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Research article

On Ulam stability of generalized Hosszú functional equation

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Abstract: We investigated Hyers-Ulam stability (H-US) of the following generalization of the Hosszú equation (HFE): $\Upsilon(x_1 + x_2 - \alpha x_1 x_2) + \Upsilon(\alpha x_1 x_2) = \Upsilon(x_1) + \Upsilon(x_2)$, in the class of maps Υ from a quadratically closed field $\mathbb{K}$ into a linear space, where $\alpha \in \mathbb{K}$ is fixed. We considered this stability in cases where the linear space is equipped with either the m-norm or the classical norm. In this way, we extended some earlier stability outcomes obtained for maps from the set of reals $\mathbb{R}$ into a Banach space. We also proved some auxiliary stability results for the Cauchy additive equation $\psi(x+y) = \psi(x) + \psi(y)$ in m-Banach spaces. Finally, we discussed the symmetry issues that can be observed in these results.

Keywords: functional equations; Ulam stability; Hosszú equation; m-norm
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1. Introduction

Hyers-Ulam stability (H-US) has attracted the interest of numerous researchers due to its possible applications and close connections with other areas of mathematics (e.g., theories of fixed points, approximation, optimization, shadowing, perturbation). In short, it involves the error made by replacing exact solutions of an equation with maps that satisfy the equation in an approximate way.

This type of stability was inspired by a problem posed by Ulam in 1940 [1–3] concerning approximate solutions of the equation of homomorphisms of groups, i.e., the Cauchy functional equation

$$\mathcal{A}(x_1 + x_2) = \mathcal{A}(x_1) + \mathcal{A}(x_2). \tag{1.1}$$

The problem was soon partially answered in [1] in the following way:

Theorem 1.1. *Assume that X_1 is a normed space, $\mathfrak{B}$ is a Banach space, and $\eta \geq 0$ is a fixed real number. Let $\Theta\colon X_1 \to \mathfrak{B}$ satisfy the inequality*

$$\sup_{x_1, x_2 \in X_1} \|\Theta(x_1 + x_2) - \Theta(x_1) - \Theta(x_2)\| \leq \eta. \tag{1.2}$$

Then, there exists a unique $\mathcal{A} : X_1 \to \mathfrak{B}$ that is additive (i.e., it satisfies Eq (1.1) *for all $x_1, x_2 \in X_1$) and such that*

$$\sup_{x_1 \in X_1} \|\Theta(x_1) - \mathcal{A}(x_1)\| \leq \eta. \tag{1.3}$$

Actually, in the original result in [1], X_1 is assumed to be a Banach space, but the completeness of X_1 is not necessary in this case. Some extensions and generalizations of these results were obtained in [4–6]. More information on H-US and examples of more recent results can be found in [2, 3, 7].

In this paper, we consider a similar problem (i.e., stability) of the following Cauchy-Hosszú functional equation

$$\Upsilon(x_1 + x_2 - \alpha x_1 x_2) + \Upsilon(\alpha x_1 x_2) = \Upsilon(x_1) + \Upsilon(x_2), \tag{1.4}$$

for maps Υ from a field into a Banach space, where α is a fixed element of the field.

Note that with $\alpha = 0$, Eq (1.4) has the form

$$\Upsilon(x_1 + x_2) = \Upsilon(x_1) + \Upsilon(x_2) - \Upsilon(0), \tag{1.5}$$

which is just the inhomogeneous version of (1.1); if $\alpha = 1$, then, (1.4) becomes the Hosszú equation (HFE)

$$\Upsilon(x_1 + x_2 - x_1 x_2) + \Upsilon(x_1 x_2) = \Upsilon(x_1) + \Upsilon(x_2). \tag{1.6}$$

The Cauchy equation is very well known and, for some details about it, we refer to monographs [2,3,7] (about stability) and [8,9] (about solutions). Information on the HFE and its stability is provided in the next section.

Since (1.4) is an obvious generalization of (1.5) and (1.6), the natural question arises of whether, for (1.4), we can get stability results similar to those already obtained for (1.5) and (1.6). Our considerations are mainly inspired by the stability results proved for (1.4) in [10, 11]. Let us recall that, in general, a functional equation is said to be stable in the sense of Ulam, in a class of maps, if any map from that class that satisfies the equation approximately (in some sense) is somehow close to an exact solution of the equation. It should be mentioned here that the H-US problem has also been considered for various other equations (e.g., differential [7, 12], integral [7, 13], difference [7, 14], etc.)

There are various ways of understanding the concept of an approximate solution and the proximity between two functions. This depends on the requirements of the given situation and the available distance measurement tools. One such unconventional method for measuring distances is provided by the m-norms, which are a natural extension of the concept of the ordinary norm. The necessary details on the notion of the m-norm can be found in Section 3.

In what follows, we investigate the H-US of Eq (1.4) with respect to m-norms and ordinary norms for functions whose domains are quadratically closed fields (e.g., the field of complex numbers). In this way, we extend the results in [10, 11] with somewhat better estimations (see Theorem 2.2 in the next section). At the end of the article, we highlight some symmetries, or their absence, between the outcomes we obtain.

2. The HFE

Equation (1.6) was first presented by M. Hosszú at the First International Conference on Functional Equations in Zakopane (Poland, 1967), which now is considered to be the event initiating a prestigious (invitation only) series of annual conferences called International Symposia on Functional Equations (ISFE) (see [15, 16]). He presented a description of differentiable solutions $f : \mathbb{R} \to \mathbb{R}$ to the equation ($\mathbb{R}$ denotes the set of real numbers). Later, the solutions to Eq (1.6) were studied in [17–19] (without assuming any regularity of f) and in [20, 21] (in the domain of distributions). We also refer to [22, 23] for some preliminary outcomes.

However, the most general result was obtained by Davison [24], who proved the following theorem.

Theorem 2.1. *Let $(U, +)$ be a commutative group and $\mathbb{K}$ be a field that has at least five elements. Then, $F : \mathbb{K} \to U$ is a solution to* (1.6) *if and only if there exist $b \in U$ and an additive $A : \mathbb{K} \to U$ such that $F(x) = A(x) + b$ for $x \in \mathbb{K}$.*

We refer to [25] and the references therein for some related results concerning (1.6) and some generalizations of it (see also [9, 26]).

Theorem 2.1 yields the next simple corollary, which provides a description of the solutions to Eq (1.4).

Corollary 2.1. *Let $(U, +)$ be a commutative group, $\mathbb{K}$ be a field with at least five elements, and $\alpha \in \mathbb{K}$ be fixed. Then, a map $F : \mathbb{K} \to U$ fulfills Eq* (1.4) *if and only if there exist $c \in U$ and an additive $A : \mathbb{K} \to U$ such that*

$$F(x) = A(x) + c, \qquad x \in \mathbb{K}. \tag{2.1}$$

Proof. Let $F : \mathbb{K} \to U$ be a solution to Eq (1.4). First, consider the case where $\alpha = 0$. Then

$$F(x + y) + F(0) = F(x) + F(y), \qquad x, y \in \mathbb{K}. \tag{2.2}$$

We write $A(x) := F(x) - F(0)$ for $x \in \mathbb{K}$. Then, $F(x) = A(x) + F(0)$ for $x \in \mathbb{K}$ and, by (2.2),

$$\begin{aligned} A(x + y) &= F(x + y) - F(0) = F(x) + F(y) - F(0) - F(0) \\ &= A(x) + A(y), \qquad x \in \mathbb{K}. \end{aligned}$$

Clearly, it is enough to take $c := F(0)$.

Now, assume that $\alpha \neq 0$. Let $\varkappa(x) := F(\alpha^{-1}x)$ for $x \in \mathbb{K}$. Then

$$\begin{aligned} \varkappa(x + y - xy) + \varkappa(xy) &= F(\alpha^{-1}(x + y - xy)) + F(\alpha^{-1}xy) \\ &= F(\alpha^{-1}x + \alpha^{-1}y - \alpha\alpha^{-1}x\alpha^{-1}y) + F(\alpha\alpha^{-1}x\alpha^{-1}y) \\ &= F(\alpha^{-1}x) + F(\alpha^{-1}y) = \varkappa(x) + \varkappa(y), \qquad x \in \mathbb{K}. \end{aligned}$$

According to Theorem 2.1, there exist $b \in U$ and an additive $A_0 : \mathbb{K} \to U$ such that $\varkappa(x) = A_0(x) + b$ for $x \in \mathbb{K}$. Clearly, it is enough to take $c := b$ and $A(x) := A_0(\alpha x)$ for $x \in \mathbb{K}$.

The converse implication (i.e., the sufficient condition) is easy to check in both cases.

The early H-US results for (1.6) were obtained in [27] and then improved in [10] and [11] (see also [3]). H-US of some modification of (1.6) (called Jensen-Hosszú equation) was considered in [28–30]. The next theorem was proved in [11] and is the most general stability result published so far for (1.6).

Theorem 2.2. *Let $\mathfrak{B}$ be a Banach space, and let $\varkappa : \mathbb{R} \to \mathfrak{B}$ be such that*

$$\epsilon := \sup_{x_1,x_2\in\mathbb{R}} \|\varkappa(x_1 + x_2 - x_1x_2) + \varkappa(x_1x_2) - \varkappa(x_1) - \varkappa(x_2)\| < \infty.$$

Then, there exists a unique additive $A : \mathbb{R} \to \mathfrak{B}$ such that

$$\sup_{x_1\in\mathbb{R}} \|\varkappa(x_1) - A(x_1) - (\varkappa(1) - A(1))\| \leq 9\epsilon. \tag{2.3}$$

In the subsequent corollary, we extend the stability result depicted by Theorem 2.2 to the case of Eq (1.4). Namely, we have the following.

Corollary 2.2. *Let V be a Banach space, and $\epsilon, \alpha \in \mathbb{R}$, $\epsilon > 0$. If $F : \mathbb{R} \to V$ is such that*

$$\epsilon := \sup_{x_1,x_2\in\mathbb{R}} \|F(x_1 + x_2 - \alpha x_1x_2) + F(\alpha x_1x_2) - F(x_1) - F(x_2)\| < \infty, \tag{2.4}$$

then, there exists a unique additive $A_0 : \mathbb{R} \to V$ such that

$$\sup_{x_1\in\mathbb{R}} \|F(x_1) - A_0(x_1) - b\| \leq \epsilon_0, \tag{2.5}$$

where

$$\epsilon_0 := \begin{cases} \epsilon, & \text{if } \alpha = 0, \\ 9\epsilon, & \text{if } \alpha \neq 0, \end{cases} \quad \text{and} \quad b := \begin{cases} F(0), & \text{if } \alpha = 0, \\ F(\alpha^{-1}) - A_0(\alpha^{-1}), & \text{if } \alpha \neq 0. \end{cases}$$

Proof. First, consider the case where $\alpha = 0$. Then, (2.4) becomes the inequality

$$\|F(x_1 + x_2) + F(0) - F(x_1) - F(x_2)\| \leq \epsilon, \quad x_1, x_2 \in \mathbb{R}. \tag{2.6}$$

Let $\Theta(x_1) := F(x_1) - F(0)$ for $x_1 \in \mathbb{R}$. It is easily seen that (1.2) holds and consequently, by Theorem 1.1, there is an additive $\mathcal{A} : \mathbb{R} \to V$ satisfying (1.3) (with $\eta = \epsilon$). Clearly, (1.3) implies (2.5) with $A_0 = \mathcal{A}$.

Assume now that $\alpha \neq 0$. Let $\varkappa(x) := F(\alpha^{-1}x)$ for $x \in \mathbb{F}$. Then

$$\begin{aligned} &\|\varkappa(x + y - xy) + \varkappa(xy) - \varkappa(x) - \varkappa(y)\| \\ &= \|F(\alpha^{-1}(x + y - xy)) + F(\alpha^{-1}xy) - F(\alpha^{-1}x) - F(\alpha^{-1}y)\| \\ &\leq \|F(\alpha^{-1}x + \alpha^{-1}y - \alpha\alpha^{-1}x\alpha^{-1}y) + F(\alpha\alpha^{-1}x\alpha^{-1}y) \\ &\quad - F(\alpha^{-1}x) + F(\alpha^{-1}y)\| \leq \epsilon, \quad x, y \in \mathbb{R}. \end{aligned}$$

Hence, by Theorem 2.2, there exists a unique additive $A : \mathbb{R} \to V$ such that (2.3) is valid, whence $\|F(\alpha^{-1}x_1) - A(x_1) - (F(\alpha^{-1}) - A(1))\| \leq 9\epsilon$ for $x_1 \in \mathbb{R}$. Define $A_0 : \mathbb{R} \to V$ by $A_0(x_1) := A(\alpha x_1)$ for $x_1 \in \mathbb{R}$. Then, it is easily seen that A_0 is additive and (2.5) is fulfilled (just replace x_1 by αx_1).

It remains to show the uniqueness of A_0. So, suppose that also $b_1 \in V$ and an additive $A_1 : \mathbb{R} \to V$ satisfy the inequality

$$\|F(x_1) - A_1(x_1) - b_1\| \le \epsilon_0, \quad x_1 \in \mathbb{R}.$$

Then, (2.5) implies that $\|A_1(x_1) + b_1 - A_0(x_1) - b\| \le 2\epsilon_0$ for $x_1 \in \mathbb{R}$. Hence, $\|A_1(x_1) - A_0(x_1)\| \le 2\epsilon_0 + \|b_1 - b\|$ for $x_1 \in \mathbb{R}$, which means that

$$\|A_1(x_1) - A_0(x_1)\| = \frac{1}{n}\|A_1(nx_1) - A_0(nx_1)\| \le \frac{1}{n}(2\epsilon_0 + \|b_1 - b\|)$$

for every $x_1 \in \mathbb{R}$ and $n \in \mathbb{N}$. Consequently, $A_1 = A_0$.

In Section 4, we prove an extension of Corollary 2.2 to the case of m-Banach spaces (for maps whose domain is a quadratically closed field with characteristic not equal to 2). Some basic information on m-norms is given in the next section.

Let us yet remark that HFE can also be considered, e.g., for $f : (0,1) \to \mathbb{R}$ (see [31]). In this case, the stability of HFE has been studied by Tabor [32], who proved a result stating a lack of stability.

3. Auxiliary information on m-normed spaces

Fix $m \in \mathbb{N}$. The notions of m-norm and m-normed space are natural extensions of those of norm and normed space. Namely, 1-norm is just a norm, and 1-normed space is a normed space (see, e.g., [33]). But let us start with a formal definition.

Let $\mathfrak{A}$ be a real linear space that is at least m-dimensional. Let $\|\cdot, \dots, \cdot\|$ be a map from $\mathfrak{A}^m$ into $\mathbb{R}$ such that, for all $a_1, a_2, w_1, \dots, w_m \in \mathfrak{A}$ and $v \in \mathbb{R}$, the subsequent four conditions are fulfilled:

(A) $\|w_1, \dots, w_m\| = 0$ if and only if vectors $w_1, \dots, w_m$ are linearly dependent;

(B) $\|w_1, \dots, w_m\|$ is invariant under permutations of the set $\{w_1, \dots, w_m\}$;

(C) $\|vw_1, \dots, w_m\| = |v|\|w_1, \dots, w_m\|$;

(D) $\|a_1 + a_2, w_2, \dots, w_m\| \le \|a_1, w_2, \dots, w_m\| + \|a_2, w_2, \dots, w_m\|$.

Then, $\|\cdot, \dots, \cdot\|$ is said to be an m-norm on $\mathfrak{A}$, and the pair $(\mathfrak{A}, \|\cdot, \dots, \cdot\|)$ is called an m-normed space (see, e.g., [34–36]).

In the rest of this section, we assume that $(\mathfrak{A}, \|\cdot, \dots, \cdot\|)$ is an m-normed space. If $(\mathfrak{A}, < \cdot, \cdot >)$ is a real inner product space and $m > 1$, then, an m-norm on $\mathfrak{A}$ can be defined by

$$\|w_1, \dots, w_m\| = \mathrm{abs}\left(\begin{vmatrix} < w_1, w_1 > & < w_1, w_2 > & \dots & < w_1, w_m > \\ \vdots & \vdots & \ddots & \vdots \\ < w_m, w_1 > & < w_m, w_2 > & \dots & < w_m, w_m > \end{vmatrix}^{1/2}\right)$$

for $w_1, \dots, w_m \in \mathfrak{A}$, where $\mathrm{abs}(r)$ means the module (absolute value) of $r \in \mathbb{R}$.

If $\mathfrak{A} = \mathbb{R}^m$ (with the usual inner product), then, such an m-norm has the form (called the Euclidean m-norm)

$$\|w_1, \dots, w_m\|_E = |\det(w_{ij})|, \qquad w_i = (w_{i1}, \dots, w_{im}) \in \mathbb{R}^m, \;\; i \in 1, \dots, m,$$

with

$$\det(w_{ij}) = \begin{vmatrix} w_{11} & w_{12} & \dots & w_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \dots & w_{mm} \end{vmatrix}.$$

Remark 3.1. As it is shown in [33, Theorem 1], in each real linear space X of dimension at least m, there is an m-norm (however, in the case where the dimension of X is not finite, the axiom of choice is necessary).

Moreover, according to [37], every m-norm yields an $(m-1)$-norm and finally also a norm. A somewhat similar observation can be found in [38, Remark 2].

In what follows, to simplify some formulas, we write

$$\|a, v\| := \|a, v_1, \dots, v_{m-1}\|, \qquad a \in \mathfrak{A}, \ v = (v_1, \dots, v_{m-1}) \in \mathfrak{A}^{m-1}. \tag{3.1}$$

Using this notation, we can now formulate some definitions and properties that are necessary in the rest of this paper.

Definition 3.1. *Given a sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathfrak{A}$, we say that it is a Cauchy sequence if*

$$\lim_{n,k\to\infty} \|u_n - u_k, v\| = 0, \qquad v \in \mathfrak{A}^{m-1}.$$

We say that a sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathfrak{A}$ is convergent, if there is $\mu \in \mathfrak{A}$ with

$$\lim_{k\to\infty} \|u_k - \mu, v\| = 0, \qquad v \in \mathfrak{A}^{m-1}.$$

Such μ is unique, called the limit of $(u_k)_{k\in\mathbb{N}}$, and is usually denoted by $\lim_{k\to\infty} u_k$ (i.e., $\lim_{k\to\infty} u_k := \mu$).

An m-Banach space is an m-normed space in which every Cauchy sequence is convergent. Moreover, in [39], the following properties have been stated (see also [40]).

Lemma 3.1. (*i*) *If $(u_k)_{k\in\mathbb{N}}$ is a convergent sequence in $\mathfrak{A}$, then*

$$\lim_{k\to\infty} \|u_k, v\| = \left\| \lim_{k\to\infty} u_k, v \right\|, \qquad v \in \mathfrak{A}^{m-1}.$$

(*ii*) *If $x_1 \in \mathfrak{A}$ and $\|x_1, v\| = 0$ for $v \in \mathfrak{A}^{m-1}$, then $x_1 = 0$.*

In what follows, we also need the following simple stability result (it can be easily derived, e.g., from [41, Corollary 10]).

Lemma 3.2. *Let T be a nonempty subset of a linear space W and $2T := \{2t : t \in T\} \subset T$. Let $\xi : \mathfrak{A}^{m-1} \to [0, \infty)$ and $f : T \to \mathfrak{A}$ satisfy*

$$\left\| \frac{1}{2} f(2t) - f(t), w \right\| \le \xi(w), \quad t \in T, \ w \in \mathfrak{A}^{m-1}.$$

Then, there is a unique $A : T \to \mathfrak{A}$ such that

$$\|f(t) - A(t), w\| \le 2\xi(w), \quad t \in T, \ w \in \mathfrak{A}^{m-1},$$

and

$$\frac{1}{2}A(2t) = A(u), \quad t \in T.$$

Moreover, A is given by

$$A(t) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n t), \quad t \in T.$$

For more information on m-normed spaces and examples of stability results obtained for them, we also refer to [42–44].

4. Main results for m-Banach spaces

In the rest of this paper, $\mathbb{K}$ stands for a field, which has the characteristic not equal to 2 and is quadratically closed (i.e., every element of $\mathbb{K}$ has a square root in $\mathbb{K}$), $\alpha \in \mathbb{K}$ is fixed, and $(\mathfrak{A}, \|\cdot, \ldots, \cdot\|)$ always denotes an m-Banach space with a fixed $m \in \mathbb{N}$, $m > 1$, unless explicitly stated otherwise. Moreover, we use the simplification defined in (3.1).

Let us start with the following auxiliary result.

Theorem 4.1. *Let V be a linear space over a field $\mathbb{F}$, $h : V \to \mathfrak{A}$, and $\eta : \mathfrak{A}^{m-1} \to [0, \infty)$. Let $D \subset V$ be nonempty, $2D := \{2u : u \in D\} \subset D$, and*

$$\|h(x_1 + x_2) - h(x_1) - h(x_2), w\| \le \eta(w), \quad x_1, x_2 \in D,\ x_1 + x_2 \in D,\ w \in \mathfrak{A}^{m-1}. \tag{4.1}$$

Then, there exists a unique $A_0 : D \to \mathfrak{A}$ that is additive on D, i.e.,

$$A_0(x_1 + x_2) = A_0(x_1) + A_0(x_2), \quad x_1, x_2 \in D,\ x_1 + x_2 \in D, \tag{4.2}$$

and fulfills the inequality

$$\|h(x_1) - A_0(x_1), w\| \le \eta(w), \quad x_1 \in D,\ w \in \mathfrak{A}^{m-1}. \tag{4.3}$$

Proof. From (4.1), with $x_2 = x_1$, we get

$$\left\|\frac{1}{2}h(2x_1) - h(x_1), w\right\| \le \frac{1}{2}\eta(w), \quad x_1 \in D,\ w \in \mathfrak{A}^{m-1}. \tag{4.4}$$

Hence, by Lemma 3.2, (4.3) holds with $A_0 : V \to \mathfrak{A}$, given by

$$A_0(u) := \lim_{n\to\infty} \frac{h(2^n u)}{2^n}, \quad u \in D.$$

We show that A_0 is additive on D, i.e., fulfills condition (4.2). Let $x_1, x_2 \in D$ and $x_1 + x_2 \in D$. Then, for every $n \in \mathbb{N}$ and $w \in \mathfrak{A}^{m-1}$,

$$\begin{aligned}\|A_0(x_1 + x_2) - A_0(x_1) - A_0(x_2), w\| \le{}& \|A_0(x_1 + x_2) - 2^{-n}h(2^n(x_1 + x_2)), w\| \\ &+ \|A_0(x_1) - 2^{-n}h(2^n x_1), w\| + \|A_0(x_2) - 2^{-n}h(2^n x_2), w\| \\ &+ \|2^{-n}h(2^n(x_1 + x_2)) - 2^{-n}h(2^n x_1) - 2^{-n}h(2^n x_2), w\|,\end{aligned} \tag{4.5}$$

and, by (4.1),

$$\|2^{-n}h(2^n(x_1+x_2)) - 2^{-n}h(2^n x_1) - 2^{-n}h(2^n x_2), w\| \le 2^{-n}\eta(w),$$

whence, with $n \to \infty$ in (4.5), we obtain $\|A_0(x_1+x_2) - A_0(x_1) - A_0(x_2), w\| = 0$ for $w \in \mathfrak{A}^{m-1}$. This means that $A_0(x_1+x_2) - A_0(x_1) - A_0(x_2) = 0$ (see Lemma 3.1(ii)).

For the proof of the uniqueness of A_0, suppose that $A_1 : D \to \mathfrak{A}$ is also additive on D and

$$\sup_{x_1 \in D} \|h(x_1) - A_1(x_1), w\| \le \eta(w), \quad w \in \mathfrak{A}^{m-1}.$$

Then

$$\begin{aligned}\|A_0(x_1) - A_1(x_1), w\| &= 2^{-n}\|A_0(2^n x_1) - A_1(2^n x_1), w\| \\ &\le 2^{-n}(\|A_0(2^n x_1) - h(2^n x_1), w\| + \|(A_1(2^n x_1) - h(2^n x_1)), w\|) \\ &\le 2^{-n}(\eta(w) + \eta(w)), \quad x_1 \in D,\ n \in \mathbb{N},\ w \in \mathfrak{A}^{m-1}.\end{aligned}$$

Consequently, letting $n \to \infty$, we obtain $A_0(x_1) - A_1(x_1) = 0$ for $x_1 \in D$ (see Lemma 3.1(ii)).

The main result is as follows.

Theorem 4.2. *Let $\epsilon : \mathfrak{A}^{m-1} \to [0,\infty)$ and $\Upsilon : \mathbb{K} \to \mathfrak{A}$ be such that*

$$\|\Upsilon(s + t - \alpha st) + \Upsilon(\alpha st) - \Upsilon(s) - \Upsilon(t), w\| \le \epsilon(w), \quad s,t \in \mathbb{K},\ w \in \mathfrak{A}^{m-1}. \tag{4.6}$$

Then, there exists a unique additive $A : \mathbb{K} \to \mathfrak{A}$ with

$$\|\Upsilon(s) - A(s) - b, w\| \le \epsilon_0(w), \quad s \in \mathbb{K},\ w \in \mathfrak{A}^{m-1}, \tag{4.7}$$

where

$$\epsilon_0(w) := \begin{cases} \epsilon(w), & \textit{if } \alpha = 0, \\ 3\epsilon(w), & \textit{if } \alpha \neq 0, \end{cases} \quad \textit{and} \quad b := \begin{cases} \Upsilon(0), & \textit{if } \alpha = 0, \\ \Upsilon(\alpha^{-1}) - A(\alpha^{-1}), & \textit{if } \alpha \neq 0. \end{cases}$$

Proof. First, let $\alpha = 1$. Note that, with $st = 1$, (4.6) takes the form

$$\|\Upsilon(s + s^{-1} - 1) + \Upsilon(1) - \Upsilon(s) - \Upsilon(s^{-1}), w\| \le \epsilon(w), \quad s \in \mathbb{K}_0 := \mathbb{K} \setminus \{0\},\ w \in \mathfrak{A}^{m-1}.$$

Next, replacing s by st and t by s^{-1} in (4.6), we get

$$\|\Upsilon(st + s^{-1} - t) + \Upsilon(t) - \Upsilon(st) - \Upsilon(s^{-1}), w\| \le \epsilon(w), \quad s,t \in \mathbb{K}_0,\ w \in \mathfrak{A}^{m-1}.$$

Since

$$\begin{aligned}&\Upsilon(s + t - st) + \Upsilon(st + s^{-1} - t) - \Upsilon(s + s^{-1} - 1) - \Upsilon(1) \\ &= (\Upsilon(s + t - st) + \Upsilon(st) - \Upsilon(s) - \Upsilon(t)) - (\Upsilon(s + s^{-1} - 1) + \Upsilon(1) - \Upsilon(s) - \Upsilon(s^{-1})) \\ &\quad + (\Upsilon(st + s^{-1} - t) + \Upsilon(t) - \Upsilon(st) - \Upsilon(s^{-1})), \quad s,t \in \mathbb{K}_0,\end{aligned}$$

(4.6) yields

$$
\begin{aligned}
&\|\Upsilon(s+t-st)+\Upsilon(st+s^{-1}-t)-\Upsilon(s+s^{-1}-1)-\Upsilon(1),w\| \\
&\le \|\Upsilon(s+t-st)+\Upsilon(st)-\Upsilon(s)-\Upsilon(t),w\|+\|\Upsilon(s+s^{-1}-1)+\Upsilon(1)-\Upsilon(s)-\Upsilon(s^{-1}),w\| \\
&\quad+\|\Upsilon(st+s^{-1}-t)+\Upsilon(t)-\Upsilon(st)-\Upsilon(s^{-1}),w\|\le 3\epsilon(w)
\end{aligned}
\tag{4.8}
$$

for $s,t\in\mathbb{K}_0$, $w\in\mathfrak{A}^{m-1}$.

We show that

$$
\|\Upsilon(u)+\Upsilon(v)-\Upsilon(u+v-1)-\Upsilon(1),w\|\le 3\epsilon(w),\quad u,v\in\mathbb{K},\ u+v\neq 2,\ w\in\mathfrak{A}^{m-1}. \tag{4.9}
$$

So, fix $u,v\in\mathbb{K}$ with $u+v\neq 2$. It is easy to check that there exists $s\in\mathbb{K}\setminus\{0,1\}$ such that $s+s^{-1}=u+v$. Let $t:=(u-s)(1-s)^{-1}$. Then,

$$
u=s+t-st,\qquad v=(u+v)-u=st+s^{-1}-t.
$$

Hence, (4.8) yields (4.9), which with $\Psi(u):=\Upsilon(u+1)-\Upsilon(1)$ for $u\in\mathbb{K}$ (and u,v replaced by $u+1,v+1$, respectively), takes the form

$$
\|\Psi(u)+\Psi(v)-\Psi(u+v),w\|\le 3\epsilon(w),\quad u,v\in\mathbb{K},\ u+v\neq 0,\ w\in\mathfrak{A}^{m-1}. \tag{4.10}
$$

Now, note that if $D:=\mathbb{K}\setminus\{0\}$, then, $2D\subset D$. Hence, from Theorem 4.1 (with $h=\Psi$, $D=\mathbb{K}_0:=\mathbb{K}\setminus\{0\}$ and $\eta=3\epsilon$), we derive that there exists an additive on $\mathbb{K}_0$, map $A_0:\mathbb{K}_0\to\mathfrak{A}$, such that

$$
\|\Psi(u)-A_0(u),w\|\le 3\epsilon(w),\quad u\in\mathbb{K}_0,\ w\in\mathfrak{A}^{m-1},
$$

that is,

$$
\|\Upsilon(u+1)-A_0(u)-\Upsilon(1)\|\le\epsilon_0(w),\quad u\in\mathbb{K}_0,\ w\in\mathfrak{A}^{m-1}. \tag{4.11}
$$

Define $A:\mathbb{K}\to\mathfrak{A}$ by

$$
A(u):=\begin{cases}A_0(u), & \text{if } u\neq 0,\\ 0, & \text{if } u=0.\end{cases} \tag{4.12}
$$

Then, by (4.11),

$$
\|\Upsilon(u+1)-A(u)-\Upsilon(1)\|\le\epsilon_0(w),\quad u\in\mathbb{K},\ w\in\mathfrak{A}^{m-1}, \tag{4.13}
$$

and, for every $s,t\in\mathbb{K}_0$ with $s+t\neq 0$, we have

$$
A_0(s)=A_0(s+t-t)=A_0(s+t)+A(-t)=A_0(s)+A_0(t)+A_0(-t),
$$

which implies $A_0(t)+A_0(-t)=0=A(0)$. Consequently, A is additive and it is easily seen that replacing $u+1$ by s in (4.13), we obtain (4.7).

If $\alpha\neq 1$, then arguing analogously as in the proof of Corollary 2.2, we can deduce the statement from case $\alpha=1$ (when $\alpha\neq 0$) and Theorem 4.1 with $D=V$ (when $\alpha=0$).

For the proof of uniqueness of A, suppose that $A_1:\mathbb{K}\to\mathfrak{A}$ is also additive and

$$
\sup_{s\in D}\|h(s)-A_1(s)-b_1,w\|\le\epsilon_0(w),\quad w\in\mathfrak{A}^{m-1},
$$

with some $b_1 \in \mathfrak{A}$. Then, for each $w \in \mathfrak{A}^{m-1}$, we have

$$\begin{aligned}\|A(s) - A_1(s), w\| &\le \|b - b_1, w\| + \|A(s) - h(s) - b, w\| + \|A_1(s) - h(s) - b_1, w\| \\ &\le \|b - b_1, w\| + 2\epsilon_0(w), \quad s \in \mathbb{K},\ n \in \mathbb{N},\end{aligned}$$

which implies that

$$\|A(s) - A_1(s), w\| = 2^{-n}\|A(2^n s) - A_1(2^n s), w\| \le 2^{-n}\left(\|b - b_1, w\| + 2\epsilon_0(w)\right), \quad s \in \mathbb{K},\ n \in \mathbb{N}.$$

Hence, letting $n \to \infty$, we get

$$\|A(s) - A_1(s), w\| = 0, \quad s \in \mathbb{K},\ w \in \mathfrak{A}^{m-1},$$

and consequently, $A(s) - A_1(s) = 0$ for $s \in \mathbb{K}$ (see Lemma 3.1(ii)).

Remark 4.1. *On the request of one of the reviewers, let us explain that the assumption that $\mathbb{K}$ is a quadratically closed field with the characteristic different from 2 is necessary in the proof of Theorem 4.2 for the existence of s (see the lines after* (4.9)*) and the inclusion $2D \subset D$. Otherwise, such an s might not exist or $2D = \{0\} \not\subset D$. These steps are important in the proof.*

The subsequent very simple observation shows that, if ϵ satisfies some additional condition, then, we obtain a result that is much stronger than Theorem 4.2.

Proposition 4.1. *Let $\epsilon : \mathfrak{A}^{m-1} \to [0, \infty)$ and $\Upsilon : \mathbb{K} \to \mathfrak{A}$ be such that* (4.6) *holds. Assume that there is a sequence $(l_k)_{k\in\mathbb{N}}$ in $(\mathbb{R} \setminus \{0\})^{m-1}$ such that*

$$\lim_{k\to\infty} \pi(l_k)^{-1}\epsilon(l_k w) = 0, \qquad w \in \mathfrak{A}^{m-1}, \tag{4.14}$$

where

$$az := (a_1 z_1, \ldots, a_{m-1} z_{m-1}), \qquad \pi(a) = \prod_{i=1}^{m-1} a_i$$

for $a = (a_1, \ldots, a_{m-1}) \in \mathbb{R}^{m-1}$, $z = (z_1, \ldots, z_{m-1}) \in \mathfrak{A}^{m-1}$. Then, Υ is a solution of Eq (1.4).

Proof. Fix $x_1, x_2 \in \mathbb{K}$. Since, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}&\|\Upsilon(x_1 + x_2 - \alpha x_1 x_2) + \Upsilon(\alpha x_1 x_2) - \Upsilon(x_1) - \Upsilon(x_2), w\| \\ &= \pi(l_k)^{-1}\|\Upsilon(x_1 + x_2 - \alpha x_1 x_2) + \Upsilon(\alpha x_1 x_2) - \Upsilon(x_1) - \Upsilon(x_2), l_k w\| \\ &\le \pi(l_k)^{-1}\epsilon(l_k w), \quad w \in \mathfrak{A}^{m-1},\end{aligned}$$

letting $k \to \infty$, in view of (4.14), we obtain

$$\left\|\Upsilon(x_1 + x_2 - \alpha x_1 x_2) + \Upsilon(\alpha x_1 x_2) - \Upsilon(x_1) - \Upsilon(x_2), w\right\|_\beta = 0, \quad w \in \mathfrak{A}^{m-1},$$

which implies that (1.4) holds (by Lemma 3.1(ii)).

Remark 4.2. *Note that if ϵ is a bounded map, then* (4.14) *holds, e.g., with $l_k = (k, k, \ldots, k)$ for $k \in \mathbb{N}$. Then, as Proposition 4.1 states, hyperstability results occur, i.e., maps satisfying inequality* (4.6) *must be, in fact, solutions to Eq* (1.4).

5. Results for Banach spaces

In nearly the same way as in the previous section, we can also obtain similar results for Banach spaces. We present them below. Because the proofs of them are very analogous to the reasonings in m-Banach spaces, we do not present them in detail, but only point out more significant differences.

As we have mentioned in Section 3, norms can in fact be considered as 1-norms (see, e.g., [33]), so the results presented in this part for Banach spaces complement the stability results obtained in this article for m-Banach spaces with $m > 1$.

In the sequel, $(\mathfrak{B}, \|\cdot\|)$ always denotes a Banach space. The next theorem is an analogue of Theorem 4.1.

Theorem 5.1. *Let V be a linear space over a field $\mathbb{F}$, $h : V \to \mathfrak{B}$, and $\overline{\eta} \in [0, \infty)$. Let $D \subset V$ be nonempty, $2D \subset D$, and*

$$\|h(x_1 + x_2) - h(x_1) - h(x_2)\| \le \overline{\eta}, \quad x_1, x_2 \in D,\ x_1 + x_2 \in D.$$

Then, there exists a unique map $A_0 : D \to \mathfrak{B}$ that is additive on D and such that

$$\sup_{x_1 \in D} \|h(x_1) - A_0(x_1)\| \le \overline{\eta}. \tag{5.1}$$

Proof. It suffices to argue as in the proof of Theorem 4.1, but we replace Lemma 3.2 with the following lemma, which can be easily derived from, e.g., [45, Theorem 2.1] (and its proof) or [46, Proposition 1].

Lemma 5.2. *Let W be a linear space, $T \subset W$ be nonempty, $2T \subset T$, and $f : T \to \mathfrak{B}$ be such that*

$$\xi := \sup_{t \in T} \left\| \frac{1}{2} f(2t) - f(t) \right\| < \infty.$$

Then, there is a unique $A : T \to \mathfrak{B}$ such that

$$\sup_{t \in T} \|f(t) - A(t)\| \le 2\xi$$

and

$$\frac{1}{2} A(2t) = A(u), \quad t \in T.$$

Moreover, A is given by

$$A(t) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n t), \quad t \in T.$$

The next theorem corresponds to Theorem 4.2.

Theorem 5.3. *Let $\overline{\epsilon} \in [0, \infty)$ and $\Upsilon : \mathbb{K} \to \mathfrak{B}$ be such that*

$$\sup_{s,t \in \mathbb{K}} \|\Upsilon(s + t - \alpha st) + \Upsilon(\alpha st) - \Upsilon(s) - \Upsilon(t)\| \le \overline{\epsilon}. \tag{5.2}$$

Then, there exists a unique additive $A : \mathbb{K} \to \mathfrak{B}$ with

$$\sup_{s \in \mathbb{K}} \|\Upsilon(s) - A(s) - b\| \le \overline{\epsilon}_0, \tag{5.3}$$

where

$$\overline{\epsilon}_0 := \begin{cases} \overline{\epsilon}, & \text{when } \alpha = 0, \\ 3\overline{\epsilon}, & \text{when } \alpha \neq 0, \end{cases} \quad \text{and} \quad b := \begin{cases} \Upsilon(0), & \text{if } \alpha = 0, \\ \Upsilon(\alpha^{-1}) - A(\alpha^{-1}), & \text{if } \alpha \neq 0. \end{cases}$$

Proof. It suffices to argue as in the proof of Theorem 4.2, with Theorem 4.1 replaced by Theorem 5.1.

The next example shows that Proposition 4.1 does not have an analogue in Banach spaces (see Remark 4.2).

Example 5.4. Let $\Upsilon : \mathbb{K} \to \mathfrak{B}$ be a bounded map that does not satisfy Eq (1.4). Then, (5.2) holds with

$$\overline{\epsilon} := \sup_{s,t\in\mathbb{K}} \|\Upsilon(s+t-\alpha st) + \Upsilon(\alpha st) - \Upsilon(s) - \Upsilon(t)\| \leq 4 \sup_{s\in\mathbb{K}} \|\Upsilon(s)\|.$$

6. Conclusions

In this paper, we have studied the stability of a generalization of the HFE for maps from the set of complex numbers into m-Banach spaces. We have also shown that similar results are possible for functions that take values in a Banach space. In this way, we have complemented some earlier outcomes in [10, 11] obtained only for maps from the set of reals and taking values in a Banach space.

The results obtained in Corollary 2.2 (in the "real case") and in Theorem 2.2 (in the case of quadratically closed field that includes the case of complex numbers) look a bit similar, but with better estimation in the latter (e.g., complex) case. It would be interesting to see whether estimation (2.5) can be made somewhat similar to (5.3).

The symmetries between the results in the cases of Banach space and m-Banach space are easily visible (even in the proofs); however, Proposition 4.1, Remark 4.2, and Example 5.4 show that such analogies may exist only when ϵ is not a bounded map, because otherwise (in view of condition (4.14)), we obtain hyperstability results as in Proposition 4.1 (i.e., functions satisfying the stability inequality must be, in fact, solutions to the corresponding equation). Moreover, the similarities between the cases $\alpha = 0$ and $\alpha \neq 0$ are also very clear, although Eqs (1.5) and (1.6) look different.

Potential future research could involve the case of fields that are not quadratically closed, and the stability of this generalization of the HFE for maps taking values in some other spaces, such as dq-metric spaces, b-metric spaces, etc. Also, it would be interesting to see whether (4.7) and (5.3) can be improved any further, or even to find the best possible estimates in these situations (see [3, 14, 47] for more recent information on the issue of best estimates in H-US).

Author contributions

S. Alsaeed, El-sayed El-hady and J. Brzdęk: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Visualization; El-sayed El-hady and J. Brzdęk: Data curation, Writing–original draft preparation, Writing–review and editing, Project administration; J. Brzdęk: Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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