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# Necessary and sufficient conditions and special cases of Drużkowski matrices

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**Abstract:** Drużkowski maps play a significant role in the study of the Jacobian conjecture. To investigate Drużkowski maps, we identified more Drużkowski matrices via quasi-D-nilpotent matrices, and then studied the corresponding Drużkowski maps. Some necessary and sufficient conditions in which quasi-D-nilpotent matrices were Drużkowski matrices were given. In addition, several classes of special Drużkowski matrices were characterized.

**Keywords:** Drużkowski matrix; Jacobian conjecture; quasi-D-nilpotent matrix
**Mathematics Subject Classification:** 13F20, 14R15

## 1. Introduction

Let $K$ be a field of characteristic 0 and $K[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over $K$ in the variables $x_1, x_2, \ldots, x_n$. A polynomial map is a map $F = (F_1, F_2, \ldots, F_n) : K^n \to K^n$ with each $F_i \in K[x_1, x_2, \ldots, x_n]$ defined by

$$x \mapsto (F_1(x), F_2(x), \ldots, F_n(x))$$

for all $x \in K^n$. The Jacobian conjecture, posed by Keller in 1939 in [1], states that a polynomial mapping $F = (F_1, F_2, \ldots, F_n)$ is invertible if its Jacobian determinant $\det JF$ is a nonzero constant. This conjecture was listed as one of 18 famous open problems in [2], and it is still open for $n \geq 2$ up to now. The Jacobian conjecture has been reduced to the cases of polynomial maps $F$ of degree 3 such that the Jacobian matrix of $F - X$ is nilpotent, where $X = (x_1, x_2, \ldots, x_n)$ is the identity map [3,4], and further to the cases of Drużkowski maps [5–7].

Let $H = (L_1^3, L_2^3, \ldots, L_n^3)$ with $JH$ nilpotent and $L_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$ for $i = 1, 2, \ldots, n$. Then, the polynomial map $F = X + H$ is called a Drużkowski map, and the matrix $A = (a_{ij})$ is called a Drużkowski matrix [8,9].

The Jacobian matrix of $H$

$$JH = 3\mathrm{diag}(L_1^2, L_2^2, \ldots, L_n^2)A,$$

and the nilpotency of $JH$ is equivalent to

$$(\mathrm{diag}(L_1(x)^2, L_2(x)^2, \dots, L_n(x)^2)A)^n = 0 \quad \text{for all } x \in K^n, \tag{1.1}$$

where $\mathrm{diag}(u)$ denotes the diagonal matrix with the vector $u$ as the diagonal. Clearly, upper triangular matrices with zero diagonal are Drużkowski matrices.

Drużkowski maps are difficult to describe and construct. Drużkowski matrices are essential to verify Jacobian conjecture for Drużkowski maps, but they are also difficult to find and classify. Hubbers [9] classified Drużkowski matrices up to dimension 5 with respect to cubic-similarity. To study Drużkowski maps, Gorni et al. [8] introduced the D-nilpotent matrices. An $n \times n$ matrix $A$ is called D-nilpotent if $DA$ is nilpotent for all $n \times n$ diagonal matrices $D$. It is proved in [8] that a matrix $A$ is D-nilpotent, if and only if, $A$ is permutation-similar to an upper triangular matrix with zero diagonal, so the corresponding Drużkowski maps are linearly triangularizable. Tian et al. [10] considered a larger class than D-nilpotent matrices. They call an $n \times n$ matrix $A$ quasi-D-nilpotent if there exists a linear subspace $V$ of dimension $n - 1$ consisting of $n \times n$ diagonal matrices such that $DA$ is nilpotent for all $D \in V$. Unlike D-nilpotent matrices, a quasi-D-nilpotent matrix is unnecessarily a Drużkowski matrix and converse. Meanwhile, they determined irreducible quasi-D-nilpotent matrices and Frobenius normal forms of quasi-D-nilpotent matrices under permutation similarity. The author has achieved some results in Jacobian conjecture [11].

The reference [10] primarily utilizes principal minors to generalize D-nilpotent matrices and determines the canonical form of quasi-D-nilpotent matrices. Based on the structural characteristics of the Frobenius canonical form of quasi-D-nilpotent matrices, this paper discusses the nilpotency of block matrices and aims to identify more Drużkowski matrices via quasi-D-nilpotent matrices. However, identifying Drużkowski matrices via quasi-D-nilpotent remains highly nontrivial. Some sufficient and necessary conditions for quasi-D-nilpotent matrices to be Drużkowski matrices are presented, and several special Drużkowski matrices are characterized.

## 2. Preliminaries

Let $A$ be a matrix with order $n$, and denote by $A_i$ the ith row of $A$. Let $X$ be the column vector of variables $x_1, x_2, \dots, x_n$. For the row vectors $\alpha = (a_1, a_2, \dots, a_s)$, $\beta = (b_1, b_2, \dots, b_s)$, let $\alpha * \beta$ be a Hadamard product, that is, $\alpha * \beta = (a_1 b_1, a_2 b_2, \dots, a_s b_s)$. For the row vector $u = (u_1, u_2, \dots, u_r)$, denote by $u^{*k}$ the kth Hadamard power of $u$, that is, $u^{*k} = (u_1^k, u_2^k, \dots, u_r^k)$. Let $H$ be a cubic power linear mapping determined by $A$, $H = (AX)^{*3}$. Then,

$$J((AX)^{*3}) = 3\mathrm{diag}((A_1X)^2, (A_2X)^2, \dots, (A_nX)^2)A.$$

Let $A$ be a quasi-D-nilpotent matrix of order $n$. Without loss of generality, we assume its Frobenius normal form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix},$$

where $A_{11}, A_{33}$ are strictly upper triangular matrices, and $A_{22}$ is the normal form of the irreducible quasi-D-nilpotent matrix. Consider the Jacobian matrix of $(AX)^{*3}$,

$$J((AX)^{*3}) = \operatorname{diag}((AX)^{*2})A = \begin{pmatrix} D_1A_{11} & D_1A_{12} & D_1A_{13} \\ 0 & D_2A_{22} & D_2A_{23} \\ 0 & 0 & D_3A_{33} \end{pmatrix},$$

where $D_1 = \operatorname{diag}(A_{11}X_1 + A_{12}X_2 + A_{13}X_3)^{*2}$, $D_2 = \operatorname{diag}(A_{22}X_2 + A_{23}X_3)^{*2}$, $D_3 = \operatorname{diag}(A_{33}X_3)^{*2}$, $X = (X_1, X_2, X_3)$, $X_1 = (x_1, x_2, \ldots, x_p)$, $X_2 = (x_{p+1}, x_{p+2}, \ldots, x_{p+q})$, $X_3 = (x_{p+q+1}, x_{p+q+2}, \ldots, x_n)$, $p$ is the order of $A_{11}$, and $q$ is the order of $A_{22}$. Therefore, $A$ is Drużkowski matrix, if and only if, the last block matrix above is nilpotent, that is,

$$\begin{pmatrix} D_1A_{11} & D_1A_{12} & D_1A_{13} \\ 0 & D_2A_{22} & D_2A_{23} \\ 0 & 0 & D_3A_{33} \end{pmatrix}^n = 0.$$

$A_{11}, A_{33}$ are strictly upper triangular matrices, so the equation holds, if and only if, $(D_2A_{22})^2 = 0$. That is,

$$\begin{pmatrix} (\operatorname{diag}(au^TvX_{21} + u^T\alpha X_{22} + A_{23}X_3)^{*2})au^Tv & (\operatorname{diag}(au^TvX_{21} + u^T\alpha X_{22} + A_{23}X_3)^{*2})u^T\alpha \\ (\operatorname{diag}(\beta^TvX_{21} + UX_{22} + A_{23}X_3)^{*2})\beta^Tv & (\operatorname{diag}(\beta^TvX_{21} + UX_{22} + A_{23}X_3)^{*2})U \end{pmatrix}^n = 0,$$

where $X_2 = (X_{21}, X_{22})^T$. If $A_{23}X_3 = 0$, then the above equation is equivalent to

$$A_{22} = \begin{pmatrix} au^Tv & u^T\alpha \\ \beta^Tv & U \end{pmatrix}$$

which is a Drużkowski matrix.

Therefore, we consider when the matrix in the following form is a Drużkowski matrix:

$$A = \begin{pmatrix} au^Tv & u^T\alpha \\ \beta^Tv & U \end{pmatrix} \tag{2.1}$$

is a matrix of order $n$, where $a \in K$, $U$ is a strictly upper triangular matrix, $u, v \in K^r, \alpha, \beta \in K^{n-r}$, and $u = (u_1, u_2, \ldots, u_r)$, $v = (v_1, v_2, \ldots, v_r)$, $\alpha = (a_1, a_2, \ldots, a_{n-r}), \beta = (b_1, b_2, \ldots, b_{n-r})$.

## 3. The condition of Drużkowski matrices

**Lemma 3.1.** *Let $A$ be an $n \times n$ matrix of the form (2.1). Then, $A$ is nilpotent, if and only if, $a \neq 0, vu^T = 0$, or $a = 0$ and $\alpha U^i \beta^T = 0, i = 0, 1, \ldots, n-r-1$.*

*Proof.* Calculate the determinant of matrix $A$

$$\begin{aligned} &\det(\lambda I - A) \\ = &\det\begin{pmatrix} \lambda I - au^TV & -U^T\alpha \\ -\beta^Tv & \lambda I - U \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \det\begin{pmatrix} \lambda I - au^T v - u^T \alpha(\lambda I - U)^{-1}\beta^T v & -0 \\ 0 & \lambda\lambda I - U \end{pmatrix} \\
&= \det(\lambda I - U)\det[\lambda I - au^T v - u^T \alpha(\lambda I - U)^{-1}\beta^T v] \\
&= \lambda^{n-r}\det(\lambda I - (a + \alpha(\lambda I - U)^{-1}\beta^T)u^T v) \\
&= \lambda^{n-r}\lambda^{r-1}(\lambda - (a + \alpha(\lambda I - U)^{-1}\beta^T)vu^T) \\
&= \lambda^n - \lambda^{n-1}avu^T - \lambda^{n-2}\alpha(I - \frac{U}{\lambda})^{-1}\beta^T vu^T \\
&= \lambda^n - \lambda^{n-1}avu^T - \lambda^{n-2}\alpha\sum_{i=0}^{n-r-1}(-\frac{U}{\lambda})^i\beta^T vu^T \\
&= \lambda^n - \lambda^{n-1}avu^T - \sum_{i=0}^{n-r-1}(-1)^i\lambda^{n-2-i}\alpha U^i\beta^T vu^T.
\end{aligned}$$

Case 1. $a \neq 0$. Then, $A$ is nilpotent, if and only if, the characteristic polynomial of $A$ is $\lambda^n$, if and only, if $avu^T = 0$ and $\alpha U^i\beta^T vu^T = 0$, and if and only, if $vu^T = 0$.

Case 2. $a = 0$. Then, $\det(\lambda I - A) = \lambda^n - \sum_{i=0}^{n-r-1}(-1)^i\lambda^{n-2-i}\alpha U^i\beta^T vu^T$. $A$ is nilpotent, if and only if, the characteristic polynomial of $A$ is $\lambda^n$, if and only if, $\alpha U^i\beta^T vu^T = 0$, and if and only if, $\alpha U^i\beta^T = 0, i = 0, 1, \ldots, n-r-1$. □

**Theorem 3.2.** *Let $A$ be an $n \times n$ matrix of the form (2.1). Then, $A$ is a Drużkowski matrix, if and only if, one of the following holds true.*

*(1) $v(u^T)^{*3} = 0$, where $(u^T)^{*3} = (u_1^3, u_2^3, \ldots, u_r^3)^T$;*

*(2) $a = 0$ and $\alpha(D_2U)^iD_2\beta^T = 0$, $i = 0, 1, \ldots, n-r-1$, where $D_2 = diag(\beta^T vX_1 + UX_2)^{*2}, X_1 = (x_1, x_2, \ldots, x_r)^T, X_2 = (x_{r+1}, x_{r+2}, \ldots, x_n)^T$.*

*Proof.* The sufficiency is obvious. We only need to prove the necessity. Let

$$D_1 = \mathrm{diag}((au_1vX_1 + u_1\alpha X_2)^2, (au_2vX_1 + u_2\alpha X_2)^2, \ldots, (au_rvX_1 + u_r\alpha X_2)^2),$$

$$D_2 = \mathrm{diag}((b_1vX_1 + U_1X_2)^2, (b_1vX_1 + U_2X_2)^2, \ldots, (b_svX_1)^2).$$

Then,

$$J((AX)^{*3}) = 3\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} au^T v & u^T\alpha \\ \beta^T v & U \end{pmatrix} = 3\begin{pmatrix} aD_1u^T v & D_1u^T\alpha \\ D_2\beta^T v & D_2U \end{pmatrix}.$$

By Lemma 3.1, $J((AX)^{*3})$ is nilpotent, if and only if,

$$avD_1u^T = 0 \quad or \tag{3.1}$$

$$a = 0, \;\; \alpha(D_2U)^iD_2\beta^T vD_1u^T = 0, i = 0, 1, \ldots, n-r-1. \tag{3.2}$$

Since

$$\begin{aligned}
avD_1u^T &= a(v_1u_1(au_1vX_1 + u_1\alpha X_2)^2 + \cdots + v_ru_r(au_rvX_1 + u_r\alpha X_2)^2) \\
&= a(\sum_{i=1}^{s} v_iu_i^3)(avX_1 + \alpha X_2)^2,
\end{aligned}$$

then (3.1) is established, if and only if, $a = 0$ or $v(u^T)^{*3}$ or $(avX_1 + \alpha X_2)^2 = 0$. (3.2) is established, if and only if, $a = 0$, and $vD_1u^T = 0$ or $\alpha(D_2U)^iD_2\beta^T = 0, i = 0, 1, \ldots, n-r-1$, if and only if $a = 0$, and $v(u^T)^{*3} = 0$ or $(avX_1 + \alpha X_2)^2 = 0$ or $\alpha(D_2U)^iD_2\beta^T = 0, i = 0, 1, \ldots, n-r-1$.

Therefore, $A$ is a Drużkowski matrix, if and only if, $v(u^T)^{*3} = 0$, or $a = 0$ and $\alpha(D_2U)^iD_2\beta^T = 0, i = 0, 1, \ldots, n-r-1$. □

For convenience in calculation, consider a row or a column vector $\alpha$, and let $\langle\alpha\rangle = \mathrm{diag}\alpha$. For $\alpha, \beta \in K^r$, it is easy to see that

$$\langle\alpha^T\rangle = \langle\alpha\rangle,$$

$$\langle\alpha\rangle\beta = \langle\beta\rangle\alpha, \text{ where } \alpha, \beta \text{ are column vectors,}$$

$$\alpha\langle\beta\rangle = \beta\langle\alpha\rangle, \text{ where } \alpha, \beta \text{ are row vectors.}$$

**Corollary 3.3.** *Let $A$ be an $n \times n$ matrix of the form (2.1). If $A$ is a Drużkowski matrix, then $v(u^T)^{*3} = 0$, or $a = 0$, $\alpha(\beta^T)^{*3} = 0$, $(\alpha * (\beta^T)^{*2})U = 0$, $U^T\mathrm{diag}(\alpha * \beta)U = 0$.*

*Proof.* Assume $v(u^T)^{*3} \neq 0$, then by Theorem 3.2, we have

$$a = 0 \text{ and } \alpha(D_2U)^iD_2\beta^T = 0, i = 0, 1, \ldots, n-r-1.$$

Considering the case where $i = 0$, we have $\alpha D_2\beta^T = 0$. That is to say, $\alpha\mathrm{diag}(\beta^T vX_1 + UX_2)^{*2}\beta^T = 0$,

$$\alpha\langle\beta^T vX_1\rangle^2\beta^T + 2\alpha\langle\beta^T vX_1\rangle\langle UX_2\rangle\beta^T + \alpha\langle UX_2\rangle^2\beta^T = 0.$$

Since the variables of $\langle\beta^T vX_1\rangle$ and $\langle UX_2\rangle$ are different, then

$$\alpha\langle\beta^T vX_1\rangle^2\beta^T = 0; \tag{3.3}$$
$$\alpha\langle\beta^T vX_1\rangle\langle UX_2\rangle\beta^T = 0; \tag{3.4}$$
$$\alpha\langle UX_2\rangle^2\beta^T = 0. \tag{3.5}$$

Since $v(u^T)^{*3} \neq 0$, we have $v \neq 0$, hence, (3.3) is equivalent to $\alpha(\beta^T)^{*3} = 0$. Since

$$\alpha\langle\beta^T vX_1\rangle\langle UX_2\rangle\beta^T = vX_1\alpha\langle\beta^T\rangle\langle\beta^T\rangle(UX_2) = vX_1\alpha\langle\beta^T\rangle^2(UX_2),$$

(3.4) is equivalent to $\alpha\langle(\beta^T)^{*2}\rangle U = 0$, that is, $(\alpha * (\beta^T)^{*2})U = 0$. Notice that

$$\begin{aligned}\alpha\langle UX_2\rangle^2\beta^T &= \alpha\langle UX_2\rangle\langle UX_2\rangle\beta^T\\ &= X_2^TU^T\langle\alpha\rangle\langle\beta^T\rangle(UX_2)\\ &= X_2^TU^T\langle\alpha * \beta\rangle(UX_2).\end{aligned}$$

Since $U^T\langle\alpha * \beta\rangle U$ is symmetric, the equivalence in (3.5) holds if and only if $U^T\langle\alpha * \beta\rangle U = 0$. □

**Corollary 3.4.** *Let $A$ be an $n \times n$ matrix of the form (2.1). Let $v(u^T)^{*3} \neq 0$ and $a = 0$, then $A$ is a Drużkowski matrix, if and only if,*

$$\begin{pmatrix} 0 & \alpha \\ \beta & U \end{pmatrix}$$

*is a Drużkowski matrix.*

*Proof.* By Theorem 3.2, $A$ is a Drużkowski matrix, if and only if,

$$\alpha(D_2U)^iD_2\beta^T = 0, i = 0,1,\ldots,n-r-1,$$

and if and only if,

$$\alpha((\mathrm{diag}((b_1vX_1+uX_2)^2,\ldots,(b_svX_1)^2))U)^i(\mathrm{diag}((b_1vX_1+uX_2)^2,\ldots,(b_svX_1)^2))\beta^T = 0.$$

Since $\begin{pmatrix}0 & \alpha\\ \beta & U\end{pmatrix}$ is a Drużkowski matrix, if and only if,

$$\alpha(D_2U)^iD_2\beta^T = 0, i = 0,1,\ldots,n-r-1,$$

and, if and only if,

$$\alpha((\mathrm{diag}((b_1x_1+uX_2)^2,\ldots,(b_sx_1)^2))U)^i(\mathrm{diag}((b_1x_1+uX_2)^2,\ldots,(b_sx_1)^2))\beta^T = 0.$$

Then, $A$ is a Drużkowski matrix, if and only if, $\begin{pmatrix}0 & \alpha\\ \beta & U\end{pmatrix}$ is a Drużkowski matrix. □

**Remark 3.5.** *If $a \neq 0$, then the fact that A is a Drużkowski matrix and $\begin{pmatrix}0 & \alpha\\ \beta & U\end{pmatrix}$ is a Drużkowski matrix does not imply any relationship between them.*

**Definition 3.6.** *[12, Definition 6.6.1] Let $H = (AX)^{*d}$ be a power linear mapping. Then H is called ditto linear triangularizable, if there exists $T \in GL_n(\mathbb{K})$ such that $T^{-1}H(Tx)$ is also a power linear mapping, and $JT^{-1}H(Tx)$ is triangularizable.*

**Theorem 3.7.** *Let A be an $n \times n$ matrix of the form (2.1). Then, polynomial map $F = X + (AX)^{*3}$ is ditto linear triangularizable.*

*Proof.* Let $X_1 = (x_1, x_2, \ldots, x_r)^T$, $X_2 = (x_{r+1}, x_{r+2}, \ldots, x_n)^T$. Then,

$$AX = \begin{pmatrix}(au^TvX_1 + u^T\alpha X_2)\\ \beta^TvX_1 + UX_2\end{pmatrix} = \begin{pmatrix}u^T(avX_1 + \alpha X_2)\\ \beta^TvX_1 + UX_2\end{pmatrix}.$$

Let $F = (F_1, F_2, \ldots, F_n)$. Then,

$$F_i = \begin{cases} x_i + u_i^3(avX_1 + \alpha X_2)^3, & 1 \leq i \leq r;\\ x_i + (b_ivX_1 + U_iX_2)^3, & r+1 \leq i \leq n.\end{cases}$$

Take

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \frac{u_2^3}{u_1^3} & 1 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \frac{u_r^3}{u_1^3} & 0 & \cdots & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Let $TL^{-1}FL(x) = X + H$. Since $v(u^T)^{*3} = 0$, we have

$$H_i = \begin{cases} (u_1^3(a\sum_{j=2}^{r} v_jx_j) + \sum_{j=r+1}^{n} a_{j-r}x_j)^3, & i = 1; \\ 0, & 2 \leq i \leq r; \\ (b_{i-r}(\sum_{j=2}^{r} v_jx_j) + \sum_{j=r+1}^{n} d_{i-r,j-r}x_j)^3, & r+1 \leq i \leq n. \end{cases}$$

Take a permutational matrix

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Then, $P^TL^{-1}FLP(x) = Y + \bar{H}$, where

$$\bar{H}_i = \begin{cases} (u_1^3(\sum_{j=2}^{n-r+1} a_{j-1}y_j) + a\sum_{j=n-r+2}^{n} v_{j-(n-r)}y_j)^3, & i = 1; \\ (\sum_{j=i+1}^{n-r+1} d_{i-1,j-1}y_j + b_{i-1}\sum_{j=n-r+2}^{n} v_{j-(n-r)}y_j)^3, & 2 \leq i \leq n-r; \\ 0, & n-r+1 \leq i \leq n. \end{cases}$$

It follows that $P^TL^{-1}FLP$ is ditto linear triangularizable. □

## 4. Some special Drużkowski matrix

**Theorem 4.1.** *Let A be an $n \times n$ matrix of the form (2.1) and* rank $U = n - r - 1$. *If A is a Drużkowski matrix, then $v(u^T)^{*3} = 0$ or $\alpha * \beta = 0$.*

*Proof.* Let $s = n - r$. Then rank$U = n - r - 1 = s - 1$. Let $v(u^T)^{*3} \neq 0$. By Theorem 3.2, we have $a = 0$ and $\alpha(D_2U)^iD_2\beta^T = 0, i = 0, 1, \ldots, n - r - 1$. By Corollary 3.3, we have

$$\alpha(\beta^T)^{*3} = 0, \tag{4.1}$$

$$(\alpha * \beta^{*2})U = 0, \tag{4.2}$$

$$U^T diag(\alpha * \beta)U = 0. \tag{4.3}$$

Since $U_s = 0$, by $(\alpha * (\beta)^{*2})U = 0$, we have

$$a_1b_1^2U_1 + a_2b_2^2U_2 + \cdots + a_{s-1}b_{s-1}^2U_{s-1} = 0.$$

By rank $U = n - r - 1$, we have $U_1, U_2, \ldots, U_{s-1}$ are linearly independent, hence, $a_ib_i = 0, i = 1, 2, \ldots, s - 1$. By $\alpha(\beta^T)^{*3} = 0$, we have $a_sb_s = 0$. Thus, $\alpha * \beta = 0$. □

**Theorem 4.2.** *Let A be an $n \times n$ matrix of the form (2.1). If $U = \omega z$, where $\omega$ is a column vector, z is a row vector. Then, A is a Drużkowski matrix, if and only if, $v(u^T)^{*3} = 0$ or $a = 0$, $\alpha(\beta^T)^{*3} = 0, (\alpha * \beta^{*2})\omega z = 0$, $\omega^T \mathrm{diag}(\alpha * \beta)\omega = 0$, and one of the following three conditions holds true.*

*(1) $(\alpha * \beta)\omega^{*2} = 0$ and $\alpha\omega^{*3} = 0$;*

*(2) $z(\beta^T)^{*3} = 0$ and $\alpha\omega^{*3} = 0$;*

*(3) $z(\beta^T)^{*3} = 0$, $z(\mathrm{diag}\omega)(\beta^T)^{*2} = 0$ and $z(\mathrm{diag}\omega)^2\beta^T = 0$.*

*Proof.* Without loss of generality, let $s = n - r$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_s)^T, z = (z_1, z_2, \ldots, z_s)$. Since $U = \omega z$ is a strictly triangularizable matrix, we have $(D_2U)^2 = 0$. Hence, $A$ is a Drużkowski matrix, if and only if,

$$\alpha D_2 \beta^T = 0, \tag{4.4}$$

$$\alpha D_2 U D_2 \beta^T = 0. \tag{4.5}$$

By Corollary 3.3, the above equations hold and are equivalent to the last set of conditions Corollary 3.3:

$$\alpha (\beta^T)^{*3} = 0, \tag{4.6}$$

$$(\alpha * \beta^{*2})\omega z = 0, \tag{4.7}$$

$$\omega^T \operatorname{diag}(\alpha * \beta)\omega = 0. \tag{4.8}$$

Now we consider $\alpha D_2 U D_2 \beta^T = 0$, that is,

$$\alpha \langle (vX_1)\beta^T + UX_2 \rangle^2 U \langle (vX_1)\beta^T + UX_2 \rangle^2 \beta^T = 0.$$

Let $(vX_1)\beta^T = \Delta_1$, $UX_2 = \Delta_2$. Then,

$$\alpha \langle \Delta_1 + \Delta_2 \rangle^2 U \langle \Delta_1 + \Delta_2 \rangle^2 \beta^T = 0.$$

We obtain

$$\begin{aligned}
&\alpha\langle\Delta_1\rangle^2 U\langle\Delta_1\rangle^2\beta^T + 2\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_1\rangle^2\beta^T + 2\alpha\langle\Delta_1\rangle^2 U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T \\
&+ \alpha\langle\Delta_2\rangle^2 U\langle\Delta_1\rangle^2\beta^T + 4\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T + \alpha\langle\Delta_1\rangle^2\langle\Delta_2\rangle^2\beta^T \\
&+ 2\alpha\langle\Delta_2\rangle^2 U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T + 2\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_2\rangle^2\beta^T + \alpha\langle\Delta_2\rangle^2 U\langle\Delta_2\rangle^2\beta^T = 0.
\end{aligned}$$

Since the variables of $\Delta_1$ and $\Delta_2$ are different, by comparing the degrees of $\Delta_1$, we can obtain

$$\alpha\langle\Delta_1\rangle^2 U\langle\Delta_1\rangle^2\beta^T = 0; \tag{4.9}$$

$$\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_1\rangle^2\beta^T + \alpha\langle\Delta_1\rangle^2 U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T = 0; \tag{4.10}$$

$$\alpha\langle\Delta_2\rangle^2 U\langle\Delta_1\rangle^2\beta^T + 4\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T + \alpha\langle\Delta_1\rangle^2\langle\Delta_2\rangle^2\beta^T = 0; \tag{4.11}$$

$$\alpha\langle\Delta_2\rangle^2 U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T + \alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_2\rangle^2\beta^T = 0; \tag{4.12}$$

$$\alpha\langle\Delta_2\rangle^2 U\langle\Delta_2\rangle^2\beta^T = 0. \tag{4.13}$$

Since $\alpha\langle\Delta_1\rangle^2 U\langle\Delta_1\rangle^2\beta^T = (vX_1)^2\alpha\langle\beta^T\rangle^2\omega z\langle\beta^T\rangle^2\beta^T(vX_1)^2 = (vX_1)^2\alpha * (\beta^T)^{*2}\omega z(\beta^T)^{*3}(vX_1)^2$, (4.9) is equivalent to $\alpha * (\beta^T)^{*2}\omega z(\beta^T)^{*3} = 0$. By (4.7), we have that (4.9) is clearly held.

Since

$$\begin{aligned}
&\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_1\rangle^2\beta^T + \alpha\langle\Delta_1\rangle^2 U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T \\
=&\alpha(vX_1)\langle\beta^T\rangle\langle\omega(zX_2)\rangle\omega z(vX_1)^2\langle\beta^T\rangle^2\beta^T + \alpha(vX_1)^2\langle\beta^T\rangle^2\omega z(vX_1)\langle\beta^T\rangle\langle\omega(zX_2)\rangle\beta^T \\
=&(vX_1)^3(zX_2)\alpha\langle\beta^T\rangle\omega^{*2}z(\beta^T)^{*3} + (vX_1)^3(zX_2)\alpha\langle\beta^T\rangle^2\omega z\langle\omega\rangle(\beta^T)^{*2},
\end{aligned}$$

we have that (4.10) is equivalent to

$$(\alpha * \beta^T)\omega^{*2}z(\beta^T)^{*3} + (\alpha * (\beta^T)^{*2}\omega z\langle\omega\rangle(\beta^T)^{*2}) = 0.$$

By (4.7), we have $(\alpha * (\beta^T)^{*2})\omega z\langle\omega\rangle(\beta^T)^{*2} = 0$. Hence, (4.10) is equivalent to $(\alpha * \beta^T)\omega^{*2}z(\beta^T)^{*3} = 0$. That is, $(\alpha * \beta^T)\omega^{*2} = 0$ or $z(\beta^T)^{*3} = 0$. Since

$$\begin{aligned}
&\alpha\langle\Delta_2\rangle^2U\langle\Delta_1\rangle^2\beta^T + 4\alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_1\rangle\langle\Delta_2\rangle\beta^T + \alpha\langle\Delta_1\rangle^2\langle\Delta_2\rangle^2\beta^T\\
=&\alpha(zX_2)^2\langle\omega\rangle^2\omega z(vX_1)^2(\beta^T)^{*3} + 4\alpha(vX_1)\langle\beta^T\rangle(zX_2)\langle\omega\rangle\omega z(vX_1)\langle\beta^T\rangle(zX_2)\langle\omega\rangle\beta^T\\
&+ \alpha(vX_1)^2\langle\beta^T\rangle^2\omega z(zX_2)\langle\omega\rangle^2\beta^T\\
=&(zX_2)^2(vX_1)^2(\alpha\omega^{*3}z(\beta^T)^{*3} + 4\alpha\langle\beta^T\rangle\omega^{*2}z\langle\omega\rangle(\beta^T)^{*2} + \alpha\langle\beta^T\rangle\omega z\langle\omega\rangle^2\beta^T),
\end{aligned}$$

we have that (4.11) holds and is equivalent to

$$\alpha\omega^{*3}z(\beta^T)^{*3} + 4\alpha\langle\beta^T\rangle\omega^{*2}z\langle\omega\rangle(\beta^T)^{*2} + \alpha\langle\beta^T\rangle^2\omega z\langle\omega\rangle^2\beta^T = 0,$$

and equivalent to

$$\alpha\omega^{*3}z(\beta^T)^{*3} + 4\omega^T(\alpha * \beta)\omega z\langle\omega\rangle(\beta^T)^{*2} + (\alpha * (\beta^T)^{*2})\omega z\langle\omega\rangle^2\beta^T = 0.$$

From (4.7) and (4.8), we respectively obtain that $(\alpha * (\beta^T)^{*2})\omega z\langle\omega\rangle^2\beta^T = 0$, $4\omega^T(\alpha * \beta)\omega z\langle\omega\rangle(\beta^T)^{*2} = 0$. Therefore, (4.11) is equivalent to $\alpha\omega^{*3}z(\beta^T)^{*3} = 0$. That is, $\alpha\omega^{*3} = 0$, or $z(\beta^T)^{*3} = 0$.

$$\begin{aligned}
&\alpha\langle\Delta_2\rangle^2U\langle\Delta_1\rangle^2\beta^T + \alpha\langle\Delta_1\rangle\langle\Delta_2\rangle U\langle\Delta_2\rangle^2\beta^T\\
=&(zX_2)^3(vX_1)\alpha\langle\omega\rangle^2\omega z\langle\beta^T\rangle\langle\omega\rangle\beta^T + (zX_2)^3(vX_1)\alpha\langle\beta^T\rangle\langle\omega\rangle\omega z\langle\omega\rangle^2\beta^T\\
=&(zX_2)^3(vX_1)\alpha\omega^{*3}z\langle\omega\rangle(\beta^T)^{*2} + (zX_2)^3(vX_1)\alpha\langle\omega\rangle^2\beta^Tz\langle\omega\rangle^2\beta^T,
\end{aligned}$$

hence (4.12) is equivalent to $\alpha\omega^{*3}z\langle\omega\rangle(\beta^T)^{*2} + \alpha\langle\omega\rangle^2\beta^Tz\langle\omega\rangle^2\beta^T = 0$. From (4.8), we have $\alpha\langle\omega\rangle^2\beta^T = 0$, hence $\alpha\langle\omega\rangle^2\beta^Tz\langle\omega\rangle^2\beta^T = 0$. Therefore, (4.12) is equivalent to $\alpha\omega^{*3}z\langle\omega\rangle(\beta^T)^{*2} = 0$. That is, $\alpha\omega^{*3} = 0$ or $z\langle\omega\rangle(\beta^T)^{*2} = 0$.

Since $\alpha\langle\Delta_2\rangle^2U\langle\Delta_2\rangle^2\beta^T = (zX_2)^2\alpha\omega^{*3}z\langle\omega\rangle^2\beta^T$, we have that (4.13) holds and is equivalent to $\alpha\omega^{*3}z\langle\omega\rangle^2\beta^T = 0$. That is, $\alpha\omega^{*3} = 0$ or $z\langle\omega\rangle^2\beta^T = 0$. □

## 5. Conclusions

This paper focuses on Jacobian conjecture—a central problem in affine algebraic geometry—by advancing the theoretical characterization of Drużkowski matrices, a critical class of matrices linked to the conjecture. Building on the relationship between Drużkowski matrices and quasi-D-nilpotent matrices, this paper investigates Drużkowski matrices via quasi-D-nilpotent matrices, and gives some sufficient and necessary conditions for quasi-D-nilpotent matrices to be Drużkowski matrices. Furthermore, this paper characterizes several special Drużkowski matrices. This paper presents a new perspective for studying Drużkowski matrices and offers a modicum of assistance in the research on Drużkowski mappings.

## Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the composition of this article.

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## Conflict of interest

The author declares no conflict of interest.

## References

1. O. H. Keller, Ganze Cremona-transformationen, *Monatsh. Math. Phys.*, **47** (1939), 299–306. https://doi.org/10.1007/BF01695502
2. S. Smale, Mathematical problems for the next century, *Math. Intell.*, **20** (1998), 7–15. https://doi.org/10.1007/BF03025291
3. H. Bass, E. H. Connell, D. Wright, The Jacobian conjecture: Reduction of degree and formal of expansion of the inverse, *Bull. Aust. Math. Soc.*, **7** (1982), 287–330.
4. A. V. Yagzhev, On Keller's problem, *Sib. Math. J.*, **21** (1980), 747–754. https://doi.org/10.1007/BF00973892
5. L. M. Drużkowski, An effective approach to Keller's Jacobian conjecture, *Math. Ann.*, **264** (1983), 303–313. https://doi.org/10.1007/BF01459126
6. A. Essen, *Polynomial automorphisms and the Jacobian conjecture*, Basel: Birkhäuser, 2000. https://doi.org/10.1007/978-3-0348-8440-2
7. G. Gorni, G. Zampieri, On cubic-linear polynomial mappings, *Indag. Math.*, **8** (1997), 471–492. https://doi.org/10.1016/S0019-3577(97)81552-2
8. G. Gorni, H. Tutaj-Gasińska, G. Zampieri, Drużkowski matrix search and D-nilpotent automorphisms, *Indag. Math.*, **10** (1999), 235–245. https://doi.org/10.1016/S0019-3577(99)80019-6
9. E. M. G. M. Hubbers, *Nilpotent Jacobians*, Radboud University Nijmegen, 1998.
10. Y. Tian, X. Du, Y. Li, Matrices with few nonzero principal minors, *Linear Multilinear Algebra*, **66** (2018), 1362–1379. https://doi.org/10.1080/03081087.2017.1354971
11. Y. Tian, C. Sun, A note on Kaliman's weak Jacobian conjecture, *AIMS Mathematics*, **9** (2024), 30406–30412. https://doi.org/10.3934/math.20241467
12. M. De Bondt, *Homogeneous Keller maps*, Radboud University Nijmegen, 2009.

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