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Research article

On Ulam stability and analysis of Atangana-Baleanu-Caputo and Caputo-Fabrizio-impulsive neutral fractional integro-differential equations

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Abstract: This article focuses on the Ulam-Hyers stability (U-HS) of impulsive neutral fractional integro-differential equations (INFIDEs) involving the Atangana-Baleanu-Caputo (ABC) and Caputo-Fabrizio (C-F) fractional derivatives (FDs) in a Banach space. The Banach contraction mapping principle (BCMP) and Krasnoselskii's fixed point theorem (KFPT) are used to prove the existence and uniqueness of solutions (E-US). To highlight the usefulness of the theoretical insights, carefully crafted examples are introduced, enhancing and building upon prior scholarly contributions.

Keywords: fixed points; AB-Caputo; Caputo-Fabrizio fractional integro-differential equation; Ulam stability; boundary value problems; impulsive

1. Introduction

Fractional-order derivatives (FODs) [1] are particularly effective in capturing memory effects due to their inherent nonlocal characteristics. Unlike traditional integer-order derivatives (IODs), which rely solely on local information, FODs incorporate the whole evolutionary history of a system, making them especially suitable for modeling real-world processes with memory-dependent dynamics. This makes FODs a powerful tool in fields such as viscoelasticity, control theory, and anomalous diffusion. Over the years, numerous formulations of FODs have been developed to address the diverse nature of physical, biological, and engineering systems (see [2–4]). Each formulation offers unique advantages in terms of mathematical structure and applicability, enabling researchers to tailor models to specific phenomena with greater fidelity.

Moreover, any proposed FOD should preserve the fundamental properties of classical IODs. Among the widely adopted classical FODs are the Riemann-Liouville (R-LFD) and Caputo fractional derivatives (CFD). However, FODs with singular kernels often fall short in accurately capturing the nonlocal behavior inherent in many real-world systems. To address this limitation, novel FODs with nonsingular kernels have been introduced–most notably, the fractional C-F exponential derivative [5] and the fractional ABC derivative based on the Mittag-Leffler function (M-LF) [6].

To date, extensive research has been conducted on various classes of fractional differential equations (FDEs), including fractional ordinary, partial, neutral, functional, and impulsive types. Notably, fractional partial differential equations exhibit broad applicability and diverse structural forms, encompassing models derived from key physical and mathematical equations such as Laplace, Langevin, Schrodinger, Navier-Stokes, wave, diffusion, Fokker-Planck, and Heisenberg frameworks [7–9].

In [10] K. Shri Akiladevi et al., an extensive study was conducted on neutral fractional integro-differential equations (NFIDEs) subject to fractional integral boundary conditions (FIBCs) of the form:

$$\begin{cases} {}^{C}D^{\delta}[z(\rho) - \Upsilon(\rho, z(\rho))] = \mathcal{F}(\rho, z(\rho)), \ \rho \in [0,1], \ 0 < \delta \leq 1, \\ z(0) = \beta I^{p} z(\eta), \ 0 < \eta < 1, \end{cases}$$

and

$$\begin{cases} {}^{C}D^{\delta}[z(\rho) - \Upsilon(\rho, z(\rho))] = \mathcal{F}(\rho, z(\rho), Bz(\rho)), \ \rho \in [0,1], \ 0 < \delta \leq 1, \\ z(0) = \beta I^{p} z(\eta), \ 0 < \eta < 1, \end{cases}$$

where ${}^{C}D^{\delta}$ denotes the CFD ($\delta \in (0,1]$).

Neutral functional differential equations (NFDEs) play a vital role in modeling complex systems encountered in biology, physics, and engineering, as in [11–13]. Particularly noteworthy are initial value problems (IVPs) involving fractional NFDEs with finite delays, which have gained considerable attention owing to their intrinsic ability to represent memory effects and hereditary characteristics. Numerous researchers have developed a variety of analytical techniques to investigate qualitative aspects of such equations involving classical FODs, as documented in [14–16].

It is increasingly recognized that many evolutionary processes are susceptible to short-term disturbances, often undergoing sudden and significant changes at discrete moments in time. These abrupt events typically unfold over durations so brief that they can be considered instantaneous, or impulsive, in nature. Modeling such phenomena calls for the use of impulsive differential equations, which provide a more realistic and refined mathematical framework for capturing these dynamics. Over recent years, these equations have garnered substantial interest as a vibrant research topic within the realm of nonlinear science, and their applications have proliferated across diverse fields, including physics, biology, engineering, economics, and control theory [17–19].

In [20], K. Kaliraj, et al., the CFD was analyzed in the context of a nonlocal NFDE with impulsive conditions ($\rho \in [0, T]$):

$$\begin{cases} {}^{C}D^{\delta}[x(\rho) - \Upsilon(\rho, z(\rho - h_1))] = -Bz(\rho) + \mathcal{F}(\rho, z(\rho), z(\rho - h_1)), \ \ \rho \neq \rho_i, \\ \Delta x|_{\rho_i} = I_i(x(\rho_i)), \ i = 1, 2, ..., m, \\ x(\rho) = \psi(\rho), \ \ [-h, 0], \ h > 0, \end{cases}$$

where ${}^{C}D^{\delta}$ is denoted as a CFD of order $\delta \in (0,1)$.

The study of initial boundary value problems (IBVPs) has gained popularity in recent decades, significantly contributing to the development of applied mathematical models that describe complex phenomena in the physical sciences and engineering. Notably, Tian and Bai [21] presented key existence results for IBVPs incorporating FDs of the Caputo type. Building on this foundation, further research has established E-US outcomes through the application of FPTs. A growing body of literature has emerged focusing primarily on FDEs involving CFD and R-LFDs under diverse conditions-including time delays, impulsive effects, and varied boundary value specifications–highlighting the versatility and depth of these mathematical frameworks in modeling systems with memory and nonlocal characteristics.

In order to overcome certain shortcomings of traditional definitions like the CFD and R-LFDs, the ABC as an FD (see e.g., [22, 23]) is a relatively new formulation of FODs. In contrast to the conventional operators, it is constructed using a non-local and non-singular kernel based on the M-LF, which enables it to more accurately mimic memory effects without adding singular behavior at the origin. It is especially helpful for explaining physical processes in which the impact of previous states gradually diminishes, as opposed to suddenly. Additionally, the benefit of adopting standard initial conditions written in terms of integer-order derivatives is preserved by its formulation in the Caputo sense, which is crucial for applications. Because of this, the ABC derivative has drawn interest in simulating intricate systems in physics, engineering, and biological processes where traditional fractional operators would not be accurate enough.

On the other hand, the C-F as an FD (see e.g., [7–9]) is a different definition of FODS that was developed to get over the problems with classical operators using solitary kernels. It is defined with a smooth, singularity-free exponential kernel rather than power-law kernels, resulting in more regular behavior close to the beginning point. Because of this characteristic, it is especially useful for simulating processes where memory effects gradually and realistically deteriorate, like in viscoelastic materials or diffusion phenomena with fading memory. Another benefit is that it is more practical for real-world applications since, similar to CFD, it permits initial conditions to be described in terms of ordinary integer-order derivatives. These characteristics have made the C-F derivative a valuable tool in a number of applied disciplines where the usage of traditional fractional models may result in mathematical or physical errors.

In the first part of this paper, inspired by past studies, the project seeks to investigate neutral impulsive and ABC-F I-DEs with BCs:

$$\begin{cases} {}^{ABC}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho),)] = \mathcal{F}(\rho, x(\rho), Bx(\rho)),\ \rho \in J := [a, T],\ 0 < \delta \leq 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(a) - x'(a) = \int_a^T \Pi(\lambda, x(\lambda))d\lambda. \end{cases} \tag{1}$$

In the second part of this paper, the project seeks to investigate neutral impulsive and C-F IDE's with BCs:

$$\begin{cases} {}^{CF}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho))] = \mathcal{F}(\rho, x(\rho), Bx(\rho)), \rho \in [a, T], 0 < \delta \leq 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(a) - x'(a) = \int_a^T \Pi(\lambda, x(\lambda))d\lambda, \end{cases} \tag{2}$$

where ${}^{ABC}D^{\delta}$, and ${}^{CF}D^{\delta}$-are the order δ, and $\mathcal{F} : [a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [a, T] \times \mathbb{R} \to \mathbb{R}$ are continuous. Moreover, $Bx(\rho) = \int_0^{\rho} z(\vartheta, \lambda, x(\lambda))d\lambda$, and $k \in C(D, \mathbb{R}^+)$, $D = \{(\vartheta, \lambda) \in \mathbb{R}^2 : a \leq \lambda \leq \vartheta \leq$

$T\}$, where $a = \rho_0 < \rho_1 < \rho_2 < ... < \rho_m = 1, \Delta x|_{\rho=\rho_k} = x(\rho_k^+) - x(\rho_k^-)$, $x(\rho_k^+) = lim_{h\to 0^+} x(\rho_k + h)$, and $x(\rho_k^-) = lim_{h\to 0^-} x(\rho_k + h)$–the left- and right- hand limits of $x(\rho)$ at $\rho = \rho_k$.

Motivated by the aforementioned issues, our work strives to bridge the identified research gap:

(1) We examined U-HS of the INFIDE involving the ABC and C-F FD in a Banach space.
(2) We proved the E-UR of the solutions for the INFIDE using the ABC and C-F FD utilizing Banach and Krasnoselskii's FPT.
(3) We presented two examples to support our findings.

The remaining parts of the article are organized as follows: Section 2 introduces foundational notions; Section 3 investigates the E-US for the problem formulations specified in Eqs (1) and (2), and in Section 4, examples of the established results are discussed, demonstrating their relevance to real-world models.

2. Preliminaries

Let $\mathscr{PC}([a, T], \mathbb{R})$ denote the space of functions defined on the interval $[a, T]$, such that each function is continuous on every subinterval $(\rho_k, \rho_{k+1}]$, for $k = 1, \ldots, m$. For any function $x \in \mathscr{PC}([a, T], \mathbb{R})$, the left- and right-hand limits at the impulsive points ρ_k, denoted by $x(\rho_k^-)$ and $x(\rho_k^+)$, are assumed to exist. The space $\mathscr{PC}([a, T], \mathbb{R})$ is equipped with the norm

$$||x||_{\mathscr{PC}} = \sup_{\rho\in[a,T]} |x(\rho)|.$$

Definition 2.1. *[1, 2] The CFD of order $\delta > 0,\ n - 1 < \delta < n \in \mathbb{N}$ is described as*

$${}^{C}D_{0^+}^{\delta}\mathcal{F}(\rho) = I^{n-\delta}D^n\mathcal{F}(\rho) = \frac{1}{\Gamma(n-\delta)}\int_0^{\rho}(\rho-\lambda)^{n-\delta-1}\mathcal{F}^{(n)}(\lambda)d\lambda.$$

Definition 2.2. *Let $\delta \in (0, 1]$; the fractional C-FD (see [1, 2]) of order δ for a function $\mathcal{F}$ is defined by*

$${}^{CF}D_{\rho}^{\delta}(\mathcal{F}(\rho)) = \frac{1}{1-\delta}\int_b^{\rho}\mathcal{F}'(\rho)exp\Big[-\delta\frac{\rho-x}{1-\delta}\Big]d\rho,\ \rho > b.$$

Definition 2.3. *Let $\delta \in (0, 1]$; the fractional C-F integral of order δ for a function $\mathcal{F}$ is defined by*

$${}^{CF}I_{\rho}^{\delta}(\mathcal{F}(\rho)) = \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho) + \frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_0^{\rho}\mathcal{F}(\rho)d\rho,\ \rho > b.$$

Definition 2.4. *The ABC-FD of a function $\mathcal{F}$ (with δ a fractional order) is given by*

$${}^{ABC}D^{\delta}x(\rho) = \frac{\mathscr{N}(\delta)}{1-\delta}\int_a^{\rho}\mathscr{E}_{\delta}(\frac{-\delta}{\delta-1}(\rho-\varphi)^{\delta})x'(\varphi)d\varphi, \rho > a,$$

where $\mathscr{E}_{\delta}$ is called the M-LF and described by

$$\mathscr{E}_{\delta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\varphi\delta+1)},\ \ Re(\delta) > 0,\ x \in \mathbb{C}.$$

The associated AB fractional integral is specified by

$${}^{AB}I_{a^+}^{\delta}x(\rho) = \frac{1-\delta}{\mathscr{N}(\delta)}x(\rho) + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho}(\rho-\varphi)^{\delta-1}x(\varphi)d\varphi,\quad \rho > a,$$

where $\mathscr{N}(\delta) > 0$ is some normalization function satisfying $\mathscr{N}(0) = \mathscr{N}(1) = 1$.

As required by one of the reviewers, we compare the FDs utilized in this investigation in Table 1. While the ABC model is better suited for complicated physical systems with memory and is more ideal for modeling systems with lengthy memory and hereditary features, the C-F model is mathematically simpler and is preferred for situations needing simplicity and fast-decaying memory. Generally speaking, it depends on the application; ABC is used when the system has long memory and inherited effects, and C-F is utilized when the underlying system has quick fading memory. Finally, there does not exist any operator that is always better. Consider the function $f(t) = t$. Then $f'(t) = 1$. The C-F is

$$^{CF}D_t^{\alpha}(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\Big(-\frac{\alpha}{1-\alpha}(t-s)\Big)\,ds = \frac{M(\alpha)}{\alpha}\Big(1-\exp\Big(-\frac{\alpha}{1-\alpha}t\Big)\Big),$$

and the ABC is

$$^{ABC}D_t^{\alpha}(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t E_{\alpha}\Big(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\Big)\,ds.$$

This expression involves the M-LF and does not simplify to an elementary closed form.

Table 1. A brief comparison of ABC and C-F FDs.

Feature	C-F	ABC
kernel	Exponential	Mittag-Leffler
Memory	Short Memory	Long Memory
Physical Realism	Moderate	Higher
Complexity	Easier	More Complex
Decay	Fast	Slow

Lemma 2.1. *A function $x \in \mathscr{PC}([a,T],\mathbb{R})$ is a solution of the problem*

$$\begin{cases} ^{ABC}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))] = \gamma(\rho),\ \rho\in[a,T],\ 0<\delta\le 1,\\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)),\\ x(a)-x'(a) = \int_a^T \Pi(\lambda,x(\lambda))d\lambda, \end{cases} \tag{3}$$

if and only if

$$x(\rho)=\begin{cases} \Upsilon(\rho,x(\rho))+\int_a^T \Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\gamma(\rho)+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi,\ for\ \rho\in[a,\rho_1),\\ y_1+\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\gamma(\rho)+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi,\ for\ \rho\in(\rho_1,\rho_2),\\ y_1+y_2+\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\gamma(\rho)+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi,\ for\ \rho\in(\rho_2,\rho_3),\\ .\\ .\\ .\\ \Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\gamma(\rho)+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi\\ \quad+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_{t_{i-1}}^{t_i}(\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi+\sum_{k=1}^m I_k(x(\rho_k^-)),\ for\ \rho\in(\rho_k,\rho_{k+1}). \end{cases} \tag{4}$$

Proof. Let x satisfy (3), and integrate the first Eq (3). If $\rho\in[a,\rho_1)$, then

$$^{ABC}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))] = \gamma(\rho),\rho\in[a,T],$$

$$x(a) - x'(a) = \int_a^T \Pi(\lambda, x(\lambda))d\lambda.$$

We get

$$x(\rho) = \Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho) + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi.$$

If $\rho \in (\rho_1, \rho_2)$, then

$$\begin{aligned} x(\rho) =& y(\rho_1^+) + \Upsilon(\rho, x(\rho)) - \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_1} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi \\ &+ \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho) \\ =& y(\rho_1^+) + y_1 + \Upsilon(\rho, x(\rho)) - \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_1} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi \\ &+ \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho) \\ =& y_1 + \Upsilon(\rho, x(\rho)) + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{T} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho). \end{aligned}$$

If $\rho \in (\rho_2, \rho_3)$, then

$$\begin{aligned} x(\rho) =& y(\rho_2^+) + \Upsilon(\rho, x(\rho)) - \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_2} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi \\ &+ \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho) \\ =& y(\rho_2^+) + y_2 + \Upsilon(\rho, x(\rho)) - \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_2} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi \\ &+ \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho) \\ =& y_1 + y_2 + \Upsilon(\rho, z(\rho)) + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{T} (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho). \end{aligned}$$

If $\rho \in (\rho_m, T)$, then

$$x(\rho) = \Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\gamma(\rho) + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T (\rho-\varphi)^{\delta-1}\gamma(\varphi)d\varphi + \sum_{k=1}^{m} I_k(x(\rho_k^-)). \quad (5)$$

□

Lemma 2.2. *A function $x \in \mathscr{PC}([a,T], \mathbb{R})$ is a solution of the problem*

$$\begin{cases} {}^{CF}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho))] = \gamma(\rho), \ \rho \in [a,T], \ 0 < \delta \le 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(a) - x'(a) = \int_a^T \Pi(\lambda, x(\lambda))d\lambda, \end{cases} \quad (6)$$

if and only if

$$x(\rho)=\begin{cases}\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\gamma(\rho),\ \rho\in[b,t_1],\\ y_1+\Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\gamma(\rho)d\rho,\ \rho\in(\rho_1,\rho_2),\\ y_1+y_2+\Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\gamma(\rho)d\rho,\ \rho\in(\rho_2,\rho_3),\\ .\\ .\\ .\\ \Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\gamma(\rho)d\rho\\ +\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_{t_{i-1}}^{t_i}\gamma(\rho)d\rho+\sum_{k=1}^m I_k(x(\rho_k^-)),\ for\ \rho\in(\rho_k,\rho_{k+1}).\end{cases}\tag{7}$$

Proof. Let x satisfy (6), and integrate first Eq (6). If $\rho\in[a,\rho_1)$, then

$$\begin{aligned}&{}^{CF}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))]=\gamma(\rho),\rho\in[a,T],\\ &x(a)-x'(a)=\int_a^T g(\varphi,z(\varphi))d\varphi.\end{aligned}$$

We can obtain

$$x(\rho)=\Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\Pi(\rho)d\rho+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\Pi(\rho)d\rho.$$

If $\rho\in(\rho_1,\rho_2)$, then

$$\begin{aligned}x(\rho)=&y(\rho_1^+)+\Upsilon(\rho,x(\rho))-\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho\\ &+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho\\ =&y(\rho_1^+)+y_1+\Upsilon(\rho,x(\rho))-\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho\\ &+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho\\ =&y_1+\Upsilon(\rho,x(\rho))+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^T\gamma(\rho)d\rho+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho.\end{aligned}$$

If $\rho\in(\rho_2,\rho_3)$, then

$$\begin{aligned}x(\rho)=&y(\rho_2^+)+\Upsilon(\rho,x(\rho))-\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho\\ &+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_2}\gamma(\rho)d\rho+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho\\ =&y(\rho_2^+)+y_2+\Upsilon(\rho,x(\rho))-\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{\rho_1}\gamma(\rho)d\rho+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho\\
=&y_1+y_2+\Upsilon(\rho,x(\rho))+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^{T}\gamma(\rho)d\rho+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho.
\end{aligned}
$$

If $\rho\in(\rho_m,T)$, then

$$
\begin{aligned}
x(\rho)=&\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\gamma(\rho)d\rho\\
&+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\gamma(\rho)d\rho+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_{t_{i-1}}^{t_i}\gamma(\rho)d\rho+\sum_{k=1}^{m}I_k(x(\rho_k^-)).
\end{aligned}
\tag{8}
$$

□

3. Main results

The following hypotheses are needed throughout the study:

(H_1): A function $\mathcal{F}$ is continuous, and there exists $K_1,K_2,N>0$ such that

(i)$\left|\mathcal{F}(\rho,\tau_1,\mathrm{v})-\mathcal{F}(\rho,\tau_2,\overline{\mathrm{v}})\right|\le K_1\left|\tau_1-\tau_2\right|+K_2\left|\mathrm{v}-\overline{\mathrm{v}}\right|,\ \forall\,\tau_1,\tau_2,\mathrm{v},\overline{\mathrm{v}}\in\mathbb{R},\ \rho\in[1,2]$.

(ii)$|z(\rho,\varphi,\tau_1)-z(\rho,\varphi,\tau_2)|\le N\left|\tau_1-\tau_2\right|$.

(H_2): $|\Upsilon(\rho,\tau_1)-\Upsilon(\rho,\tau_2)|\le L_G\left|\tau_1-\tau_2\right|$, for $\tau_1,\tau_2\in\mathbb{R}$.

(H_3): A positive constant exists: $K_G>0$: $|\Pi(\rho,\tau)|\le K_G\ \forall\,\rho\in J,\tau\in\mathbb{R}$.

(H_4): Let $K^*>0$ be some constant: $|I_k(x(\rho))-I_k(\overline{x}(\rho))|\le K^*\ \forall\,\rho\in J,\rho\in\mathbb{R}$.

(H_5): Let $K_F>0$ be some constant: $|\mathcal{F}(\rho,\tau,\mathrm{v})|\le K_F\ \forall\,\rho\in J,\tau,\mathrm{v}\in\mathbb{R}$.

Theorem 3.1. *Let* (H_1)–(H_4) *be fulfilled and*

$$
\Theta:=\left[L_G+K_G(T-a)+\left(\frac{1-\delta}{\Gamma(\delta)}+\frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1+K_2N)+K^*\right]<1.
\tag{9}
$$

Then, (1) *has a unique solution on the interval J:=[a,T].*

Proof. Let the operator P be as follows:

$$
\begin{aligned}
(Px)(\rho)=&\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))\\
&+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\mathcal{F}(\varphi,x(\varphi),Bx(\rho))d\varphi+\sum_{k=1}^{m}I_k(x(\rho_k^-)).
\end{aligned}
$$

Now, we aim to prove that P is contractive.

Let $x,y\in\mathbb{R}$ and $\rho\in J$ be given. Then,

$$
\begin{aligned}
&|(Px)(\rho)-(Py)(\rho)|\\
\le&|\Upsilon(\rho,x(\rho))-\Upsilon(\rho,y(\rho))|+\int_a^T|\Pi(\lambda,x(\lambda))-\Pi(\lambda,y(\lambda))|d\lambda+\frac{1-\delta}{\Gamma(\delta)}|\mathcal{F}(\vartheta,x(\vartheta),Bx(\vartheta))-\mathcal{F}(\vartheta,y(\vartheta),Bx(\vartheta))|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\left|\mathcal{F}(\rho,x(\rho),Bx(\rho))-\mathcal{F}(\vartheta,y(\vartheta),Bx(\vartheta))\right|d\varphi+\sum_{k=1}^{m}\left|I_k(x(\rho_k^-))-I_k(y(\rho_k^-))\right|\\
\le& L_G\left|x(\rho)-y(\rho)\right|+K_G\int_a^T\left|x(\lambda)-y(\lambda)\right|d\lambda+\frac{1-\delta}{\Gamma(\delta)}(K_1+K_2N)\left|x(\rho)-y(\rho)\right|\\
&+\frac{\delta}{\mathscr{N}(\delta)}(K_1+K_2N)\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi^{\delta-1})\left|x(\rho)-y(\rho)\right|d\varphi+K^*\left|x(\rho)-y(\rho)\right|\\
\le&\left[L_G+K_G(T-a)+\left(\frac{1-\delta}{\Gamma(\delta)}+\frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1+K_2N)+K^*\right]\|x-y\|\,.
\end{aligned}
$$

Therefore, P is a contraction. Consequently, the problem (1) admits a unique solution on $[a,T]$. □

Theorem 3.2. *Suppose* (H_4) *and* (H_5) *are fulfilled. Then, there exists at least one solution of* (1).

Proof. Consider, $\mathcal{B}_r=\{x\in\mathscr{PC}(J,\mathbb{R}):|x|\le \mathrm{r}\}$.

Let P_1 and P_2 be operators defined on B_r, as follows:

$$
(P_1x)(\rho)=\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))+\sum_{k=1}^{m}I_k(x(\rho_k^-)),\tag{10}
$$

and

$$
(P_2x)(\rho)=\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho.\tag{11}
$$

Correspondingly, if $x,y\in B_r$, then $P_1x+P_2y\in B_r$. As we now show,

$$
\begin{aligned}
|P_1x+P_2y|\le&\Big\|\Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))\\
&+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho+\sum_{i=1}^{m}y_i\Big\|\\
\le&\left|\Upsilon(\rho,x(\rho))\right|+\left|\int_a^T\Pi(\lambda,x(\lambda))d\lambda\right|+\left|\frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))\right|\\
&+\left|\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\right|+\left|\sum_{i=1}^{m}y_i\right|\\
\le&\left|\Upsilon(\rho,x(\rho))\right|+\int_a^T\left|\Pi(\lambda,x(\lambda))\right|d\lambda+\frac{1-\delta}{\Gamma(\delta)}\left|\mathcal{F}(\rho,x(\rho),Bx(\rho))\right|\\
&+\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\left|\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\right|+\left|\sum_{i=1}^{m}y_i\right|\\
\le&\left[L_G+K_G(T-a)+\left(\frac{1-\delta}{\Gamma(\delta)}+\frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1+K_2N)+K^*\right]\\
\le&\mathrm{r}.
\end{aligned}
$$

Thus, $P_1x+P_2y\in B_r$ for all $x,y\in B_r$.

Given that x and the operator $(P_2x)(\rho)$ are continuous, it follows that (P_1) is a contraction mapping. Moreover, observe that

$$|(P_2x)(\rho)| \leq \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T (\rho-\varphi)^{\delta-1}\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho \leq \frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}K_F.$$

It follows that (P_2) is uniformly bounded throughout the domain B_r.

Let $\rho_1, \rho_2 \in J$ with $\rho_2 \leq \rho_1$, and $x \in B_r$. Since $\mathcal{F}$ is bounded on compact sets, $(P_2x)(\rho)$ is equicontinuous on J.

$$\sup_{(\rho,x,y)\in J\times B_r} |\mathcal{F}(\varphi, x(\varphi), Bx(\rho))| := C_0 < \infty.$$

We will get

$$\begin{aligned}
|(P_2x)(\rho_2)-(P_2y)(\rho_1)| =& \Bigg|\frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_2}(\rho_2-\varphi)^{\delta-1}\mathcal{F}(\varphi, x(\varphi), Bx(\varphi))d\varphi \\
&- \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_1}(\rho_1-\varphi)^{\delta-1}\mathcal{F}(\varphi, y(\varphi), By(\varphi))d\varphi\Bigg| \\
\leq& \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_2}(\rho_2-\varphi)^{\delta-1}\left|\mathcal{F}(\varphi, x(\varphi), Bx(\varphi))\right|d\varphi \\
&+ \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_1}(\rho_1-\varphi)^{\delta-1}\left|\mathcal{F}(\varphi, y(\varphi), By(\varphi))d\varphi\right|d\varphi \\
\leq& \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_2}(\rho_2-\varphi)^{\delta-1}C_0 d\varphi + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^{\rho_1}(\rho_1-\varphi)^{\delta-1}C_0 d\varphi \\
\leq& \frac{1}{\mathscr{N}(\delta)}\frac{C_0}{\Gamma(\delta)}[2(\rho_2-\rho_1)^{\delta}+(\rho_1-a)^{\delta}-(\rho_2-a)^{\delta}] \\
\leq& \frac{2C_0}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}(\rho_2-\rho_1)^{\delta} \quad \to 0 \quad \text{as} \quad \rho_2 \to \rho_1.
\end{aligned}$$

Since $P_2(B_r)$ is relatively compact by equicontinuity and boundedness, P_2 is compact by the Ascoli-Arzela theorem. Hence, the problem (1) has at least one FP on J. □

Theorem 3.3. *Assume* (H_1)–(H_3) *hold, and*

$$\left[L_G + K_G(T-b) + \left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} + \frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1 + K_2N)) + K^*\right] < 1. \tag{12}$$

Then, the problem (2) *has a unique solution on* J.

Proof. Define the operator P as follows:

$$\begin{aligned}
(Px)(\rho) =& \Upsilon(\rho, x(\rho)) + \int_b^T \Pi(\lambda, x(\lambda))d\lambda + \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho \\
&+ \frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T \mathcal{F}(\rho, x(\rho), Bx(\rho)) + \sum_{k=1}^{m} I_k(x(\rho_k^-)).
\end{aligned}$$

Now, we show that P satisfies the contraction property.

Let $x, y \in \mathbb{R}$ and $\rho \in J$ be given. Then,

$$
\begin{aligned}
&|(Px)(\rho) - (Py)(\rho)| \\
&\leq |\Upsilon(\rho, x(\rho)) - \Upsilon(\rho, y(\rho))| + \int_a^T |\Pi(\lambda, x(\lambda)) - \Pi(\lambda, y(\lambda))|\, d\lambda \\
&\quad + \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} |\mathcal{F}(\rho, x(\rho), Bx(\rho)) - \mathcal{F}(\rho, y(\rho), By(\rho))| \\
&\quad + \frac{2\delta}{\mathscr{P}(\delta)(2-\delta)} \int_a^T |\mathcal{F}(\rho, x(\rho), Bx(\rho)) - \mathcal{F}(\rho, y(\rho), By(\rho))|\, d\rho + \sum_{k=1}^{m} |I_k(x(\rho_k^-)) - I_k(y(\rho_k^-))| \\
&\leq L_G\, |x(\varphi) - y(\varphi)| + K_G \int_b^T |x(\lambda) - y(\lambda)|\, d\lambda + \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} (K_1 + K_2 N)\, |x(\lambda) - y(\lambda)|\, d\lambda \\
&\quad + \frac{2\delta}{\mathscr{P}(\delta)(2-\delta)} (K_1 + K_2 N) \frac{1}{\Gamma(\delta)} \int_b^T (\rho - \lambda)^{\delta - 1} |x(\lambda) - y(\lambda)|\, d\lambda + K^* \|x(\rho) - y(\rho)\|, \\
&\leq \left[L_G + K_G(T-b) + \left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} + \frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)} \right) (K_1 + K_2 N) \right] \|x - y\|_{\mathscr{PC}}.
\end{aligned}
$$

Therefore, P is a contraction. Consequently, the problem (2) has a unique solution on $[a, T]$. □

Theorem 3.4. *On the assumptions* (H_4) *and* (H_5)*, the problem* (2) *has at least one solution on the interval* J*.*

Proof. Consider, $\mathcal{B}_r = \{x \in \mathscr{PC}(J, \mathbb{R}) : |x| \leq \mathrm{r}\}$.

Let P_1 and P_2 be operators defined on B_r, as follows:

$$
(P_1 x)(\rho) = \Upsilon(\rho, x(\rho)) + \int_b^T \Pi(\lambda, x(\lambda)) d\lambda + \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} \mathcal{F}(\rho, x(\rho), Bx(\rho)) d\rho + \sum_{k=1}^{m} I_k(x(\rho_k^-)), \tag{13}
$$

and

$$
(P_2 x)(\rho) = \frac{2\delta}{\mathscr{P}(\delta)(2-\delta)} \int_b^T \mathcal{F}(\rho, x(\rho), Bx(\rho)) d\rho. \tag{14}
$$

Correspondingly, if $x, y \in B_r$, then $P_1 x + P_2 y \in B_r$. As we now show,

$$
\begin{aligned}
|P_1 x + P_2 y| \leq & \|\Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda)) d\lambda + \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} \mathcal{F}(\rho, x(\rho), Bx(\rho)) \\
& + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_a^T \mathcal{F}(\rho, x(\rho), Bx(\rho)) d\rho + \sum_{k=1}^{m} I_k(x(\rho_k^-))\| \\
\leq & |\Upsilon(\rho, x(\rho))| + \left| \int_a^T \Pi(\lambda, x(\lambda)) d\lambda \right| + \left| \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} \mathcal{F}(\rho, x(\rho), Bx(\rho)) \right| \\
& + \left| \frac{2\delta}{\mathscr{P}(\delta)(2-\delta)} \int_a^T \mathcal{F}(\varphi, x(\varphi), Bx(\varphi)) d\varphi \right| + \left| \sum_{k=1}^{m} I_k(x(\rho_k^-)) \right| \\
\leq & |\Upsilon(\rho, x(\rho))| + \int_a^T |\Pi(\lambda, x(\lambda))|\, d\lambda + \frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} |\mathcal{F}(\rho, x(\rho), Bx(\rho))|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^T |\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho| + \left|\sum_{k=1}^{m} I_k(x(\rho_k^-))\right| \\
\leq &\left[L_G + K_G(T-a) + \left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} + \frac{(2\delta)(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1 + K_2 N) + K^*\right] \\
\leq &r.
\end{aligned}
$$

Thus, $P_1x + P_2y \in B_r$ for all $x, y \in B_r$.

Given that x and the operator $(P_2x)(\rho)$ are continuous, it follows that (P_1) is a contraction mapping. Moreover, observe that

$$
|(P_2x)(\rho)| \leq \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_b^T \mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho, \leq \frac{(2\delta)(T-b)}{\mathscr{P}(\delta)(2-\delta)}(K_F).
$$

It follows that (P_2) is uniformly bounded throughout the domain B_r.

Let $\rho_1, \rho_2 \in J$ with $\rho_2 \leq \rho_1$, and $x \in B_r$. Since $\mathcal{F}$ is bounded on compact sets, $(P_2x)(\rho)$ is equicontinuous on J.

$$
\sup_{(\rho,x,y)\in J\times B_r} |\mathcal{F}(\rho, x(\rho), Bx(\rho))| := C_0 < \infty.
$$

We will get $|(P_2x)(\rho_2) - (P_2y)(\rho_1)| \to 0$ as $\rho_2 \to \rho_1$. Since $P_2(B_r)$ is relatively compact by equicontinuity and boundedness, P_2 is compact by the Ascoli-Arzela theorem. Hence, the problem (2) admits at least one FP on J. □

4. Stability in the sense of Ulam

In both theoretical and practical contexts, a system's stability is essential for assuring dependability in applications like transportation systems, which must run smoothly and safely. The analysis of DEs regulating such systems, which has been thoroughly investigated by researchers and produced useful findings, depends critically on mathematical stability (see [24–26]). The U-HS stability technique is well-known; see e.g., [27]. U-HS of FDEs has a long history and nice outputs (see e.g., [28, 29], and see stability results for some integro-differential equations with delay [30–32]). In general, it answers the issue of how near an approximate solution to an equation is to its exact solution.

Consider the problem

$$
\begin{cases}
{}^{ABC}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho))] = \mathcal{F}(t, x(\rho), Bx(\rho)),\ \rho \in [a, T],\ 0 < \delta \leq 1, \\
\Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\
x(a) - x'(a) = \int_a^T \Pi(\lambda, x(\lambda))d\lambda.
\end{cases}
\tag{15}
$$

Definition 4.1. *Problem* (15) *is called U-HS if there exists some constant* $\mathfrak{Y}_{UHS} > 0$ *such that for any* $\eta > 0$ *and any function* $x(\rho)$ *satisfying the inequality*

$$
\left|{}^{ABC}D^{\delta}x(\rho) - \Upsilon(\rho, x(\rho)) - \mathcal{F}(\rho, x(\rho), Bx(\rho))\right| \leq \eta,
\tag{16}
$$

there exists a solution $\overline{x}(\rho)$ *such that*

$$
|x(\rho) - \overline{x}(\rho)| \leq \mathfrak{Y}_{UHS}\,\eta.
$$

Remark 4.1. *A function $x \in \mathscr{PC}([a, T], \mathbb{R})$ is a solution of (16) if and only if there exists a function $g \in \mathscr{PC}([a, T], \mathbb{R})$:*

(i) $|g(\rho)| \leq \eta, \rho \in [a, T]$,
(ii) ${}^{ABC}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho))] = \mathcal{F}(\rho, x(\rho), Bx(\rho)) + g(\rho),\ \rho \in [a, T]$.

Lemma 4.1. *Consider the following neutral impulsive and ABC-FIDEs with BCs:*

$$\begin{cases} {}^{ABC}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho))] = \mathcal{F}(\rho, x(\rho), Bx(\rho)),\ \rho \in [a, T],\ 0 < \delta \leq 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(a) - x'(a) = \int_a^T \Pi(\lambda, x(\lambda))d\lambda. \end{cases} \tag{17}$$

Then,

$$x(\rho) = \begin{cases} \Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho, x(\rho), Bx(\rho)) \\ +\frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T (\rho - \varphi)^{\delta-1}\mathcal{F}(\vartheta, x(\vartheta), Bx(\vartheta))d\vartheta,\ for\ \vartheta \in [a, \vartheta_1), \\ y_1 + \Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho, x(\rho), Bx(\rho)) \\ +\frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T (\rho - \varphi)^{\delta-1}\mathcal{F}(\vartheta, x(\vartheta), Bx(\vartheta))d\vartheta,\ for\ \vartheta \in (\vartheta_1, \vartheta_2), \\ y_1 + y_2 + \Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho, x(\rho), Bx(\rho)) \\ +\frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T (\rho - \varphi)^{\delta-1}\mathcal{F}(\vartheta, x(\vartheta), Bx(\vartheta))d\vartheta,\ for\ \vartheta \in (\vartheta_2, \vartheta_3), \\ . \\ . \\ . \\ \Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho, x(\rho), Bx(\rho)) \\ +\frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}(\rho - \varphi)^{\delta-1}\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho \\ +\frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_{t_{i-1}}^{t_i}(\rho - \varphi)^{\delta-1}\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho + \sum_{k=1}^m I_k(x(\rho_k^-)),\ for\ \rho \in (\rho_k, \rho_{k+1}). \end{cases} \tag{18}$$

Moreover, from (8), we get

$$\begin{aligned} &\Big| y(\rho) - \Upsilon(\rho, x(\rho)) - \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho, x(\rho), Bx(\rho)) \\ &- \frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(\rho - \varphi)^{\delta-1}\mathcal{F}(\varphi, x(\varphi), Bx(\varphi))d\varphi \\ &- \frac{\delta}{\mathcal{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_{t_{i-1}}^{t_i}(\rho - \varphi)^{\delta-1}\mathcal{F}(\varphi, x(\varphi), Bx(\varphi))d\varphi + \sum_{k=1}^{m} I_k(x(\rho_k^-))\Big|, \\ \leq &\left[\frac{(1-\delta)}{\Gamma(\delta)} + \frac{(T-a)^{\delta}}{\mathcal{N}(\delta)\Gamma(\delta)}\epsilon + K^*\right]\eta. \end{aligned}$$

Theorem 4.1. *Let hypothesis (H_1–H_4) hold. Then, the following problem (problem* (1)*)*

$$\begin{cases} {}^{ABC}D^{\delta}[x(\rho) - \Upsilon(\rho, x(\rho))] = \mathcal{F}(\rho, x(\rho), Bx(\rho)),\ \rho \in [a, T],\ 0 < \delta \leq 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(b) - x'(b) = \int_b^T \Pi(\lambda, x(\lambda))d\lambda, \end{cases} \tag{19}$$

is U-HS, if

$$\Omega := \left[L_G + K_G(T-a) + \left(\frac{1-\delta}{\Gamma(\delta)} + \frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1 + K_2N) + K^*\right] \leq 1. \tag{20}$$

Proof. Let $y \in \mathscr{PC}(J,\mathbb{R})$ be a solution of (19); then,

$$\begin{aligned}|y(\rho) - x(\rho)| \leq & \Big|y(\rho) - \Big(\Upsilon(\rho, x(\rho)) + \int_a^T \Pi(\lambda, x(\lambda))d\lambda + \frac{1-\delta}{\Gamma(\delta)}\mathcal{F}(\rho, x(\rho), Bx(\vartheta)) \\ & + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(\rho-\varphi)^{\delta-1}\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho \\ & + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_{t_{i-1}}^{t_i}(\rho-\varphi)^{\delta-1}\mathcal{F}(\rho, x(\rho), Bx(\rho))d\rho + \sum_{k=1}^{m} I_k(x(\rho_k^-))\Big)\Big| \\ & + \Big(|\Upsilon(\rho, y(\rho)) - \Upsilon(\rho, x(\rho))| + \int_a^T |\Pi(\lambda, y(\lambda)) - \Pi(\lambda, x(\lambda))|\,d\lambda \\ & + \frac{1-\delta}{\Gamma(\delta)}\,|\mathcal{F}(\rho, y(\rho), By(\rho)) - \mathcal{F}(\rho, x(\rho), Bx(\rho))| \\ & + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}\int_a^T(\rho-\varphi)^{\delta-1}\,|\mathcal{F}(\rho, y(\rho), By(\rho)) - \mathcal{F}(\vartheta, x(\vartheta), Bx(\vartheta))|\,d\varphi \\ & + \sum_{k=1}^{m}|I_k(y(\rho_k^-)) - I_k(x(\rho_k^-))|\Big) \\ \leq & \gamma_1\eta + L_G\,\|y-x\| + K_G(T-a)\,\|y-x\| \\ & + \frac{1-\delta}{\Gamma(\delta)}(K_1 + K_2N)\,\|y-x\| + \frac{\delta}{\mathscr{N}(\delta)}\frac{1}{\Gamma(\delta)}(K_1 + K_2N)\,\|y-x\| + K^*\,\|y-x\| \\ \leq & \gamma_1\eta + \left[L_G + K_G(T-a) + \left(\frac{1-\delta}{\Gamma(\delta)} + \frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1 + K_2N) + K^*\right]|y-x|_{\mathscr{PC}},\end{aligned}$$

where

$$\gamma_1 = \frac{(1-\delta)}{\Gamma(\delta)} + \frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\epsilon + K^*.$$

Then,

$$\begin{aligned}&|y(\rho)-x(\rho)| \leq \gamma_1\eta + \left[L_G + K_G(T-a) + \left(\frac{1-\delta}{\Gamma(\delta)} + \frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1+K_2N)+K^*\right]|y-x|_{\mathscr{PC}}, \\ &|y(\rho)-x(\rho)| - \left[L_G + K_G(T-a) + \left(\frac{1-\delta}{\Gamma(\delta)} + \frac{(T-a)^{\delta}}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1+K_2N)+K^*\right]|y-x| \leq \gamma_1\eta, \\ &(1-\Omega)\,|y(\rho) - x(\rho)| \leq \gamma_1\eta. \end{aligned} \tag{21}$$

Hence, from (21), we get

$$|y(\rho) - x(\rho)| \leq \frac{\gamma_1}{1-\Omega}\eta. \tag{22}$$

Hence, the solution of (1) is U-HS. □

Consider the problem

$$\begin{cases} {}^{CF}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))]=\mathcal{F}(\rho,x(\rho),Bx(\rho)), \rho\in[a,T], 0<\delta\leq 1,\\ \Delta x|_{\rho=\rho_k}=I_k(x(\rho_k^-)),\\ x(a)-x'(a)=\int_a^T\Pi(\lambda,x(\lambda))d\lambda. \end{cases} \tag{23}$$

Definition 4.2. *Problem* (23) *is called U-HS if some constant* $\mathfrak{Y}_{UHS}>0$ *exists such that for any* $\eta>0$ *and any function* $x(\rho)$ *satisfying*

$$|{}^{CF}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))]-\mathcal{F}(\rho,x(\rho),Bx(\rho))|<\eta, \tag{24}$$

there exists a solution $\overline{x}(\rho)$ *such that*

$$|x(\rho)-\overline{x}(\rho)|\leq\mathfrak{Y}_{UHS}\eta.$$

Remark 4.2. *A function* $x\in\mathscr{PC}([a,T],\mathbb{R})$ *is a solution of (24) if and only if there exists a function* $g\in\mathscr{PC}([a,T],\mathbb{R})$*:*

(*i*) $|g(\rho)|\leq\eta,\rho\in[a,T]$,
(*ii*) ${}^{CF}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))]=\mathcal{F}(\rho,x(\rho),Bx(\rho))+g(\rho),\ \rho\in[a,T]$.

Lemma 4.2. *A function* $x\in\mathscr{PC}([a,T],\mathbb{R})$ *is a solution of the following CF-FIDEs with BCs:*

$$\begin{cases} {}^{CF}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))]=\mathcal{F}(\rho,x(\rho),Bx(\rho)),\ \rho\in[a,T],\ 0<\delta\leq 1,\\ \Delta x|_{\rho=\rho_k}=I_k(x(\rho_k^-)),\\ x(b)-x'(b)=\int_b^T\Pi(\lambda,x(\lambda))d\lambda, \end{cases} \tag{25}$$

if and only if

$$x(\rho)=\begin{cases} \Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho,x(\rho),Bx\rho))d\rho\\ +\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho,\ \rho\in[b,\rho_1],\\ y_1+\Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\\ +\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho,\ \rho\in(\rho_1,\rho_2),\\ y_1+y_2+\Upsilon(\rho,x(\rho))+\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\\ +\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho,\ \rho\in(\rho_2,\rho_3),\\ .\\ .\\ .\\ \Upsilon(\rho,x(\rho))+\int_a^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\\ +\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\\ +\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_{t_{i-1}}^{t_i}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\vartheta+\sum_{k=1}^{m}I_k(x(\vartheta_k^-)),\ for\ \vartheta\in(\vartheta_k,\vartheta_{k+1}). \end{cases} \tag{26}$$

Moreover, from (11), we get

$$\Big|y(\rho)-\Upsilon(\rho,x(\rho))\int_b^T\Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho$$

$$+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_b^T \mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho+\sum_{k=1}^{m} I_k(x(\rho_k^-))\Big|$$
$$\leq\left[L_G+K_G(T-b)+\left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}+\frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1+K_2N))+K^*\right]\eta.$$

Theorem 4.2. *Let the hypothesis* (H_1–H_4) *hold. Then, the problem*

$$\begin{cases} {}^{ABC}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))]=\mathcal{F}(\rho,x(\rho),Bx(\rho)),\ \rho\in[a,T],\ 0<\delta\leq 1,\\ \Delta x|_{\rho=\rho_k}=I_k(x(\rho_k^-)),\\ x(a)-x'(a)=\int_a^T \Pi(\lambda,x(\lambda))d\lambda, \end{cases} \tag{27}$$

is U-HS, if

$$\omega:=\left[L_G+K_G(T-b)+\left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}+\frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1+K_2N))+K^*\right]\leq 1. \tag{28}$$

Proof. Let $y\in\mathscr{PC}(J,\mathbb{R})$ be a solution of (27); then

$$\begin{aligned}
&|y(\rho)-x(\rho)|\\
\leq&\Big|y(\rho)-\Big(\Upsilon(\rho,x(\rho))+\int_a^T \Pi(\lambda,x(\lambda))d\lambda+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}\mathcal{F}(\rho,x(\rho),Bx(\rho))\\
&+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho\\
&+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_{t_{i-1}}^{t_i}\mathcal{F}(\rho,x(\rho),Bx(\rho))d\rho+\sum_{k=1}^{m} I_k(x(\rho_k^-))\Big)\Big|\\
&+\Big(|\Upsilon(\rho,y(\rho))-\Upsilon(\rho,x(\rho))|+\int_a^T|\Pi(\lambda,y(\lambda))-\Pi(\lambda,x(\lambda))|\,d\lambda\\
&+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}|\mathcal{F}(\rho,y(\rho),By(\rho))-\mathcal{F}(\rho,x(\rho),Bx(\rho))|\\
&+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}\int_a^T|\mathcal{F}(\rho,y(\rho),By(\rho))-\mathcal{F}(\rho,x(\rho),Bx(\rho))|\,d\rho+\sum_{k=1}^{m}|I_k(y(\rho_k^-))-I_k(x(\rho_k^-))|\Big)\\
\leq&\gamma_1\eta+L_G\,\|y-x\|+K_G(T-b)\,\|y-x\|+\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}(K_1+K_2N))\,\|y-x\|+\frac{2\delta}{\mathscr{P}(\delta)(2-\delta)}(K_1+K_2N))\,\|y-x\|\\
\leq&\gamma_1\eta+\left[L_G+K_G(T-b)+\left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}+\frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1+K_2N))+K^*\right]|y-x|_{\mathscr{PC}},
\end{aligned}$$

where

$$\gamma_1=\left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}+\frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1+K_2N))+K^*.$$

Then,

$$|y(\rho)-x(\rho)|\leq\gamma_1\eta+\left[L_G+K_G(T-b)+\left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}+\frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1+K_2N))+K^*\right]|y-x|,$$

$$|y(\rho)-x(\rho)|-\left[L_G+K_G(T-b)+\left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)}+\frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1+K_2N))+K^*\right]|y-x|\leq\frac{\gamma_1}{1-\omega}\eta,$$
$$(1-\omega)\,|y(\rho)-x(\rho)|\leq\frac{\gamma_1}{1-\omega}\eta. \tag{29}$$

Hence, from (29), we get

$$|y(\rho)-x(\rho)|\leq\frac{\gamma_1}{1-\omega}\eta. \tag{30}$$

Hence, (27) is U-HS. □

5. Examples

Example 5.1. *We examine the ABC fractional problem that follows:*

$$\begin{cases} {}^{ABC}D^{1/2}x(\rho)-\frac{tan^{-1}|x(\rho)|}{35}=\frac{\rho^3+sin|x(\rho)|}{45}+\int_0^\rho\frac{1}{50}(\rho^2+\varphi^2)\rho(\varphi)d\varphi,\\ \Delta x(\rho)=\frac{\rho(1^-/2)}{3+\rho(1^-/2)},\\ x(a)-x'(a)=\int_1^2\frac{|x(\lambda)|}{10+|x(\lambda)|}d\lambda. \end{cases} \tag{31}$$

Set

$$\Upsilon(\rho,x(\rho))=\frac{tan^{-1}|x(\rho)|}{35},$$
$$\mathcal{F}(\rho,x(\rho),Bx(\rho))=\frac{\rho^3+sin\,|x(\rho)|}{45},$$
$$Bx(\rho))=\int_0^\rho\frac{1}{50}(\rho^2+\varphi^2)\rho(\varphi)d\varphi,$$

and

$$\Pi(\lambda,x(\lambda))=\int_1^2\frac{|x(\lambda)|}{10+|x(\lambda)|}.$$

Therefore, the assumptions (H_1–H_4) *holds. Where* $0<\delta\leq 1,\ \ a=0.5,\ \ T=1,\ \ K_1=K_2=\frac{1}{45},\ N=\frac{1}{750},\ L_G=\frac{1}{35},\ K_G=\frac{1}{10},\ \delta=\frac{1}{2},\ \mathscr{N}(\delta)=1,\ K^*=\frac{1}{3},\ \Gamma(\delta)=1.77,$ *and* $(T-a)^\delta=0.71$. *By using Theorem 3.1, we determine that*

$$\Theta=\left[L_G+K_G(T-a)+\left(\frac{1-\delta}{\Gamma(\delta)}+\frac{(T-a)^\delta}{\mathscr{N}(\delta)\Gamma(\delta)}\right)(K_1+K_2N)+K^*\right]<0.6836\approx 1.$$

According to Theorem 3.1, the problem (31) *has a unique solution on the interval [1, 2]. Furthermore, the solution x to problem (23) can be obtained as a consequence of this result and is given explicitly below:*

$$x(\rho)=\frac{|x(\rho)|}{2(1+|x(\rho)|)}+\frac{1}{\Gamma(\frac{1}{2})}\int_\rho^0(\rho-\lambda)^{\frac{1}{2}}+\frac{1}{5e^{\rho+2}(1+|x|)}+\int_0^\rho\frac{e^{-(\lambda-\rho)}}{10}x(\lambda)d\lambda\int_1^2\frac{|x(\lambda)|}{10+|x(\lambda)|}d\lambda,\rho\in(b,1],$$
$$x(\rho)=\frac{|x(\rho)|}{2(1+|x(\rho)|)},\ \rho\in(1,2],$$

$$
\begin{aligned}
x(b) - x'(b) &= \frac{|x(\rho)|}{2(1+|x(\rho)|)} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\rho}^{0} (\rho-\lambda)^{\frac{1}{2}} + \frac{1}{5e^{\rho+2}(1+|x|)} \\
&+ \int_0^{\rho} \frac{e^{-(\lambda-\rho)}}{10} x(\lambda)d\lambda - \frac{1}{\Gamma(\frac{1}{2})}\int_{\rho}^{0} (\rho-\lambda)^{\frac{1}{2}} + \frac{1}{5e^{\rho+2}(1+|x|)} + \int_0^{\rho} \frac{e^{-(\lambda-\rho)}}{10} x(\lambda)d\lambda, \iota \in (1,2].
\end{aligned}
$$

Where $\gamma_1 = 0.74,\ \ \Omega = 0.6836,$ *and* $\eta = \frac{3}{2}$, *we obtain*

$$
\frac{\gamma_1}{1-\Omega}\eta \leq 1.6323,
$$

which shows that (31) *is U-HS.*

Example 5.2. *We consider the C-F problem as*

$$
\begin{cases}
{}^{CF}D^{1/2}x(\rho) - \frac{tan^{-1}|x(\rho)|}{35} = \frac{1}{5e^{\rho+2}(1+|x|)} + \int_0^{\rho} \frac{e^{-(\lambda-\rho)}}{10} x(\lambda)d\lambda, \\
\Delta|x(\rho) = \frac{\rho(1^-/2)}{3+\rho(1^-/2)}, \\
x(b) - x'(b) = \int_1^2 \frac{|x(\lambda)|}{10+|x(\lambda)|} d\lambda.
\end{cases}
\tag{32}
$$

Set

$$
\begin{aligned}
\Upsilon(\rho, x(\rho)) &= \frac{tan^{-1}|x(\rho)|}{35}, \\
F(\rho, x(\rho), Bx(\rho)) &= \frac{1}{5e^{\rho+2}(1+|x|)} + Bx(\rho),
\end{aligned}
$$

and

$$
\begin{aligned}
Bx(\rho) &= \int_0^{\rho} \frac{e^{-(\lambda-\rho)}}{10} x(\lambda)d\lambda, \\
\Pi(\lambda, x(\lambda)) &= \int_1^2 \frac{|x(\lambda)|}{10+|x(\lambda)|}.
\end{aligned}
$$

Hence, the assumptions (H_1–H_3) *holds. Where* $1 < \delta \leq 2,\ \ b = 1,\ \ T = 2,\ \ K_1 = K_2 = \frac{1}{5e^3},$ $K_G = \frac{1}{10},\ \ N = \frac{1}{10},\ \ \delta = \frac{3}{2},\ \ \mathscr{P}(\delta) = 1,$ *and* $L_G = \frac{1}{35}$. *By using Theorem 3.3, we conclude that*

$$
L_G + K_G(T-b) + \left(\frac{2(1-\delta)}{\mathscr{P}(\delta)(2-\delta)} + \frac{2\delta(T-b)}{\mathscr{P}(\delta)(2-\delta)}\right)(K_1 + K_2N)) + K < 0.8973 \approx 1.
$$

According to Theorem 3.3, there is a unique solution of (32) on $[1,2]$. *Further, we get the solution x of the problem* (32) *as*

$$
\begin{aligned}
x(\rho) &= \frac{|x(\rho)|}{2(1+|x(\rho)|)} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\rho}^{0} (\rho-\lambda)^{\frac{1}{2}} + \frac{1}{5e^{\rho+2}(1+|x|)} + \int_0^{\rho} \frac{e^{-(\lambda-\rho)}}{10} x(\lambda)d\lambda \int_1^2 \frac{|x(\lambda)|}{10+|x(\lambda)|} d\lambda, \rho \in (b,1], \\
x(\rho) &= \frac{|x(\rho)|}{2(1+|x(\rho)|)},\ \rho \in (1,2], \\
x(b) - x'(b) &= \frac{|x(\rho)|}{2(1+|x(\rho)|)} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\rho}^{0} (\rho-\lambda)^{\frac{1}{2}} + \frac{1}{5e^{\rho+2}(1+|x|)} + \int_0^{\rho} \frac{e^{-(\lambda-\rho)}}{10} x(\lambda)d\lambda
\end{aligned}
$$

$$-\frac{1}{\Gamma(\frac{1}{2})}\int_{\rho}^{0}(\rho-\lambda)^{\frac{1}{2}}+\frac{1}{5e^{\rho+2}(1+|x|)}+\int_{0}^{\rho}\frac{e^{-(\lambda-\rho)}}{10}x(\lambda)d\lambda,\rho\in(1,2].$$

Where $\gamma_1 = 0.73,\ \ \omega = 0.8973,$ *and* $\eta = \frac{3}{2}$*, we obtain*

$$\frac{\gamma_1}{1-\omega}\eta \leq 1.2303,$$

which shows that (32) *is U-HS.*

6. Conclusions

We discussed the Ulam stability for INFIDEs incorporating the ABC and CF-FD within a Banach space. Furthermore, findings regarding E-US are proven. Banach's FPTs is used to show the uniqueness result, while Krasnoselskii's FPTs is used to examine the existence results. Lastly, we provide examples that verify the theoretical findings. In the future, a numerical approach for a generalized version of ABC and CF-BVP with various FD types will be developed. Additionally, we plan to do the following:

1) Examine E-UR and U-HS for a generalized version of (1) and (2) with various FD types using a novel approach.
2) Prove U-HS for the following generalized version of (1):

$$\begin{cases} {}^{ABC}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho),)] = \sum_{i=1}^{n}A_iF_i(\rho,x(\rho),Bx(\rho)),\ \rho\in[a,T],\ 0<\delta\leq 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(a)-x'(a) = \int_a^T \Pi(\lambda,x(\lambda))d\lambda, \end{cases} \tag{6.1}$$

and for the following generalized form of (2):

$$\begin{cases} {}^{CF}D^{\delta}[x(\rho)-\Upsilon(\rho,x(\rho))] = \sum_{i=1}^{n}A_iF_i(\rho,x(\rho),Bx(\rho)),\rho\in[a,T],0<\delta\leq 1, \\ \Delta x|_{\rho=\rho_k} = I_k(x(\rho_k^-)), \\ x(a)-x'(a) = \int_a^T \Pi(\lambda,x(\lambda))d\lambda. \end{cases} \tag{6.2}$$

Author contributions

E. El-hady, K. Venkatachalam, and E. Aljarallah: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Visualization; K. Venkatachalam, and E. El-hady: Data curation, Writing–original draft, preparation, Writing–review and editing, Project administration; E. El-hady: Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No conflict of interest.

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