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Research article

The stability of predictor-corrector methods of Runge-Kutta type for uncertain differential equations

Qiaohong Liu, Zhi Li and Liping Xu*

School of Information and Mathematics, Yangtze University, Jingzhou 434023, China

* **Correspondence:** Email: xlp211@126.com; Tel: +8618727419739.

Abstract: This paper studies the asymptotically mean-square stability of a Runge–Kutta type predictor–corrector numerical scheme for uncertain differential equations. The method consists of a fourth-order Runge–Kutta predictor coupled with a one-step implicit correction, which leads to a fully discrete scheme for the associated α-path equations. For a linear test equation driven by a Liu process, the corresponding growth factor of the numerical solution is derived. By employing vectorization techniques and Kronecker product representations, a recursive relation is established for the second-moment matrix of the numerical solution. It is shown that the asymptotically mean-square stability of the proposed scheme is equivalent to a necessary and sufficient condition on the modulus of the growth factor. Numerical examples are provided to illustrate the theoretical results and to demonstrate the influence of the step size on the stability.

Keywords: uncertain differential equations; Runge-Kutta predictor-corrector method; asymptotically mean-square stability; Liu process; growth factor
Mathematics Subject Classification: 34D20, 65L20

1. Introduction

The theory of uncertainty was systematically proposed by Liu [1]. A Liu process, whose increments are normal uncertain variables, is a special uncertain process. An uncertain differential equation (UDE) driven by a Liu process was proposed by Liu [2]. Since then, many scholars have studied UDEs both in theory and in application.

In theory, depending on the Lipschitz condition and linear growth condition, the existence and uniqueness theorem of solution for UDEs were proven by Chen and Liu [3]. An UDE has a unique solution if its coefficients satisfy the Lipschitz and linear growth conditions. Yang and Shen [4] proposed a method to solve UDEs using the fourth-order Runge-Kutta method.

Stability is an important indicator to evaluate the impact of disturbance factors on the dynamic system. The stability of UDEs was presented by Liu [5]. Various types of stability of UDEs have been studied, including stability in measure [5], stability in moment [6], almost sure stability [7], stability in mean [8], exponential stability [9], stability in distribution, and stability in inverse distribution [10]. Sufficient and necessary conditions attractively of uncertain differential system were presented by Tao and Zhu [11]. Recent work by Peng et al. [12] investigated the stability of Caputo–Hadamard fractional UDEs, highlighting the growing interest in stability issues for more general classes of uncertain differential systems.

Yao and Chen [13] proposed the Yao-Chen formula and proved that the inverse uncertain distribution of the solution of the UDE is its α-path. For several special cases, the analytic solution of the UDE can be obtained. In most instances, similar to a stochastic differential equation (SDE), analytical solutions are hard to obtain.

The development of numerical methods for UDEs has attracted increasing attention in recent years. Liu [14] proposed an analytical method for specific UDEs, but its applicability is limited to simple structures. Yao [15] first constructed a basic numerical method for UDEs, while Yao [16] further studied the extreme values and integrals of numerical solutions. These early methods, mostly Euler-type schemes, suffer from low precision and narrow stable regions, thus restricting their use in complex systems.

The application of the uncertainty theory in financial modeling is one of the important branches in this field. Peng [17] proposed an option pricing model in an uncertain market, thereby breaking through the dependence of the classic Black-Scholes model on the probability distribution. Sun [18] further studied the mean-reverting stock model and the no-arbitrage theorem, thus providing a mathematical basis for the theoretical construction of uncertain financial markets.

As a core tool in the uncertainty theory, UDEs driven by Liu processes have been widely applied in finance, engineering, management, and other fields to model dynamic systems with human uncertainty—an inherent ambiguity that cannot be adequately characterized by the probability theory or fuzzy set theory. However, most practical UDEs (e.g., portfolio selection models in uncertain finance [19] and optimal control systems [20]) lack analytical solutions, thus making efficient and stable numerical methods indispensable for their application.

To solve stochastic differential equations (SDEs), various numerical methods have been proposed. Chen et al. [21] studied numerical solutions for SDEs with Markovian switching and jumps under non-Lipschitz conditions. Izgi and Cem [22] developed semi-implicit split-step methods for nonlinear SDEs with non-Lipschitz drift terms. Komori et al. [23] proposed S-rock methods for stochastic delay differential equations with fixed delay. Lan et al. [24] investigated the polynomial stability of numerical solutions for SDEs with time-dependent delay. Mao et al. [25] focused on numerical methods for stationary distributions of SDEs with Markovian switching. Mirzaee and Alipour [26] introduced a cubic B-spline approximation for fractional stochastic integro-differential equations. For UDEs, early researchers raised the Euler Method, Runge-Kutta (RK) method [4], Adams method [27], Milne method [28], Adam-Simpson method [29], and Hamming method [30]. In practical applications, we have higher requirements for the accuracy and convergence of the method.

RK methods, which are known for a higher strong convergence order, have been gradually extended to UDEs; however, existing RK schemes for UDEs are primarily explicit, which may exhibit instability when dealing with stiff systems or large step sizes. Predictor-corrector methods (PCMs), which combine the ease of implementation of explicit schemes with the superior stability of implicit schemes, have shown promising performances in SDEs and stochastic delay differential equations (SDDEs). However, the extension of RK-type PCMs to UDEs and the analysis of their stability remain underdeveloped—a critical gap in the current literature.

A stability analysis is a key criterion to evaluate numerical methods, as unstable numerical solutions can significantly deviate from the true solution and lead to misleading conclusions. For UDEs, the existing stability results focused on simple numerical schemes: Yao et al. [31] derived sufficient conditions for the mean square stability of basic UDE numerical methods; Sheng and Wang [6] studied the p-th moment stability; Liu et al. [7] investigated the almost sure stability; and Sheng and Gao [9] analyzed the exponential stability. There is no systematic research on the stability of RK-PCMs for UDEs, especially regarding how the corrector step's regulatory parameters affect stable regions.

The motivation for introducing the RK4 predictor–corrector scheme lies in combining the advantages of high-order accuracy and improved stability. The RK4 predictor provides a fourth-order accurate approximation, while the implicit corrector step enhances the numerical stability, especially for relatively large step sizes. This combination is particularly beneficial for UDEs, where both accuracy and stability are crucial due to the presence of uncertainty. Existing RK methods for UDEs are mainly explicit and may exhibit limited stability. In contrast, the proposed RK4 predictor–corrector method incorporates an implicit correction step and yields a rational growth factor, which results in an improved stability and a larger stability region.

Here, we highlight two key contributions of this paper:

(1) The fourth-order RK Prediction-Correction method (RK4-PCM) was proposed to solve UDEs;

(2) The necessary and sufficient conditions for the RK4-PCM method of UDEs in terms of asymptotically mean-square stability were obtained.

This paper is structured as follows: in Section 2, a RK4 prediction-correction method is constructed, thereby developing a numerical scheme based on the fourth-order RK prediction step and the implicit Euler correction step, and its growth factor form under the test equation is derived; in Section 3, a stability analysis is performed through the second-order moment recursive relationship and spectral radius analysis, thus proving that the RK4-PCM method is asymptotically mean-square stable if and only if the modulus of the growth factor is less than 1; and in Section 4, numerical examples are provided, thus verifying the theoretical stability conclusions and demonstrating the significant influence of the step size on the second moment stability of the numerical solution.

2. Preliminaries

Definition 2.1. *[5] An uncertain process is said to be a Liu process if the following hold:*

- *(i)* $C_0 = 0$ *and almost all sample paths are Lipschitz continuous;*
- *(ii)* C_t *has stationary and independent increments;*
- *(iii) every increment* ΔC_t *is a normal uncertain variable with an expected value of 0 and a variance of* Δt^2.

It is clear that a Liu process is a normal uncertain process with an expected value of 0 and a variance of t^2 (i.e., $C_t \sim N(0, t)$). Furthermore, C_t has an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \tag{2.1}$$

and an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = \frac{\sqrt{3}t}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{2.2}$$

The α-path representation plays a key role in numerical computations. It transforms the UDE into a family of deterministic differential equations parameterized by $\alpha \in (0,1)$. As a result, classical numerical methods can be applied to each α-path, and the statistical properties of the solution can be obtained by integrating over all α.

Definition 2.2. *[2] An uncertain process is said to have independent increments if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$ are independent uncertain variables for any times $t_0 < t_1 < \cdots < t_k$. An uncertain process is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.*

Definition 2.3. *[3] Suppose C_t is a Liu process, and f and g are continuous functions. Then,*

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dC_t \tag{2.3}$$

is called a UDE. A solution is an uncertain process that satisfies (2.3) *identically in t.*

Theorem 2.1. *[3] The UDE* (2.3) *has a unique solution if the coefficients $f(x,t)$ and $g(x,t)$ satisfy the Lipschitz condition*

$$|f(x,t) - f(y,t)| + |g(x,t) - g(y,t)| \le L|x-y|, \quad \forall x, y \in R, t \ge 0 \tag{2.4}$$

and the linear growth condition

$$|f(x,t)| + |g(x,t)| \le L(1+|x|), \quad \forall x \in R, t \ge 0 \tag{2.5}$$

for some constant L. Moreover, the solution is sample-continuous.

Definition 2.4. *(a) The vectorization* vec(A) *of an $m \times n$ matrix A_{ij}, $i = 1, \ldots, m, j = 1, \ldots, n$ transforms it into an $mn \times 1$ column vector given by stacking columns of matrix A on the top of the one another.*

(b) The Kronecker product of an $m \times n$ matrix A and a $p \times q$ matrix B_{ij}, $i = 1, \ldots, p, j = 1, \ldots, q$ means the $mn \times pq$ matrix is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}. \tag{2.6}$$

Lemma 2.1. *When the matrices A, B, C are given so as the matrix product ABC is available, the vectorization operation holds the identity*

$$\text{vec}(ABC) = \left(C^T \otimes A\right) \text{vec}(B). \tag{2.7}$$

Lemma 2.2. *If $\rho(A) = \mu, \rho(B) = \sigma$, then $\rho(A \otimes B) = \mu\sigma$, where ρ represents the spectral radius.*

For Eq (2.3) with an initial value of X_0, and its α-path equations are as follows:

$$dX_t^\alpha = f\left(t, X_t^\alpha\right) dt + \left|g\left(t, X_t^\alpha\right)\right| \Phi^{-1}(\alpha) dt, \tag{2.8}$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{2.9}$$

Theorem 2.2. *[4] Let X_t and X_t^α be the solution and the α-path of the UDE* (2.3)*, respectively. If the linear growth, Lipschitz, and regular conditions hold, then*

$$E[X_t] = \int_0^1 X_t^\alpha d\alpha. \tag{2.10}$$

Lemma 2.3. *[4] Let X_t and X_t^α be the solution and the α-path of the UDE, respectively, and let J be a continuous and monotone (increasing or decreasing) function. Assume the linear growth, Lipschitz, and regular conditions hold. Then,*

$$E\left[J\left(X_t\right)\right] = \int_0^1 J\left(X_t^\alpha\right) d\alpha. \tag{2.11}$$

3. The construction step for RK4-PCMs

For clarity, the construction of the proposed method is presented in two steps. First, a fourth-order RK method is applied to obtain a predicted value. Then, an implicit correction step is introduced to improve the stability, thus leading to the final predictor–corrector scheme.

The following introduces the RK method [4].

When solving the above α-path equation using the fourth-order RK method, the iterative format is as follows: given the step size h, the i go to the next step $i+1$

$$X_{i+1}^\alpha = X_i^\alpha + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right). \tag{3.1}$$

The slope terms for each stage are defined as follows:

$$\begin{aligned}
k_1 &= \left(f\left(t_i, X_i^\alpha\right) + \left|g\left(t_i, X_i^\alpha\right)\right| \Phi^{-1}(\alpha)\right),\\
k_2 &= \left(f\left(t_i + \frac{h}{2}, X_i^\alpha + \frac{h}{2}k_1\right) + \left|g\left(t_i + \frac{h}{2}, X_i^\alpha + \frac{h}{2}k_1\right)\right| \Phi^{-1}(\alpha)\right),\\
k_3 &= \left(f\left(t_i + \frac{h}{2}, X_i^\alpha + \frac{h}{2}k_2\right) + \left|g\left(t_i + \frac{h}{2}, X_i^\alpha + \frac{h}{2}k_2\right)\right| \Phi^{-1}(\alpha)\right),\\
k_4 &= \left(f\left(t_i + h, X_i^\alpha + hk_3\right) + \left|g\left(t_i + h, X_i^\alpha + hk_3\right)\right| \Phi^{-1}(\alpha)\right).
\end{aligned}$$

By solving the above RK iteration for each value of $\alpha \in \{0.01, 0.02, \ldots, 0.99\}$, the value of X_s^α (at the target time s) can be obtained. Then, based on the uncertainty theory, the inverse uncertain distribution of X_s can be derived.

Take (3.1) as the predictor to obtain the predicted value as follows:

$$\widetilde{X}_{i+1}^\alpha = X_i^\alpha + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right). \tag{3.2}$$

To analyze the stability analysis of the solution of UDE, we need to consider the following linear test equation to explore its behavior:

$$dX_t = \mu X_t\, dt + \sigma X_t\, dC_t; \tag{3.3}$$

the corresponding α-path equation is as follows:

$$dX_t^\alpha = \mu X_t^\alpha dt + \left|\sigma X_t^\alpha\right| \Phi^{-1}(\alpha) dt, \tag{3.4}$$

where μ and σ are constants that represent the drift coefficient and the diffusion coefficient (the intensity of uncertain disturbance), respectively.

Since $\left|\sigma X_t^\alpha\right| = |\sigma| \left|X_t^\alpha\right|$, if only considering $X_t^\alpha > 0$, then it can be simplified to the following:

$$dX_t^\alpha = \left(\mu + |\sigma|\Phi^{-1}(\alpha)\right) X_t^\alpha dt \triangleq \lambda_\alpha X_t^\alpha dt, \tag{3.5}$$

where $\lambda_\alpha = \mu + |\sigma|\Phi^{-1}(\alpha)$.

Calculate the slopes k_1, k_2, k_3, and k_4 of the fourth-order RK method, where $F(t, X) = \lambda_\alpha X_i^\alpha$, as follows:

$$\begin{aligned}
k_1 &= F\left(t_i, X_i^\alpha\right) = \lambda_\alpha X_i^\alpha, \\
k_2 &= F\left(t_i + \frac{h}{2}, X_i^\alpha + \frac{h}{2}k_1\right) = \lambda_\alpha X_i^\alpha \left(1 + \frac{\lambda_\alpha h}{2}\right), \\
k_3 &= F\left(t_i + \frac{h}{2}, X_i^\alpha + \frac{h}{2}k_2\right) = \lambda_\alpha X_i^\alpha \left(1 + \frac{\lambda_\alpha h}{2} + \frac{(\lambda_\alpha h)^2}{4}\right), \\
k_4 &= F\left(t_i + h, X_i^\alpha + hk_3\right) = \lambda_\alpha X_i^\alpha \left(1 + \lambda_\alpha h + \frac{(\lambda_\alpha h)^2}{2} + \frac{(\lambda_\alpha h)^3}{4}\right).
\end{aligned}$$

Substituting k_1, k_2, k_3, k_4 into the iterative form (3.2), expanding, and rearranging yields the following:

$$\widetilde{X}_{i+1}^\alpha = X_i^\alpha \left[1 + \lambda_\alpha h + \frac{(\lambda_\alpha h)^2}{2} + \frac{(\lambda_\alpha h)^3}{6} + \frac{(\lambda_\alpha h)^4}{24}\right]. \tag{3.6}$$

Introduce the growth factor $R(z)$ as follows:

$$R\left(\lambda_\alpha h\right) = 1 + \lambda_\alpha h + \frac{(\lambda_\alpha h)^2}{2} + \frac{(\lambda_\alpha h)^3}{6} + \frac{(\lambda_\alpha h)^4}{24}. \tag{3.7}$$

Then,

$$\widetilde{X}_{i+1}^\alpha = X_i^\alpha \cdot R\left(\lambda_\alpha h\right). \tag{3.8}$$

Next, the implicit Euler method is employed for correction. The implicit Euler method is chosen as the corrector due to its well-known unconditional stability properties. Compared with explicit schemes, it can effectively suppress the numerical instability, especially for relatively large step sizes. Therefore, incorporating the implicit correction step helps to enhance the overall stability of the proposed predictor–corrector method.

The implicit Euler corrector is as follows:

$$X_i^\alpha = \widetilde{X}_{i+1}^\alpha + \lambda_\alpha h X_i^\alpha. \tag{3.9}$$

Define

$$G(X) = X - \widetilde{X}_{i+1}^{\alpha} - \lambda_\alpha h X, \tag{3.10}$$

whose derivative is

$$G'(X) = 1 - \lambda_\alpha h. \tag{3.11}$$

The Newton iteration with an initial guess of

$$X_{i+1}^{\alpha,0} = \widetilde{X}_{i+1}^{\alpha} \tag{3.12}$$

reads

$$X_{i+1}^{\alpha,m+1} = X_{i+1}^{\alpha,m} - \frac{X_{i+1}^{\alpha,m} - \widetilde{X}_{i+1}^{\alpha} - \lambda_\alpha h X_{i+1}^{\alpha,m}}{1 - \lambda_\alpha h}. \tag{3.13}$$

Because the drift is linear, Newton converges in one iteration and the corrector is exact:

$$X_{i+1}^{\alpha} = \frac{\widetilde{X}_{i+1}^{\alpha}}{1 - \lambda_\alpha h}. \tag{3.14}$$

Thus, the full predictor-corrector formula becomes the following:

$$X_{i+1}^{\alpha} = \frac{R(\lambda_\alpha h)}{1 - \lambda_\alpha h} X_i^{\alpha}. \tag{3.15}$$

Let

$$r(\lambda_\alpha h) = \frac{R(\lambda_\alpha h)}{1 - \lambda_\alpha h}; \tag{3.16}$$

then,

$$X_{i+1}^{\alpha} = r(\lambda_\alpha h) X_i^{\alpha}. \tag{3.17}$$

To clarify the derivation, the RK4 method first provides a predicted value with a growth factor of $R(\lambda_\alpha h)$. Then, an implicit Euler step is applied as a corrector, which can be explicitly solved due to the linearity of the test equation. Substituting the predictor into the corrector yields the final predictor–corrector form with a growth factor of $r(\lambda_\alpha h) = \frac{R(\lambda_\alpha h)}{1-\lambda_\alpha h}$, which will play a central role in the subsequent stability analysis.

4. Stability analysis

The vectorization and Kronecker product techniques are introduced to transform the matrix recursion into a linear vector form. This allows the second-moment dynamics to be analyzed using standard spectral radius arguments, which greatly simplifies the derivation of the mean-square stability condition.

Define the state vector as follows:

$$U_i^{\alpha} = \begin{pmatrix} X_i^{\alpha} \\ \widetilde{X}_{i+1}^{\alpha} \end{pmatrix} \in \mathbb{R}^2, \tag{4.1}$$

where X_i^{α} denotes the corrected value of the α-path at step i, and $\widetilde{X}_{i+1}^{\alpha}$ denotes the predicted value of the α-path at step $i + 1$ obtained by the RK4 method.

Since

$$X_{i+1}^{\alpha} = X_i^{\alpha} \cdot r(\lambda_\alpha h), \quad \widetilde{X}_{i+2}^{\alpha} = X_{i+1}^{\alpha} \cdot R(\lambda_\alpha h) = R(\lambda_\alpha h)\, r(\lambda_\alpha h)\, X_i^{\alpha}, \tag{4.2}$$

the exact linear update for the state vector is

$$U_{i+1}^{\alpha} = G(\lambda_\alpha h)\, U_i^{\alpha}, \quad G(\lambda_\alpha h) = \begin{pmatrix} r(\lambda_\alpha h) & 0 \\ R(\lambda_\alpha h)\, r(\lambda_\alpha h) & 0 \end{pmatrix}. \tag{4.3}$$

Define the second-moment matrix as follows:

$$P_i^{\alpha} = E\left[U_i^{\alpha} (U_i^{\alpha})^T\right] = \begin{pmatrix} a_i & b_i \\ b_i & c_i \end{pmatrix} = \begin{pmatrix} E\left[\left(X_i^{\alpha}\right)^2\right] & E\left[X_i^{\alpha} \widetilde{X}_{i+1}^{\alpha}\right] \\ E\left[X_i^{\alpha} \widetilde{X}_{i+1}^{\alpha}\right] & E\left[\left(\widetilde{X}_{i+1}^{\alpha}\right)^2\right] \end{pmatrix}. \tag{4.4}$$

Set $U_{i+1}^{\alpha} = G(\lambda_\alpha h)\, U_i^{\alpha}$; then, we have the following:

$$P_{i+1}^{\alpha} = E\left[U_{i+1}^{\alpha} (U_{i+1}^{\alpha})^T\right] = G(\lambda_\alpha h)\, P_i^{\alpha} \left(G(\lambda_\alpha h)\right)^T. \tag{4.5}$$

From Lemma 2.1, we can obtain the following:

$$\operatorname{vec}(P_{i+1}^{\alpha}) = (G(\lambda_\alpha h) \otimes G(\lambda_\alpha h)) \operatorname{vec}(P_i^{\alpha}). \tag{4.6}$$

Therefore, the mean-square stability is equivalent to the following:

$$\rho(G(\lambda_\alpha h) \otimes G(\lambda_\alpha h)) < 1. \tag{4.7}$$

From Lemma 2.2, we can obtain the following:

$$\rho(G(\lambda_\alpha h) \otimes G(\lambda_\alpha h)) = \rho(G(\lambda_\alpha h))^2. \tag{4.8}$$

The eigenvalues of $G(\lambda_\alpha h)$ follow from

$$\begin{aligned} \det(G(\lambda_\alpha h) - \lambda I) &= \det \begin{pmatrix} r(\lambda_\alpha h) - \lambda & 0 \\ R(\lambda_\alpha h)\, r(\lambda_\alpha h) & -\lambda \end{pmatrix} \\ &= -\lambda (r(\lambda_\alpha h) - \lambda) = 0; \end{aligned} \tag{4.9}$$

thus,

$$\operatorname{spec}(G(\lambda_\alpha h)) = \{0, r(\lambda_\alpha h)\}, \quad \rho(G(\lambda_\alpha h)) = |r(\lambda_\alpha h)|. \tag{4.10}$$

Therefore,

$$\rho(G(\lambda_\alpha h) \otimes G(\lambda_\alpha h)) < 1 \Leftrightarrow |r(\lambda_\alpha h)| < 1. \tag{4.11}$$

Consider the linear test equation for the α-path, where $X_t^{\alpha} > 0$ is the solution to the uncertain differential equation, in the form

$$dX_t^{\alpha} = \lambda_\alpha X_t^{\alpha} dt, \tag{4.12}$$

where

$$\lambda_\alpha = \mu + |\sigma| \Phi^{-1}(\alpha), \tag{4.13}$$

and

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right). \tag{4.14}$$

The growth factor $r(z)$ for the RK4-PCM method applied to this α-path equation is given by the following:

$$r(z) = \frac{R(z)}{1-z}, \tag{4.15}$$

where

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}, \quad z = \lambda_\alpha h. \tag{4.16}$$

Theorem 4.1 (Mean-Square Stability of RK4-PCM)**.** *If the step size $h > 0$ and $\alpha \in (0,1)$ are fixed, then the necessary and sufficient condition for the asymptotically mean-square stability of the RK4-PCM method is*

$$-4 < \lambda_\alpha h < 0, \tag{4.17}$$

which can be rewritten as

$$-\frac{4}{h} < \lambda_\alpha < 0. \tag{4.18}$$

Proof. Let $z = \lambda_\alpha h \in \mathbb{R}$. By Eqs (43) and (44) of the main text, the growth factor is as follows:

$$r(z) = \frac{R(z)}{1-z}, \quad R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}.$$

The asymptotically mean-square stability is equivalent to $|r(z)| < 1$.

Because $z \in \mathbb{R}$ and $z < 1$ implies $1 - z > 0$, on the region $z < 1$, we have the equivalence

$$|r(z)| < 1 \iff -1 < \frac{R(z)}{1-z} < 1. \tag{A}$$

Now, we treat the two inequalities separately.

(i) The inequality $\frac{R(z)}{1-z} < 1$. For $z < 1$, multiplying by the positive number $1-z$ gives the following:

$$\frac{R(z)}{1-z} < 1 \iff R(z) < 1 - z \iff R(z) - (1-z) < 0.$$

Using the expression of $R(z)$,

$$\begin{aligned} R(z) - (1-z) &= \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right) - (1-z) \\ &= 2z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}. \end{aligned}$$

Factorizing yields the following:

$$2z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = \frac{1}{24} z(z+4)(z^2+12).$$

Since $z^2 + 12 > 0$ for all real z, the sign of this expression is determined by $z(z+4)$. Hence,

$$R(z) - (1-z) < 0 \iff z(z+4) < 0 \iff -4 < z < 0. \tag{B}$$

(ii) The inequality $\dfrac{R(z)}{1-z} > -1$. Again for $z < 1$, multiplying by $1 - z > 0$ gives the following:

$$\frac{R(z)}{1-z} > -1 \iff R(z) > -(1-z) = z - 1 \iff R(z) - (z-1) > 0.$$

Compute the following:

$$\begin{aligned} R(z) - (z-1) &= \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right) - (z-1) \\ &= 2 + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}. \end{aligned}$$

Moreover, this polynomial is strictly positive for all real z, since it can be rewritten as a sum of nonnegative terms as follows:

$$2 + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} = \frac{1}{24}\left((z^2+2z)^2 + 8z^2 + 48\right) > 0, \quad \forall z \in \mathbb{R}. \tag{C}$$

Therefore, $\dfrac{R(z)}{1-z} > -1$ holds for all real $z < 1$.

Combining (A), (B), and (C), we conclude that for a real z,

$$|r(z)| < 1 \iff -4 < z < 0.$$

Recalling $z = \lambda_\alpha h$, this is equivalent to

$$-4 < \lambda_\alpha h < 0,$$

which completes the proof. □

Since $\lambda_\alpha = \mu + |\sigma|\Phi^{-1}(\alpha)$ depends on $\alpha \in (0,1)$, the stability condition should be interpreted pointwise with respect to α. In other words, for the asymptotically mean-square stability, the condition $-4 < \lambda_\alpha h < 0$ is required to hold for all $\alpha \in (0,1)$.

Corollary 4.2. *The stability condition in terms of the original parameters μ and σ is given by the following:*

$$-\frac{4}{h} < \mu + |\sigma|\frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) < 0. \tag{4.19}$$

This defines the stability region for the parameters μ and σ in the RK4-PCM method.

Proof. Substituting the expression for λ_α from Eq (4.13) and $\Phi^{-1}(\alpha)$ from Eq (4.14) into inequality (4.18) yields the desired condition. □

5. Numerical experiments

Separating variables in (3.3) and integrating from 0 to t yields

$$\int_{X_0}^{X_t^\alpha} \frac{dx}{x} = \int_0^t \lambda_\alpha ds, \tag{5.1}$$

which gives the closed-form solution

$$X_t^\alpha = X_0 \exp(\lambda_\alpha t). \tag{5.2}$$

The uncertain expectation of a function of X_t is defined by the following:

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha)\, d\alpha. \tag{5.3}$$

Applying this to $J(x) = x^2$ and using (5.2) gives the following:

$$\begin{aligned} E[X_t^2] &= \int_0^1 (X_t^\alpha)^2 d\alpha = X_0^2 \int_0^1 \exp(2\lambda_\alpha t)\, d\alpha \\ &= X_0^2 e^{2\mu t} \int_0^1 \exp\left(2|\sigma| t \Phi^{-1}(\alpha)\right) d\alpha. \end{aligned} \tag{5.4}$$

By substituting the expression of $\Phi^{-1}(\alpha)$ from (2.9) into (5.4), we obtain the following:

$$E[X_t^2] = X_0^2 e^{2\mu t} \int_0^1 \left(\frac{\alpha}{1-\alpha}\right)^k d\alpha, \quad k = \frac{2\sqrt{3}}{\pi}|\sigma| t. \tag{5.5}$$

Using the Beta function and the Gamma function, the final result can be calculated as follows:

$$E[X_t^2] = X_0^2 e^{2\mu t} \frac{\pi k}{\sin(\pi k)}, \quad k = \frac{2\sqrt{3}|\sigma|}{\pi} t, \quad |k| < 1. \tag{5.6}$$

It is easy to see that if $\sigma \neq 0$, then the second moment is divergent. In other words, the analytical solution of the linear test equation (3.3) is unstable.

To verify the effectiveness and stability of the proposed RK4-PCM method to solve UDEs, we conducted numerical experiments that focused on the behavior and stability of the numerical solutions under different step sizes.

To verify the effectiveness of the proposed RK4-PCM method to solve UDEs in numerical experiments, we selected the following parameter settings:

- Drift coefficient: $\mu_1 = 0.1, \mu_2 = -0.2$;
- Diffusion coefficient: $\sigma_1 = -0.1, \sigma_2 = 0.3$;
- Initial value: $X_0 = 1$;
- Maximum time: $t_{\max 1} = 100, t_{\max 2} = 10$.

The chosen parameter sets are selected to represent both stable and unstable scenarios of the system. The values of σ are chosen to reflect different intensities of uncertainty, which allows us to illustrate how the noise level influences the mean-square behavior and stability of the numerical solution.

To study the influence of different step sizes on the stability of the numerical solution, we chose step sizes $h = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5$. For each step size, we performed integration within the time range $t \in [0,\ t_{\max}]$ and calculated the second moment of the solution (i.e., $E[X_t^2]$).

We fix the parameters:

$$\mu = -0.2, \quad \sigma = 0.3, \quad X_0 = 1, \quad t \in [0, 10],$$

and choose a moderate step size $h = 0.1$. We consider the following representative confidence levels:

$$\alpha = 0.01,\ 0.1,\ 0.5,\ 0.9,\ 0.99.$$

For each α, we compute the numerical solution using the RK4-PCM method and evaluate the second moment as follows:

$$E[X_t^2] \approx \int_0^1 (X_t^\alpha)^2 d\alpha,$$

where the α-paths are discretized accordingly.

The stability analysis presented in the previous text is based on the establishment of a linear test equation. Next, we will briefly discuss the applicability of the proposed RK4-PCM method to more general forms of nonlinear UDEs.

Consider a general UDE of the form

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dC_t,$$

and its corresponding α-path equation

$$dX_t^\alpha = \left[f(t, X_t^\alpha) + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) \right] dt.$$

Although an explicit stability condition similar to the linear case cannot generally be derived, useful insights can still be obtained via local linearization. Specifically, for a sufficiently smooth nonlinear system, we may linearize the drift term around an equilibrium point X^*:

$$f(t, X) \approx f_X(t, X^*)(X - X^*), \quad g(t, X) \approx g_X(t, X^*)(X - X^*).$$

Under this approximation, the α-path equation locally behaves like a linear equation:

$$dX_t^\alpha \approx \lambda_\alpha (X_t^\alpha - X^*) dt,$$

where

$$\lambda_\alpha = f_X(t, X^*) + |g_X(t, X^*)| \Phi^{-1}(\alpha).$$

Therefore, the stability of the numerical method applied to the nonlinear system can be characterized by the same type of condition as in the linear case:

$$-4 < \lambda_\alpha h < 0.$$

In summary, although the rigorous stability theory is derived for linear test equations, the proposed RK4-PCM method can be extended to nonlinear UDEs, and its stability can be reasonably assessed via local linearization.

The numerical results show that the stability behavior significantly varies with respect to α: for moderate values such as $\alpha = 0.5$, the term $\Phi^{-1}(\alpha) \approx 0$, and thus $\lambda_\alpha \approx \mu$. The numerical solution remains stable and the second moment decays smoothly.For $\alpha = 0.1$ and $\alpha = 0.9$, the magnitude of $\Phi^{-1}(\alpha)$ increases, which leads to a larger $|\lambda_\alpha|$. As a result, the stability condition $-4 < \lambda_\alpha h < 0$ becomes more restrictive. Slight oscillations or a slower decay of the second moment can be observed.For extreme values $\alpha = 0.01$ and $\alpha = 0.99$, the logarithmic term in $\Phi^{-1}(\alpha)$ becomes very large in magnitude. This causes $\lambda_\alpha h$ to fall outside the stability interval, which leads to rapid growth of the second moment and numerical instability.

As can be seen from Figure 1, in the case of $\mu_1 = 0.1$, $\sigma_1 = -0.1$, the second-order moment of the numerical solution is highly sensitive to the step size. When the step size is small (e.g., $h = 0.05$), the second-order moment changes smoothly over time, and the numerical solution remains relatively stable; as the step size increases, the second-order moment rapidly grows in a short time, which indicates that the numerical error is significantly amplified and the stability obviously decreases. This shows that under this parameter configuration, an overly large step size will cause the numerical solution to deviate from the mean-square stability region.

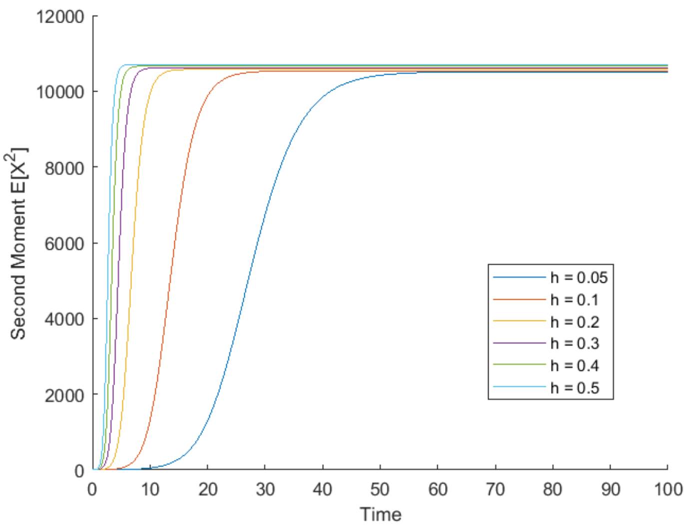

Figure 1. Second moment of the solution vs time for different step sizes ($\mu_1 = 0.1$, $\sigma_1 = -0.1$).

Figure 2 corresponds to the case of $\mu_2 = -0.2$, $\sigma_2 = 0.3$. It can be observed that when the step size is small, the second-order moment rapidly decays over time and tends to zero, and the numerical results well reflect the stable behavior of the system; when the step size increases, the decay rate significantly slows down, and in some cases, it deviates from the theoretical decay trend in the initial stage, thus indicating that a large step size weakens the numerical stability of the method.

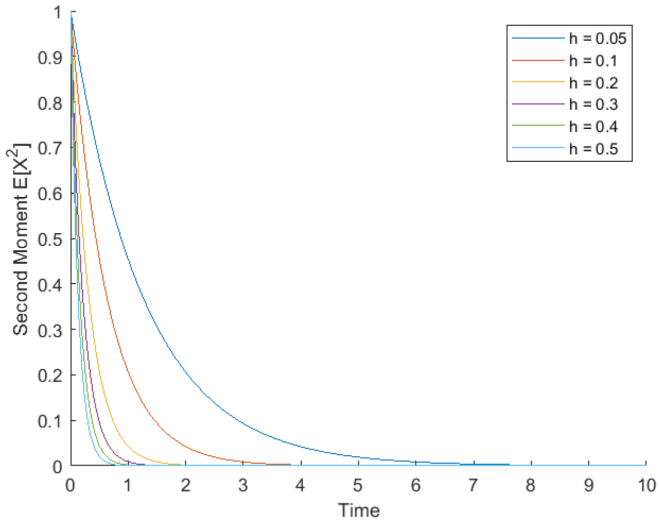

Figure 2. Second moment of the solution vs time for different step sizes ($\mu_2 = -0.2$, $\sigma_2 = 0.3$).

Combining the results of Figures 1 and 2, it can be concluded that the mean-square stability of the RK4-PCM method largely depends on the choice of the step size. A small step size can ensure that the growth factor satisfies the stability condition $-4 < \lambda_\alpha h < 0$, thus effectively controlling the growth of the second-order moment, while an overly large step size may lead to numerical instability or even divergence. This phenomenon is completely consistent with the theoretical stability analysis given in Theorem 4.1.

In addition to illustrating the influence of the step size, comparisons with existing numerical methods for UDEs (e.g., Euler and classical Runge–Kutta methods) can further highlight the advantages of the proposed scheme. In particular, the RK4-PCM method shows an improved stability performance under relatively large step sizes.

The numerical experiments confirm the theoretical findings presented in Section 4. The stability condition $-4 < \lambda_\alpha h < 0$ provides a clear guideline to select appropriate step sizes in practical applications. The sensitivity of the numerical solution to step size variations observed in Figures 1 and 2 aligns perfectly with the theoretical predictions, thus demonstrating the reliability of the proposed RK4-PCM method and its stability analysis framework.

6. Conclusions

In this paper, we analyzed the stability of the prediction-correction method based on the RK scheme in UDEs. We have established the necessary and sufficient conditions for an asymptotically mean-square stability. Numerical experiments verified the theoretical findings, which indicated that the RK4-PCM method can maintain stability when choosing appropriate step sizes, thus providing an effective approach to solve UDEs.

Despite its effectiveness, the proposed RK4-PCM method has some limitations. The current analysis was mainly restricted to linear test equations, and its performance for more general nonlinear

UDEs requires further investigation. In addition, the method involves an implicit correction step, which may increase the computational cost. Despite its effectiveness, the proposed RK4-PCM method has some limitations. Future research will focus on extending the method to nonlinear systems and developing more efficient implementations.

Author contributions

Qiaohong Liu: Conceptualization, methodology, formal analysis, writing – original draft; Zhi Li: Investigation, formal analysis, validation, data curation; Liping Xu: Conceptualization, supervision, funding acquisition, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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