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Research article

On nonlinear (p, q)-fractional hybrid pantograph equations: Existence, and uniqueness

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Abstract: We consider a nonlinear class of hybrid pantograph equations governed by Caputo-type (p, q)-fractional derivatives and proportional delay arguments. By rewriting the problem in an equivalent integral form, we first established the existence of solutions through Dhage's hybrid fixed point theorem in a Banach algebra. Under additional Lipschitz conditions, the Banach contraction principle was employed to guarantee the existence and uniqueness of solutions. The results generalize and extend studies on fractional and hybrid pantograph equations within the (p, q)-fractional setting.

Keywords: (p, q)-fractional derivative; hybrid pantograph equation; fixed point methods; Banach contraction principle; Dhage's theorem; proportional delays
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1. Introduction

During the last decades, fractional differential equations have emerged as powerful tools for modeling dynamical processes with memory, hereditary effects, and nonlocal interactions. Such models arise naturally in viscoelasticity, anomalous diffusion, control theory [1–3], and biological systems [4, 5]. The need to describe deformation-based and discrete-scale effects has motivated the development of generalized operators based on q- and (p, q)-calculus (see [6–8]).

In this context, the (p, q)-calculus provides a two-parameter generalization of the q-calculus, and researchers have studied this field, such as [9–11]. This yields (p, q)-analogues of the classical Gamma and Beta functions and leads to new fractional integral and derivative operators. These operators enrich the modeling capacity of nonlocal systems and have found applications in approximation theory, difference equations [12–14], and the study of (p,q)-difference systems [15,16].

Pantograph-type differential equations constitute another important class of functional differential equations, characterized by the appearance of the unknown function in proportionally shifted arguments. Such equations arise in electrodynamics, population dynamics, control theory, and the modeling of the collection for electric locomotives [17,18]. The combination of fractional derivatives with pantograph-type terms has received considerable attention, and numerous existence, uniqueness, and stability results have been established [19–21].

In particular, Karimov et al. [22] investigated a fractional hybrid pantograph problem with Caputo derivatives of the form

$$\begin{cases} D_{0^+}^{\alpha}\left[\dfrac{u(t)}{\mathcal{G}(t,u(t),u(\theta(t)))}\right] = \mathcal{F}(t,u(t),u(\rho(t))), \quad 0 < t < 1, \\ u(0) = 0, \end{cases}$$

where θ and ρ are proportional delay functions. The hybrid nature of this system arises from the nonlinear term inside the fractional operator combined with pantograph-type delayed arguments.

Motivated by [22,23], and the increasing interest in (p,q)-fractional operators and their ability to capture discrete-type memory and quantum-related effects, we consider a (p,q)-fractional analogue of hybrid pantograph systems. More precisely, we study the problem

$$\begin{cases} {}^{c}D_{p,q}^{\alpha}\left[\dfrac{u(t)}{\mathcal{G}(t,u(t),u(\kappa(t)))}\right] = \mathcal{F}(p^{\alpha}t,u(p^{\alpha}t),u(\rho(p^{\alpha}t))), \quad t \in [0,1], \\ u(0) = 0, \end{cases} \tag{1.1}$$

where $0 < q < p \leq 1$, $\alpha \in (0,1]$, and $\kappa,\rho : [0,1] \to [0,1]$ are assumed to be continuous and proportional-type mappings. The simultaneous presence of the unknown function inside the Caputo-type (p,q)-fractional operator, in nonlinear hybrid terms, and in pantograph arguments places this model in the class of (p,q)-fractional hybrid pantograph equations. The nonlinear functions $\mathcal{F}$ and $\mathcal{G}$ are assumed to be continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$ and measurable in the first argument. The initial condition is imposed in the Caputo-type (p,q) fractional sense and is consistent with the associated integral formulation.

The major contributions of this paper are summarized as follows:

- Using (p,q)-fractional calculus, we convert the problem into an equivalent nonlinear integral equation.
- By employing Dhage's hybrid fixed point theorem, we establish the existence of a solution without requiring contractive conditions.
- Under suitable Lipschitz assumptions, we prove existence and uniqueness via the Banach $\mathcal{G}$ principle.

The paper is organized as follows: In Section 2, we recall basic elements of (p,q)-fractional calculus and derive the equivalent integral formulation of the problem. Section 3 is devoted to proving the existence of solutions in a noncontractive framework using Dhage's hybrid fixed point theorem with an illustrative example. In Section 4, we establish existence and uniqueness results using the Banach principle and provide a corresponding example. Finally, in Section 5, we conclude the paper with remarks and future directions.

2. Preliminaries

In this section, we recall the basic definitions and tools from (p, q)-calculus that will be used throughout the paper. We also present several auxiliary lemmas concerning (p, q)-fractional integrals and derivatives. Throughout this work, we assume that $0 < q < p \leq 1$.

For $n \in \mathbb{N}$, the (p, q)-integer is defined by

$$[n]_{p,q} = \begin{cases} \dfrac{p^n - q^n}{p - q}, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

The (p, q)-factorial is given by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \qquad [0]_{p,q}! = 1.$$

For $\psi > \varphi > 0$ and $n \in \mathbb{N}$, the (p, q)-power of order n is

$$(\psi - \varphi)^{(n)}_{p,q} = \prod_{i=0}^{n-1} (\psi p^i - \varphi q^i).$$

For a real exponent $\gamma \in \mathbb{R}$, the generalized (p, q)-power is defined by

$$(\psi - \varphi)^{(\gamma)}_{p,q} = p^{\binom{\gamma}{2}} (\psi - \varphi)^{(\gamma)}_{q/p}.$$

Definition 2.1. *[9] For $t \neq 0$, the (p, q)-derivative of a function h is defined as*

$$D_{p,q}h(t) = \frac{h(pt) - h(qt)}{(p - q)t},$$

and $D_{p,q}h(0) = \lim_{t \to 0} D_{p,q}h(t)$ whenever the limit exists.

Higher-order derivatives are defined recursively by

$$D^n_{p,q}h(t) = D_{p,q}\Big(D^{n-1}_{p,q}h(t)\Big), \qquad n \in \mathbb{N}.$$

Definition 2.2. *[9] The (p, q)-integral of a function h on $[0, t]$ is given by*

$$I_{p,q}h(t) = (p - q)t \sum_{i=0}^{\infty} \frac{q^i}{p^{i+1}} h\left(\frac{q^i}{p^{i+1}} t\right),$$

whenever the series converges.

Definition 2.3. *[10] The (p, q)-Gamma function is defined by*

$$\Gamma_{p,q}(\gamma) = \frac{(p - q)^{(\gamma-1)}_{p,q}}{(p - q)^{\gamma-1}},$$

and satisfies the recurrence identity

$$\Gamma_{p,q}(\gamma+1) = [\gamma]_{p,q}\,\Gamma_{p,q}(\gamma).$$

The (p, q)-Beta function is defined by

$$B_{p,q}(\gamma, \delta) = \int_0^1 s^{\gamma-1}(1-qs)_{p,q}^{(\delta-1)}\, d_{p,q}s,$$

and satisfies

$$B_{p,q}(\gamma, \delta) = p^{(\delta-1)(2\gamma+\delta-2)/2}\, \frac{\Gamma_{p,q}(\gamma)\Gamma_{p,q}(\delta)}{\Gamma_{p,q}(\gamma+\delta)}.$$

Definition 2.4. *[11] For $\alpha > 0$, the (p, q)-fractional integral of order α is defined as*

$$I_{p,q}^{\alpha} h(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s,$$

and $I_{p,q}^{0} h(t) = h(t)$.

Definition 2.5. *[11] For $\alpha \in (0, 1]$, the Caputo-type (p, q)-fractional derivative is defined by*

$${}^{c}D_{p,q}^{\alpha} h(t) = I_{p,q}^{1-\alpha}(D_{p,q}h(t)) = \frac{1}{p^{\binom{1-\alpha}{2}}\Gamma_{p,q}(1-\alpha)} \int_0^t (t-qs)_{p,q}^{(-\alpha)} D_{p,q}\left(h\left(\frac{s}{p^{-\alpha}}\right)\right) d_{p,q}s.$$

Lemma 2.1. *[11] Let $\alpha \in (k-1, k]$ with $k \in \mathbb{N}$. Then, for any sufficiently smooth h,*

$$I_{p,q}^{\alpha}({}^{c}D_{p,q}^{\alpha} h(t)) = h(t) - \sum_{j=0}^{k-1} \frac{t^j}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(j+1)} D_{p,q}^{j} h(0).$$

In particular, if ${}^{c}D_{p,q}^{\alpha} h(t) = 0$, then

$$h(t) = c_0 + c_1 t + \cdots + c_{k-1}t^{k-1}.$$

Lemma 2.2. *[11] For $\alpha, \beta \geq 0$, and continuous function h,*

$$I_{p,q}^{\alpha}\left(I_{p,q}^{\beta} h\right)(t) = I_{p,q}^{\alpha+\beta} h(t), \qquad {}^{c}D_{p,q}^{\alpha}\left(I_{p,q}^{\alpha} h\right)(t) = h(t).$$

Lemma 2.3. *[16] For $0 < \alpha < 1$,*

$$\int_0^t (t-qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s = t^{\alpha}\, B_{p,q}(1, \alpha),$$

and

$$\int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha}\, d_{p,q}s = B_{p,q}(\alpha, 1-\alpha).$$

Remark 2.1. *If $p = q = 1$, then the (p, q)-fractional operators reduce to the classical fractional operators and the integral formulation coincides with the standard fractional integral framework used in fractional variational approaches.*

The natural functional setting is the Banach algebra $C([0,1])$ endowed with pointwise multiplication. Each solution $u \in C([0,1])$ of (1.1) satisties that, the (p,q)-fractional integral $I_{p,q}^{1-\alpha}u$ exists and belongs to $C^1([0,1],\mathbb{R})$. We denote this class by

$$\mathcal{AC}_{p,q}^{\alpha}(J) := \{u \in C([0,1],\mathbb{R}) : I_{p,q}^{1-\alpha}u \in C^1([0,1],\mathbb{R})\}.$$

Lemma 2.4. *Let* $\alpha \in (0,1]$, $0 < q < p \leq 1$, *and let* $\kappa, \rho : [0,1] \to [0,1]$ *be proportional delay functions. Assume that* $\mathcal{G}, \mathcal{F} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous and that* $u(0) = 0$. *Then* $u \in C([0,1])$ *is a solution of problem* (1.1) *if and only if it satisfies*

$$u(t) = \frac{\mathcal{G}(t,u(t),u(\kappa(t)))}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps,u(ps),u(\rho(ps)))\, d_{p,q}s. \tag{2.1}$$

Proof. (⇒) Assume that u solves (1.1). Applying the (p,q)-fractional integral operator $I_{p,q}^{\alpha}$ to both sides of (1.1) and using Lemma 2.1 (the inversion formula between the Caputo-type (p,q)-derivative and the (p,q)-integral for $\alpha \in (0,1]$ and $u(0) = 0$), we obtain

$$\frac{u(t)}{\mathcal{G}(t,u(t),u(\kappa(t)))} = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps,u(ps),u(\rho(ps)))\, d_{p,q}s.$$

Multiplying both sides by $\mathcal{G}(t,u(t),u(\kappa(t)))$ yields (2.1).

(⇐) Conversely, assume that u satisfies (2.1). Dividing both sides by $\mathcal{G}(t,u(t),u(\kappa(t)))$ and applying the Caputo-type (p,q)-fractional derivative ${}^cD_{p,q}^{\alpha}$ to both sides, we obtain

$${}^cD_{p,q}^{\alpha}\left[\frac{u(t)}{\mathcal{G}(t,u(t),u(\kappa(t)))}\right] = \mathcal{F}(pt,u(pt),u(\rho(pt))),$$

where we again use Lemma 2.1, namely that

$${}^cD_{p,q}^{\alpha} I_{p,q}^{\alpha}\varphi(t) = \varphi(t)$$

for continuous φ. Hence, u satisfies (1.1). □

This integral equation will serve as the basis for the analysis in the coming sections. Although the analysis is performed via an equivalent integral equation, the problem remains fractional because the integral representation is generated by the Caputo-type (p,q)-fractional operator and involves a fractional kernel of order α. Such integral reformulations are standard in fractional calculus and are mathematically equivalent to the original fractional differential equation.

3. Existence of solutions via Dhage's hybrid fixed point theorem

In this section, we prove the existence of solutions for (1.1). Using Lemma 2.4, we define the operators $R, S : C([0,1]) \to C([0,1])$ by

$$(Ru)(t) := \mathcal{G}(t,u(t),u(\kappa(t))), \tag{3.1}$$

$$(Su)(t) := \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps, u(ps), u(\rho(ps)))\, d_{p,q}s. \tag{3.2}$$

With these definitions, the differential problem (1.1) is equivalent to the hybrid equation

$$u = Ru \cdot Su, \tag{3.3}$$

which is the exact structure required in Dhage's theorem. We recall a fundamental result from [24,25].

Theorem 3.1. *Let X be a Banach algebra and $Q \subset X$ be a nonempty, closed, bounded, and convex subset. Suppose that R and $S : Q \to X$ satisfy:*

(D1) R is Lipschitz with constant $\eta > 0$;

(D2) S is completely continuous;

(D3) $Rv \cdot Sw \in Q$ for all $v, w \in Q$.

Define $M := \sup_{u \in Q} \|Ru\|$. If $\eta M < 1$, then the hybrid equation $u = Ru \cdot Su$ admits at least one solution in Q.

For $r > 0$, define the closed ball

$$Q := \{u \in C([0,1]) : \|u\|_\infty \le r\},$$

and we impose the following hypotheses:

(H1) For all $t \in [0,1]$ and $x, y, \tilde{x}, \tilde{y} \in Q$, there exists $L_{\mathcal{G}} > 0$, such that

$$|\mathcal{G}(t,x,y) - \mathcal{G}(t,\tilde{x},\tilde{y})| \le L_{\mathcal{G}}(|x-\tilde{x}| + |y-\tilde{y}|).$$

(H2) For all $t \in [0,1]$ and $x, y \in Q$, there exist $a_0, a_1 \ge 0$, such that

$$|\mathcal{G}(t,x,y)| \le a_0 + a_1(|x| + |y|).$$

(H3) For all $t \in [0,1]$ and $x, y \in Q$, there exist $b_0, b_1 \ge 0$, such that

$$|\mathcal{F}(t,x,y)| \le b_0 + b_1(|x| + |y|).$$

Theorem 3.2. *Assume (H1)–(H3) hold. If there exists $r > 0$, such that*

$$(a_0 + 2a_1 r)\frac{b_0 + 2b_1 r}{\Gamma_{p,q}(\alpha+1)} \le r \tag{3.4}$$

holds, and

$$\eta M < 1, \qquad \eta = 2L_{\mathcal{G}}, \quad M = a_0 + 2a_1 r,$$

then problem (1.1) admits at least one solution $u \in C([0,1])$ with $\|u\|_\infty \le r$.

Proof. **Step1.** Under (H1), the operator R is Lipschitz on Q with constant $\eta = 2L_{\mathcal{G}}$, because

$$|Ru(t) - Rv(t)| \le L_{\mathcal{G}}\left(|u(t) - v(t)| + |u(\kappa(t)) - v(\kappa(t))|\right) \le 2L_{\mathcal{G}}\|u - v\|_\infty,$$

and using (H2), we get

$$\|Ru\|_\infty \le a_0 + 2a_1 r.$$

Step2. Under (H2) and (H3), operator S is completely continuous. Indeed, for $u \in Q$,

$$|\mathcal{F}(ps, u(ps), u(\rho(ps)))| \leq b_0 + 2b_1 r.$$

Thus,

$$|S\,u(t)| \leq \frac{b_0 + 2b_1 r}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s.$$

Using Lemma 2.3, $\int_0^t (t - qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s \leq B_{p,q}(1, \alpha)$, so

$$|S\,u(t)| \leq \frac{(b_0 + 2b_1 r)\, B_{p,q}(1, \alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} = \frac{b_0 + 2b_1 r}{\Gamma_{p,q}(\alpha + 1)},$$

which shows that $S\,(Q)$ is uniformly bounded. Since the kernel $(t-qs)_{p,q}^{(\alpha-1)}$ is continuous on the compact set $[0, 1] \times [0, 1]$, and $\mathcal{F}$ is uniformly bounded on Q, the integral operator S maps bounded sets into equicontinuous families. Hence, by the Arzelà–Ascoli theorem, S is completely continuous.

Step3. For $v, w \in Q$,

$$|Rv(t)\, S\,w(t)| \leq (a_0 + 2a_1 r)\frac{b_0 + 2b_1 r}{\Gamma_{p,q}(\alpha + 1)} \leq r.$$

Thus, $Rv \cdot S\,w \in Q$.

Conditions (D1)–(D3) of Theorem 3.1 are verified. Hence, the hybrid equation $u = Ru \cdot S\,u$ admits at least one fixed point in Q, which is a solution of (1.1). □

Example 3.1. *Consider the* (p, q)*-fractional hybrid pantograph problem*

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{\frac{1}{4},\frac{1}{5}}\left[\dfrac{u(t)}{\mathcal{G}(t, u(t), u(\kappa(t)))}\right] = \mathcal{F}(\left(\tfrac{1}{4}\right)^{\frac{1}{2}} t,\; u(\left(\tfrac{1}{4}\right)^{\frac{1}{2}} t),\; u(\rho(\left(\tfrac{1}{4}\right)^{\frac{1}{2}} t))), \quad t \in [0, 1], \\ u(0) = 0, \end{cases} \tag{3.5}$$

where

$$\mathcal{G}(t, x, y) = 1 + t^2 + \frac{1}{5}(|x| + |y|), \qquad \mathcal{F}(t, x, y) = \frac{t^2}{5} + \frac{|x|}{5\,(1 + |x|)} + \frac{|y|}{5\,(1 + |y|)},$$

and the proportional delays

$$\kappa(t) = \frac{t}{2}, \qquad \rho(t) = \frac{t}{3}, \qquad t \in [0, 1].$$

Then, problem (3.5) *admits at least one solution* $u \in C([0, 1])$.

Proof. We work in the closed ball

$$Q := \{u \in C([0, 1]) : \|u\|_\infty \leq 1\}, \qquad r = 1.$$

Checking (H1)–(H3) conditions: For all $t \in [0, 1]$ and $x, y, \tilde{x}, \tilde{y} \in Q$,

$$|\mathcal{G}(t, x, y) - \mathcal{G}(t, \tilde{x}, \tilde{y})| = \frac{1}{5}(|x - \tilde{x}| + |y - \tilde{y}|) \leq L_{\mathcal{G}}(|x - \tilde{x}| + |y - \tilde{y}|),$$

so (H1) holds with

$$L_{\mathcal{G}} = \frac{1}{5}.$$

Moreover,

$$|\mathcal{G}(t,x,y)| \le 1 + t^2 + \frac{1}{5}(|x| + |y|) \le 2 + \frac{1}{5}(|x| + |y|),$$

so (H2) holds with

$$a_0 = 2, \qquad a_1 = \frac{1}{5}.$$

For $\mathcal{F}$, we have for all $t \in [0,1]$ and $x, y \in Q$,

$$|\mathcal{F}(t,x,y)| \le \frac{1}{5} + \left|\frac{|x|}{5(1+|x|)}\right| + \left|\frac{|y|}{5(1+|y|)}\right| \le \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5},$$

so

$$|\mathcal{F}(t,x,y)| \le \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}.$$

Thus, (H3) is satisfied, and we may take, for instance,

$$b_0 = \frac{3}{5}, \qquad b_1 = 0.$$

Checking Dhage's conditions: On the ball Q with $r = 1$,

$$M := a_0 + 2a_1 r = 2.4 \text{ and } \eta = 2L_{\mathcal{G}} = 0.4.$$

Hence,

$$\eta M = \frac{2}{5} \cdot \frac{12}{5} = \frac{24}{25} < 1,$$

so the condition $\eta M < 1$ in Theorem 3.1 is fulfilled.

Next, the radius condition (3.4) reads, since

$$\frac{1}{\Gamma_{p,q}(\alpha+1)} = \frac{1}{\Gamma_{1/4,1/5}\left(\frac{3}{2}\right)} \approx 0.4474,$$

we obtain

$$(a_0 + 2a_1 r)\frac{b_0 + 2b_1 r}{\Gamma_{p,q}(\alpha+1)} \approx 0.644 < 1 = r,$$

so (3.4) holds as well. Finally, for any $v, w \in Q$ and $t \in [0,1]$,

$$|Rv(t)\, Sw(t)| \le (a_0 + 2a_1 r)\,\frac{M_{\mathcal{F}}}{\Gamma_{p,q}(\alpha+1)} \le r,$$

which shows that $Rv \cdot Sw \in Q$. All hypotheses of Theorem 3.2 are satisfied for problem (3.5) on the ball Q. Therefore, by Dhage's hybrid fixed point theorem, problem (3.5) admits at least one solution $u \in C([0,1])$ with $\|u\|_\infty \le 1$. □

4. Existence and uniqueness results

In this section, we establish the existence and uniqueness of solutions to problem (1.1) by applying the Banach $\mathcal{G}$ principle.

Let $J := [0,1]$, $(C(J,\mathbb{R}), \|\cdot\|_\infty)$ denote the Banach space of real-valued continuous functions on J, equipped with the supremum norm

$$\|u\|_\infty = \sup_{t\in J} |u(t)|.$$

We impose the following assumptions:

(C1) $\mathcal{G}, \mathcal{F} : J \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, and there exist constants $L_\mathcal{G}, L_\mathcal{F} > 0$, such that, for all $t \in J$ and all $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}$,

$$|\mathcal{G}(t,x,y) - \mathcal{G}(t,\tilde{x},\tilde{y})| \le L_\mathcal{G}(|x-\tilde{x}| + |y-\tilde{y}|),$$
$$|\mathcal{F}(t,x,y) - \mathcal{F}(t,\tilde{x},\tilde{y})| \le L_\mathcal{F}(|x-\tilde{x}| + |y-\tilde{y}|).$$

(C2) There exist constants $M_\mathcal{G}, M_\mathcal{F} > 0$, such that

$$|\mathcal{G}(t,x,y)| \le M_\mathcal{G}, \qquad |\mathcal{F}(t,x,y)| \le M_\mathcal{F}, \qquad t \in J,\ x, y \in \mathbb{R}.$$

From the integral representation (2.1), define the operator $\mathcal{T} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$(\mathcal{T}u)(t) = \frac{\mathcal{G}(t,u(t),u(\kappa(t)))}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps,\ u(ps),\ u(\rho(ps)))\, d_{p,q}s, \qquad t \in J. \tag{4.1}$$

Thus, u is a solution of problem (1.1) if and only if $\mathcal{T}u = u$.

Theorem 4.1. *Assume (C1) and (C2) hold and*

$$K = \frac{2L_\mathcal{G}M_\mathcal{F} + 2M_\mathcal{G}L_\mathcal{F}}{\Gamma_{p,q}(\alpha+1)} < 1.$$

Then, the operator $\mathcal{T}$ defined in (4.1) is a $\mathcal{G}$ on $C(J,\mathbb{R})$, and problem (1.1) admits a unique solution in $C(J,\mathbb{R})$.

Proof. Let $u, v \in C(J,\mathbb{R})$ and $t \in J$. From (4.1), and by adding and subtracting the term

$$\frac{\mathcal{G}(t,v(t),v(\kappa(t)))}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps,\ u(ps),\ u(\rho(ps)))\, d_{p,q}s,$$

we obtain

$$(\mathcal{T}u)(t) - (\mathcal{T}v)(t) = A_1(t) + A_2(t),$$

where

$$A_1(t) = \frac{\mathcal{G}(t,u(t),u(\kappa(t))) - \mathcal{G}(t,v(t),v(\kappa(t)))}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps,u(ps),u(\rho(ps)))\, d_{p,q}s,$$

$$A_2(t) = \frac{\mathcal{G}(t,v(t),v(\kappa(t)))}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \left[\mathcal{F}(ps,u(ps),u(\rho(ps))) - \mathcal{F}(ps,v(ps),v(\rho(ps)))\right] d_{p,q}s.$$

First, we estimate $A_1(t)$. By (C1) and (C2),

$$\left| \int_0^t (t - qs)_{p,q}^{(\alpha-1)} \mathcal{F}(ps, u(ps), u(\rho(ps)))\, d_{p,q}s \right| \leq M_{\mathcal{F}} \int_0^t (t - qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s.$$

Using Lemma 2.3,

$$\int_0^t (t - qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s = t^{\alpha} B_{p,q}(1, \alpha) \leq B_{p,q}(1, \alpha),$$

since

$$B_{p,q}(1, \alpha) = \frac{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha + 1)}.$$

Hence,

$$\begin{aligned} |A_1(t)| &\leq \frac{2L_{\mathcal{G}} M_{\mathcal{F}}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \|u - v\|_{\infty} \int_0^t (t - qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s \\ &\leq \frac{2L_{\mathcal{G}} M_{\mathcal{F}} B_{p,q}(1, \alpha)}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \|u - v\|_{\infty} = \frac{2L_{\mathcal{G}} M_{\mathcal{F}}}{\Gamma_{p,q}(\alpha + 1)} \|u - v\|_{\infty}. \end{aligned} \tag{4.2}$$

Now, for $A_2(t)$, in the same way

$$|A_2(t)| \leq \frac{2M_{\mathcal{G}} L_{\mathcal{F}} B_{p,q}(1, \alpha)}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \|u - v\|_{\infty} = \frac{2M_{\mathcal{G}} L_{\mathcal{F}}}{\Gamma_{p,q}(\alpha + 1)} \|u - v\|_{\infty}. \tag{4.3}$$

By combining (4.2) and (4.3), we get

$$|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| \leq \frac{2L_{\mathcal{G}} M_{\mathcal{F}} + 2M_{\mathcal{G}} L_{\mathcal{F}}}{\Gamma_{p,q}(\alpha + 1)} \|u - v\|_{\infty}.$$

Taking the supremum,

$$\|\mathcal{T}u - \mathcal{T}v\|_{\infty} \leq K \|u - v\|_{\infty}.$$

Since $K < 1$, then $\mathcal{T}$ is a $\mathcal{G}$. By the Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point u^*, which is the unique solution of (1.1). □

Example 4.1. *Consider the (p, q)-fractional hybrid pantograph problem*

$$\begin{cases} {}^{c}D_{\frac{1}{4}, \frac{1}{5}}^{\frac{1}{2}} \left[\dfrac{u(t)}{\mathcal{G}(t, u(t), u(\kappa(t)))} \right] = \mathcal{F}\left(\left(\frac{1}{4}\right)^{\frac{1}{2}} t,\ u\left(\left(\frac{1}{4}\right)^{\frac{1}{2}} t\right),\ u\left(\rho\left(\left(\frac{1}{4}\right)^{\frac{1}{2}} t\right)\right)\right), \quad t \in J, \\ u(0) = 0, \end{cases} \tag{4.4}$$

where

$$\mathcal{G}(t, x, y) = 1 + \frac{1}{100}(\tanh x + \tanh y), \qquad \mathcal{F}(t, x, y) = \frac{t}{5} + \frac{\tanh x}{40} + \frac{\tanh y}{20},$$

and

$$\kappa(t) = \frac{t}{2}, \qquad \rho(t) = \frac{t}{3}.$$

Then, by Theorem 4.1, problem (4.4) *admits a unique solution* $u^* \in C(J, \mathbb{R})$.

Proof. We work in the Banach space $C(J,\mathbb{R})$ and verify assumptions (C1) and (C2).

Since $|\tanh z| \le 1$ and $|\tanh'(z)| \le 1$ for all $z \in \mathbb{R}$, we obtain

$$|\mathcal{G}(t,x,y) - \mathcal{G}(t,\tilde{x},\tilde{y})| \le \frac{1}{100}(|x-\tilde{x}| + |y-\tilde{y}|),$$

so

$$L_{\mathcal{G}} = \frac{1}{100}.$$

Similarly,

$$|\mathcal{F}(t,x,y) - \mathcal{F}(t,\tilde{x},\tilde{y})| \le \frac{1}{40}|x-\tilde{x}| + \frac{1}{20}|y-\tilde{y}| \le L_{\mathcal{F}}(|x-\tilde{x}| + |y-\tilde{y}|),$$

with

$$L_{\mathcal{F}} = \max\left\{\frac{1}{40}, \frac{1}{20}\right\} = \frac{1}{20}.$$

Moreover, for all $t \in J$ and $x, y \in \mathbb{R}$,

$$|\mathcal{G}(t,x,y)| \le 1 + \frac{1}{100} = \frac{101}{100}, \qquad |\mathcal{F}(t,x,y)| \le \frac{1}{5} + \frac{1}{40} + \frac{1}{20} = \frac{11}{40}.$$

Thus, we may take

$$M_{\mathcal{G}} = \frac{101}{100}, \qquad M_{\mathcal{F}} = \frac{11}{40}.$$

The contraction constant from Theorem 4.1 is

$$K = \frac{2L_{\mathcal{G}}M_{\mathcal{F}} + 2M_{\mathcal{G}}L_{\mathcal{F}}}{\Gamma_{p,q}(1.5)} = \frac{2\frac{1}{100}\frac{11}{40} + 2\frac{101}{100}\frac{1}{20}}{2.2361} \simeq 0.31.$$

Then, the operator $\mathcal{T}$ is a contraction on $C(J,\mathbb{R})$ provided that $K < 1$. Therefore, by Theorem 4.1, problem (4.4) admits a unique solution $u^* \in C(J,\mathbb{R})$. □

5. Conclusions

In this paper, we introduced and investigated a class of (p,q)-fractional hybrid pantograph equations involving nonlinear perturbations inside a Caputo-type (p,q)-fractional derivative together with proportional delay arguments. Using tools from the (p,q)-calculus, we derived an equivalent integral formulation that enables the application of fixed point techniques.

Two complementary approaches were applied. By means of Dhage's hybrid fixed point theorem, we obtained the existence of solutions in a noncontractive setting. In addition, under Lipschitz-type conditions, the Banach contraction principle yielded the existence and uniqueness of solutions.

The analysis relied on several structural assumptions, including continuity, boundedness, and global Lipschitz-type conditions on the nonlinear terms, as well as proportional delay functions. The results were obtained for Caputo-type (p,q)-fractional operators and were based on fixed point methods in Banach spaces, which did not provide explicit solution formulas or numerical error estimates. The presented results extended several earlier contributions on fractional, q-fractional, and hybrid pantograph equations. Possible future directions include numerical approximation methods for (p,q)-fractional hybrid models, multi-term and higher-order problems, and variants with impulsive or stochastic effects.

Author contributions

Mouataz Billah Mesmouli: Writing-review and editing; Yasir A. Madani: Writing-review and editing, project administration; Ioan-Lucian Popa: Writing-review and editing, supervision; Taher S. Hassan: Writing-review and editing, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest.

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