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***Research article***

# Covering morphisms of soft groupoids

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**Abstract:** Soft set theory has been developed as an effective mathematical tool for handling uncertainty through parameterization, leading to more practical solutions to complex problems. In parallel, the theory of covering spaces and their algebraic interpretation via groupoids has played a fundamental role in algebraic topology. Previous studies have established important categorical equivalences between coverings of spaces, coverings of groupoids, and groupoid actions on a set. In this paper, these ideas are extended to the framework of soft set theory. We introduce the notion of soft covering groupoids and define the category $SGdCov(H)$ of soft coverings of a soft groupoid $H$. Additionally, we define the actions of soft groupoids and the category $SGdOp(H)$ of soft actions of soft groupoid $H$. The main result of this study is the proof of a categorical equivalence between these two categories, thereby providing a new algebraic perspective on soft covering structures and contributing to the interaction between soft set theory and categorical topology.

**Keywords:** soft set; category; groupoid; covering; action; equivalent categories
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## 1. Introduction

Traditional methods in mathematics are insufficient to solve some uncertainty problems of our age. One of the important modern theories developed to overcome this situation is undoubtedly soft set theory introduced by Molodtsov [1]. Since its inception, soft set theory has been successfully applied to a wide range of mathematical structures, including groups [2], rings [3], groupoids [4], and topological spaces [5], leading to the development of various soft algebraic systems [6–8].

The concept of soft groupoids arises from the integration of soft group theory with groupoid theory. A soft groupoid can be viewed as a soft set over a universe where each parameter corresponds to a subgroupoid of a given groupoid. This combination allows the study of algebraic properties under parameterized uncertainty, providing a flexible and robust framework for handling situations where algebraic information depends on varying parameters. As a result, soft groupoids offer promising applications in decision-making, information systems, and other areas where algebraic structures interact with uncertain or parameter-dependent data.

The notion of a covering plays a fundamental role in mathematics, particularly in topology and algebra, as it allows the transfer of structural properties from one object to another while preserving local behavior. In the context of groupoids, coverings generalize classical covering spaces and group coverings, offering a unified approach to studying local–global relationships within groupoid structures. In other words, a covering of groupoids is typically defined as a morphism between groupoids that is locally bijective on objects and morphisms, preserving the underlying algebraic structure [9–11]. Such coverings enable the lifting of morphisms and algebraic properties from the base groupoid to the covering groupoid. This concept provides a natural framework for analyzing groupoid actions and fundamental groupoids arising in both algebraic and topological settings. In this context, R. Brown [12] made significant contributions to the application of covering spaces to groupoid theory. He proved the existence of a covering morphism of the groupoid $\pi p : \pi\tilde{Y} \to \pi Y$ for any given space $Y$ by defining the fundamental groupoid $\pi Y$. More generally, he showed the equivalence of categories between the category $TCov(Y)$ of coverings of a space $Y$ and the category $GdCov(\pi Y)$ of covering morphisms of the fundamental groupoids $\pi Y$ [12]. One of the important equivalences for covering groupoids is the category $GdCov(H)$ of the groupoid coverings of $H$, which is equivalent to the category $GdOp(H)$ of the actions of $H$ on sets [13]. The topological version of this equivalence was proved in [14]. Many mathematicians have proven equivalences of various categories by taking different algebraic structures [15–17]. Consequently, the coverings of groupoids offer a rich and versatile area of study, bringing algebraic and topological perspectives and providing deeper insight into the structural behavior of groupoids and their morphisms.

The aim of this article is to expose the basic definitions and theories given for covering groupoids to soft groupoid theory. First, the concepts of soft star and soft costar are defined for the soft groupoids, the definition of soft covering morphisms among soft groupoids is given, and the relevant theorems are proven. In addition, for a given soft groupoid $H$, a category $SGdCov(H)$ of soft coverings is obtained whose objects are soft covering morphisms and whose morphisms are homomorphisms between soft groupoids. Second, a soft action groupoid is obtained by defining the action of a soft groupoid on a soft set. For a soft groupoid $H$, the category $SGdOp(H)$ of soft actions is defined, where the objects are soft actions and the morphisms are soft action morphisms between objects. Third, the equivalence between the category $SGdCov(H)$ of soft coverings and the category $SGdOp(H)$ of soft actions is proven.

## 2. Preliminaries

A groupoid is a (small) category in which each morphism is an isomorphism. So, a groupoid $H$ has a set $H$ of morphisms, which we simply call elements of $H$, a set $H_0$ of objects together with maps $\alpha, \beta : H \to H_0$ and $1_{()} : H_0 \to H$ such that $x \mapsto 1_x$. The maps $\alpha, \beta$ are called source and target point maps, respectively, and the $1_{()}$ is called object map. If $a, b \in H$ and $\alpha(b) = \beta(a)$, then the composite $b \circ a$ exists such that $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$. So, there exist a partial composition defined by $\circ : H_\alpha \times_\beta H \to H$, $(b, a) \mapsto b \circ a$, where $H_\alpha \times_\beta H$ is the pullback of $\alpha$ and $\beta$. Further, each element $a$ has an inverse $a^{-1}$ such that $\alpha(a) = \beta(a^{-1})$, $\beta(a) = \alpha(a^{-1})$, $a^{-1} \circ a = 1_{\alpha(a)}$, and $a \circ a^{-1} = 1_{\beta(a)}$. The map $H \to H$, $a \mapsto a^{-1}$ is called inversion. In a groupoid $H$, for $x, y \in H_0$, we write $H(x, y)$ for the set of all morphisms with source point $x$ and target point $y$ [12].

Let $Y$ be an initial universe set and let $E$ be a set of parameters. Let $P(Y)$ denote the power set of $Y$ and $B \subset E$. In this case, a soft set is defined as follows.

**Definition 1.** *A pair* $(R, B)$ *is called a soft set on* $Y$*, where* $R$ *is a mapping defined by* $R : B \to P(Y)$ *[1].*

According to this definition, we can say that a soft set defined on $Y$ is a parameterized family of subsets of the set $Y$. In this case, for all $\theta \in B$, the family $R(\theta)$ can be expressed as a set of $\theta$ approximating elements of the soft set $(R, B)$. A soft set $(R, B)$ defined on $Y$ will sometimes be denoted by $(Y, R, B)$ for convenience.

**Definition 2.** *Let* $(R, B)$ *and* $(H, C)$ *be two soft sets on the common universe* $Y$*. Then,* $(H, C)$ *is called a soft subset of* $(R, B)$ *if*
***i)*** $C \subset B$ *and*
***ii)*** *for all* $\theta \in C$*,* $H(\theta)$ *and* $R(\theta)$ *are identical approximations.*
*We denote it as* $(H, C)\widetilde{\subset}(R, B)$ *[18].*

**Definition 3.** *Let* $H$ *be a group,* $B$ *be a non-empty set, and* $(R, B)$ *be a soft set on* $H$*. If* $R(\theta) \leq H$ *for all* $\theta \in B$*, then* $(R, B)$ *is called a soft group on* $H$*. More generally speaking, the soft group* $(R, B)$ *can be defined as a parameterized family of subgroups of group* $H$*. A soft group* $(R, B)$ *above group* $H$ *will sometimes be denoted by* $(H, R, B)$ *[2].*

**Definition 4.** *Let* $(R, B)$ *and* $(G, C)$ *be two soft groups on* $H$*. Then,* $(G, C)$ *is said to be a soft subgroup of* $(R, B)$*, denoted by* $(G, C)\tilde{<}(R, B)$*, if*
***i)*** $C \subset B$ *and* ***ii)*** $G(\theta) \leq R(\theta)$ *for all* $\theta \in C$ *[2].*

**Definition 5.** *Let* $(R, B)$ *and* $(G, C)$ *be two soft groups on* $H$ *and* $K$*, respectively, and let* $f : H \to K$ *and* $g : B \to C$ *be two functions. Then,* $(f, g)$ *is called a soft homomorphism if*
***i)*** $f$ *is a surjective homomorphism,*
***ii)*** $g$ *is a surjective mapping, and*
***iii)*** $f(R(\theta)) = G(g(\theta))$ *for all* $\theta \in B$*.*
*Thus,* $(R, B)$ *is called soft homomorphic to* $(G, C)$*, and this situation is denoted by* $(R, B) \sim (G, C)$ *[2].*

**Definition 6.** *Let* $(R, B)$ *be a soft group on* $H$ *and* $(R', B)$ *be a soft set on* $Y$*. Then, a (left) soft group action of* $H$ *on* $Y$ *is a binary operation*

$$\pi_\theta : R(\theta) \times R'(\theta) \to R'(\theta) \tag{2.1}$$

*This satisfies the following two axioms for all* $\theta \in B$ *:*
***i)*** $\pi_\theta(e, x) = x$ *for all* $x \in R'(\theta)$ *and*
***ii)*** $\pi_\theta(a, \pi_\theta(b, x)) = \pi_\theta(ab, x)$ *for all* $x \in R'(\theta)$ *and* $a, b \in R(\theta)$*.*
*Here, for all* $\theta \in B$*,* $\pi_\theta$ *is a (left) group action mapping. Then, the soft set* $Y$ *is called an* $H$*-soft set [19].*

**Definition 7.** *Let* $H$ *be a groupoid and* $P(H)$ *denote the set of all subgroupoids of* $H$*. A triple* $(H, R, B)$ *is called a soft groupoid if* $R(\theta)$ *is a subgroupoid of* $H$ *for all* $\theta \in B$*, where* $R : B \to P(H)$ *is a mapping and* $B$ *is a set of parameters [4].*

Also, we can define a soft groupoid $(H, R, B)$ on the soft set $(H_0, R_0, B)$, where $Ob(H) = H_0$. Let $(H, R, B)$ and $(G, R', C)$ be two soft groupoids. Then, $(G, R', C)$ is called a soft subgroupoid of $(H, R, B)$, if $C \subseteq B$ and $R'(\theta)$ is a subgroupoid of $R(\theta)$ for all $\theta \in C$ [4].

**Example 1.** *Consider the soft group $(H, R, B)$. Indeed, since $(H, R, B)$ is a soft group, $R(\theta)$ is a subgroup of $H$ for all $\theta \in B$. We know from the groupoid theory that every group is a groupoid with only one object. For this reason, the group $H$ and $R(\theta)$ are groupoids for all $\theta \in B$. From here, it is easy to verify that $R(\theta)$ is a subgroupoid of the groupoid $H$ for all $\theta \in B$. Therefore, the $(H, R, B)$ is a soft groupoid. Thus, every soft group is a soft groupoid.*

**Example 2.** *A category $TGrp$ whose objects are topological groups and whose morphisms are linear homeomorphisms between topological groups is a groupoid. If we take the parameter set $B = \{Abelian\ \ group, finite\ \ group, cyclic\ \ group\}$, then $R : B \to P(TGrp)$ defines a soft groupoid and is denoted by $(TGrp, R, B)$.*

**Definition 8.** *Let $(H, R, B)$ and $(G, R', C)$ be two soft groupoids, $g : B \to C$ be a surjective morphism, and $\mathfrak{R} : H \to G$ be a functor. If the following conditions exist, the $(\mathfrak{R}, g)$ pair is called a soft groupoid homomorphism:*
***i)*** *$\mathfrak{R}$ is full (i.e., $\mathfrak{R}$ is surjective over morphisms) and*
***ii)*** *$\mathfrak{R}(R(\theta)) = R'(g(\theta))$ for all $\theta \in B$.*

*From this definition, we obtain a new category whose objects are soft groupoids and whose morphisms are soft groupoid homomorphisms. This category is called the category of soft groupoids and is denoted by $SGd$ [4].*

## 3. Soft covering groupoids

For a groupoid $H$ and for each $x \in H_0$, the star of the point at $x$, denoted by $St_H x$, is the union of the sets $H(x, y)$ for all objects $y$ of the groupoid $H$. In other words, $St_H x$ consists of all morphisms of the groupoid $H$ with source point $x$. Similarly, the costar of the point at $x$, denoted by $CSt_H x$, consists of all morphisms of the groupoid $H$ with target point $x$. Let $p : \tilde{H} \to H$ be a morphism of groupoids. We say $p$ is a covering morphism if, for each object $\tilde{x}$ of $\tilde{H}$, the restriction of $p : St_{\tilde{H}}\tilde{x} \to St_H p(\tilde{x})$ is bijective; in such case, we call $\tilde{H}$ a covering groupoid of $H$ [12].

**Definition 9.** *Let $H$ be a groupoid on $Y$. $SSt_H : Y \to P(H)$, $x \mapsto SSt_H x = St_H x \subseteq H$ is a soft set. This soft set is called the soft star set and denoted by $(H, SSt_H, Y)$. Similarly, $SCSt_H : Y \to P(H)$, $x \mapsto SCSt_H x = CSt_H x \subseteq H$ is a soft set. This soft set is called the soft costar set and denoted by $(H, SCSt_H, Y)$.*

**Definition 10.** *Let $(H, R, Y)$ and $(G, R', Z)$ be soft groupoids, and $\varphi : H \to G$ be a soft groupoid homomorphism. If the restriction of $\varphi$ to the soft star sets $(H, SSt_H, Y) \to (G, SSt_G, Z)$ is injective, then $\varphi$ is called a soft covering morphism and then soft groupoid $(H, R, Y)$ is called a soft covering groupoid of $(G, R', Z)$. This soft covering groupoid is denoted by $(H, \varphi, G)$.*

In this definition, it is sufficient to show injectivity of the soft covering groupoid. In fact, $\varphi' : (H, SSt_H, Y) \to (G, SSt_G, Z)$ is the restriction of $\varphi$ to soft star sets, a soft groupoid homomorphism, and naturally surjective. Indeed, Surjectivity of $\varphi'$ is obtained from condition (ii) of Definition 8

$(\varphi'(SSt_G x) = SSt_H \varphi_0(x))$. This situation is obviously shown in the commutative diagram below (see Figure 1).

$$\begin{tikzcd}
Y \arrow[r, "SSt_H"] \arrow[d, "\varphi_0"'] & P(H) \arrow[d, "\varphi'"] \\
Z \arrow[r, "SSt_G"] & P(G)
\end{tikzcd}$$

**Figure 1.** $\varphi'$ is surjective.

**Example 3.** *Similar to Example 2, the category Grp is also a groupoid, where the objects are groups and the morphisms are group isomorphisms. If we take the parameter set* $B = \{Abelian\ group, finite\ group, cyclic\ group\}$, *then* $R' : B \to P(Grp)$ *defines a soft groupoid, denoted by* $(Grp, R', B)$. *A 'Discrete Topological Functor' between soft groupoids* $(Grp, R', B)$ *and* $(TGrp, R, B)$ *defines a soft groupoid homomorphism. Specifically,* $D : (Grp, R', B) \to (TGrp, R, B)$ *maps every group to a discrete topological group and every group homomorphism to a continuous group homomorphism. The restriction of the homomorphism to soft stars, denoted by* $D' : (Grp, SSt_{Grp}, (Grp)_0) \to (TGrp, SSt_{TGrp}, (TGrp)_0)$, *is injective because distinct group homomorphisms remain distinct as morphisms between topological groups. Consequently,* $(Grp, R', B)$ *becomes a soft covering groupoid of* $(TGrp, R, B)$.

If $(H, R, Y)$ and $(G, R', Z)$ soft groupoids are both soft connected (or soft 1-connected), $\varphi$ soft covering morphism is called soft connected (or soft 1-connected) covering morphism and $(H, R, Y)$ is called the soft connected (or soft 1-connected) covering groupoid of $(G, R', Z)$.

Let $(H, R, B)$ be a soft groupoid on the soft set $(H_0, R_0, B)$ and $(G, R', B)$ be a soft groupoid on the soft set $(G_0, R'_0, B)$; let $\varphi : H \to G$ and $\varphi_0 : H_0 \to G_0$ be the soft covering morphisms of soft groupoids. Let the following pullback diagram be given for all $\theta \in B$:

$$R'(\theta)_\alpha \times_{\varphi_0} R_0(\theta) = \{(a, x) \in R'(\theta) \times R_0(\theta) : \alpha(a) = \varphi_0(x)\}, \tag{3.1}$$

where $\alpha$ is a source map of groupoid $H$. $\pi_\theta(sp) : R'(\theta)_\alpha \times_{\varphi_0} R_0(\theta) \to R(\theta)$ is a soft function that takes the pair $(a, x)$ satisfying the condition $\varphi(h) = a$ to a single $h \in SSt_H x$ starting at $x$. It is clear that $\pi_\theta(sp)$ is the inverse of the soft homomorphism $(\varphi, \alpha) : R(\theta) \to R'(\theta)_\alpha \times_{\varphi_0} R_0(\theta)$.

**Proposition 1.** $\varphi : H \to G$ *is a soft covering morphism of soft groupoids if and only if* $(\varphi, \alpha) : R(\theta) \to R'(\theta)_\alpha \times_{\varphi_0} R_0(\theta)$ *is injective.*

*Proof.* Let $\varphi : H \to G$ be a soft covering morphism. Then, $\varphi$ defines an injective. Accordingly, for all $h \in SSt_H x$, there is only one $(a, x) \in R'(\theta) \times R_0(\theta)$ that satisfies the condition $\varphi(h) = a$ and $\alpha(a) = \varphi_0(x)$. Then, $(\varphi, \alpha)$ is injective. Conversely, let $(\varphi, \alpha)$ be injective. Then, for all a pair of $(a, x) \in R'(\theta) \times R_0(\theta)$, there is a unique $h \in SSt_H x$ such that $\alpha(a) = \varphi_0(x)$ and satisfies the condition $\varphi(h) = a$. Accordingly, the restriction of $\varphi$ to soft star sets is injective. Then, $\varphi$ is a soft covering morphism. □

**Example 4.** *Let* $(H, R, B)$ *be a soft groupoid on the soft set* $(H_0, R_0, B)$. *The identity homomorphism* $\varphi : H \to H$ *is the soft covering morphism of groupoids. Here, since* $\pi_\theta(sp) : R(\theta)_\alpha \times_{\varphi_0} R_0(\theta) \to R(\theta)$ *is a projection morphism, it is easily seen that it is bijective.*

**Definition 11.** *Let $H$ be a groupoid on $Y$ and $H\{x\}$ be the vertex group of the groupoid $H$. $SH\{x\} : Y \to P(H)$, $x \mapsto SH\{x\} = H\{x\} \subseteq H$ is a soft set. This soft set is called a soft vertex group and is denoted by $(H, SH\{x\}, Y)$. Here, $SH\{x\}$ is specifically the single-object subgroupoid of the groupoid $H$.*

**Definition 12.** *Let $(H, R, Y)$ be a soft groupoid on the soft set $(H_0, R_0, Y)$, $(G, R', Z)$ be a soft groupoid on the soft set $(G_0, R'_0, Z)$, and $\varphi : H \to G$ be a soft groupoid homomorphism. For all $x \in H_0$, the soft subgroup $\varphi(SH\{x\})$ of the soft group $SG\{\varphi(x)\}$ is called the soft characteristic group of $\varphi$ at $x$.*

**Corollary 1.** *If $\varphi$ is a soft covering morphism, then $\varphi$ transforms $SH\{x\}$ isomorphically into the soft characteristic group $\varphi(SH\{x\})$. If $a \in SG\{\varphi(x)\}$, $SSt_H x$ has a single element $a'$ such that $\varphi(a') = a$, but $a'$ does not have to be a loop. It is $a' \in SH\{x\}$ if and only if $a \in \varphi(SH\{x\})$.*

**Proposition 2.** *Let $\vartheta : K \to H$ and $\varphi : H \to G$ each be soft groupoid homomorphisms. In this situation:*

*(1) If $\varphi$ and $\vartheta$ are soft covering morphisms, $\varphi\vartheta$ is also a soft covering morphism.*
*(2) If $\varphi$ and $\varphi\vartheta$ are soft covering morphisms, $\vartheta$ is also a soft covering morphism.*
*(3) If $\vartheta$ and $\varphi\vartheta$ are soft covering morphisms, $\varphi$ is also a soft covering morphism.*

*Proof.* The proof is clear from the definition of the composition operation of the soft groupoid homomorphism. □

Let $\varphi : H \to G$ be the soft covering morphism of soft groupoids. Each element $\tilde{a}$ of $H$ is called a soft lifting of $\varphi(\tilde{a})$ or a soft cover of $\varphi(\tilde{a})$. Similarly, the $\tilde{a} = \tilde{a}_n \circ ... \circ \tilde{a}_1$ composition in the $H$ soft groupoid is called a soft lifting of the $\varphi(a_n) \circ ... \circ \varphi(a_1)$ composition or a soft cover of $\varphi(a_n) \circ ... \circ \varphi(a_1)$.

**Proposition 3.** *Let $\varphi : H \to G$ be the soft covering morphism. Let $\tilde{x}$ be an object of $H$ and $\varphi(\tilde{x}) = x$. If there is $a = a_n \circ ... \circ a_1$ composition for each element of $SSt_G x$, there is only one element $\tilde{a}_n, ..., \tilde{a}_1$ of $H$ that satisfies the following conditions:*

*(1) $\varphi(\tilde{a}_i) = a_i, i = 1, ..., n$.*
*(2) $\tilde{a} = \tilde{a}_n \circ ... \circ \tilde{a}_1$ is defined and belongs to $SSt_H \tilde{x}$.*

*Proof.* Since $SSt_H \tilde{x} \to SSt_G \varphi(\tilde{x})$ is injective for all object $\tilde{x}$ of $H$, $\varphi(\tilde{a}_i) = a_i$, $\tilde{a}_1 \in SSt_H \tilde{x}_1$, $\tilde{a}_2 \in SSt_H \tilde{x}_2$, ... , $\tilde{a}_i \in SSt_H \tilde{x}_i$ is defined uniquely by induction. Here, $\tilde{x} = \tilde{x}_1$ and for all $i > 1$, $\tilde{x}_i$ is the end point of $\tilde{a}_{i-1}$. Then, for all $\tilde{a} \in SSt_H \tilde{x}$, there exists $\tilde{a} = \tilde{a}_n \circ ... \circ \tilde{a}_1$. □

**Definition 13.** *Let $(H, R, Y)$ be a soft groupoid on the soft set $(H_0, R_0, Y)$. For all $x \in H_0$, $(H, R, Y, x)$ is called a soft pointed groupoid and denoted by $S(H, x)$.*

A homomorphism between the soft pointed groupoids is a $\varphi : H \to G$ homomorphism of soft groupoids that satisfies the condition $\varphi(x) = y$.

The characteristic group of a homomorphism of soft pointed groupoids is the characteristic group of the $\varphi : H \to G$ groupoid homomorphism of soft groupoids.

If the $p : \tilde{H} \to H$ homomorphism of soft groupoids is a soft covering morphism, then the $p : S(\tilde{H}, \tilde{x}) \to S(H, x)$ soft pointed homomorphism is also a soft covering morphism.

According to the commutative diagram of soft pointed groupoids given below (see Figure 2), $\widetilde{\varphi}$ is called the soft lifting of $\varphi$ or the soft covering of $\varphi$.

$$\begin{tikzcd}
& S(\tilde{H},\tilde{x}) \arrow[d, "p"] \\
S(F,z) \arrow[ur, "\widetilde{\varphi}"] \arrow[r, "\varphi"] & S(H,x)
\end{tikzcd}$$

**Figure 2.** Soft lifting.

**Definition 14.** *Let $(H,R,B)$ be a soft groupoid. If for all $\theta \in B$, $R(\theta)$ is connected, then $(H,R,B)$ is called a soft connected groupoid; if $R(\theta)$ has a single morphism, $(H,R,B)$ is called a soft $1$-connected groupoid; if $R(\theta)$ does not have more than one morphism, $(H,R,B)$ is called a soft simply $1$-connected groupoid; and if $R(\theta)$ consists of only unit morphism, $(H,R,B)$ is called a soft discrete groupoid.*

**Proposition 4.** *Let $p : S(\tilde{H},\tilde{x}) \to S(H,x)$ be a soft covering morphism, $\varphi : S(F,z) \to S(H,x)$ be a soft groupoid homomorphism, and $F$ be a soft connected groupoid. In this case, $\varphi$ is lifted to a $\widetilde{\varphi} : S(F,z) \to S(\tilde{H},\tilde{x})$ soft homomorphism if and only if the soft characteristic group of $\varphi$ is contained in the soft characteristic group of $p$; that is, $\varphi(SF\{z\}) \subseteq p(S\tilde{H}\{\tilde{x}\})$. If there is such a lifting, then it is unique.*

*Proof.* Let $\widetilde{\varphi}$ be the lifting of $\varphi$ to a soft homomorphism. Then, the equation $p\widetilde{\varphi} = \varphi$, $\varphi(SF\{z\}) \subseteq p(S\tilde{H}\{\tilde{x}\})$ is obtained, which proves the necessity of this condition. Now suppose that the soft characteristic group of $\varphi$ is contained by a soft characteristic group $D$ of $p$; that is, let $\varphi(SF\{z\}) \subseteq p(S\tilde{H}\{\tilde{x}\})$.

Since $p$ is restricted to the $\tilde{H}\{\tilde{x}\} \to D$ isomorphism, the $\varphi : SF\{z\} \to S(H,x)$ soft groupoid homomorphism is uniquely lifted to the $\widetilde{\varphi} : SF\{z\} \to S\tilde{H}\{\tilde{x}\}$ soft homomorphism. Let $c \in F(z,v)$ for all object $v$ of $F$. If $a \in F(u,v)$, then $a$ can uniquely be written as: $a = c \circ a' \circ d^{-1}$, $a' \in SF\{z\}$. If one takes it one step further, $b \in F(v,w)$ becomes $(b \circ a)' = b' \circ a'$. Here, each element $\varphi(c)$ covers the element $\widetilde{\varphi}(c)$ of $SSt_{\tilde{H}}\tilde{x}$ uniquely. We define this as: $\widetilde{\varphi}(a) = \widetilde{\varphi}(c) \circ \widetilde{\varphi}(a') \circ \widetilde{\varphi}(d^{-1})$ and hence $\widetilde{\varphi}(b) \circ \widetilde{\varphi}(a) = \widetilde{\varphi}(e) \circ \widetilde{\varphi}(b') \circ \widetilde{\varphi}(a') \circ \widetilde{\varphi}(d^{-1}) = \widetilde{\varphi}(e) \circ \widetilde{\varphi}(b \circ a)' \circ \widetilde{\varphi}(d^{-1}) = \widetilde{\varphi}(b \circ a)$. Here $\widetilde{\varphi}$ is the soft lifting of $\varphi$. Moreover, any soft homomorphism that lifts $\varphi$ must be compatible with $\widetilde{\varphi}$ on the elements $SF\{z\}$ and $c$; accordingly, such a lifting must coincide with $\widetilde{\varphi}$. This shows the uniqueness of such a lifting. □

If $\varphi(SF\{z\} = p(S\tilde{H}\{\tilde{x}\})$, this proposition requires that the soft groupoids $F$ and $\tilde{H}$ are soft isomorphic.

**Corollary 2.** *Let $p : S(\tilde{H},\tilde{x}) \to S(H,x)$ and $q : S(\tilde{G},\tilde{y}) \to S(H,x)$ be soft-connected covering morphisms with soft characteristic groups $C$ and $D$, respectively. If $C \subseteq D$, there is only one soft covering morphism $r : S(\tilde{H},\tilde{x}) \to S(\tilde{G},\tilde{y})$ such that $p = q \circ r$. If $C = D$, $r$ is a soft isomorphism.*

**Corollary 3.** *The 1-connected soft covering groupoid of $H$ covers all soft covering groupoids of $H$.*

The 1-connected soft covering groupoid of $H$ is called the soft universal covering groupoid of $H$.

Let $(H,R,B)$ be a soft groupoid on the soft set $(H_0,R_0,B)$. Then, we define the category $SGdCov(H)$ of the soft coverings, whose objects are the soft covering morphisms of groupoids $(G,p,H)$ and morphisms from $(G, p, H)$ object to $(M, q, H)$ object is a soft homomorphism $r : G \to M$ such that $p = q \circ r$. From Proposition 2, it is clear that $r$ is also a soft covering morphism.

Let $r : G \to M$ and $r' : M \to M'$ be two soft covering morphisms. Then, composition is defined as follows (see Figure 3):

$$\begin{tikzcd}
G \arrow[r, "r"] \arrow[dr, "q"'] & M \arrow[r, "r'"] \arrow[d, "p"'] & M' \arrow[dl, "q'"] \\
& H &
\end{tikzcd}$$

**Figure 3.** Composition of morphisms.

## 4. Soft actions of soft groupoids

During this section, let $\alpha, \beta$ be denoted by the source and target maps of the groupoid, respectively.

**Definition 15.** *Let* $(H, R, B)$ *be a soft groupoid on the soft set* $(H_0, R_0, B)$ *and* $(Y, R', B)$ *be a soft set. A soft action of soft groupoid* $(H, R, B)$ *on the soft set* $(Y, R', B)$*, for all* $\theta \in B$*,* $w_\theta : R'(\theta) \to R_0(\theta)$ *map,* $\pi_\theta : R(\theta)_\alpha \times_w R'(\theta) \to R'(\theta)$*,* $(a, x) \mapsto \pi_\theta(a, x) =^{a} x$*, where* $R(\theta)_\alpha \times_w R'(\theta) = \{(a, x) \in R(\theta)_\alpha \times_w R'(\theta) : \alpha(a) = w(x)\}$ *that satisfies the following conditions:*

***i.*** $\pi_\theta(1_{wx}, x) =^{1_{wx}} x = x;$

***ii.*** $\pi_\theta(b, \pi_\theta(a, x)) = \pi_\theta(b \circ a, x) =^{b \circ a} x;$

***iii.*** $w(\pi_\theta(a, x)) = \beta(a).$

*This action is denoted by* $(\pi, Y, w)$*. We also say the soft groupoid* $(H, R, B)$ *acts on* $(Y, R', B)$ *via* $w_\theta$*, and that* $(Y, R', B)$ *is a (left) soft H-set.*

**Example 5.** *Let* $(H, R, B)$ *be a soft groupoid on the soft set* $(H_0, R_0, B)$ *and for all* $\theta \in B$*,* $w_\theta : R_0(\theta) \to R_0(\theta)$ *be a map. Accordingly, each H soft groupoid has action on itself via* $w_\theta$ *as follows;*

$$\pi_\theta : R(\theta)_\alpha \times_w R_0(\theta) \to R_0(\theta),$$
$$(a, x) \mapsto \pi_\theta(a, x) =^{a} x = x.$$

*Now, let's show that the soft action conditions are satisfied. For all* $\theta \in B$*,* $x \in R_0(\theta)$ *and* $a, b \in R(\theta)$*,*

***i.*** $\pi_\theta(1_{wx}, x) =^{1_{wx}} x = x;$

***ii.*** $\pi_\theta(b, \pi_\theta(a, x)) = \pi_\theta(b, x) = x$ *and* $\pi_\theta(b \circ a, x) =^{b \circ a} x = x;$

***iii.*** $w(\pi_\theta(a, x)) = w(x) = \beta(a).$

*This action is called the trivial soft groupoid action of H on itself (Here,* $\pi_\theta$ *is the projection morphism).*

**Example 6.** *Let* $(H, R, B)$ *be a soft groupoid on the soft set* $(H_0, R_0, B)$*,* $(Y, R', B)$ *be a soft set, and for all* $\theta \in B$*,* $w_\theta : R'(\theta) \to R_0(\theta)$ *be a map. Let the soft groupoid action of* $(H, R, B)$ *on* $(Y, R', B)$ *be as follows:*

$$\pi_\theta : R(\theta)_\alpha \times_w R'(\theta) \to R'(\theta),$$
$$(a, x) \mapsto \pi_\theta(a, x).$$

*An arbitrary soft set $(Z, R'', B)$, where $f : Y \to Z$ is an injective function, $R'' = f(R')$ and for all $\theta \in B$ the function defined below is also a soft action:*

$$\pi'_\theta : R(\theta)_\alpha \times_w f(R'(\theta)) \to f(R'(\theta)),$$
$$(a, f(x)) \mapsto \pi'_\theta(a, f(x)) = f(\pi_\theta(a, x)).$$

*With this soft groupoid action, $Z$ is also a soft $H$-set that satisfies Figure 4:*

$$\begin{tikzcd}
B \arrow[r, "R'"] \arrow[dr, "R''=f\circ R'"'] & P(Y) \arrow[d, "f"] \\
 & P(Z)
\end{tikzcd}$$

**Figure 4.** $Z$ is a soft $H$-set.

**Example 7.** *Let $(\tilde{H}, R, B)$ be a soft groupoid on the soft set $(\tilde{H}_0, R_0, B)$ and $(H, R', B)$ be a soft groupoid on the soft set $(H_0, R'_0, B)$ ; let $\varphi : \tilde{H} \to H$ be the soft covering morphism of soft groupoids and $w_\theta = \varphi_0 : R_0(\theta) \to R'_0(\theta)$ for all $\theta \in B$. Thus, the $\pi_\theta : R'(\theta)_\alpha \times_w R_0(\theta) \to R_0(\theta)$, $(a, \tilde{x}) \mapsto \pi_\theta(a, \tilde{x}) = \widetilde{\beta}(\tilde{a})$ soft action of $H$ on $R_0(\theta)$ is obtained via $w_\theta$. Here, since $\varphi$ is a soft covering morphism, it leads to a unique $\tilde{a}$ soft lifting of $a$, whose source is $\tilde{x}$, such that $\varphi(\tilde{a}) = a$ and $w(\tilde{x}) = x$ for $\tilde{x} \in R_0(\theta)$ and $a \in SSt_H w(\tilde{x})$. Now, let's show that the soft action conditions are satisfied:*

***i.*** $\pi_\theta(1_{w\tilde{x}}, \tilde{x}) = \widetilde{\beta}(\tilde{c}) = \tilde{x}$ *where* $\widetilde{c}$ *is the soft lifting of* $1_{w\tilde{x}}$.

***ii.*** $\pi_\theta(b, \pi_\theta(a, \tilde{x})) = \pi_\theta(b, \widetilde{\beta}(\tilde{a})) = \widetilde{\beta}(\tilde{b})$, $\pi_\theta(b \circ a, \tilde{x}) = \widetilde{\beta}(\tilde{b} \circ \tilde{a}) = \widetilde{\beta}(\tilde{b})$, *then* $\pi_\theta(b, \pi_\theta(a, \tilde{x})) = \pi_\theta(b \circ a, \tilde{x})$.

***iii.*** $w(\pi_\theta(a, \tilde{x})) = w(\widetilde{\beta}(\tilde{a})) = \beta(a)$. *This action groupoid is denoted by $(\pi, \tilde{H}_0, \varphi_0)$.*

*(A loop in $H$ does not have to be a loop in $\tilde{H}$, but in accordance with the soft covering definition, a morphism starting at $x$ in $H$ corresponds to a morphism starting at $\tilde{x}$ in $\tilde{H}$.)*

Let $(H, R, B)$ be a soft groupoid on the soft set $(H_0, R_0, B)$ and $(Y, R', B)$ be a soft $H$-set via $w_\theta$. From the rules for a soft action that an element $a \in H(x, y)$ defines a bijective $a^\star : w^{-1}(x) \to w^{-1}(y)$, $x \mapsto^a x$. The action is said to be transitive if, for all $x, y \in R_0(\theta)$, $x' \in w^{-1}(x)$, and $y' \in w^{-1}(y)$, there is a $a \in H(x, y)$ such that $\pi_\theta(a, x') = y'$.

**Proposition 5.** *Let $(H, R, B)$ be a soft groupoid on the soft set $(H_0, R_0, B)$ and let $(Y, R', B)$ be a soft $H$-set via $w_\theta$. Then, there is a soft groupoid $(H \bowtie Y, R'', B)$ on the soft set $(Y, R', B)$.*

*Proof.* We can define $R'' : B \to P(H \bowtie Y)$, $\theta \mapsto R''(\theta) = R(\theta)_\alpha \times_w R'(\theta) = R(\theta) \bowtie R'(\theta) \leq H \bowtie Y$, where the set of objects of this groupoid is $R'(\theta)$ and the set of morphisms of this groupoid is the set $R(\theta) \bowtie R'(\theta)$. For all $\theta \in B$, $R(\theta) \bowtie R'(\theta)$ is a subgroupoid of $H \bowtie Y$. In fact, the morphism of the groupoid is the set $R(\theta) \bowtie R'(\theta)$. This means that a morphism from an object $x$ to an object $y$ is a pair $(a, x)$ that satisfies the $\pi_\theta(a, x) =^a x = y$ condition. Also, we define the source map $\alpha_\theta(a, x) = x$, the target map $\beta_\theta(a, x) =^a x = y$, the object map $x \mapsto (1_{wx}, x)$, the inverse map $(a, x)^{-1} = (a^{-1}, {}^a x)$, and the composition operation $(b, y) \circ (a, x) = (b \circ a, x)$. □

Thus, we obtain the groupoid $(H \bowtie Y, R'', B)$ is called the soft action groupoid.

**Proposition 6.** *The following are true:*

*(1) Let* $(H \bowtie Y, R'', B)$ *be a soft groupoid. Accordingly, the projection* $\pi^*_\theta : R(\theta) \bowtie R'(\theta) \to R(\theta)$*, given on objects by* $w_\theta : R'(\theta) \to R_0(\theta)$ *and on elements by* $(a, x) \mapsto \pi^*_\theta(a, x) = a$*, is a soft covering morphism of soft groupoids.*

*(2) The soft groupoid* $(H \bowtie Y, R'', B)$ *is transitive if and only if the action is connected.*

*(3) If* $x \in R'(\theta)$*, then* $(R(\theta) \bowtie R'(\theta))\{x\}$ *soft object group can be soft isomorphically mapped to* $R(\theta)\{x\}$ *with* $\pi^*_\theta$*.*

*Proof.* **(1)** $\pi^*_\theta((b, y) \circ (a, x)) = \pi^*_\theta(b \circ a, x) = b \circ a = \pi^*_\theta((b, y)) \circ \pi^*_\theta((a, x))$ is found from the definition of $\pi^*_\theta$. Moreover, since $\pi^*_\theta$ is given by $w$ on objects, it is $\pi^*_\theta(1_{w(x)}, x) = 1_{w(x)} = 1_{\pi^*_\theta(1_{w(x)}, x)}$, and $\pi^*_\theta$ is a soft groupoid homomorphism. Moreover, $\pi^*_\theta$ is a soft covering morphism, because if $a \in H(x, y)$ and $x' \in w^{-1}(x)$, then $(a, x')$ is a unique element of $R(\theta) \bowtie R'(\theta)$, whose source is $x'$ and whose projection is $a$. Accordingly, $(\pi^*_\theta, \alpha) : R(\theta) \bowtie (R(\theta) \bowtie R'(\theta))_0 \to R(\theta) \bowtie R'(\theta)$ is injective.

**(2)** Let $(R(\theta) \bowtie R'(\theta))(x, y)$ be non-empty. Then, there is $a \in R(\theta)$ such that $(a, x) \mapsto \pi^*_\theta(a, x) =^a x = y$. Therefore, the soft action here is transitive. Conversely, let the action here be transitive. Then, there is $(a, x) \mapsto \pi^*_\theta(a, x) =^a x = y$ such that $y \in R'(\theta)$. Then $(R(\theta) \bowtie R'(\theta))(x, y)$ is non-empty.

**(3)** Since it is defined as $(R(\theta) \bowtie R'(\theta))\{x\} = \{(a, x) : \alpha(a, x) = x, \beta(a, x) = x\}$ and $\pi^*_\theta(a, x) =^a x = \beta(a, x)$, and from the definition of the soft object group, it is clear that $\pi^*_\theta(a, x) = x$. □

**Proposition 7.** *Let* $(H, R, B)$ *be a soft connected groupoid on the soft set* $(H_0, R_0, B)$*. Let* $SL\{x\}$ *be a subgroup of* $SH\{x\}$*, where* $x \in R_0(\theta)$ *and* $Y = \{a \circ SL\{x\} : a \in SSt_H x\}$*. For all* $\theta \in B$*, let the map* $w_\theta : R'(\theta) \to R_0(\theta)$ *be defined by* $a \circ SL\{x\} \mapsto \beta(a)$*. Then,* $(H, R, B)$ *acts transitively on* $(Y, R', B)$ *by* $\pi_\theta : R(\theta)_\alpha \times_w R'(\theta) \to R'(\theta)$, $\pi_\theta(b, a \circ SL\{x\}) = b \circ a \circ SL\{x\}$*. The corresponding soft connected covering morphism* $S(H \bowtie Y, \tilde{x}) \to S(H, x)$ *has the soft characteristic group* $SL\{x\}$*, where* $\tilde{x}$ *is the coset* $SL\{x\}$*.*

*Proof.* Let's show that the soft action conditions are satisfied:

**i.** $\pi_\theta(1_y, (a \circ SL\{x\})) = 1_y \circ a \circ SL\{x\} = a \circ SL\{x\}$.

**ii.** Suppose $a \in H(x, y), b \in H(y, z)$. Then, $w_\theta(a \circ SL\{x\}) = y$ and $w_\theta(b \circ a \circ SL\{x\}) = z$ become $w_\theta(\pi_\theta(b, a \circ SL\{x\})) = z = \beta(b)$.

**iii.** For $b' \in H(z, z')$, it is $\pi_\theta(b', \pi_\theta(b, a \circ SL\{x\})) = \pi_\theta(b', b \circ a \circ SL\{x\}) = (b' \circ b \circ a \circ SL\{x\}) = (b' \circ b) \circ (a \circ SL\{x\}) = \pi_\theta(b' \circ b, a \circ SL\{x\})$, and the last condition is satisfied. The action is transitive; if $a \circ SL\{x\}$, $a' \circ SL\{x\}$ are cosets and $b = a' \circ a^{-1}$, then $(b \circ a \circ SL\{x\}) = (a' \circ SL\{x\})$. It follows that $H \bowtie Y$ is soft connected.

The required soft characteristic group consists of the elements of $R(\theta)$, such that $SL\{x\}$ and $a \circ SL\{x\}$ is defined, and this holds if and only if $a \in SL\{x\}$. Hence, the soft characteristic group is $SL\{x\}$. □

The soft pointed groupoid given in Proposition 7 is denoted by $STr(H, SL\{x\})$, and this soft groupoid is called the soft covering groupoid of $(H, R, B)$ based on the soft subgroup $SL\{x\}$ of $SH\{x\}$.

**Corollary 4.** *Every soft connected groupoid has a soft universal covering groupoid.*

*Proof.* Let $(H, R, B)$ be a soft connected groupoid and $x \in R(\theta)$, $STr(H, \{1_x\})$ be a soft universal covering groupoid of $H$ such that $\{1_x\}$ is a trivial soft subgroup of $(R(\theta), x)$. Notice that in this case, the set on which $H$ acts is simply $SSt_H x$. □

**Proposition 8.** *Let* $(H, R, B)$ *be a soft connected groupoid on the soft set* $(H_0, R_0, B)$*; let* $x \in R_0(\theta)$ *and* $SL\{x\}$ *be a subgroup of the soft object group* $SH\{x\}$*. In this case, there is a soft connected groupoid*

*$(G, R', B)$ such that a soft covering morphism $\varphi : G \to H$ of the soft groupoids, and $\tilde{x} \in R'_0(\theta)$ satisfy the condition $\varphi(SG\{\tilde{x}\}) = SL\{x\}$.*

*Proof.* Let $(G, R', B)$ be a soft connected groupoid on the soft set $(G_0, R'_0, B)$. From Proposition 6, there is a soft covering morphism $\varphi : G \to H$. Then, since $\varphi$ will be bijective, morphisms starting at $x$ in $H$ correspond to morphisms starting from a point $\tilde{x}$ in $G$. Additionally, $\varphi$ transforms $SG\{\tilde{x}\}$ isomorphically into the soft characteristic group $\varphi(SG\{\tilde{x}\})$. From Corollary 1, if $a \in SL\{\varphi(x)\}$, there is only one element $a'$ of $SSt_G\tilde{x}$, which is $\varphi(a') = a$, and if there is $a' \in SG\{\tilde{x}\}$, there is $a \in \varphi(SG\{\tilde{x}\}) = SL\{x\}$. □

Let $(H, R, B)$ be a soft groupoid on the soft set $(H_0, R_0, B)$ and let $(Y, R', B)$ and $(Z, R'', B)$ be soft sets. We define the category $SGdOp(H)$ of the soft actions, whose objects are soft actions $(\pi, Y, w)$ and morphisms $f : (\pi, Y, w) \to (\pi', Z, w')$ such that $w'_\theta \circ f_\theta = w_\theta$ and $f_\theta({}^a x) ={}^a f_\theta(x)$ for all $\theta \in B$, where $f : Y \to Z$ is a soft map (see Figure 5).

$$\begin{tikzcd}
R'(\theta) \arrow[r, "f_\theta"] \arrow[dr, "w_\theta"'] & R''(\theta) \arrow[d, "w'_\theta"] \\
 & R_0(\theta)
\end{tikzcd}
\qquad
\begin{tikzcd}
R(\theta)_\alpha \times_w R'(\theta) \arrow[r, "\pi_\theta"] \arrow[d, "1\times f_\theta"'] & R'(\theta) \arrow[d, "f_\theta"] \\
R(\theta)_\alpha \times_{w'} R''(\theta) \arrow[r, "\pi'_\theta"] & R''(\theta)
\end{tikzcd}$$

**Figure 5.** The morphism in the category $SGdOp(H)$.

Here, the source map is defined as $\alpha(f_\theta) = (\pi, Y, w)$, the target map is $\beta(f_\theta) = (\pi', Z, w')$, and the object map is $1_{(\pi,Y,w)} : (\pi, Y, w) \to (\pi, Y, w)$. The composition operation is defined by the commutative diagram below (see Figure 6).

$$\begin{tikzcd}
R'(\theta) \arrow[r, "f_\theta"] \arrow[dr, "w_\theta"'] & R''(\theta) \arrow[r, "f'_\theta"] \arrow[d, "w'_\theta"] & R'''(\theta) \arrow[dl, "w''_\theta"] \\
 & R_0(\theta) &
\end{tikzcd}
\qquad
\begin{tikzcd}
R(\theta)_\alpha \times_w R'(\theta) \arrow[r, "\pi_\theta"] \arrow[d, "1\times f_\theta"'] & R'(\theta) \arrow[d, "f_\theta"] \\
R(\theta)_\alpha \times_{w'} R''(\theta) \arrow[r, "\pi'_\theta"] \arrow[d, "1\times f'_\theta"'] & R''(\theta) \arrow[d, "f'_\theta"] \\
R(\theta)_\alpha \times_{w''} R'''(\theta) \arrow[r, "\pi''_\theta"] & R'''(\theta)
\end{tikzcd}$$

**Figure 6.** The composition operation.

After these preliminaries, we can now give the main result of this paper regarding the equivalence of categories.

**Theorem 1.** *Let $(H, R, B)$ be a soft groupoid on the soft set $(H_0, R_0, B)$. Then, the categories $SGdCov(H)$ and $SGdOp(H)$ are equivalent.*

*Proof.* Let us define $\Gamma : SGdOp(H) \rightarrow SGdCov(H)$ as follows:

Let $(\pi, Y, w)$ be a soft $H$-set. We define a soft action of soft groupoid $(H, R, B)$ on the soft set $(Y, R', B)$ by $\pi_\theta : R(\theta)_\alpha \times_w R'(\theta) \rightarrow R'(\theta)$, $(a, x) \mapsto \pi_\theta(a, x) =^a x$. From Proposition 5, there is a soft action groupoid $(H \bowtie Y, R'', B)$. From Proposition 6, the projection $\pi^*_\theta : R(\theta) \bowtie R'(\theta) \rightarrow R(\theta)$, given on objects by $w_\theta : R'(\theta) \rightarrow R_0(\theta)$ and on elements by $(a, x) \mapsto \pi^*_\theta(a, x) = a$, is a soft covering morphism of soft groupoids. That is, $\Gamma(\pi, Y, w) = (H \bowtie Y, \pi^*, H)$ is a soft covering of soft groupoids (see Figure 7).

$$\begin{tikzcd}
(\pi, Y, w) \arrow[r, "\Gamma"] \arrow[d, "f"'] & \Gamma(\pi, Y, w) = (H \bowtie Y, \pi^*_1, H) \arrow[d, "\Gamma(f)=1\times f"] \\
(\pi', Y', w') \arrow[r, "\Gamma"] & \Gamma(\pi', Y', w') = (H \bowtie Y', \pi^*_2, H')
\end{tikzcd}
\qquad
\begin{tikzcd}
H \bowtie Y \arrow[r, "1\times f"] \arrow[dr, "\pi^*_1"] & H \bowtie Y' \arrow[d, "\pi^*_2"] \\
& H
\end{tikzcd}$$

**Figure 7.** $\Gamma$ is a functor.

Since $\pi^*_1$ and $\pi^*_2$ are soft covering morphisms, $1 \times f$ is also a soft covering morphism from Proposition 2. Then, $\Gamma$ is a functor.

Conversely, let us define $\Phi : SGdCov(H) \rightarrow SGdOp(H)$ as follows:

Let $(\tilde{H}, \varphi, H)$ be a soft covering morphism of soft groupoids and let $(\tilde{H}, \tilde{R}, B)$ be a soft groupoid on the soft set $(\tilde{H}_0, \tilde{R}_0, B)$. Let $R'(\theta) = \tilde{R}_0(\theta)$ and $w_\theta = \varphi_0 : \tilde{R}_0(\theta) \rightarrow R_0(\theta)$, for all $\theta \in B$. From Example 7, the soft groupoid $(H, R, B)$ defines an action $\pi_\theta : R(\theta)_\alpha \times_w \tilde{R}_0(\theta) \rightarrow \tilde{R}_0(\theta)$, $(a, \tilde{x}) \mapsto \pi_\theta(a, \tilde{x}) = \widetilde{\beta}(\tilde{a})$ on $R'(\theta) = \tilde{R}_0(\theta)$ via $w_\theta = \varphi_0$. We obtain this action from the composition of $\pi_\theta(sp) : R(\theta)_\alpha \times_w \tilde{R}_0(\theta) \rightarrow \tilde{R}(\theta)$ and target map $\beta : \tilde{R}(\theta) \rightarrow \tilde{R}_0(\theta)$. That is, $\Phi(\tilde{H}, \varphi, H) = (\pi, \tilde{H}_0, \varphi_0)$ is the soft action of the soft groupoid $(H, R, B)$ (see Figure 8).

$$\begin{tikzcd}
(\tilde{H}, \varphi, H) \arrow[r, "\Phi"] \arrow[d, "f"'] & \Phi(\tilde{H}, \varphi, H) = (\pi, \tilde{H}_0, \varphi_0) \arrow[d, "\Phi(f)=1\times f_0"] \\
(\tilde{H}', \varphi', H) \arrow[r, "\Phi"] & \Phi(\tilde{H}', \varphi', H) = (\pi', \tilde{H}'_0, \varphi'_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
H \bowtie \tilde{H}_0 \arrow[r, "\pi"] \arrow[d, "1\times f_0"'] & \tilde{H}_0 \arrow[d, "f_0"] \\
H \bowtie \tilde{H}'_0 \arrow[r, "\pi'"] & \tilde{H}'_0
\end{tikzcd}$$

**Figure 8.** $\Phi$ is the functor.

Thus, $\Phi$ is the functor.

We now show that $\Phi\Gamma \cong 1_{SGdOp(H)}$ is a natural equivalence. Let $(\pi, Y, w)$ be a soft action. By Proposition 5, there exists a soft groupoid $(H \bowtie Y, R'', B)$ with the object set $R'(\theta)$ as a soft set. According to Proposition 6, there is a covering morphism of soft groupoids $\varphi : (H \bowtie Y, R'', B) \rightarrow (Y, R', B)$. Furthermore, following Example 7, $\Phi(\Gamma(H \bowtie Y, R'', B))$ endows each soft set $R'(\theta)$ with a soft action via the soft set morphism $\varphi_0 = w : (H \bowtie Y)_0 \rightarrow H_0$ over the soft groupoid $(H, R, B)$. This implies that $\Phi(\Gamma(\pi, Y, w)) = (\pi, Y, w)$.

We construct a natural transformation $\eta : \Phi \circ \Gamma \to Id_{SGdOp(H)}$. We show that $\eta$ is a natural isomorphism via the following commutative diagram (see Figure 9):

$$\begin{tikzcd}
(\pi, Y, w) \arrow[d, "f"'] & (\Phi \circ \Gamma)(\pi, Y, w) \arrow[r, "\eta(\pi,Y,w)"] \arrow[d, "(\Phi\circ\Gamma)(f)"'] & Id(\pi, Y, w) \arrow[d, "f"] \\
(\pi', Y', w') & (\Phi \circ \Gamma)(\pi', Y', w') \arrow[r, "\eta(\pi',Y',w')"] & Id(\pi', Y', w')
\end{tikzcd}$$

**Figure 9.** $\eta$ is a natural isomorphism.

Since $\eta_{(\pi,Y,w)}$ and $\eta_{(\pi',Y',w')}$ are isomorphisms, it follows that $\eta : \Phi \circ \Gamma \to Id_{SGdOp(H)}$ is a natural isomorphism. Consequently, we obtain the natural equivalence $\Phi\Gamma \cong 1_{SGdOp(H)}$. Another direction $\varepsilon : Id_{SGdCov(H)} \to \Gamma \circ \Phi$ is shown similarly. □

## 5. Conclusions

In this study, we defined the concepts of soft star and soft costar and a soft covering morphism for the soft groupoids. We obtained the category $SGdCov(H)$ of soft coverings for a soft groupoid $H$, whose objects are soft covering morphisms and whose morphisms are homomorphisms between soft groupoids. We also obtained a soft action groupoid by defining the action of a soft groupoid on a soft set. We defined the category $SGdOp(H)$ of soft actions, where the objects are soft actions and the morphisms are soft morphisms between objects for a soft groupoid $H$. Finally, we proved the equivalence between the category $SGdCov(H)$ of soft coverings and the category $SGdOp(H)$ of soft actions. Looking at the articles, you can find many references to the equivalence of covering spaces obtained using different algebraic structures. Applying the method we used in this study to these equivalences will open up new areas of research using soft algebraic and topological approaches. Further, the study of soft covering maps contributes to a deeper understanding of soft groupoids and their interrelationships, and it establishes a foundation for further theoretical developments.

In conclusion, the structural relationship between classical covering groupoid morphisms and soft covering groupoid morphisms can be summarized as follows: A classical covering morphism is defined as a deterministic morphism that provides a bijective mapping between star sets for each object, thereby preserving the local structure. This structure possesses a strong lifting property, particularly within covering theory and topological groupoids, offering a crisp algebraic framework free from ambiguity.

In contrast, soft covering groupoid morphisms rely on families of subgroupoid defined over a parameter set. Satisfying the covering condition for each parameter confers a parametric locality on the structure, enabling the representation of uncertainty, incomplete information, or multiple modeling scenarios. In this respect, the soft covering approach constitutes a more flexible yet more complex generalization of classical covering theory.

Structurally, while every classical covering groupoid can be interpreted as a single-parameter soft covering groupoid, the converse is not true; not every soft covering groupoid can be reduced to a classical one due to the potential for varying local structures across different parameters. Therefore, classical covering theory can be considered a special case of soft covering theory. Categorically, soft groupoids can be treated as parametric families of groupoids, where soft coverings exhibit a structure similar to fiberwise coverings, providing a fibration-like extension.

Ultimately, soft covering groupoid theory provides a more general framework that encompasses classical theory. This generalization has the potential to open new avenues of research in both theoretical algebra and mathematical modeling involving uncertainty.

## Author contributions

All authors contributed equally in this paper. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare we have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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