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Research article

Development of high-order multistep iterative schemes for nonlinear equations with applications in applied sciences

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Abstract: This article investigates a high-order convergent multistep method for solving nonlinear equations, with particular emphasis on the method's local and semi-local convergence properties in general Banach space settings. Conventional convergence analyses typically rely on Taylor series expansions and require the computation of higher-order derivatives, which are not part of the method. Such schemes may be costly, impractical, or nonexistent and are restricted to Euclidean spaces of finite dimension. Other constraints of such analyses involve the absence of a priori error estimates as well as uniqueness of the solution results. This is why, in this article, the convergence is established using only the operators of the method in combination with the concept of generalized continuity required to control the derivative. This is how the applicability of the method is extended. Due to its generality the same methodology is applicable to other methods. Numerical examples are provided to support and illustrate the theoretical findings.

Keywords: multi-step method; Fréchet derivative; Banach space; nonlinear equation; convergence of any iterative method
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1. Introduction

Let $\mathbb{X}$, $\mathbb{Y}$ stand for Banach spaces and $\mathbb{D}$ denote an open and convex subset of $\mathbb{X}$. A plethora of applications from computational sciences using mathematical modeling can be written as a nonlinear equation of the form

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet differentiable operator mapping $\mathbb{D}$ into $\mathbb{Y}$. It is desirable to find a solution $x^* \in \mathbb{D}$ of the equation $F(x) = 0$ in closed form. However, this task can only be achieved in rare cases. Therefore, researchers and practitioners have worked to develop iterative methods to generate a sequence which, under certain conditions, converges to x^*.

A popular iterative method with a convergence order of two is Newton's [1, 10], defined by

$$x_0 \in \mathbb{D},\ x_{\xi+1} = x_\xi - F'(x_\xi)^{-1}F(x_\xi) \quad \text{for each} \quad \xi = 0, 1, 2 \ldots \tag{1.2}$$

Numerous methods have been developed to increase the convergence order and reduce the computational cost.

Let us mention the following two third-order methods by Xiao and Yin [14, 15]:

$$\begin{aligned} y_\xi &= x_\xi - s\,F'(x_\xi)^{-1}F(x_\xi),\ s \neq 0 \\ z_\xi &= x_\xi - \left[\frac{1}{2s}F'(y_\xi)^{-1} + \left(1 - \frac{1}{2s}\right)F'(x_\xi)^{-1}\right]F(x_\xi), \end{aligned}$$

and

$$\begin{aligned} y_\xi &= x_\xi - sF'(x_\xi)^{-1}F(x_\xi), \\ z_\xi &= x_\xi - \left[\left(2 - \frac{1}{2s}\right)I + \left(1 - \frac{1}{2s}\right)F'(y_\xi)^{-1}F'(x_\xi)\right]F'(y_\xi)^{-1}F(x_\xi), \end{aligned}$$

where I is an identity matrix.

In addition, we consider the following fourth-order method by Sharma [11, 12, 14]:

$$\begin{aligned} y_\xi &= x_\xi - \frac{2}{3}F'(x_\xi)^{-1}F(x_\xi), \\ z_\xi &= x_\xi - \frac{1}{2}\left[-I + \frac{9}{4}F'(y_\xi)^{-1}F'(x_\xi) + \frac{3}{4}F'(x_\xi)^{-1}F'(y_\xi)\right]F'(x_\xi)^{-1}F(x_\xi). \end{aligned}$$

Inspired by the work in [16], the local convergence of the k-step method of order $3k + p$, for $k \geq 2$ and $p \geq 3$ is defined by

$$\begin{aligned} y_\xi &= x_\xi - F'(x_\xi)^{-1}F(x_\xi), \\ z_\xi^{(0)} &= z_\xi = \Phi(x_\xi, y_\xi), \\ z_\xi^{(1)} &= z_\xi - \Psi(x_\xi, y_\xi)\,F\Big(z_\xi\Big), \\ &\ldots \\ z_\xi^{(k-1)} &= z_\xi^{(k-2)} - \Psi(x_\xi, y_\xi)F\left(z_\xi^{(k-2)}\right) \\ &\text{and} \\ x_{\xi+1} &= z_\xi^{(k)} = z_\xi^{(k-1)} - \Psi(x_\xi, y_\xi)\,F\Big(z_\xi^{(k-1)}\Big), \end{aligned} \tag{1.3}$$

where Φ is an iteration function of order k, and

$$\Phi(x,y) = \frac{1}{2}\Big(F'(y)^{-1}F'(x)F'(y)^{-1} + F'(x)^{-1}\Big).$$

Taylor series expansions are used to establish the convergence of method (1.3). There are drawbacks with this approach, however, which restrict the applicability of the method. These drawbacks constitute the motivation for writing this article.

Motivational issues

(I_1) The existence of the fourth derivative is required in [16] for the local convergence analysis and for $\mathbb{X} = \mathbb{Y} = \mathbb{R}^{\xi}$, where ξ is a natural number. However, the first derivative is only present on the method (1.3).
Let us look at an example for $\mathbb{D} = [-1.5, 1.5]$ and $f : \mathbb{D} \to \mathbb{R}$, defined by

$$f(t) = \begin{cases} l_1 t^4 \log t + l_2 t^5 + l_3 t^6, & t \neq 0 \\ 0, & t = 0, \end{cases}$$

where l_1, l_2, l_3 are real parameters with $l_1 \neq 0$ and $l_2 + l_3 = 0$. It follows by these definitions that $x^* = 1 \in \mathbb{D}$ solves the equation $f(t) = 0$. However, the fourth derivative of the function f is unbounded at $t = 0 \in \mathbb{D}$. Hence, the results in [16] cannot assure the convergence of the method (1.2) to x^*. But, the method (1.3) converges to x^*, if, say, $x_0 = 0.95 \in \mathbb{D}$, $a = 3$, $k = 3$, and

$$\varphi(x,y) = x - \frac{1}{2}\Big(F'(y)^{-1} + F(x)^{-1}\Big)F(x).$$

These observations indicate that the convergence conditions in [14–16] can be switched by weaker ones.

(I_2) There is no advance knowledge on the number of iterations to reach a desired error tolerance.
(I_3) There is no information about a set containing only one solution of the equation $F(x) = 0$.
(I_4) The convergence of the method is restricted on $\mathbb{R}^d$, where d denotes a natural integer.
(I_5) The more challenging semi-local convergence is not considered.

We have addressed issues (I_1)–(I_5) in this article as follows:

Novelty of the Taylor free technique

(I_1)' The convergence is based only on the first derivative which appears on the method. Moreover, the concept of generalized continuity is used to control F' [1–3].
(I_2)' The number of iterations to be carried out is known in advance.
(I_3)' A region is specified with only one solution of Eq (1.1).
(I_4)' The convergence (local and semi-local) is given on Banach spaces.
(I_5)' The semi-local convergence is presented using majorizing sequences.

This is how we extend the applicability of the method (1.3). The proposed technique is very general. Hence, it can be used to extend the applicability of other methods analogously [4–26].

Notice that the objective of this article is not to provide a way other than the Taylor series approach for obtaining the convergence order. Instead, we address the drawbacks (I_1)–(I_5). However, it is worth noting that the same convergence order is recovered using a computational order of convergence in the last three numerical examples. In addition, the computational and efficiency advantages of (1.3) over other methods have been reported in [11–16]. That is why we refer the reader to these references, and we deal with the issues (I_1)–(I_5).

The rest of the article is structured as follows: The local and semi-local convergence of the method (1.3) appear in Sections 2 and 3, respectively. The numerical problems can be found in Section 4, and the conclusions are reported in Section 5.

2. Convergence I: Local

Some functions are introduced to study the local convergence analysis of the method (1.3).
Let $\mathbb{A} = [0, +\infty)$.

It is convenient for the presentation of the convergence analysis to introduce some abstractions:
$S(\mathbb{A}, \mathbb{B})$ denotes the set of continuous and nondecreasing functions mapping $\mathbb{A}$ into a set $\mathbb{B}$;
the symbol (ms) stands for the minimal solution of an equation.

Suppose the following:

(H_1) There exists a function $g_0 \in S(\mathbb{A}, \mathbb{A})$ such that the equation $g_0(t) - 1 = 0$ has (ms) denoted by $s_0 \in \mathbb{A} - \{0\}$.
Define the interval $\mathbb{A}_0 = [0, s_0)$.

(H_2) There exists a function $g_0 \in S(A_0, \mathbb{A})$ such that for $h_{-1} : \mathbb{A}_0 \to \mathbb{A}$ defined by

$$h_{-1}(t) = \frac{\int_0^1 g((1-\theta)t)d\theta}{1 - g_0(t)},$$

the equation $h_{-1}(t) - 1 = 0$ has (ms) denoted by $\epsilon_{-1} \in A_0 - \{0\}$.

(H_3) There exists a function $h_0 \in S(\mathbb{A}_0, \mathbb{A})$ such that the equation $h_0(t) - 1 = 0$ has (ms) denoted by $\epsilon_0 \in A_0 - \{0\}$.

(H_4) For each $t \in \mathbb{A}_0,\ i = 1, 2, \dots, k$, define the functions

$$\alpha(t) = \begin{cases} g\Big((1 + h_{-1}(t))t\Big) \\ or \\ g_0(t) + g_0(h_{-1}(t)t), \end{cases}$$

$$\beta^{(i-1)}(t) = \begin{cases} g\Big((1 + h_{i-1}(t))t\Big) \\ or \\ g_0(t) + g_0(h_{i-1}(t)t), \end{cases}$$

$$\gamma^{(i-1)}(t) = \begin{cases} g\Big(h_0(t) + h_{i-1}(t)t\Big) \\ or \\ g_0(h_0(t)t) + g_0(h_{i-1}(t)t), \end{cases}$$

$$h_\sigma(t) = \Bigg[\frac{\int_0^1 g((1-\theta)h_{\sigma-1}(t)t)d\theta}{1-g_0(h_{\sigma-1}(t)t)} + \frac{1}{2}\Big(\frac{\beta^{(\sigma-1)}(t)}{(1-g_0(t))(1-g_0(h_{\sigma-1}(t)t))}$$
$$+ \frac{\gamma^{(\sigma-1)}(t)}{(1-g_0(h_{-1}(t)t))(1-g_0(h_{i-1}(t)t))}$$
$$+ \frac{\alpha(t)}{(1-g_0(h_{-1}(t)t))^2}\Big)(1+\int_0^1 g_0(\theta h_{\sigma-1}(t))d\theta)\Bigg]h_{\sigma-1}(t);$$

the equations $h_\sigma(t) - 1 = 0, \quad \sigma = 1, 2, 3, \cdots, k$ have (ms) denoted by ϵ_σ, respectively in $\mathbb{A}_0 - \{0\}$. Define the parameter ϵ by

$$\epsilon = \min\{\epsilon_j\}, \quad j = -1, 0, 1, 2, \ldots, k \tag{2.1}$$

and the interval $\mathbb{A}_1 = [0, \epsilon)$. Then, for each $t \in \mathbb{A}_1$,

$$0 \le g_0(t) < 1, \tag{2.2}$$
$$0 \le g_0(h_{i-1}(t)t) < 1, \tag{2.3}$$
$$0 \le h_j(t) < 1. \tag{2.4}$$

Next, we relate the functions g_0 and g to the operators on the method (1.3).

By $\mathbb{B}(x,\rho)$, $\mathbb{B}[x,\rho]$, we denote the open and closed balls in $\mathbb{X}$ with center $x \in \mathbb{X}$, and the symbol $L(\mathbb{X},\mathbb{Y})$ stands for the space of bounded linear operators mapping $\mathbb{X}$ into $\mathbb{Y}$.

(H_5) There exists a point $x^* \in \mathbb{D}$ solving the equation $F(x) = 0$ and a linear operator $M \in L(\mathbb{X},\mathbb{Y})$, which is invertible such that for each $u \in \mathbb{D}$,

$$\left\|M^{-1}\Big(F'(u) - M\Big)\right\| \le g_0(\|u - x^*\|).$$

Define the region $\mathbb{D}_0 = \mathbb{D} \cap \mathbb{B}(x_0, \epsilon_0)$.

(H_6) $\left\|M^{-1}\Big(F'(u_2) - F'(u_1)\Big)\right\| \le g(\|u_2 - u_1\|)$, and

$$\|\varphi(u_1,u_2) - x^*\| \le h_0(\|u_1 - x^*\|)\|u_1 - x^*\|, \text{ for each } u_2, u_1 \in \mathbb{D}_0.$$

(H_7) $B[x^*, \epsilon] \subset \mathbb{D}$.

Remark 2.1. *The linear operator M may or may not depend on x^*. Some possible selections for M are $M = I$, the identity operator or $M = F'(\bar{x})$ for some $\bar{x} \in \mathbb{D}$ with $\bar{x} \ne x^*$, or $M = F'(x^*)$. Notice that in the latter case, it follows that x^* is a simple solution, which is the usual assumption in local convergence results. However, such an assumption is not made or implied here by the conditions* (H_5) *and* (H_6).

Define the set $\mathbb{B}_0 = \mathbb{B}(x^*, \epsilon) - \{x^*\}$.

The local convergence analysis follows next.

Theorem 2.1. *Suppose that the conditions* (H_1)–(H_7) *hold. Then, the sequence $\{x_\xi\}$ generated by the method* (1.3)*, for $x_0 \in \mathbb{B}_0$ is well-defined in the ball $\mathbb{B}(x^*, \epsilon)$, and it stays in $\mathbb{B}(x^*, \epsilon)$ and converges to the solution x^* of the equation $F(x) = 0$. Moreover, the following assertions hold for $i = 1, 2, \ldots, k$ and each $\xi = 0, 1, 2, \ldots$:*

$$\|y_\xi - x^*\| \le h_{-1}\Big(\Big\|x_\xi - x^*\Big\|\Big)\|x_\xi - x^*\| \le \|x_\xi - x^*\| \le \epsilon, \tag{2.5}$$

$$\|z_\xi - x^*\| \le h_0\Big(\Big\|x_\xi - x^*\Big\|\Big)\|x_\xi - x^*\| \le \|x_\xi - x^*\|, \tag{2.6}$$

$$\|z_\xi^{(1)} - x^*\| \le h_1\Big(\Big\|x_\xi - x^*\Big\|\Big)\|x_\xi - x^*\| \le \|x_\xi - x^*\|, \tag{2.7}$$

$$\dots$$

$$\|z_\xi^{(i)} - x^*\| \le h_i\Big(\Big\|x_\xi - x^*\Big\|\Big)\|x_\xi - x^*\| \le \|x_\xi - x^*\|. \tag{2.8}$$

In particular, if i = k,

$$\|x_{\xi+1} - x^*\| = \|z_\xi^{(k)} - x^*\| \le h_k\Big(\Big\|x_\xi - x^*\Big\|\Big)\|x_\xi - x^*\| \le \|x_\xi - x^*\|. \tag{2.9}$$

Proof. Mathematical induction shall establish the existence of the sequence $\{x_\xi\}$ in the ball and the validity of the assertions (2.5)–(2.9).

Let $v \in \mathbb{B}(x^*, \epsilon)$. It follows by conditions (H_1), (H_5), (2.1), and (2.2) that

$$\Big\|M^{-1}\Big(F'(v) - M\Big)\Big\| \le g_0(\|r - x^*\|) \le g_0(\epsilon) < 1. \tag{2.10}$$

Then, Estimate (2.10) and the Banach lemma on linear and invertible linear operators [1,10] assure the invertibility of $F'(v)$ and

$$\Big\|F'(v)^{-1}M\Big\| \le \frac{1}{1 - g_0(\|v - x^*\|)}. \tag{2.11}$$

If $v = x_0$, (2.11) holds, because $x_0 \in \mathbb{B}_0$, and we can write by the first substep of the method (1.3),

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= \Big[F'(x_0)^{-1}M\Big]\Big[\int_0^1 M^{-1}\Big(F'(x^* + \theta(x_0 - x^*)\Big) - F'(x_0)\Big)d\theta\Big](x_0 - x^*). \end{aligned} \tag{2.12}$$

Using (2.1), (2.4) for $j = -1$ (2.11) for $v = x_0$ and (H_6), we have by (2.12)

$$\begin{aligned} \|y_0 - x^*\| &\le \frac{\int_0^1 g\Big((1-\theta)\|x_0 - x^*\|\Big)d\theta\|x_0 - x^*\|}{1 - g_0(\|x_0 - x^*\|)} \\ &\le h_{-1}\Big(\Big\|x_0 - x^*\Big\|\Big)\|x_0 - x^*\| \le \|x_0 - x^*\| < \epsilon. \end{aligned} \tag{2.13}$$

Thus, the assertion (2.5) holds if $\xi = 0$, and the iterate $g_0 \in \mathbb{B}(x^*, \epsilon)$.

Then, by the second condition in (H_6), (2.1), and (2.4) for $j = 0$, we get

$$\|z_0 - x^*\| = \|\Phi(x_0, y_0) - x^*\| \le h_0\Big(\Big\|x_0 - x^*\Big\|\Big)\|x_0 - x^*\| \le \|x_0 - x^*\|. \tag{2.14}$$

Hence, the assertion (2.6) holds, and the iterate $z_0 \in \mathbb{B}(x^*, \epsilon)$ holds. Notice also that (2.11) holds if $v = z_0$.

Next, we can write by the third substep of the method (1.3), for $n = 0$

$$z_0^{(1)} - x^* = z_0 - x^* - F'(z_0)^{-1}F(z_0) + F'(z_0)^{-1}F(z_0) - \frac{1}{2}\Phi(x_0, z_0)F(z_0). \tag{2.15}$$

We need the estimate

$$\begin{aligned}F'(y_0)^{-1}F'(x_0)F'(y_0)^{-1} &= F'(y_0)^{-1}\Big(F'(x_0)-F'(y_0)+F'(y_0)\Big)F'(y_0)^{-1}\\ &= F'(y_0)^{-1}\Big(F'(x_0)-F'(y_0)\Big)F'(y_0)^{-1}+F'(y_0)^{-1}.\end{aligned} \tag{2.16}$$

Then, by the definition of the function ψ and (2.16), the identity (2.15) gives

$$\begin{aligned}z_0^{(1)}-x^* &= z_0-x^*-F'(z_0)^{-1}F(z_0)+\frac{1}{2}\Big[\Big(F'(z_0)^{-1}-F'(y_0)^{-1}\Big)\\ &+\Big(F'(z_0)^{-1}-F'(x_0)^{-1}\Big)+F'(y_0)^{-1}\Big(F'(y_0)-F'(x_0)\Big)F'(y_0)^{-1}\Big]F(z_0).\end{aligned} \tag{2.17}$$

An upper bound is needed for each term in (2.17). As in (2.13), we have

$$\|z_0^{(1)}-x^*\| \le \frac{\int_0^1 g\Big((1-\theta)\|z_0-x^*\|\Big)d\theta\|z_0-x^*\|}{1-g_0(\|z_0-x^*\|)}, \tag{2.18}$$

$$\begin{aligned}F(z_0) = F(z_0)-F(x^*) &= \int_0^1 F'\Big(x^*+\theta(z_0-x^*)\Big)d\theta(z_0-x^*),\\ \Big\|M^{-1}F(z_0)\Big\| &= \Big\|\int_0^1 M^{-1}\Big(F'(x^*+\theta(z_0-x^*))-M+M\Big)d\theta(z_0-x^*)\Big\|\\ &\le (1+\int_0^1 g_0\Big(\theta\|z_0-x^*\|)d\theta\Big)\|z_0-x^*\|,\end{aligned} \tag{2.19}$$

$$\begin{aligned}\|(F'z_0)^{-1}-F'(y_0)F(z_0)\| &= [F'(z_0)^{-1}M]\Big[M^{-1}(F'(y_0)-F'(z_0)\Big][F'(y_0)^{-1}M]\Big[M^{-1}F(z_0)\Big]\\ &\le \frac{\gamma_0^{(0)}(1+\int_0^1 g_0(\theta\|z_0-x^*\|))\|z_0-x^*\|}{(1-g_0(h_{-1}(\|x_0-x^*\|))(1-g_0(h_0(x_0-x^*\|)\|x_0-x^*\|))}.\end{aligned} \tag{2.20}$$

Similarly, we get

$$\begin{aligned}\|\Big(F'(z_0)^{-1}-F'(x_0)^{-1}\Big)F(z_0)\| &= \Big\|[F'(z_0)^{-1}M][M^{-1}(F'(x_0)-F'(z_0))][F'(x_0)^{-1}M][M^{-1}F(z_0)]\Big\|\\ &\le \Big\|F'(z_0)^{-1}M\Big\|\Big\|M^{-1}(F'(x_0)-F'(z_0))\Big\|\Big\|F'(x_0)^{-1}M\Big\|\Big\|M^{-1}F(z_0)\Big\|\\ &\le \frac{\gamma_0^{(0)}(1+\int_0^1 g_0(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\|)}{(1-g_0(\|x_0-x^*\|))(1-g_0(h_0(\|x_0-x^*\|)\|x_0-x^*\|))},\\ &= \|F'(y_0)^{-1}(F'(y_0)-F'(x_0))F'(y_0)^{-1}F(z_0)\|\\ &= \|[F'(y_0)^{-1}M][M^{-1}(F'(y_0)-F'(x_0))][F'(y_0)^{-1}M][M^{-1}F(z_0)]\|\\ &\le \frac{\alpha_0\Big(1+\int_0^1 g_0(\theta\|z_0-x^*\|d\theta\Big)\|z_0-x^*\|}{1-g_0(h_{-1}(\|x_0-x^*\|)\|x_0-x^*\|)}.\end{aligned} \tag{2.21}$$

Summing up (2.18)–(2.21) and using (1.2), (2.4) for $j=1$, estimate (2.17) gives:

$$\|z_0^{(1)}-x^*\| \le h_1\Big(\Big\|x_0-x^*\Big\|\Big)\|x_0-x^*\| \le \|x_0-x^*\|. \tag{2.22}$$

So, the assertion (2.7) holds, and the iterate $z_0^{(1)} \in \mathbb{B}(x^*, \epsilon)$ holds. Simply exchange $x_0, y_0, z_0, z_0^{(1)}$ for $x_m, y_m, z_m, z_m^{(\sigma)}(\sigma = 2, 3, ..., k)$, $m = 1, 2, \ldots$, respectively in the estimations (2.17)–(2.22) to complete the induction for (2.8). In particular, if $i = m = k$,

$$\|x_{m+1} - x^*\| = \|z_m^{(k)} - x^*\| \leq h_k\Big(\Big\|x_m - x^*\Big\|\Big)\|x_m - x^*\| \leq \|x_m - x^*\|. \tag{2.23}$$

Notice that by (1.2) and (2.4) for $j = k$, $C : h_k(\|x_0 - x^*\|) \in [0, 1)$. Estimate (2.24) implies

$$\|x_{m+1} - x^*\| \leq C\|x_m - x^*\| \leq C^{m+1}\|x_0 - x^*\| < \epsilon. \tag{2.24}$$

Therefore, we obtain from (2.24) that $\lim\limits_{m\to+\infty} x_m = x^*$, and the iterate $x_{m+1} \in \mathbb{B}(x^*, \epsilon)$ holds. □

Remark 2.2. *The second condition in (H_6) can be dropped if the function φ is specialized. Let us choose as an example $a = 3$ and $\varphi(x, y) = x - \frac{1}{2}\Big(F'(y)^{-1} + F'(x)^{-1}\Big)F(x)$. Then, by the second substep of the method* (1.3)*, we get in turn*

$$\begin{aligned} z_\xi - x^* &= x_\xi - x^* - F'(x_\xi)^{-1}F(x_\xi) + \frac{1}{2}\Big(F'(x_\xi)^{-1} - F'(y_\xi)^{-1}\Big)F(x_\xi) \\ &= x_\xi - x^* - F'(x_\xi)^{-1}F(x_\xi) - \frac{1}{2}F'(y_\xi)^{-1}\Big(F'(x_\xi) - F'(y_\xi)\Big)F'(x_\xi)^{-1}F(x_\xi), \end{aligned}$$

leading to

$$\begin{aligned} \|z_\xi - x^*\| \leq & \left[\frac{\int_0^1 g\Big((1-\theta)\|x_\xi - x^*\|\Big)d\theta}{1 - g_0(\|x_\xi - x^*\|)} + \frac{1}{2}\frac{\alpha_\xi\Big(1 + \int_0^1 g_0(\theta\|x_\xi - x^*\|)d\theta\Big)}{(1 - g_0(\|x_\xi - x^*\|))(1 - g_0(\|y_\xi - x^*\|))}\right] \\ & \times \|x_\xi - x^*\|. \end{aligned}$$

This estimate suggests that if

$$h_0(t) = \frac{\int_0^1 g((1-\theta)t)d\theta}{1 - g_0(t)} + \frac{1}{2}\frac{\alpha(t)\Big(1 + \int_0^1 g_0(\theta t)d\theta\Big)}{(1 - g_0(t))(1 - g_0(h_{-1}(t)t))}, \tag{2.25}$$

the second condition in (H_6) holds.

Next, a region is determined which contains only x^* as a solution of the equation $F(x) = 0$.

Proposition 2.1. *Suppose the condition (H_5) holds in the ball $\mathbb{B}(x^*, s)$ for some $s > 0$, and there exists $s_1 \geq s$ such that*

$$\int_0^1 g_0(\theta s)d\theta < 1. \tag{2.26}$$

Define the region $\mathbb{D}_1 = \mathbb{D} \cap \mathbb{B}[x^*, s]$. Then, x^* is the only solution of the equation $F(x) = 0$ in the region $\mathbb{D}_1$.

Proof. Suppose that there exists $w \in \mathbb{D}_1$ solving the equation $F(x) = 0$ and not satisfying $w \neq x^*$. Define the linear operator $T_1 = \int_0^1 F'\Big(x^* + \theta(w - x^*)\Big)d\theta$. Then, it follows from the condition (H_5) and (2.26) that

$$\Big\|M^{-1}\Big(T_1 - M\Big)\Big\| \leq \int_0^1 g_0\Big(\theta(\|w - x^*\|\Big)d\theta \leq \int_0^1 g_0(\theta s)d\theta < 1.$$

So, the linear operator T_1 is invertible. Finally, from the identity

$$w - x^* = T_1^{-1}\Big(F(w) - F(x^*)\Big) = T_1^{-1}(0) = 0,$$

we deduce that $w = x^*$. □

Remark 2.3. *Under all the conditions (H_1)–(H_7), we can set $s = \epsilon$ in the Proposition* (2.1).

3. Convergence II: Semi-local

The items x^*, g_0, and g are replaced by x_0, f_0, and f, respectively. However, the calculations and formulas remain essentially the same.

Suppose the following:

(C_1) There exists a function $f_0 \in S(\mathbb{A}, \mathbb{A})$ such that the equation $f_0(t) - 1 = 0$ has (ms) denoted by R_0 in the interval $A - \{0\}$. Define the interval $\mathbb{A}_2 = [0, R_0)$.

(C_2) There exists a function $f \in S(\mathbb{A}_2, \mathbb{A})$ and a non-negative sequence $\{\chi_\xi\}$. Define the sequences for $a_0^{(-2)} = 0$, some $a_0^{(-1)} \geq 0$, for some $j = 0, 1, \ldots, k$, and $\xi = 0, 1, 2, \ldots$ by

$$\begin{aligned}
w_\xi &= \begin{cases} f\Big(a_\xi^{(-1)} - a_\xi^{(-2)}\Big) \\ f_0\Big(a_\xi^{(-1)}\Big) + f_0\Big(a_\xi^{(-2)}\Big), \end{cases} \\
\lambda_\xi &= \int_0^1 f\Big((1-\theta)\Big(a_\xi^{(-1)} - a_\xi^{(-2)}\Big)\Big) d\theta \Big(a_\xi^{(-1)} - a_\xi^{(-2)}\Big), \\
\mu_\xi &= \Big(1 + \int_0^1 f_0\Big(a_\xi^{(-1)} + \theta\Big(a_\xi^0 - a_\xi^{(-1)}\Big)\Big) d\theta\Big)(a_\xi^0 - a_\xi^{(-1)}) + \lambda_\xi, \\
a_\xi^0 &= a_\xi^{(-1)} + \chi_\xi, \\
a_\xi^{(-1)} &= a_\xi^0 + \frac{1}{1 - f_0(a_\xi^{(-1)})}\Big(1 + \frac{w_\xi}{1 - f_0(a_\xi^{(-1)})}\Big)\mu_\xi, \\
q_\xi &= \frac{1}{2}\left[\frac{1}{1 - f_0(a_\xi^{(-1)})} + \frac{1}{1 - f_0(a_\xi^{(-2)})} + \frac{w_\xi}{(1 - f_0(a_\xi^{(-1)}))^2}\right], \\
a_\xi^{(j)} &= a_\xi^{(j-1)} + q_\xi \mu_\xi,\ a_{\xi+1}^{(-2)} = a_\xi^{(k)} \\
b_{\xi+1} &= \int_0^1 f\Big((1-\theta)\Big(a_{\eta+1}^{(-1)} - a_{\eta+1}^{(-2)}\Big)\Big) d\theta \Big(a_{\eta+1}^{(-1)} - a_{\eta+1}^{(-2)}\Big) \\
&\quad + \Big(1 + f_0(a_{\xi+1}^{(-2)})\Big)\Big(a_{\xi+1}^{(-1)} - a_{\xi+1}^{(-2)}\Big) \\
&\text{and} \\
a_{\xi+1}^{(-1)} &= a_{\xi+1}^{(-2)} + \frac{b_{\xi+1}}{1 - f_0(a_{\xi+1}^{(-2)})}.
\end{aligned} \tag{3.1}$$

Sequence (1.3) can also be written in a condensed form. for $i \in \{1, 2, \ldots, k\}$. Define the sequence $\{\gamma_n\}$ for $\gamma_{-2} = 0$, $\gamma_{-1} = a_{(-1)}^0$ and each $i = 1, 2, \ldots, k,\ \xi = 0, 1, 2, \ldots$, by

$$\gamma_{\xi+1} = a_\xi^{(k)}. \tag{3.2}$$

The sequence $\{a_\xi^{(j)}\}$ for $j = -2, -1, 0, \ldots, k, \xi = 0, 1, 2, \ldots$ is shown to majorize the sequence $\{x_\xi\}$, $\{y_\xi\}$, $\{z_\xi^{(i)}\}$, $i = 0, 1, 2, \ldots$, $\xi = 1, 2, \ldots$, in Theorem 3.1. But first, a general convergence condition is presented.

(C_3) There exist $R \in [0, R_0)$, such that for each $j = -2, -1, 0, \ldots, k$, and $\xi = 0, 1, 2, \ldots$,

$$f_0(\alpha_\xi^{(-2)}) < 1, \ f_0(\alpha_\xi^{(-1)}) < 1 \text{ and } \alpha_\xi^{(j)} \leq R.$$

This condition and (1.3) imply that the sequence $\{\alpha_\xi^{(j)}\}$ is nondecreasing and bounded for above by R. Hence, there exists $\alpha^* \in [0, R]$ such that $\lim\limits_{\xi \to +\infty} \alpha_\xi^{(k)} = \alpha^*$. It is known that the limit point α^* is the unique least upper bound of the sequence $\{\alpha_\xi^{(k)}\}$. The functions f_0 and f and connect to the operators on the method (1.3).

(C_4) There exist an initial point $x_0 \in \mathbb{D}$ and a linear operator $M \in L(\mathbb{X}, \mathbb{Y})$ which are invertible such that for each $u \in \mathbb{D}$,

$$\left\| M^{-1}\Big(F'(u) - M\Big) \right\| \leq f_0(\|u - x_0\|).$$

Notice that if $u = x_0$, this condition gives by (C_1) that $\|M^{-1}(F'(x_0) - M)\| \leq f_0(0) < 1$. Thus, the operator $F'(x_0)$ is invertible, and we can set $a_0^{-1} = \|F'(x_0)^{-1}F(x_0)\|$ and $\mathbb{D}_2 = \mathbb{D} \cap B(x_0, R_0)$

(C_5) $\left\| M^{-1}\Big(F'(u_2) - F'(u_1)\Big) \right\| \leq f(\|u_2 - u_1\|)$ for each $u_2, u_1 \in \mathbb{D}_2$ and $\|z_n - y_n\| < \chi_n$.

(C_6) $\mathbb{B}[x_0, \alpha^*] \subset \mathbb{D}$.

Remark 3.1. *As in Remark* (2.1) *possible choices for $M = I$ or $M = F'(\bar{x})$ for some $\bar{x} \in \mathbb{D}$ are an auxiliary point or some other choice. The semi-local convergence analysis follows next.*

Theorem 3.1. *Suppose that the conditions $(C_1) - (C_6)$ hold. Then, the sequence $\{x_\xi\}$ generated by the method* (1.3) *is well-defined in $\mathbb{B}(x_0, \alpha^*)$, and it stays in $\mathbb{B}(x_0, \alpha^*)$ and converges to a solution $x^* \in \mathbb{B}[x_0, \alpha^*]$ of the equation $F(x) = 0$. Moreover, the following assertions hold for each $\xi = 0, 1, 2, \ldots$:*

$$\|x^* - x_\xi\| \leq \alpha^* - a_\xi. \tag{3.3}$$

Proof. The formulas and calculations remain the same as in the local case by the (H) conditions; functions g_0 and g are replaced by the (C) conditions and functions f_0 and f, respectively. The following assertions are shown using induction:

$$\|y_\xi - x_\xi\| \leq a_\xi^{(-1)} - a_\xi^{(-2)}, \tag{3.4}$$

$$\|z_\xi - y_\xi\| \leq a_\xi^0 - a_\xi^{(-1)}, \tag{3.5}$$

$$\left\| z_\xi^{(i)} - z_\xi^{(i-1)} \right\| \leq a_\xi^{(i)} - a_\xi^{(i-1)}, \tag{3.6}$$

for each $n = 0, 1, 2, \ldots$ and $i = 1, 2, 3, \ldots, k$. By the choice of $a_0^{(-2)}, a_0^{(-1)}$ and the first substep of the method 1.3 we have $\|y_0 - x_0\| = \|F'(x_0)^{(-1)}F(x_0)\| \leq a_0^{(-1)} - a_0^{(-2)} < \alpha^*$. So, Assertion (3.4), holds if $n = 0$ and the iterate $y_0 \in B(x_0, \alpha^*)$ holds. If we subtract the first from the second substep of the method (1.3), and use the second condition in (C_5), we get

$$\|z_0 - y_0\| \leq \chi_0 = a_0^{(0)} - a_0^{(-1)} < \alpha^*.$$

Thus, the assertion (3.5) holds if $\xi = 0$ and the iterate $z_0 \in B(x_0, \alpha^*)$. We need some estimates:

By the first substep of the method (1.3) we can write in turn

$$\begin{aligned}F(y_m) &= F(y_m) - F(x_m) - F'(y_m)(y_m - x_m)\\ &= \int_0^1 \Big[F'\Big(x_m + \theta(y_m - x_m)\Big) - F'(x_m)\Big]d\theta(y_m - x_m),\end{aligned}$$

so,

$$\begin{aligned}\Big\|M^{-1}F(y_m)\Big\| &\le \int_0^1 f\Big(\theta\|y_m - x_m\|\Big)d\theta\|y_m - x_m\|\\ &\le \int_0^1 f\Big(\theta\Big(a_m^{(-1)} - a_m^{(-2)}\Big)\Big)d\theta\Big(a_m^{(-1)} - a_m^{(-2)}\Big) = \lambda_m\\ F(z_m) &= F(z_m) - F(y_m) + F(y_m)\\ &= \int_0^1 F'\Big(y_m + \theta(z_m - y_m)\Big)(z_m - y_m) + F(y_m),\end{aligned}$$

but

$$\begin{aligned}&\int_0^1 M^{-1}\Big(F'\Big(y_m + \theta(z_m - y_m)\Big) - M + M\Big)d\theta(z_m - y_m)\\ &\le \Big(1 + \int_0^1 f_0\Big(\|y_m - x_0\| + \theta\|z_m - y_m\|\Big)d\theta\Big)\|z_m - y_m\|\\ &\le \Big(1 + \int_0^1 f_0\Big(a_m^{(-1)} + \theta(a_m^{(0)} - a_m^{-1})\Big)d\theta\Big)(a_m^{(0)} - a_m^{(-1)}).\end{aligned}$$

Thus, we get

$$\Big\|F(z_m^{(0)})\Big\| \le \mu_m^{(0)}.$$

Then, we similarly have from the ith step of the method (1.3)

$$\Big\|z_m^{(i)} - z_m^{(i-1)}\Big\| = \Big\|\phi(x_m, y_m)F(z_m^{(i-1)})\Big\| \le q_m\mu_m^{(i)} = a_m^{(i)} - a_m^{(i-1)}$$

and

$$\Big\|z_m^{(i)} - x_0\Big\| \le \Big\|z_m^{(i)} - z_m^{(i-1)}\Big\| + \Big\|z_m^{(i-1)} - x_0\Big\| \le a_m^{(i)} - a_m^{(i-1)} + a_m^{(i-1)} - a_0^{(-2)} < \alpha^*,$$

where we also used

$$\begin{aligned}\phi(x_m, y_m) &= \frac{1}{2}\Big[F'(y_m)^{-1}(F'(x_m) - F'(y_m))F'(y_m)^{-1} + F'(x_m)^{-1})\Big]\\ &= \frac{1}{2}\Big[F'(y_m)^{-1}(F'(x_m) - F'(y_m))F'(y_m)^{-1} + F'(x_m)^{-1})\Big].\end{aligned}$$

So, we have

$$\Big\|\varphi(x_m, y_m)F(z_m^{(i-1)})\Big\| \le \Big\|\varphi(x_m, y_m)M\Big\|\Big\|M^{-1}F(z_m^{(i-1)})\Big\|.$$

Thus, the assertion (3.6) holds, and the iterate $z_m^i \in \mathbb{B}(x_0, \alpha^*)$ holds.

We can also write by the first substep of the method (1.3)

$$
\begin{aligned}
F(x_{m+1}) &= F(x_{m+1}) - F(x_m) - F'(x_m)(y_m - x_m)\\
&= F(x_{m+1}) - F(x_m) - F'(x_m)(x_{m+1} - x_m) + F'(x_m)(x_{m+1} - y_m)\\
&= \int_0^1 \Big[F'\Big(x_m + \theta(x_{m+1} - x_m)\Big) - F'(x_m)\Big]d\theta(x_{m+1} - x_m)\\
&+ \Big(F'(x_m) - M + M\Big)(x_{m+1} - y_m),
\end{aligned}
$$

leading to the following expressions:

$$
\left\|M^{-1}F(x_{m+1})\right\| \le b_{m+1}, \tag{3.7}
$$

$$
\begin{aligned}
\|y_{m+1} - x_{m+1}\| &\le \left\|F'(x_{m+1})^{(-1)}M\right\|\left\|M^{(-1)}F(x_{m+1})\right\|\\
&\le \frac{b_{m+1}}{1 - f_0(a_{m+1}^{(-2)})} = a_{m+1}^{(-1)} - a_{m+1}^{(-2)},
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_{m+1} - x_0\| &\le \|y_{m+1} - x_{m+1}\| + \|x_{m+1} - x_0\|\\
&\le a_{m+1}^{(-1)} - a_{m+1}^{(-2)} + a_{m+1}^{(-2)} - a_0^{(-2)} < \alpha^*.
\end{aligned}
$$

The induction for the assertions (3.4)–(3.6) is completed, and all iterates of the method (1.3) belong in $\mathbb{B}(x_0, \alpha^*)$. It also follows that the sequence $\{x_m\}$ is Cauchy in the Banach space $\mathbb{X}$, and as such, it converges to some $x^* \in \mathbb{B}[x_0, \alpha^*]$, which is a closed set. Moreover, by letting $m \to \infty$ in (3.7), we deduce that $F(x^*) = 0$, where we also used the continuity of the operator F. Furthermore, by the triangular inequality for $\sigma = 0, 1, 2, \ldots$

$$
\|x_{m+\sigma} - x_m\| \le a_{m+\sigma}^{-2} - a_m^0. \tag{3.8}
$$

Finally, by letting $\sigma \to +\infty$ in (3.8), we show (3.3). □

Remark 3.2. *As in Remark 2.2, we can determine the sequence* $\{\chi_\xi\}$ *if the function is chosen as* $\varphi(x_\xi, y_\xi) = x_\xi - \frac{1}{2}\Big(F'(y_\xi)^{-1} + F'(x_\xi)^{-1}\Big)F(x_\xi)$.

Then, we have by subtracting the first from the second substep of the method (1.3) *in turn that*

$$
\begin{aligned}
z_\xi - y_\xi &= \varphi(x_\xi, y_\xi) - x_\xi + F'(x_\xi)^{-1}F(x_\xi)\\
&= \frac{1}{2}\Big(F'(x_\xi)^{-1}F'(y_\xi)^{-1}\Big)F(x_\xi) = -\frac{1}{2}\Big(F'(x_\xi)^{-1} - F'(y_\xi)^{-1}\Big)F(x_\xi)\\
&= -\frac{1}{2}F'(y_\xi)^{-1}\Big(F'(x_\xi) - F'(y_\xi)\Big)F'(x_\xi)^{-1}F(x_\xi)\\
&= \frac{1}{2}F'(y_\xi)^{-1}\Big(F'(x_\xi) - F'(y_\xi)\Big)(y_\xi - x_\xi),
\end{aligned}
$$

so

$$
\begin{aligned}
\|z_\xi - y_\xi\| &\le \frac{1}{2}\left\|F'(y_\xi)^{-1}M\right\|\left\|M^{-1}\Big(F'(x_\xi) - F'(y_\xi)\Big)\right\|\|y_\xi - x_\xi\|\\
&\le \frac{1}{2}\frac{a_\xi\Big(a_\xi^{-1} - a_\xi^{-2}\Big)}{1 - f_0(a_\xi^{-1})} := \chi_\xi.
\end{aligned} \tag{3.9}
$$

Next, a region is determined containing only one solution of the equation $F(x) = 0$.

Proposition 3.1. *Suppose that there exists a solution $w_\xi \in \mathbb{B}(x_0, R_1)$ of the equation $F(x) = 0$ some $R_1 > 0$ of the equation $F(x) = 0$, and there exists $R_2 \geq R_1$ such that*

$$\int_0^1 f_0\Big((1-\theta)R_1 + \theta R_2\Big)d\theta < 1. \tag{3.10}$$

Define the region $\mathbb{D}_4 = \mathbb{D} \cap \mathbb{B}[x_0, R_2]$.

Then, w_1 is the only solution of the equation $F(x) = 0$ in the region $\mathbb{D}_4$.

Proof. Suppose that there exists $w_2 \in \mathbb{D}_4$ solving the equation $F(x) = 0$ and satisfying $w_2 \neq w_1$. Define the linear operator $T_2 = \int_0^1 F'(w_1 + \theta(w_2 - w_1))d\theta$. Using the condition (C_4) and (3.10), we get

$$\begin{aligned}\Big\|M^{-1}(T_2 - M)\Big\| &\leq \int_0^1 f_0\Big((1-\theta)\|w_2 - x_0\| + \theta\|w_2 - x_0\|\Big)d\theta \\ &\leq \int_0^1 f_0\Big((1-\theta)R_1 + \theta R_2\Big)d\theta < 1.\end{aligned}$$

Hence, the linear operator T_2 is invertible. Finally, from the identity

$$F(w_2) - F(w_1) = T_2(w_2 - w_1) = 0,$$

we deduce that $w_2 = w_1$. □

Remark 3.3. *(i) The limit point α^* can be replaced in the condition (C_6) by the parameter R_0 given in (C_1).*

(ii) If all the conditions (C_1)–(C_6) hold, then we can set $w_1 = x^*$ and $R_1 = \alpha^*$ in the Proposition 3.1.

4. Numerical examples

In this section, we choose the following three methods:

$$\begin{aligned} y_\xi &= x_\xi - \frac{2}{3}F'(x_\xi)^{-1}F(x_\xi), \\ z_\xi &= x_\xi - \frac{1}{4}\Big[3F'(y_\xi)^{-1} + F'(x_\xi)^{-1}\Big]F(x_\xi), \\ x_{\xi+1} &= z_\xi - \Big[\frac{3}{2}F'(y_\xi)^{-1} - \frac{1}{2}F'(x_\xi)^{-1}\Big]F(z_\xi); \end{aligned} \tag{4.1}$$

$$\begin{aligned} y_\xi &= x_\xi - \frac{2}{3}F'(x_\xi)^{-1}F(x_\xi), \\ z_\xi &= x_\xi - \frac{1}{8}\Big[3F'(y_\xi)^{-1}F'(x_\xi)F'(y_\xi)^{-1} + 5F'(x_\xi)^{-1}\Big]F(x_\xi), \\ x_{\xi+1} &= z_\xi - \Big[\frac{3}{2}F'(y_\xi)^{-1} - \frac{1}{2}F'(x_\xi)^{-1}\Big]F(z_\xi); \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
y_\xi &= x_\xi - F'(x_\xi)^{-1}F(x_\xi),\\
z_\xi &= x_\xi - \Big[\frac{1}{2}F'(y_\xi)^{-1} + \frac{1}{2}F'(x_\xi)^{-1}\Big]F(x_\xi),\\
x_{\xi+1} &= z_\xi - \Big[\frac{1}{2}F'(y_\xi)^{-1}F'(x_\xi)F'(y_\xi)^{-1} + \frac{1}{2}F'(x_\xi)^{-1}\Big]F(z_\xi).
\end{aligned}
\tag{4.3}
$$

We refer to the above methods (4.1)–(4.3) as Cases (I)–(III), respectively, in the numerical results.

Moreover, we evaluate the computational significance of Cases (I)–(III), as theoretically proved in the previous section. We consider a total of five nonlinear problems, of which the first two demonstrate LAC (local acceleration convergence), whereas the remaining examples demonstrate SLAC (semi-local acceleration convergence). The radii of convergence for local convergence are given in Tables 1 and 2. On the other hand, Tables 4 and 5 present the results for semi-local convergence. Furthermore, we determine the coefficient of convergence (COC) [10] using the following equations:

$$
\vartheta = \frac{\ln \frac{\|x_{\tau+1}-x_*\|}{\|x_\tau - x_*\|}}{\ln \frac{\|x_\tau - x_*\|}{\|x_{\tau-1}-x_*\|}}, \qquad \text{for } \tau = 1, 2, \ldots, \tag{4.4}
$$

or approximated coefficient of convergence ($ACOC$) [5, 18] by:

$$
\vartheta^* = \frac{\ln \frac{\|x_{\tau+1}-x_\tau\|}{\|x_\tau - x_{\tau-1}\|}}{\ln \frac{\|x_\tau - x_{\tau-1}\|}{\|x_{\tau-1}-x_{\tau-2}\|}}, \qquad \text{for } \tau = 2, 3, \ldots. \tag{4.5}
$$

Table 1. Radii for Example (4.1).

Method (1.3)	s_0	ϵ_{-1}	ϵ_0	ϵ_1	ϵ_2	ϵ	
$\sigma = 1$	1	0.5819	0.3827	0.2329	0.1946	-	0.1946
$\sigma = 2$	2	0.5819	0.3827	0.2329	0.1946	0.1524	0.1524

Table 2. Radii for Example (4.2).

Method (1.3)	s_0	ϵ_{-1}	ϵ_0	ϵ_1	ϵ_2	ϵ
$\sigma = 1$	0.07407	0.03704	0.01991	0.01587	-	0.01587
$\sigma = 2$	0.07407	0.03704	0.01991	0.01587	0.01472	0.01472

Various stopping criteria are available for iterative methods. However, the most commonly used criteria in the literature are either Criterion $\|x_{\tau+1} - x_\tau\| < \Delta$, and Criterion $\|F(x_\tau)\| < \Delta$. In this study, both criteria are employed to achieve higher accuracy. This choice is motivated by the observation that there are cases in which Criterion $\|x_{\tau+1} - x_\tau\| < \Delta$, is satisfied while Criterion $\|F(x_\tau)\| < \Delta$ is not, and vice versa. Furthermore, we set $\Delta = 10^{-210}$ to minimize round-off errors and obtain more accurate numerical solutions.

All numerical calculations were carried out using Mathematica 11 with multiple-precision arithmetic. The use of multiple-precision arithmetic minimizes round-off errors and enhances computational reliability. The computational results were obtained on a personal computer with the following hardware and software specifications: an Intel® Core™ i7-4790 CPU operating at 3.60 GHz, manufactured by HP, equipped with 8.00 GB of RAM (7.89 GB usable), and running a 64-bit Windows 10 Enterprise operating system (version 22H2) on an x64-based processor.

4.1. Examples for LAC

In order to highlight the theoretical concepts of local convergence introduced earlier in Section 2, we consider two examples, denoted by (4.1) and (4.2). These examples are carefully selected to demonstrate the applicability of Method (1.3). Moreover, the radii of convergence ensure the confirmed convergence of the method. It is almost impossible to choose an initial guess close to the desired root without knowledge of the radii of convergence. In this way, these examples provide a deeper understanding of the local convergence behavior of the Method (1.3) and also confirm the theoretical results discussed earlier.

Example 4.1. *Let $k = 3$ and $\mathbb{D} = U(0,1)$. Define the mapping F on $\mathbb{R}^k$ for $s = (s_1, s_2, s_3)^T$*

$$F(s) = \left(\frac{e-1}{2}s_1^2 + s_1, \quad e^{s_2} - 1, \quad s_3\right)^T. \tag{4.6}$$

Then, the Jacobian of the mapping H is defined by

$$F'(s) = \begin{bmatrix} (e-1)s_1 + 1 & 0 & 0 \\ 0 & e^{s_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)^T$ solves the equation $H(h) = 0$, and $H'(x^*) = I$. Let us choose $M = I$. By plugging the values of F' in the conditions (H_5) and (H_6) hold, we see that they are validated if we choose the following values:

$$g_0(t) = (e-1)t \quad \text{and} \quad g(t) = e^{\frac{1}{e-1}}t.$$

In Table 1, we present radii for Example (4.1).

Example 4.2. *Let $\mathbb{X} = \mathbb{Y} = C[0,\ 1]$, and let the space of continuous functions defined on* $[0,\ 1]$ *be and equipped with the max norm. Let $\mathbb{D} = \bar{U}(0,\ 1)$. Define function F on $\mathbb{D}$ by*

$$F(\varphi)(x) = \varphi(x) - 9\int_0^1 x\tau\varphi(\tau)^3 d\tau, \tag{4.7}$$

so that we have

$$F'\Big(\varphi(\xi)\Big)(x) = \xi(x) - 27\int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in \mathbb{D}. \tag{4.8}$$

Then, for $x^* = 0 \in [0,1]$ solves the equation (4.7). *The conditions (H_5) and (H_6) are validated if we choose the following values:*

$$g_0(t) = 13.5t \quad \text{and} \quad g(t) = 27t.$$

Table 2 suggests the radii of convergence for Example (4.2).

4.2. Examples for SLAC

In order to illustrate the applicability of semi-local convergence, we chose three different problems. The first two are well-known: the Hammerstein integral equation and nutrient diffusion in a biological substrate from applied science. Detailed information is provided in Examples (4.3) and (4.4). We transformed these two examples into nonlinear systems of orders 10×10 and 169×169, respectively. Finally, we considered an academic exercise involving algebraic and exponential functions, with more information provided in (4.5). In this Example (4.5), we obtained another larger nonlinear system of order 175×175, which allows us to demonstrate the applicability of the methods, namely Case–I, Case–II, and Case–III on large systems of nonlinear equations. Moreover, Cases (I)–(III) are compared with the sixth-order family of iterative methods developed by Cordero et al. [26], from which method (M_6), denoted by (CM_6), is selected. A further comparison is carried out with sixth-order family proposed by Behl et al. [28], in which we considered Expression (23) with $g = -\frac{1}{2}$, denoted by (BM_6). Finally, we also compared the mentioned method with the efficient Jarratt-like method introduced by Sharma [27], specifically Method (18), referred to as (SM_6). Moroever, algorithmic description of case-III for nonlinear system in the Table 3.

Table 3. Algorithmic description of Case–III for nonlinear systems.

Input: Initial guess x_0, tolerance $\Delta < 10^{-210}$, maximum iterations $N_{\max}$
Step 1: Initialize set $\xi = 0$.
Step 2: First Substep (Newton-like Step) Compute $F(x_\xi)$ and the Jacobian matrix $F'(x_\xi)$, then calculate the first substep $y_\xi = x_\xi - \frac{2}{3}F'(x_\xi)^{-1}F(x_\xi)$
Step 3: Second Substep (Correction Step) Compute a second intermediate point using a correction function $z_\xi = x_\xi - \frac{1}{4}\Big[3F'(y_\xi)^{-1} + F'(x_\xi)^{-1}\Big]F(x_\xi)$
Step 4: Third Substep (Final Update) Compute the new iterate $x_{\xi+1} = z_\xi - \Big[\frac{3}{2}F'(y_\xi)^{-1} - \frac{1}{2}F'(x_\xi)^{-1}\Big]F(z_\xi)$ using a combination of previous substeps.
Step 5: Convergence Check $\lVert x_{\tau+1} - x_\tau \rVert < 10^{-210}$ and $\lVert F(x_\tau) \rVert < 10^{-210}$ then stop and return x^*.
Step 6: Iteration Set $\xi = \xi + 1$, if $k < N_{max}$, go back to Step 2; otherwise, terminate with failure.

(Similarly, we can write for Case–I and Case–II.)

Example 4.3. Hammerstein integral equation:

Here, we investigate a classical problem in applied science, specifically the Hammerstein integral equation, as presented in [10, pp. 19 and 20]. The primary objective is to evaluate and compare the efficiency, accuracy, and practical applicability of the proposed methods for semi-local convergence. The following formulation of the Hammerstein integral equation is employed as a benchmark problem for this comparative analysis:

$$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3 dt,$$

where $x \in C[0,1]$ $s,t \in [0,1]$, *and the kernel* G *is*

$$G(s,t) = \begin{cases} (1-s)t, t \le s, \\ s(1-t), s \le t. \end{cases}$$

By applying the Gauss-Legendre quadrature formula, the original equation is transformed into an equivalent finite-dimensional problem as follows:

$$\int_0^1 g(t)dt \simeq \sum_{j=1}^{10} w_j g(t_j),$$

where the abscissas t_j *and the associated weights* w_j *are computed using the Gauss-Legendre quadrature formula with* $j = 10$. *The* x_i *denote the numerical approximation of* $x(t_i)$ *at the quadrature nodes* t_i, *for* $i = 1, 2, ..., 10$. *Consequently, the integral equation is reduced to a system of nonlinear algebraic equations, as given below:*

$$5x_i - 5 - \sum_{j=1}^{10} a_{ij} x_j^3 = 0, \quad i = 1, 2, ..., 10,$$

where

$$a_{ij} = \begin{cases} w_j t_j (1-t_i), j \le i, \\ w_j t_i (1-t_j), i < j. \end{cases}$$

The values of abscissas t_j *and associated weights* w_j *are given in Expressions* (4.9) *and* (4.10)*:*

$$\begin{aligned}(t_1, t_2, t_3, \ldots, t_{10})^T =& \Big(0.01304\ldots, 0.06746\ldots, 0.1602\ldots, 0.2833\ldots, 0.4255\ldots, \\ & 0.5744\ldots, 0.7166\ldots, 0.8397\ldots, 0.9325\ldots, 0.9869\ldots\Big)^T\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}(w_1, w_2, \ldots, w_{10})^T =& \Big(0.03333\ldots, 0.07472\ldots, 0.1095\ldots, 0.1346\ldots, 0.1477\ldots, \\ & 0.1477\ldots, 0.1346\ldots, 0.1095\ldots, 0.07472\ldots, 0.03333\ldots\Big)^T,\end{aligned} \tag{4.10}$$

respectively.

Table 4 presents the computational order of convergence (COC), CPU execution time, residual errors, and the differences between successive iterates for Example (4.3). *The iterative procedure converges to the root, which is represented by the following column vector:*

$$x^* = \Big(1.001\ldots, 1.006\ldots, 1.014\ldots, 1.021\ldots, 1.026\ldots, 1.026\ldots, 1.021\ldots, 1.014\ldots, 1.006\ldots, 1.0013\ldots\Big)^T$$

In addition, we adopt the following initial approximation,

$$x_0 = \overbrace{\left(\frac{9}{10}, \frac{9}{10}, \cdots, \frac{9}{10}\right)}^{10\ times}{}^T,$$

for the computational results.

Table 4. Computational results of Example (4.3).

Cases	$\|F(x_3)\|$	$\|x_4 - x_3\|$	ξ^*	ϑ	CPU timing
Case–I	1.1×10^{-236}	2.4×10^{-237}	3	4.9995	0.326663
Case–II	1.0×10^{-375}	2.2×10^{-376}	3	5.9995	0.434044
Case–III	6.7×10^{-272}	1.4×10^{-272}	3	4.9784	0.495059
CM_6	4.9×10^{-408}	1.1×10^{-408}	3	5.9999	0.383456
BM_6	6.6×10^{-377}	1.4×10^{-377}	3	5.9995	0.399138
SM_6	1.1×10^{-363}	2.3×10^{-364}	3	5.9995	0.583254

(ξ^* : stands for the number of iterations to reach the desired accuracy.)

Example 4.4. Nutrient diffusion in a biological substrate:

Nutrient diffusion is essential in biological systems [17], affecting cellular metabolism, tissue engineering, and microbial growth. Mathematical models of nutrient diffusion enable the prediction of nutrient availability, the optimization of bioreactor performance, and the advancement of medical applications, including drug delivery. In this study, we examine a two-dimensional biological substrate that functions as a growth medium for microorganisms. Our aim is to investigate how nutrients diffuse and are distributed within this medium, as these processes are critical for the organism's development and survival. The corresponding mathematical formulation is presented below.

$$\frac{\partial^2 u}{\partial \mathfrak{x}^2} + \frac{\partial^2 u}{\partial \mathbf{y}^2} = u(\mathfrak{x}, \mathbf{y})^3 + |u(\mathfrak{x}, \mathbf{y})|, \quad \textit{for}\ \Omega = \{(\mathfrak{x}, \mathbf{y}) \in \mathbb{R}^2 : \mathfrak{x}, \mathbf{y} \in [0, 1]\},$$
$$u(\mathfrak{x}, 0) = 2\mathfrak{x}^2 - \mathfrak{x} + 1, \quad u(\mathfrak{x}, 1) = 2, \quad 0 \le \mathfrak{x} \le 1,$$
$$u(0, \mathbf{y}) = 2\mathbf{y}^2 - \mathbf{y} + 1, \quad u(1, \mathbf{y}) = 2, \quad 0 \le \mathbf{y} \le 1.$$

Let $u(\mathfrak{x}, \mathbf{y})$ denote the nutrient concentration at location $(\mathfrak{x}, \mathbf{y})$ in the substrate. The equation models nutrient diffusion, with $u(\mathfrak{x}, \mathbf{y})^3 + |u(\mathfrak{x}, \mathbf{y})|$ representing interactions with biochemical processes.

Boundary conditions define nutrient behavior at the substrate edges. Lateral boundaries ($\mathfrak{x} = 0$ and $\mathfrak{x} = 1$) reflect initial concentrations or continuous supply, while top and bottom ($\mathbf{y} = 0$) and ($\mathbf{y} = 1$)) capture uptake by roots or microorganisms [17].

Solving the model reveals nutrient distribution and its effects on organism growth and health. As an example, the equation is solved for a small system using a blockwise finite difference method on a computational mesh:

$$h = \frac{1}{n+1}, \quad k = \frac{1}{m+1},$$

along the mesh points $(\mathfrak{x}_i, \mathbf{y}_j)$ *provides as* $\mathfrak{x}_i = ih$, $i = 0, \ldots, n+1$, *and* $\mathbf{y}_j = jk$, $j = 0, \ldots, m+1$. *In this way, we obtain the following discrete form:*

$$u_{\mathfrak{x}\mathfrak{x}}(\mathfrak{x}_i, \mathbf{y}_j) + u_{\mathbf{y}\mathbf{y}}(\mathfrak{x}_i, \mathbf{y}_j) = u(\mathfrak{x}_i, \mathbf{y}_j)^3 + |u(\mathfrak{x}_i, \mathbf{y}_j)|.$$

This discretized formulation enables the numerical computation of $u(\mathfrak{x}, \mathbf{y})$ *using IMs. We use the preceding finite differences for* $i = 1, \ldots, n$, *and* $j = 1, \ldots, m$

$$\begin{aligned}
&\frac{u(\mathfrak{x}_{i+1}, \mathbf{y}_j) - 2u(\mathfrak{x}_i, \mathbf{y}_j) + u(\mathfrak{x}_{i-1}, \mathbf{y}_j)}{h^2} + \frac{u(\mathfrak{x}_i, \mathbf{y}_{j+1}) - 2u(\mathfrak{x}_i, \mathbf{y}_j) + u(\mathfrak{x}_i, \mathbf{y}_{j-1})}{k^2} \\
&= u(\mathfrak{x}_i, \mathbf{y}_j)^3 + |u(\mathfrak{x}_i, \mathbf{y}_j)|.
\end{aligned}$$

in order to obtain the following nonlinear system of equations:

$$2\left(-\frac{\Lambda^2+1}{\Lambda^2}\right)u_{i,j} - \frac{1}{\Lambda^2}(u_{i,j+1} + u_{i,j-1}) - (u_{i+1,j} + u_{i-1,j}) + h^2\left(u_{i,j}^3 + |u_{i,j}|\right) = 0,$$

where $u(\mathfrak{x}_i, \mathbf{y}_j) = u_{i,j}$. *We select the parameter values*

$$\Lambda = \frac{h}{k}, \quad i = j = 1, 2, 3 \ldots, 14,$$

which results in a nonlinear system of dimension 169×169. *All three methods, namely Case–I, Case–II, and Case–III , converge to the following approximate solution:*

$$\begin{aligned}
\xi = \Big(&0.9210\ldots, 0.9075\ldots, 0.9047\ldots, 0.9154\ldots, 0.9413\ldots, 0.9833\ldots, 1.042\ldots, 1.119\ldots, \\
&1.213\ldots, 1.326\ldots, 1.458\ldots, 1.612\ldots, 1.790\ldots, 0.9075\ldots, 0.9149\ldots, 0.9266\ldots, 0.9466\ldots, \\
&0.9774\ldots, 1.021\ldots, 1.078\ldots, 1.150\ldots, 1.239\ldots, 1.344\ldots, 1.469\ldots, 1.616\ldots, 1.790\ldots, \\
&0.9047\ldots, 0.9266\ldots, 0.9491\ldots, 0.9762\ldots, 1.011\ldots, 1.055\ldots, 1.111\ldots, 1.180\ldots, 1.263\ldots, \\
&1.363\ldots, 1.482\ldots, 1.623\ldots, 1.793\ldots, 0.9154\ldots, 0.9466\ldots, 0.9762\ldots, 1.008\ldots, 1.045\ldots, \\
&1.089\ldots, 1.143\ldots, 1.209\ldots, 1.288\ldots, 1.382\ldots, 1.496\ldots, 1.632\ldots, 1.797\ldots, 0.9413\ldots, \\
&0.9774\ldots, 1.011\ldots, 1.045\ldots, 1.083\ldots, 1.126\ldots, 1.178\ldots, 1.240\ldots, 1.314\ldots, 1.404\ldots, \\
&1.512\ldots, 1.642\ldots, 1.802\ldots, 0.9833\ldots, 1.021\ldots, 1.055\ldots, 1.089\ldots, 1.126\ldots, 1.168\ldots, \\
&1.217\ldots, 1.275\ldots, 1.344\ldots, 1.428\ldots, 1.530\ldots, 1.654\ldots, 1.808\ldots, 1.042\ldots, 1.078\ldots, \\
&1.111\ldots, 1.143\ldots, 1.178\ldots, 1.217\ldots, 1.262\ldots, 1.315\ldots, 1.379\ldots, 1.456\ldots, 1.552\ldots, \\
&1.669\ldots, 1.815\ldots, 1.119\ldots, 1.150\ldots, 1.180\ldots, 1.209\ldots, 1.240\ldots, 1.275\ldots, 1.315\ldots, \\
&1.362\ldots, 1.420\ldots, 1.491\ldots, 1.578\ldots, 1.687\ldots, 1.824\ldots, 1.213\ldots, 1.239\ldots, 1.263\ldots, \\
&1.288\ldots, 1.314\ldots, 1.344\ldots, 1.379\ldots, 1.420\ldots, 1.470\ldots, 1.533\ldots, 1.610\ldots, 1.709\ldots, \\
&1.836\ldots, 1.326\ldots, 1.344\ldots, 1.363\ldots, 1.382\ldots, 1.404\ldots, 1.428\ldots, 1.456\ldots, 1.491\ldots, \\
&1.533\ldots, 1.585\ldots, 1.652\ldots, 1.738\ldots, 1.850\ldots, 1.458\ldots, 1.469\ldots, 1.482\ldots, 1.496\ldots, \\
&1.512\ldots, 1.530\ldots, 1.552\ldots, 1.578\ldots, 1.610\ldots, 1.652\ldots, 1.705\ldots, 1.775\ldots, 1.870\ldots, \\
&1.612\ldots, 1.616\ldots, 1.623\ldots, 1.632\ldots, 1.642\ldots, 1.654\ldots, 1.669\ldots, 1.687\ldots, 1.709\ldots, \\
&1.738\ldots, 1.775\ldots, 1.826\ldots, 1.897\ldots, 1.790\ldots, 1.790\ldots, 1.793\ldots, 1.797\ldots, 1.802\ldots, \\
&1.808\ldots, 1.815\ldots, 1.824\ldots, 1.836\ldots, 1.850\ldots, 1.870\ldots, 1.897\ldots, 1.937\ldots\Big)^T,
\end{aligned}$$

corresponding to the following starting point

$$x_0 = (\overbrace{1, 1, 1, \cdots, 1}^{169\ times})^T .$$

Table 5 presents the coefficient of convergence (COC), CPU time, residual errors, the values of ϑ, and the differences in errors between successive iterations for Example (4.4)*, as computed using three iterative methods.*

Table 5. Numerical results for Example 4.4.

Cases	$\|\|F(x_3)\|\|$	$\|\|x_4 - x_3\|\|$	ξ^*	ϑ	CPU timing
Case–I	7.5×10^{-144}	5.3×10^{-143}	4	5.0019	112.344
Case—II	7.9×10^{-257}	5.5×10^{-256}	3	6.0027	100.122
Case-III	2.2×10^{-160}	1.2×10^{-159}	4	4.9586	355.113
CM_6	1.3×10^{-203}	9.0×10^{-203}	4	5.9986	495.015
BM_6	1.2×10^{-245}	8.2×10^{-245}	3	6.0023	289.251
SM_6	1.7×10^{-204}	1.2×10^{-203}	4	5.9996	1203.60

Example 4.5. *Finally, we consider an academic exercise involving algebraic and exponential functions, which is given below:*

$$F(X) = \sum_{i \neq j} \left[x_j - \exp(-x_i) \right]. \tag{4.11}$$

From the expression (4.11)*, a large nonlinear system of order* 175×175*, (with n equal to* 175*) is obtained which enables the demonstration of the applicability of the methods, namely Case–I, Case–II, and Case–III on large-scale systems of nonlinear equations. We choose the following starting point*

$$x_0 = \overbrace{\left(\frac{6}{10}, \frac{6}{10}, \frac{6}{10}, \cdots, \frac{6}{10} \right)^T}^{175\ times},$$

for the desired solution

$$\xi^* = (\overbrace{0.5671\ldots, 0.5671\ldots, 0.5671\ldots, \cdots, 0.5671\ldots}^{175\ times})^T .$$

Table 6 depicts the coefficient of convergence (COC), CPU time, residual errors, the values of ϑ, and the differences in errors between successive iterations for Example (4.5)*, as computed using three iterative methods.*

Table 6. Computational results of Example (4.5).

Cases	$\|F(x_3)\|$	$\|x_4 - x_3\|$	ξ^*	ϑ	CPU timing
Case–I	3.7×10^{-275}	1.4×10^{-277}	3	5.1032	485.153
Case–II	2.8×10^{-509}	1.0×10^{-511}	3	6.0800	674.066
Case–III	1.1×10^{-487}	4.1×10^{-490}	3	6.0836	684.648
CM_6	1.6×10^{-434}	5.9×10^{-437}	3	6.0000	586.542
BM_6	4.0×10^{-481}	1.5×10^{-483}	3	6.0000	463.841
SM_6	5.2×10^{-428}	1.9×10^{-430}	3	6.0000	899.972

5. Conclusions

Taylor series expansions have been widely used in the literature to establish the convergence order of iterative methods [4–18]. However, such expansions require assumptions on high order derivatives which are not on the methods and are costly or may not even exist. Here, the applicability of such methods is limited to solve equations where the operator is sufficiently many times differentiable. However, these methods may converge even if such high order derivatives do not exist. Moreover, the Taylor series approach does not provide isolation of the solution or computable error distances. That is why we introduce a Taylor series free technique where the convergence is established using the operators on the method and in more general setting than the Euclidean space. Additionally, a priori error distances are provided, which are computable as well as solution isolation results. The error distances are controlled by the generalized continuity of F'. The same approach can we used on other methods [1–15, 17, 18]. This is the focus of future research.

Author contributions

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Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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