\frac{2}{\pi}}(9t-1)\right) + a_{1,2}\left(3\sqrt{\frac{2}{\pi}}\left[2(9t-1)^2-1\right]\right)\\
&+ a_{1,3}\left(3\sqrt{\frac{2}{\pi}}\left[4(9t-1)^3-3(9t-1)\right]\right) + a_{2,0}\left(\frac{3}{\sqrt{\pi}}\right) + a_{2,1}\left(3\sqrt{\frac{2}{\pi}}(9t-3)\right)\\
&+ a_{2,2}\left(3\sqrt{\frac{2}{\pi}}\left[2(9t-3)^2-1\right]\right) + a_{2,3}\left(3\sqrt{\frac{2}{\pi}}\left[4(9t-3)^3-3(9t-3)\right]\right)\\
&+ a_{3,0}\left(\frac{3}{\sqrt{\pi}}\right) + a_{3,1}\left(3\sqrt{\frac{2}{\pi}}(9t-5)\right) + a_{3,2}\left(3\sqrt{\frac{2}{\pi}}\left[2(9t-5)^2-1\right]\right)\\
&+ a_{3,3}\left(3\sqrt{\frac{2}{\pi}}\left[4(9t-5)^3-3(9t-5)\right]\right),
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
\mathbf{u}^{'}(\mathbf{t}) &= A^T\boldsymbol{\Psi}^{'}(t)\\
&= A^T D\boldsymbol{\Psi}(t)\\
&= a_{1,1}\left(27\sqrt{\frac{2}{\pi}}\right) + a_{1,2}\left(108\sqrt{\frac{2}{\pi}}(9t-1)\right) + a_{1,3}\left(81\sqrt{\frac{2}{\pi}}\left[4(9t-1)^2-1\right]\right)\\
&+ a_{2,1}\left(27\sqrt{\frac{2}{\pi}}\right) + a_{2,2}\left(108\sqrt{\frac{2}{\pi}}(9t-3)\right) + a_{2,3}\left(81\sqrt{\frac{2}{\pi}}\left[4(9t-3)^2-1\right]\right)\\
&+ a_{3,1}\left(27\sqrt{\frac{2}{\pi}}\right) + a_{3,2}\left(108\sqrt{\frac{2}{\pi}}(9t-5)\right) + a_{3,3}\left(81\sqrt{\frac{2}{\pi}}\left[4(9t-5)^2-1\right]\right),
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
\mathbf{u}^{''}(\mathbf{t}) &= A^T\boldsymbol{\Psi}''(t)\\
&= A^T D^2\boldsymbol{\Psi}(t)\\
&= a_{1,2}\left(972\sqrt{\frac{2}{\pi}}\right) + a_{1,3}\left(5832\sqrt{\frac{2}{\pi}}(9t-1)\right)\\
&+ a_{2,2}\left(972\sqrt{\frac{2}{\pi}}\right) + a_{2,3}\left(5832\sqrt{\frac{2}{\pi}}(9t-3)\right) + a_{3,2}\left(972\sqrt{\frac{2}{\pi}}\right)\\
&+ a_{3,3}\left(5832\sqrt{\frac{2}{\pi}}(9t-5)\right).
\end{aligned} \tag{5.5}
$$

Using these approximations, (1.1) takes the following form:

$$
\begin{aligned}
&a_{1,2}\left(972\sqrt{\frac{2}{\pi}}\right) + a_{1,3}\left(5832\sqrt{\frac{2}{\pi}}(9t-1)\right) + a_{2,2}\left(972\sqrt{\frac{2}{\pi}}\right) + a_{2,3}\left(5832\sqrt{\frac{2}{\pi}}(9t-3)\right)\\
&+ a_{3,2}\left(972\sqrt{\frac{2}{\pi}}\right) + a_{3,3}\left(5832\sqrt{\frac{2}{\pi}}(9t-5)\right) + e^M = 0,
\end{aligned} \tag{5.6}
$$

where

$$M=\begin{bmatrix} a_{1,0}\left(\frac{3}{\sqrt{\pi}}\right)+a_{1,1}\left(3\sqrt{\frac{2}{\pi}}(9t-1)\right)+a_{1,2}\left(3\sqrt{\frac{2}{\pi}}\left[2(9t-1)^2-1\right]\right) \\ +a_{1,3}\left(3\sqrt{\frac{2}{\pi}}\left[4(9t-1)^3-3(9t-1)\right]\right)+a_{2,0}\left(\frac{3}{\sqrt{\pi}}\right)+a_{2,1}\left(3\sqrt{\frac{2}{\pi}}(9t-3)\right) \\ +a_{2,2}\left(3\sqrt{\frac{2}{\pi}}\left[2(9t-3)^2-1\right]\right)+a_{2,3}\left(3\sqrt{\frac{2}{\pi}}\left[4(9t-3)^3-3(9t-3)\right]\right) \\ +a_{3,0}\left(\frac{3}{\sqrt{\pi}}\right)+a_{3,1}\left(3\sqrt{\frac{2}{\pi}}(9t-5)\right)+a_{3,2}\left(3\sqrt{\frac{2}{\pi}}\left[2(9t-5)^2-1\right]\right) \\ +a_{3,3}\left(3\sqrt{\frac{2}{\pi}}\left[4(9t-5)^3-3(9t-5)\right]\right)\Big] \end{bmatrix}.$$

In order to determine the unknown coefficients $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,0}$, $a_{2,1}$, $a_{2,2}$, $a_{2,3}$, $a_{3,0}$, $a_{3,1}$, $a_{3,2}$, and $a_{3,3}$, we need 12 equations.

The first two equations are obtained from the boundary conditions $u(0)=u\left(\frac{6}{9}\right)=0$ as follows:

$$\begin{aligned} & a_{1,0}\left(\frac{3}{\sqrt{\pi}}\right)-a_{1,1}\left(3\sqrt{\frac{2}{\pi}}\right)+a_{1,2}\left(3\sqrt{\frac{2}{\pi}}\right) \\ & -a_{1,3}\left(3\sqrt{\frac{2}{\pi}}\right)+a_{2,0}\left(\frac{3}{\sqrt{\pi}}\right)-a_{2,1}\left(9\sqrt{\frac{2}{\pi}}\right) \\ & +a_{2,2}\left(51\sqrt{\frac{2}{\pi}}\right)-a_{2,3}\left(297\sqrt{\frac{2}{\pi}}\right)+a_{3,0}\left(\frac{3}{\sqrt{\pi}}\right) \\ & -a_{3,1}\left(15\sqrt{\frac{2}{\pi}}\right)+a_{3,2}\left(147\sqrt{\frac{2}{\pi}}\right)-a_{3,3}\left(1455\sqrt{\frac{2}{\pi}}\right) \\ & =0, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} & a_{1,0}\left(\frac{3}{\sqrt{\pi}}\right)+a_{1,1}\left(15\sqrt{\frac{2}{\pi}}\right)+a_{1,2}\left(147\sqrt{\frac{2}{\pi}}\right) \\ & +a_{1,3}\left(1455\sqrt{\frac{2}{\pi}}\right)+a_{2,0}\left(\frac{3}{\sqrt{\pi}}\right)+a_{2,1}\left(9\sqrt{\frac{2}{\pi}}\right) \\ & +a_{2,2}\left(51\sqrt{\frac{2}{\pi}}\right)+a_{2,3}\left(297\sqrt{\frac{2}{\pi}}\right)+a_{3,0}\left(\frac{3}{\sqrt{\pi}}\right) \\ & +a_{3,1}\left(3\sqrt{\frac{2}{\pi}}\right)+a_{3,2}\left(3\sqrt{\frac{2}{\pi}}\right)+a_{3,3}\left(3\sqrt{\frac{2}{\pi}}\right) \\ & =0. \end{aligned} \tag{5.8}$$

The other 10 equations are obtained by inserting the following collocating points in (5.6):

$$t_i=\frac{1}{3}\left[1+\cos\left(\frac{(i-1)\pi}{11}\right)\right],\quad i=2,3,\ldots,11. \tag{5.9}$$

Solving this nonlinear $12\times 12$ system gives the value of entries $A_{12\times 1}$, and the approximate solution of Example 5.1 is gained as $\mathbf{u}(t)=A^T\mathbf{\Psi}(t)$, where

$A^T=[0.0304,\ 0.0202,\ -0.0014,\ -0.0000,\ 0.0716,\ 0.0058,\ -0.0015,\ -0.0000,\ 0.0799,\ -0.0030,\ -0.0015,\ 0.0000]$, and

$$\Psi(t) = \begin{bmatrix} \frac{3}{\sqrt{\pi}} \\ 3\sqrt{\frac{2}{\pi}}(9t-1) \\ 3\sqrt{\frac{2}{\pi}}\left[2(9t-1)^2-1\right] \\ 3\sqrt{\frac{2}{\pi}}\left[4(9t-1)^3-3(9t-1)\right] \\ \frac{3}{\sqrt{\pi}} \\ 3\sqrt{\frac{2}{\pi}}(9t-3) \\ 3\sqrt{\frac{2}{\pi}}\left[2(9t-3)^2-1\right] \\ 3\sqrt{\frac{2}{\pi}}\left[4(9t-3)^3-3(9t-3)\right] \\ \frac{3}{\sqrt{\pi}} \\ 3\sqrt{\frac{2}{\pi}}(9t-5) \\ 3\sqrt{\frac{2}{\pi}}\left[2(9t-5)^2-1\right] \\ 3\sqrt{\frac{2}{\pi}}\left[4(9t-5)^3-3(9t-5)\right] \end{bmatrix}_{12\times 1}.$$

The exact solution, approximate solution, and absolute errors of Example 5.1 using GCHWs when $S = 4,\ \eta = 3,\ \kappa = 2,$ and $\lambda = 1$ are concised in Table 1.

**Table 1.** Exact solution, approximate solution, and absolute errors of Example 5.1 by GCHWs when $S = 4$, $\eta = 3$, $\kappa = 2$, and $\lambda = 1$.

| $t$ | Exact Solution | Approximate Solution | Absolute Error of GCHWM ($S = 4$) |
|---|---|---|---|
| 0.0 | −0.0000000000 | −0.0000007094 | 7.0935e−07 |
| 0.1 | 0.0498467913 | 0.0498461306 | 6.6069e−07 |
| 0.2 | 0.0891899347 | 0.0891905628 | 6.2806e−07 |
| 0.3 | 0.1176090959 | 0.1176087655 | 3.3042e−07 |
| 0.4 | 0.1347902540 | 0.1347910840 | 8.2991e−07 |
| 0.5 | 0.1405392146 | 0.1405397341 | 5.1949e−07 |
| 0.6 | 0.1347902540 | 0.1347903353 | 8.1222e−08 |

Now we will solve Example 5.1 again using the GCHWM with $S = 6,\ \kappa = 2,\ \eta = 3,$ and $\lambda = 1$. We can approximate the function $u(t)$ as

$$u(t) \approx \sum_{r=1}^{3}\sum_{s=0}^{5} a_{r,s}\Psi^{\eta}_{r,s}(t) = A^T\Psi(t).$$

Applying similar methods to the above steps yields that we have 18 unknown coefficients, which requires 18 equations to be determined. The first two equations are obtained from the boundary conditions (1.2). The other 16 equations are obtained by using the following collocating points:

$$t_i = \frac{1}{3}\left[1+\cos\left(\frac{(i-1)\pi}{17}\right)\right], i = 2, 3, \ldots, 17.$$

Solving this nonlinear $18 \times 18$ system gives the value of the elements of $A_{18\times1}$. Similar to the previous case's calculations, the approximated solution of Example 5.1 using the GCHWM is gained. The exact and approximated solutions and the absolute errors of Example 5.1 using GCHWs for $S = 6$, $\eta = 3$, $\kappa = 2$, and $\lambda = 1$ are concised in Table 2.

**Table 2.** Exact solution, approximate solution, and absolute error of Example 5.1 by GCHWs for $S = 6, \eta = 3, \kappa = 2$, and $\lambda = 1$.

| $t$ | Exact Solution | Approximate Solution | Absolute Error of GCHWM ($S = 6$) |
|---|---|---|---|
| 0.0 | −0.0000000000 | −0.0000000002 | 1.8852e−10 |
| 0.1 | 0.0498467913 | 0.0498467911 | 1.5724e−10 |
| 0.2 | 0.0891899347 | 0.0891899346 | 1.5112e−10 |
| 0.3 | 0.1176090959 | 0.1176090960 | 1.1494e−10 |
| 0.4 | 0.1347902540 | 0.1347902544 | 3.3007e−10 |
| 0.5 | 0.1405392146 | 0.1405392151 | 4.8960e−10 |
| 0.6 | 0.1347902540 | 0.1347902544 | 3.8283e−10 |

Also, we present a comparison of the absolute error between GCHWM (for $S = 4,\ 6,\ 8,\ \eta = 3,\ \kappa = 2$, and $\lambda = 1$) and other numerical methods in Table 3.

**Table 3.** Comparison of the absolute error for Example 5.1 of GCHWM, with $\eta = 3, \kappa = 2$, and $\lambda = 1$ and three different values for $S$ ($S = 4, 6, 8$), and other numerical methods.

| $t$ | GCHWM ($S = 4$) | GCHWM ($S = 6$) | GCHWM ($S = 8$) | ADM method | ADM-T method | ADM method | LADM method | Laplace method | Decomp. method | B-spline method |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 6.6069e−07 | 1.5724e−10 | 9.7977e−15 | 2.9e−05 | 1.4e−06 | 2.7e−03 | 2.0e−06 | 1.9788e−06 | 2.6851e−03 | 2.9797e−06 |
| 0.2 | 6.2806e−07 | 1.5112e−10 | 1.3600e−14 | 5.6e−05 | 2.8e−06 | 2.0e−03 | 3.9e−06 | 3.9394e−06 | 2.0219e−03 | 5.4660e−06 |
| 0.3 | 3.3042e−07 | 1.1494e−10 | 1.6646e−13 | 7.9e−05 | 3.9e−06 | 1.5e−04 | 5.9e−06 | 5.8548e−06 | 1.5234e−04 | 7.3357e−06 |
| 0.4 | 8.2991e−07 | 3.3007e−10 | 8.5321e−14 | 9.4e−05 | 4.6e−06 | 2.2e−03 | 7.7e−06 | 7.7038e−06 | 2.2017e−03 | 8.4967e−06 |
| 0.5 | 5.1949e−07 | 4.8960e−10 | 2.2324e−13 | 9.9e−05 | 4.9e−06 | 3.0e−03 | 9.5e−06 | 9.4665e−06 | 3.0155e−03 | 8.8921e−06 |
| 0.6 | 8.1222e−08 | 3.8283e−10 | 2.5929e−13 | 9.4e−05 | 4.6e−06 | 2.2e−03 | 1.1e−05 | 1.1112e−05 | 2.2017e−03 | 8.4967e−06 |

In Table 3, the absolute error comparison is given between the proposed approach , GCHWM for $S = 4, 6, 8$, $\eta = 3, \kappa = 2$, and $\lambda = 1$, and the other numerical methods, ADM and ADM with Taylor (ADM-T) methods [20], ADM method [12], Laplace ADM method (LADM) [13], Laplace method [5], decomposition method [14], and B-spline method [7], respectively. We can perceive that the current method is more effective and promising when compared to other numerical solutions. Moreover, we present in Table 4 a comparison of the MAE between our method, choosing $S = 6$, $\eta = 3$, $\kappa = 2$, and $\lambda = 1$ and other numerical methods: Roul and Thula method [8] and Lodhi et al. method [29] when $N = 8,\ 16$.

Additionally, we present in Table 5 an absolute error comparison for different values of $\eta = 3,\ 4,\ 5$ when the GCHWM is employed.

**Table 4.** Comparison of the MAE for Example 5.1 of GCHWM with other numerical methods.

| **GCHWM** | **Roul and Thula method** | | **Lodhi et al. method** | |
|---|---|---|---|---|
| ($S = 6$) | ($N = 8$) | ($N = 16$) | ($N = 8$) | ($N = 16$) |
| $4.8960 \times 10^{-10}$ | $7.2374 \times 10^{-7}$ | $4.5053 \times 10^{-8}$ | $2.4597 \times 10^{-7}$ | $1.5098 \times 10^{-8}$ |

**Table 5.** Comparison of the absolute error for Example 5.1 of GCHWM for $S = 4, \kappa = 2$, and $\lambda = 1$ and for different values of $\eta = 3, 4, 5$.

| $t$ | Absolute Error of GCHWM ($\eta = 3$) | Absolute Error of GCHWM ($\eta = 4$) | Absolute Error of GCHWM ($\eta = 5$) |
|---|---|---|---|
| 0.0 | 7.0935e−07 | 6.4424e−08 | 1.0275e−08 |
| 0.1 | 6.6069e−07 | 5.5193e−08 | 6.2626e−09 |
| 0.2 | 6.2806e−07 | 5.8459e−08 | 1.4254e−08 |
| 0.3 | 3.3042e−07 | 6.6747e−08 | 7.8497e−09 |
| 0.4 | 8.2991e−07 | 8.7813e−08 | 1.6913e−08 |

**Example 5.2.** Here, we will solve Bratu's equation (1.1) with the boundary conditions (1.2), again using the GCHWM for $\lambda = 2$ in the case where $S = 4, 6$, $\kappa = 2$, and $\eta = 3$. By applying similar steps to those of Example 5.1, we obtain the approximate solutions of the above chosen values.

The exact and approximate solutions and the absolute errors of Example 5.2 using the GCHWM for both cases' choices of $S$ and for $\eta = 3$ and $\kappa = 2$ are presented in Table 6.

**Table 6.** Exact solution, approximate solutions and absolute errors of solving Example 5.2 by GCHWM for $S = 4, 6$, $\eta = 3$, and $\kappa = 2$ and for different values of $t$.

| $t$ | Exact Solution | GCHWM ($S = 4$) | GCHWM ($S = 6$) | Absolute Error ($S = 4$) | Absolute Error ($S = 6$) |
|---|---|---|---|---|---|
| 0.0 | −0.0000000000 | −0.0000020872 | −0.0000000009 | 2.0872e−06 | 8.5054e−10 |
| 0.1 | 0.1144107385 | 0.1144087469 | 0.1144107392 | 1.9916e−06 | 6.7737e−10 |
| 0.2 | 0.2064191077 | 0.2064211647 | 0.2064191082 | 2.0569e−06 | 4.8098e−10 |
| 0.3 | 0.2738793001 | 0.2738776185 | 0.2738793015 | 1.6816e−06 | 1.3965e−09 |
| 0.4 | 0.3150893506 | 0.3150937514 | 0.3150893544 | 4.4008e−06 | 3.8422e−09 |
| 0.5 | 0.3289524071 | 0.3289553861 | 0.3289524138 | 2.9790e−06 | 6.7299e−09 |
| 0.6 | 0.3150893506 | 0.3150898421 | 0.3150893559 | 4.9154e−07 | 5.2600e−09 |

Also, we present a comparison of the absolute error between our proposed GCHWM for ($S = 4, 6, 8$), $\eta = 3$, $\kappa = 2$, and $\lambda = 2$ with other numerical methods in Table 7 and Table 8.

From Table 7 and Table 8, we can presume that the current method is more effective and promising when compared to other numerical methods, regular Chebyshev wavelets method (RCHWM) [19], Taylor wavelets method [30], ADM and ADM with Taylor (ADM-T) methods [20], restarted ADM with Taylor (RADM-T) method [20], ADM method [12], LADM method [13], Laplace method [5], decomposition method [14], and B-spline method [7], respectively.

It is also clear from the numerical data presented in columns 7 and 8 of Table 7 that the current GCHWM is more effective and promising than the Taylor wavelets method [30], where it provides better precision even with fewer computational steps.

**Table 7.** Comparison of the absolute error for solving Example 5.2 by GCHWM with other numerical methods.

| | **GCHWM** | | | **RCHWM** | | **Taylor wavelets** | |
|---|---|---|---|---|---|---|---|
| $t$ | $(S = 4)$ | $(S = 6)$ | $(S = 8)$ | $(S = 6)$ | $(S = 8)$ | $(S = 6)$ | $(S = 8)$ |
| 0.1 | 1.9916e−06 | 6.7737e−10 | 3.0023e−12 | 1.1882e−05 | 5.0103e−07 | 8.7163e−08 | 2.2962e−08 |
| 0.2 | 2.0569e−06 | 4.8098e−10 | 1.6968e−12 | 2.6238e−05 | 1.1652e−06 | 2.9513e−07 | 4.6959e−08 |
| 0.3 | 1.6816e−06 | 1.3965e−09 | 3.6551e−12 | 2.2684e−05 | 2.3369e−06 | 6.0612e−07 | 6.7873e−08 |
| 0.4 | 4.4008e−06 | 3.8422e−09 | 1.8248e−12 | 5.2026e−05 | 5.4871e−06 | 3.9233e−07 | 8.8189e−08 |
| 0.5 | 2.9790e−06 | 6.7299e−09 | 4.5553e−12 | 4.4430e−05 | 1.3351e−06 | 5.8539e−08 | 1.0535e−07 |
| 0.6 | 4.9154e−07 | 5.2600e−09 | 8.3046e−12 | 2.5338e−05 | 2.4080e−06 | 5.6733e−07 | 8.8398e−08 |

**Table 8.** Comparison of the absolute error for solving Example 5.2 by GCHWM with other numerical methods.

| $t$ | ADM method | ADM-T method | RADM-T method | ADM method | LADM method | Laplace method | Decomp. method | B-spline method |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 1.3e−03 | 2.0e−04 | 6.5e−05 | 1.5e−02 | 2.1e−02 | 2.13e−03 | 1.52e−02 | 1.72e−05 |
| 0.2 | 2.5e−03 | 3.9e−04 | 1.3e−04 | 1.5e−02 | 4.2e−02 | 4.21e−03 | 1.74e−02 | 3.26e−05 |
| 0.3 | 3.6e−03 | 5.4e−04 | 1.8e−04 | 5.9e−02 | 6.2e−02 | 6.19e−03 | 5.89e−03 | 4.49e−05 |
| 0.4 | 4.2e−03 | 6.5e−04 | 2.1e−04 | 3.3e−02 | 8.0e−02 | 8.00e−03 | 3.25e−03 | 5.29e−05 |
| 0.5 | 4.5e−03 | 6.8e−04 | 2.3e−04 | 7.0e−02 | 9.6e−02 | 9.60e−03 | 6.99e−03 | 5.56e−05 |
| 0.6 | 4.2e−03 | 6.5e−04 | 2.1e−04 | 3.3e−02 | 1.1e−02 | 1.09e−02 | 3.45e−03 | 5.29e−05 |

Moreover, we present a comparison of the MAE between our proposed method, GCHWM, for $S = 6, \eta = 3, \kappa = 2$, and $\lambda = 2$ and other numerical methods: Roul and Thula method [8] and Lodhi et al. method [29] when $N = 8,\ 16$ in Table 9.

**Table 9.** Comparison of the MAE for solving Example 5.2 by GCHWM when $\eta = 3, \kappa = 2$, and $\lambda = 2$ with other numerical methods.

| GCHWM $(S = 6)$ | Roul and Thula method $(N = 8)$ | Lodhi et al. method $(N = 8)$ | Roul and Thula method $(N = 16)$ | Lodhi et al. method $(N = 16)$ |
|---|---|---|---|---|
| 6.7299e−09 | 9.5527e−06 | 3.3205e−06 | 5.9226e−07 | 1.9962e−07 |

Also, we present a comparison of the absolute error for GCHWM choosing $S = 4, \kappa = 2$, and $\lambda = 2$ for different values of $\eta = 3, 4, 5$ in Table 10.

**Table 10.** Comparison of the absolute error for solving Example 5.2 by GCHWM when $S = 4, \kappa = 2$, and $\lambda = 2$ for different values of $\eta = 3, 4, 5$.

| $t$ | Absolute Error (GCHWM) $\eta = 3$ | Absolute Error (GCHWM) $\eta = 4$ | Absolute Error (GCHWM) $\eta = 5$ |
|---|---|---|---|
| 0.0 | 2.0872e − 06 | 1.4613e − 07 | 1.9792e − 08 |
| 0.1 | 1.9916e − 06 | 1.3115e − 07 | 2.0434e − 08 |
| 0.2 | 2.0569e − 06 | 2.4113e − 07 | 5.9565e − 08 |
| 0.3 | 1.6816e − 06 | 3.4135e − 07 | 3.8442e − 08 |
| 0.4 | 4.4008e − 06 | 5.0510e − 07 | 9.2685e − 08 |

**Example 5.3.** In this example, we consider solving Bratu's equation (1.1) with the boundary conditions (1.2) again using the GCHWM but when $\lambda = 3.51$ and for $S = 4, 6$, $\kappa = 2$, and $\eta = 3$. The exact solution, approximate solutions, and absolute errors of Example 5.3 using GCHWM for $S = 4, 6$ and $\eta = 3$, $\kappa = 2$, and $\lambda = 3.51$ are concised in Table 11.

**Table 11.** Exact and approximate solutions and the absolute errors of solving Example 5.3 by GCHWM when $S = 4, 6$, $\eta = 3$, $\kappa = 2$, and $\lambda = 3.51$.

| $t$ | Exact Solution | GCHWM ($S = 4$) | GCHWM ($S = 6$) | Absolute Error GCHWM ($S = 4$) | Absolute Error GCHWM ($S = 6$) |
|---|---|---|---|---|---|
| 0.0 | −0.0000000000 | 0.0000345830 | −0.0000002126 | 3.4583e−05 | 2.1260e−07 |
| 0.1 | 0.3958056982 | 0.3958372467 | 0.3958058756 | 3.1548e−05 | 1.7732e−07 |
| 0.2 | 0.7390974108 | 0.7390710315 | 0.7390975794 | 2.6379e−05 | 1.6867e−07 |
| 0.3 | 1.0087582601 | 1.0087459387 | 1.0087582938 | 1.2321e−05 | 3.3672e−08 |
| 0.4 | 1.1825366617 | 1.1825828859 | 1.1825367142 | 4.6224e−05 | 5.2526e−08 |
| 0.5 | 1.2427426922 | 1.2427945549 | 1.2427431838 | 5.1863e−05 | 4.9156e−07 |
| 0.6 | 1.1825366617 | 1.1825478190 | 1.1825370444 | 1.1157e−05 | 3.8269e−07 |

Approximated solutions by GCHWM for $S = 4, 6, 8$ and $\eta = 3, \kappa = 2$, and $\lambda = 3.51$ and B-spline method are in Table 12.

**Table 12.** Comparison of the absolute error for Example 5.3's solution by GCHWM when $S = 4,\ 6,\ 8$ and $\eta = 3, \kappa = 2$, and $\lambda = 3.51$ with B-spline method.

| $t$ | GCHWM ($S = 4$) | GCHWM ($S = 6$) | GCHWM ($S = 8$) | B-spline method |
|---|---|---|---|---|
| 0.1 | 3.1548e−05 | 1.7732e−07 | 1.8177e−10 | 3.8417e−02 |
| 0.2 | 2.6379e−05 | 1.6867e−07 | 1.9544e−10 | 7.4814e−02 |
| 0.3 | 1.2321e−05 | 3.3672e−08 | 7.6619e−10 | 1.0583e−01 |
| 0.4 | 4.6224e−05 | 5.2526e−08 | 5.2448e−10 | 1.2712e−01 |
| 0.5 | 5.1863e−05 | 4.9156e−07 | 1.3445e−09 | 1.3475e−01 |
| 0.6 | 1.1157e−05 | 3.8269e−07 | 2.2624e−09 | 1.2712e−01 |

We can presume that the current method is more effective and promising when compared to the B-spline method. Moreover, we present a comparison of the MAE between GCHWM's solution when $S = 4, 6, \eta = 3, \kappa = 2$, and $\lambda = 3.51$ and other numerical methods solutions: Roul and

Thula method [8], Lodhi et al. method [29] when $N = 8,\ 16$, Xu et al. method [31], and Buckmire method [32] in Table 13.

**Table 13.** Comparison of the MAE for Example 5.3's solutions by GCHWM for $\eta = 3$, $\kappa = 2$, and $\lambda = 3.51$ and for $S = 4, 6$ with other numerical methods.

| **GCHWM** | | **Roul and Thula** | | **Lodhi et al.** | | **Xu et al.** | **Buckmire** |
|---|---|---|---|---|---|---|---|
| ($S = 4$) | ($S = 6$) | ($N = 8$) | ($N = 16$) | ($N = 8$) | ($N = 16$) | | |
| 5.1863e−05 | 4.9156e−07 | 2.9000e−03 | 1.7339e−04 | 1.0016e−03 | 5.8761e−05 | 6.66e−04 | 4.00e−03 |

Also, we present a comparison of the absolute error between the exact solution obtained by GCHWM with $S = 4, \kappa = 2$, and $\lambda = 3.51$ for different values of $\eta = 3, 4, 5$ in Table 14.

**Table 14.** Comparison of the absolute error of the exact solution for Example 5.3 with the ones obtained by GCHWM ($S = 4, \kappa = 2$, and $\lambda = 3.51$) for different values of $\eta = 3, 4, 5$.

| $t$ | GCHWM ($\eta = 3$) | GCHWM ($\eta = 4$) | GCHWM ($\eta = 5$) |
|---|---|---|---|
| 0.0 | 3.4583e−05 | 3.8567e−06 | 6.5560e−07 |
| 0.1 | 3.1548e−05 | 3.2369e−06 | 2.7408e−07 |
| 0.2 | 2.6379e−05 | 1.0107e−06 | 2.1987e−07 |
| 0.3 | 1.2321e−05 | 2.6161e−06 | 1.7992e−07 |
| 0.4 | 4.6224e−05 | 8.9207e−06 | 1.2100e−06 |

## 6. Conclusions and future extensions

This paper establishes and verifies the novel extended Chebyshev wavelet basis with $\eta$-base ($\eta \geq 3$). Enhanced accuracy and efficiency in calculation are achieved by transitioning from conventional base-2 scaling to the adaptable parameter $\eta$-base. Simultaneously, the qualities of orthogonality and approximation are effectively preserved. The operational matrices of differentiation are constructed in this new GCHW setting.

The strategy used in this study has proven highly reliable and effective for solving the Bratu boundary-value problem. Employing this method, we achieved high-precision solutions across a broad spectrum of parameters, both far from and close to the critical bifurcation point $\lambda_c \approx 3.51383$. This was accomplished using a nonlinear, non-stanzaic, non-polynomial differential equation, solved via a collocation method to derive algebraic equations. A comparison with established numerical methods, analytical techniques, and existing wavelet approximations demonstrated that the solutions were superior in accuracy and converged rapidly for highly nonlinear scenarios.

The introduction of the $\eta$-base parameter enables dynamic adjustment of resolution levels to meet the specific requirements of the task at hand. This feature is highly beneficial for addressing issues with gradients of varying magnitudes, such as those encountered in the Bratu equation. This work primarily presents a computational method that preserves the spectral accuracy of the general approach while accommodating non-uniform meshes. The GCHWM can be readily adapted to solve higher-dimensional partial differential equations and numerous nonlinear problems in quantum mechanics and fluid dynamics.

Future areas of this research include incorporating machine learning methodologies to improve computational efficiency by leveraging optimal parameter selection, parallel computing, and real-time simulations of complex problems, such as combustion modeling. The current project will include comprehensive error analysis alongside the integration of the suggested methodology with finite element and spectral techniques to effectively address difficult, large-scale multiphysics challenges.

## Author contributions

Mohammed Z. Alqarni: Validation, Formal analysis, Review and editing, Resources; Mohamed A. Ramadan: Conceptualization, Writing–review and editing, Project administration, Methodology, Supervision; Naglaa M. El-Shazly: Conceptualization, Writing–review and editing, Software, Data curation, Writing–original draft preparation, Visualization. All authors have read and agreed to the published version of the manuscript.

## Data availability

No datasets were generated or analysed during the current study.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

There is no conflict of interest.

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