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Research article

Predator-prey model with additional food inducing competition among predators incorporating anti-predator behavior

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Abstract: The dynamical behavior of a predator-prey model with additional food inducing competition among predators incorporating anti-predator behavior is analyzed in this paper. The existence, stability, and local dynamics of the equilibria are discussed. By using bifurcation theory, we studied the existence of Hopf bifurcation and Bogdanov-Takens bifurcation. The numerical simulations are presented to illustrate the results of the theoretical analysis, and complex dynamical behaviors were found, such as limit cycles and homoclinic cycles.

Keywords: predator-prey model; Hopf bifurcation; Bogdanov-Takens bifurcation; anti-predator behavior; numerical simulations
Mathematics Subject Classification: 39A28, 39A30, 39A60

1. Introduction

Population dynamics, as an important branch of biomathematics, describes the changing process of population quantities under various interactions within biological systems, and has long attracted extensive attention from scholars at home and abroad.

In ecosystems, the interaction between predators and prey is a core process that maintains the structure and functional stability of communities [1–3], and its dynamical behavior has long been a research focus in theoretical ecology and mathematical ecology [4–6]. Moreover, according to the needs of production and daily life [7–9], through reasonable and effective control of certain interspecific interactions, better protection and utilization of biological resources can be achieved [10–12].

Classic predator-prey models have successfully characterized basic dynamics such as coexistence and extinction of the two populations by introducing parameters like functional response functions

and environmental carrying capacity. However, with the deepening of research, more real ecological mechanisms have been proven to significantly alter the dynamical characteristics of the system [1, 2].

With the rapid development of ecology, to make models more in line with actual situations, researchers have proposed predator-prey models with functional response functions, as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = rx(1 - \dfrac{x}{K}) - f(x)y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = ef(x)y - dy, \end{cases} \tag{1.1}$$

where $x(t)$ represents the population density of prey, $y(t)$ represents the population density of predators, and t is the evolution time (the same below). The absorption efficiency of predators for food is a function $f(x)$, called the **functional response function**, which indicates the number of prey captured by predators per unit time.

In population dynamics, the functional response of predators to prey density describes the characteristics of the number of prey captured by each predator per unit time as a function of prey density. It reflects the absorption of food by predators. In 1959, Holling proposed several basic functional response functions [13],

$$\begin{cases} \text{Holling Type I: } f(x) = cx, \\ \text{Holling Type II: } f(x) = \frac{cx}{a+x}, \\ \text{Holling Type III: } f(x) = \frac{cx^2}{ax^2+1}, \end{cases} \tag{1.2}$$

which are applicable to lower organisms, invertebrates, and higher vertebrates,respectively [14–16].

Biological control is a sustainable alternative to chemical pesticides [17–19]. To improve predator effectiveness, additional food sources are often supplemented [20–22]. A general model with additional food is given as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \dfrac{x}{\gamma}\right) - f(x, \xi, \alpha)y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, \xi, \alpha)y - \delta y, \end{cases}$$

where $f(x, \xi, \alpha)$ is the functional response function of predators, which depends on the pest (prey) x and explicitly on ξ and α that describe the provision of additional food. Meanwhile, $g(x, \xi, \alpha)$ is the numerical response function of predators. δ represents the natural mortality rate of predators.

Srinivasu and Prasad studied the following model from 2007 to 2011 [18–20]:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \dfrac{x}{\gamma}\right) - \dfrac{x}{1 + \alpha\xi + x}y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta(x + \xi)}{1 + \alpha\xi + x}y - \delta y, \end{cases}$$

and proved that pests and diseases can be eradicated when $\xi > \frac{\delta}{\beta - \delta a}$. Additionally, by using ξ to control the provision of additional food, it is possible to choose whether to retain the introduced predators in the ecosystem.

Srinivasu, Vamsi, and Ananth [21] studied a predator-prey model with Holling-III-type functional response, and the dynamic behavior after introducing additional food is expressed as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \dfrac{x}{\gamma}\right) - \dfrac{x^2}{1 + \alpha\xi^2 + x^2}y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta\left(x^2 + \xi^2\right)}{1 + \alpha\xi^2 + x^2}y - \delta y, \end{cases}$$

In this model, pests and diseases can also be eradicated.

Furthermore, Srinivasu, Vamsi, and Aditya [17] studied how to maintain living systems by supplementing additional food in the presence of inhibitory effects, with the model as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \dfrac{x}{\gamma}\right) - \dfrac{x}{(1 + \alpha\xi)\left(\omega x^2 + 1\right) + x}y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta\left(x + \xi\left(\omega x^2 + 1\right)\right)}{(1 + \alpha\xi)\left(\omega x^2 + 1\right) + x}y - \delta y. \end{cases}$$

However, excessive additional food may cause uncontrolled predator growth and induce intraspecific competition [23–25], which can be utilized to prevent species invasion [26–28]. Bazykin [6] established a model for intraspecific competition:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - ex^2 - \dfrac{bxy}{1 + Ax}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -cy + \dfrac{dxy}{1 + Ax} - hy^2, \end{cases}$$

where $\frac{b}{A}$ is the maximum consumption rate of predators, $\frac{1}{A}$ is the population density at which the consumption of predators is half of the maximum value, c is the natural mortality rate of predators, and $\frac{d}{b}$ is the energy conversion efficiency of predators.

Lu, Huang, et al. completed rigorous theoretical proofs for various complex dynamic behaviors of the Bazykin model, such as Bogdanov-Takens bifurcation, codimension 2 degenerate Hopf bifurcation, and codimension 3 focus-type degenerate Bogdanov-Takens bifurcation [16]. Accordingly, they classified the long-term dynamics of the model using the threshold c_0 of the natural mortality rate c of predators.

In 2023, Parshad et al. [29] proved that in the absence of intraspecific competition, additional food will lead to the uncontrolled infinite expansion of predator quantities. Therefore, simply introducing additional food as a pest management strategy is flawed. Based on the work of Srinivasu et al. [18–20], they proposed another model with Holling-II-type functional response function and additional food provision, where additional food induces intraspecific competition among predators, and no competition occurs without additional food. The intraspecific competition of predators is represented by $c\xi y^2$:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \dfrac{x}{\gamma}\right) - \dfrac{x}{1 + \alpha\xi + x}y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta(x + \xi)}{1 + \alpha\xi + x}y - \delta y - c\xi y^2. \end{cases}$$

Numerical simulations of this model revealed bifurcations such as Saddle-Node Transcritical (SNTC) and Cusp-Pitchfork Transcritical (CPTC) [30–32], suggesting that controlling additional food allows for effective predator management.

Anti-predator behavior also impacts system dynamics [2,8]. Freedman and Wolkowicz [9] proposed a group defense model,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(x,K) - yp(x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(-s + q(x)), \\ x(0) \geq 0, y(0) \geq 0, \end{cases}$$

where $x(t)$ and $y(t)$ represent the densities of prey and predators, respectively. The functions g, p, and q are continuously differentiable, and s and K are positive constants.

In 1990, Mischaikow and Wolkowicz proposed a method to study predation systems involving group defense using connection matrices based on the above model [10].

Combining previous research results [9–11], Tang and Xiao proposed the following anti-predation model considering the functional response function [8]:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = rx(1 - \dfrac{x}{K}) - \dfrac{\beta xy}{\alpha + x^2}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \mu\dfrac{\beta xy}{\alpha + x^2} - dy - \eta xy. \end{cases}$$

Here, all parameters are positive constants. r is the intrinsic growth rate of prey, k is the carrying capacity of the environment, β is the capture rate of predators, μ is the energy conversion efficiency of predators, d is the natural mortality rate of the predator population, and η is the anti-predator behavior of prey against the predator population. Note that the anti-predator behavior in this model does not directly benefit the prey population but reduces the growth of the predator population because prey do not feed on predators [12].

Therefore, based on the model with additional food and predator competition proposed by Parshad et al. [1],

$$\begin{cases} \dfrac{dx}{dt} = x\left(1 - \dfrac{x}{\gamma}\right) - f(x,\xi,\alpha)y, \\ \dfrac{dy}{dt} = g(x,\xi,\alpha)y - \delta y, \end{cases}$$

this paper further introduces the anti-predator behavior of prey to construct a new dynamical model:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \dfrac{x}{\gamma}\right) - \dfrac{x}{1 + \alpha\xi + x}y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\beta(x+\xi)}{1 + \alpha\xi + x}y - \delta y - \bar{A}xy - c\xi y^2. \end{cases} \tag{1.3}$$

Among the parameters, the newly introduced $\bar{A}$ represents the anti-predator behavior of prey against the predator population. $\frac{1}{\alpha}$ is the quality of the additional food provided to the predator, and ξ is the quantity of additional food provided to the predator. γ is the carrying capacity of the pest. We simplify

the model through transformations: let $x = \gamma u$ and $y = \frac{v}{c\xi}$, then the original system is transformed into:

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = u(1-u) - \frac{uv}{c\xi(1+\alpha\xi+\gamma u)}, \\ \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\beta(\gamma u+\xi)v}{1+\alpha\xi+\gamma u} - \gamma\bar{A}uv - \delta v - v^2. \end{cases}$$

Denote $R = R(a) = \frac{1+a\xi}{\gamma} \in (0,1)$, $K = \frac{\xi}{\gamma}$ and $A = \gamma\bar{A}$; the system becomes:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = u(1-u) - \frac{uv}{\gamma c\xi(u+R)}, \\ \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\beta(u+K)v}{u+R} - \delta v - Auv - v^2. \end{cases} \tag{1.4}$$

Let $\eta = \frac{1}{\gamma c\xi}$; the simplified model studied in this paper is as follows:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = u(1-u) - \frac{\eta uv}{u+R}, \\ \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\beta(u+K)v}{u+R} - \delta v - Auv - v^2. \end{cases} \tag{1.5}$$

In Eq (1.5), $R = R(a)$ reflects the total amount of additional food determined by the food quality, A represents the anti-predator behavior, γ is the environmental carrying capacity, β represents the conversion efficiency of predators, δ represents the natural mortality rate of predators, and $\frac{1}{a}$ represents the quality of extra feed, while ξ represents the quantity of extra food. The introduction of extra food will induce intraspecific competition among predators, and the coefficient c is introduced into the model to describe the impact of intraspecific competition on the system. Assume that $\beta - \delta a > 0$ and $\xi > \frac{\delta}{\beta-\delta a}$.

To better reflect ecological realism, it is important to further clarify the underlying mechanisms of the two key factors considered in this study. On the one hand, additional food supply, although widely used to enhance predator survival and improve biological control efficiency, may lead to unintended consequences when provided excessively. In particular, surplus food can promote rapid predator population growth, which in turn intensifies intraspecific competition due to limited environmental resources such as space and territory. This density-dependent competition can significantly regulate predator dynamics and prevent unbounded population growth, thereby playing a crucial role in maintaining ecosystem balance. On the other hand, anti-predator behavior represents an important adaptive strategy of prey species in response to predation risk. Such behaviors—including vigilance, refuge use, and group defense—effectively reduce predation success. From a modeling perspective, anti-predator behavior does not directly increase prey reproduction but instead suppresses predator growth by reducing predation efficiency, thereby altering the interaction strength between species. The incorporation of this mechanism provides a more realistic description of predator–prey interactions.

In this paper, we focus on the qualitative analysis and complex dynamics of system (1.5). Compared with existing studies, this paper develops a predator–prey model that simultaneously incorporates additional food-induced intraspecific competition and anti-predator behavior, thereby providing a more comprehensive ecological framework. Such a combination naturally arises in many real ecological systems, particularly in biological control practices where supplementary food is introduced to sustain predator populations, while prey species simultaneously evolve or exhibit anti-predator behaviors in response to increased predation pressure. Within this framework, we examine the existence and stability of equilibria and identify the conditions under which species coexistence or extinction may occur. Furthermore, a detailed bifurcation analysis is carried out, through which a variety

of complex dynamical behaviors—including transcritical, saddle-node, Hopf, and Bogdanov-Takens bifurcations—are revealed with the aid of center manifold and normal form theories. Numerical simulations are then performed to illustrate the emergence of rich dynamics such as limit cycles and homoclinic orbits. Overall, the results highlight the combined effects of food supplementation and behavioral defense, offering new insights that are relevant to both theoretical ecology and practical biological control.

The paper is organized as follows: We first analyze the existence and stability of both boundary and coexistence equilibria. Subsequently, we investigate the bifurcation phenomena, deriving the normal forms and stability conditions for the Transcritical, Saddle-node, Hopf, and Bogdanov-Takens bifurcations. Finally, numerical simulations are presented to illustrate the theoretical conclusions and discuss their biological implications.

2. Existence and stability of equilibria

The equilibrium of system (1.5) is determined by the following equations:

$$\begin{cases} u\left(1-u-\frac{\eta v}{u+R}\right)=0, \\ v\left(\frac{\beta(u+K)}{u+R}-\delta-Au-v\right)=0. \end{cases} \tag{2.1}$$

Obviously, from Eq (2.1), system (1.5) must have boundary equilibrium $E_0 = (0, 0)$ and $E_1 = (1, 0)$.

2.1. Existence of coexistence equilibria

To analyze the existence of the coexistence equilibrium (u^*, v^*) of system (1.5), consider the following equations:

$$\begin{cases} 1-u-\frac{\eta v}{u+R}=0, \\ \frac{\beta(u+K)}{u+R}-\delta-Au-v=0, \end{cases} \tag{2.2}$$

which are equivalent to

$$1-u-\frac{\eta}{u+R}\left(\frac{\beta(u+K)}{u+R}-\delta-Au\right)=0, u\in(0,1).$$

To analyze the solutions of (2.2), let

$$\begin{aligned} G(u) &= (u-1)(u+R)^2+\eta\beta(u+K)-\eta(Au+\delta)(u+R)=0. \\ &= u^3+(2R-\eta A-1)u^2+\left[R^2-2R+\eta(\beta-\delta-AR)\right]u+\left(\eta\beta K-\eta\delta R-R^2\right). \\ &:= u^3+a_2u^2+a_1u+a_0, \end{aligned} \tag{2.3}$$

where

$$a_0=\eta\beta K-\eta\delta R-R^2,$$

$$a_1=R^2-2R+\eta(\beta-\delta-AR),$$

$$a_2=2R-\eta A-1,$$

and consider the first derivative of $G(u)$,

$$G'(u) := g(u) = 3u^2 + 2a_2u + a_1, \tag{2.4}$$

and the second derivative

$$G''(u) := g'(u) = 6u + 2a_2. \tag{2.5}$$

It should be noted that $g'(u) = 6u + 2a_2$ is monotonically increasing for $u > 0$.

First, we consider the case when $a_2 \geq 0$.

Theorem 2.1. *Under the condition $a_2 \geq 0$, if $a_1 \geq 0$, system (1.5) has at most one coexistence equilibrium. Specifically, if $a_0 < 0$ and $G(1) > 0$, the system has a unique coexistence equilibrium (u_c, v_c). In other cases, the system has no coexistence equilibrium. The geometric interpretation of this result is illustrated in Figure 1.*

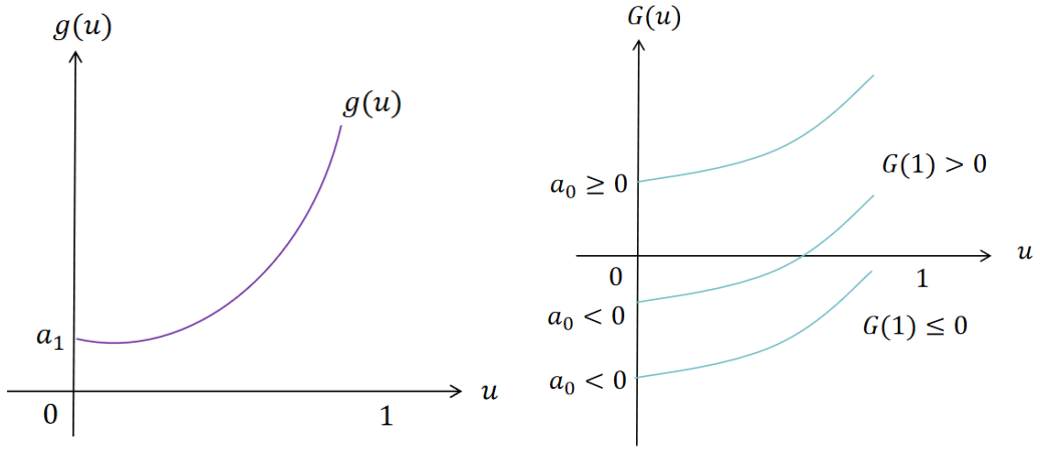
Proof. When $a_2 \geq 0$, $g'(u) > 0$ holds true for all u in $(0, 1)$, $g'(u) > 0$ is monotonically increasing in $(0, 1)$.

Given $a_1 \geq 0$, we have $g(u) > g(0) = a_1 \geq 0$, and $G(u)$ is monotonically increasing in $(0, 1)$.

If $a_0 \geq 0$, then $G(u) > 0$, and there are no zero points in $(0, 1)$. At this point, system (1.5) does not have a coexistence equilibrium.

If $a_0 < 0$ and $G(1) \leq 0$, that is, $1 + a_2 + a_1 + a_0 \leq 0$, then $G(u) < 0$ holds true for all $(0, 1)$, and there are no zero points. At this point, the system (1.5) does not have a coexistence equilibrium.

If $a_0 < 0$ and $G(1) > 0$, that is, $1 + a_2 + a_1 + a_0 > 0$, then $G(u)$ has a unique zero point u_c in $(0, 1)$, that is, there exists a coexistence equilibrium (u_c, v_c). □



(a) When $a_1 \geq 0$, the image of $g(u)$ (b) The image of $G(u)$

Figure 1. The existence of the coexistence equilibrium point of the system under the condition $a_2 \geq 0$.

Theorem 2.2. *If $a_2 \geq 0$ and $a_1 < 0$, the existence of coexistence equilibrium of system (1.5) is as follows:*

(1) *When $g(1) = 3 + 2a_2 + a_1 \leq 0$, if $a_0 > 0$ and $G(1) < 0$, system (1.5) has a unique coexistence equilibrium. The corresponding graphical illustration is shown in Figure 2.*

(2) *When* $g(1) = 3 + 2a_2 + a_1 > 0$ *, there exists* u_c *satisfying the equation* $g(u) = 3u^2 + 2a_2u + a_1 = 0$*:*

$$u_c = \frac{-2a_2 + \sqrt{4a_2^2 - 12a_1}}{6} \in (0, 1).$$

This case is further demonstrated in Figure 3.

a) *If* $a_0 \leq 0$ *and* $G(1) > 0$*, the system has a unique coexistence equilibrium* (u_c, v_c)*.*

b) *If* $a_0 > 0$ *and* $G(u_c) < 0$*, the system has two coexistence equilibria* (u_c, v_c) *and* (u_2, v_2)*.*

c) *If* $a_0 > 0$ *and* $G(u_c) = 0$*, the system has a unique coexistence equilibrium* (u_c, v_c)*.*

Proof. When $a_1 < 0$, $g(0) = a_1 < 0$.
(1) When $g(1) = 3 + 2a_2 + a_1 \leq 0$, there is $g(u) < 0$ in $(0, 1)$, and $G(u)$ is monotonically decreasing in $(0, 1)$.

If $a_0 \leq 0$, there is $G(u) < 0$ in $(0, 1)$, $G(u)$ has a unique zero points in $(0, 1)$. In this case, system (1.5) has no coexistence equilibrium.

If $a_0 > 0$ and $G(1) \geq 0$, that is, $1 + a_2 + a_1 + a_0 \geq 0$, then there is $G(u) > 0$ in $(0, 1)$, and $G(u)$ has no zero points in $(0, 1)$. In this case, system (1.5) has no coexistence equilibrium.

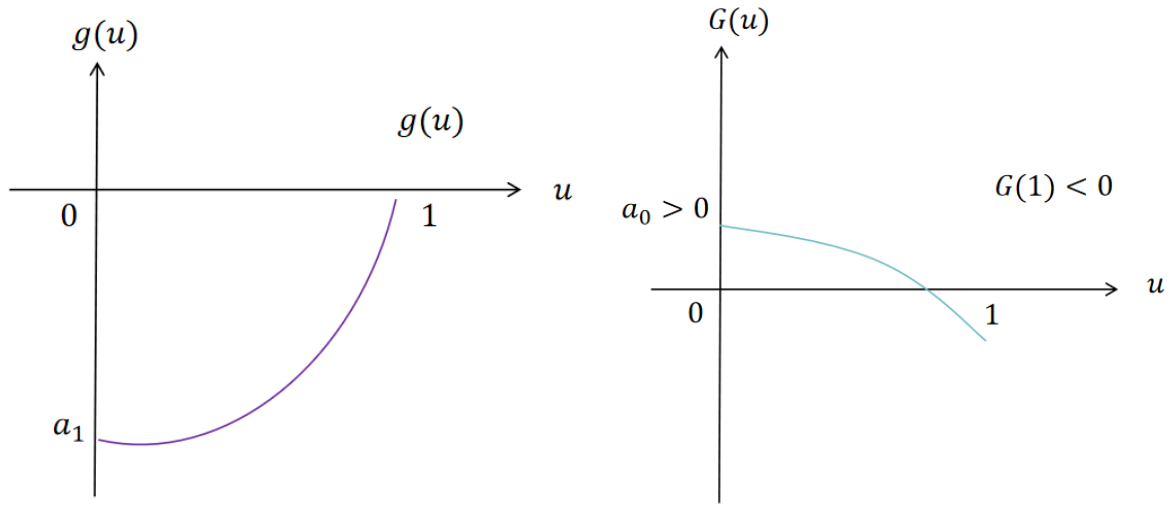
If $a_0 > 0$ and $G(1) < 0$, then $G(u)$ has one coexistence equilibrium u_c in $(0, 1)$, and system (1.5) has a unique coexistence equilibrium (u_c, v_c).

(2) When $g(1) = 3 + 2a_2 + a_1 > 0$, $g(u)$ has a unique zero point u_c in $(0, 1)$. For $(0, u_c)$, $g(u) < 0$ and $G(u)$ is monotonically decreasing. For $(u_c, 1)$, $g(u) > 0$ and $G(u)$ is monotonically increasing. u_c is determined by the root of $g(u) = 3u^2 + 2a_2u + a_1$:

$$u_c = \frac{-2a_2 + \sqrt{4a_2^2 - 12a_1}}{6} \in (0, 1).$$

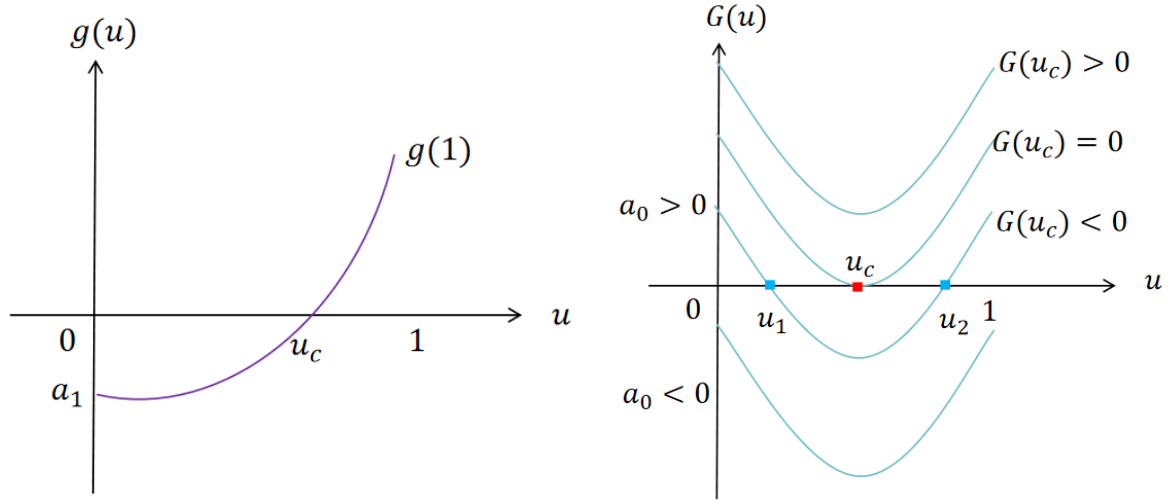
If $a_0 \leq 0$, when $G(1) \leq 0$, the system has no coexistence equilibrium; when $G(1) > 0$, $G(u)$ has a unique positive root u_c in $(u_c, 1)$, so the system has a unique coexistence equilibrium (u_c, v_c).

Under the condition $a_0 > 0$, the existence of coexistence equilibria depends on the sign of $G(u_c)$. Specifically, if $G(u_c) > 0$, the system has no coexistence equilibrium, as $G(u)$ has no roots in $(0, 1)$. Conversely, when $G(u_c) < 0$ and $G(1) > 0$, $G(u)$ admits two roots $u_1 \in (0, u_c)$ and $u_2 \in (u_c, 1)$, resulting in two coexistence equilibria (u_1, v_1) and (u_2, v_2). Lastly, if $G(u_c) = 0$, a unique equilibrium (u_c, v_c) exists, since $G(u)$ has a unique root at u_c. □



(a) When $g(1) \le 0$, the image of $g(u)$ (b) Image of $G(u)$

Figure 2. The existence of the coexistence equilibrium point of the system when $a_2 \ge 0$, $a_1 < 0$, $g(1) \le 0$, $a_0 > 0$, and $G(1) < 0$.



(a) Image of $g(u)$ when $g(1) > 0$ (b) Corresponding image of $G(u)$

Figure 3. Existence of the coexistence equilibrium point of the system when $g(1) > 0$.

Next, we investigate the case when $a_2 \le -\frac{1}{3}$. In this case, $g'(u) < 0$ holds for all $u \in (0, 1)$, and $g'(u)$ is monotonically decreasing in $(0, 1)$.

Theorem 2.3. *If $a_2 \le -\frac{1}{3}$ and $a_1 \le 0$, system (1.5) has at most one coexistence equilibrium. Specifically, if $a_0 > 0$ and $G(1) < 0$, system (1.5) has a unique coexistence equilibrium (u_c, v_c). In other cases, there is no coexistence equilibria for system (1.5).*

Proof. When $a_1 \le 0$, we have $g(u) < g(0) \le 0$, $G(u)$ is monotonically decreasing in $(0, 1)$.

If $a_0 \leq 0$, then $G(u) < 0$, there are no zero points in $(0, 1)$. At this point, system (1.5) has no coexistence equilibrium.

If $a_0 > 0$ and $G(1) \geq 0$, that is, $1 + a_2 + a_1 + a_0 \geq 0$, then $G(u) > 0$ holds for all $(0, 1)$, which implies that there are no zero points. At this point, system (1.5) has no coexistence equilibrium.

If $a_0 > 0$ and $G(1) < 0$, that is, $1 + a_2 + a_1 + a_0 < 0$, then $G(u)$ has a unique zero point u_c in $(0, 1)$, which means the system has a coexistence equilibrium (u_c, v_c). □

A summary of these results is given in Table 1.

Table 1. Existence of coexistence equilibrium of system (1.5) when $a_2 \geq 0$ and $a_1 < 0$.

Condition			Number of coexistence equilibrium
$g(1) \leq 0$	$a_0 > 0$	$G(1) < 0$	1
$g(1) > 0$	$a_0 \leq 0$	$G(1) > 0$	1
		$G(1) \leq 0$	0
$g(1) > 0$	$a_0 > 0$	$G(u_c) < 0$	2
		$G(u_c) = 0$	1
		$G(u_c) > 0$	0

Theorem 2.4. *If $a_2 \leq -\frac{1}{3}$ and $a_1 > 0$, the existence of coexistence equilibrium of system (1.5) is as follows:*

(1) *If $g(1) = 3 + 2a_2 + a_1 \geq 0$, $a_0 < 0$, and $G(1) > 0$, then system (1.5) has a coexistence equilibrium (u_c, v_c).*

(2) *If $g(1) = 3 + 2a_2 + a_1 < 0$, there exists u_c satisfying $g(u) = 3u^2 + 2a_2u + a_1 = 0$:*

$$u_c = \frac{-2a_2 + \sqrt{4a_2^2 - 12a_1}}{6} \in (0, 1).$$

a) *If $a_0 \geq 0$ and $G(1) < 0$, the system has a unique coexistence equilibrium (u_c, v_c).*

b) *If $a_0 < 0$, $G(u_c) > 0$, and $G(1) < 0$, the system has two coexistence equilibria (u_c, v_c) and (u_2, v_2).*

c) *If $a_0 < 0$ and $G(u_c) = 0$, the system has a unique coexistence equilibrium (u_c, v_c).*

Proof. When $a_1 > 0$, we have $g(0) > 0$.

(1) When $g(1) = 3 + 2a_2 + a_1 \geq 0$, we have $g(u) > 0$ in $(0, 1)$, and $G(u)$ is monotonically increasing in $(0, 1)$.

If $a_0 \geq 0$, then we have $G(u) > 0$ in $(0, 1)$, $G(u)$ has no zero points in $(0, 1)$. At this point, system (1.5) has no coexistence equilibrium.

If $a_0 < 0$ and $G(1) \leq 0$, that is, $1 + a_2 + a_1 + a_0 \leq 0$, then we have $G(u) < 0$ in $(0, 1)$, and $G(u)$ has no zero points in $(0, 1)$. System (1.5) has no coexistence equilibrium.

If $a_0 < 0$ and $G(1) > 0$, then $G(u)$ has a coexistence equilibrium u_c in $(0, 1)$. The system (1.5) has a unique coexistence equilibrium (u_c, v_c).

(2) When $g(1) = 3 + 2a_2 + a_1 < 0$, $g(u)$ must have a zero point u_c in $(0, 1)$. We have $g(u) > 0$ in $(0, u_c)$, and $G(u)$ is monotonically increasing. We have $g(u) < 0$ in $(u_c, 1)$, and $G(u)$ is monotonically

decreasing. u_c is determined by the zero point of $g(u) = 3u^2 + 2a_2u + a_1$:

$$u_c = \frac{-2a_2 + \sqrt{4a_2^2 - 12a_1}}{6} \in (0, 1).$$

If $a_0 \geq 0$, then when $G(1) \geq 0$, the system has no coexistence equilibrium. When $G(1) < 0$, $G(u)$ has a unique positive root u_c in $(u_c, 1)$, which means the system has a unique coexistence equilibrium (u_c, v_c).

If $a_0 < 0$, when $G(u_c) < 0$, system (1.5) has no coexistence equilibrium. When $a_0 < 0$ and $G(u_c) > 0$, if $G(1) < 0$, then $G(u)$ has a zero point $u_c \in (0, u_c)$, and a zero point $u_2 \in (u_c, 1)$, satisfying $g(u_c) > 0$ and $g(u_2) < 0$. The system has two equilibria (u_c, v_c) and (u_2, v_2). When $a_0 < 0$ and $G(u_c) = 0$, $G(u)$ has a unique zero point u_c in $(0, 1)$. The system has a unique coexistence equilibrium (u_c, v_c). When $a_0 < 0$ and $G(u_c) < 0$, the system has no coexistence equilibrium. □

The corresponding classification is listed in Table 2.

Table 2. Existence of coexistence equilibrium of system (1.5) when $a_2 \leq -\frac{1}{3}$ and $a_1 > 0$.

Condition		Number of coexistence equilibrium
$g(1) \geq 0$, $a_0 < 0$	$G(1) > 0$	1
$g(1) < 0$, $a_0 \geq 0$	$G(1) < 0$	1
	$G(1) \geq 0$	0
$g(1) < 0$, $a_0 < 0$	$G(u_c) > 0$ and $G(1) < 0$	2
	$G(u_c) = 0$	1
	$G(u_c) < 0$	0

Finally, we consider the case when $-\frac{1}{3} < a_2 < 0$. In this case, $g'(u)$ has a zero point $u_{cc} = -\frac{a_2}{3} \in (0, 1)$. For $(0, u_{cc})$, $g'(u) < 0$ and $g(u)$ is monotonically decreasing; for $(u_{cc}, 1)$, $g'(u) > 0$ and $g(u)$ is monotonically increasing. Consider $g(u_{cc})$, $g(0) = a_1$ and $g(1) = 3 + 2a_2 + a_1$, where

$$g(u_{cc}) = -\frac{a_2^2}{3} + a_1.$$

If $g(u_{cc}) > 0$, then $g(u) > 0$, and $G(u)$ is strictly increasing in $(0, 1)$, so system (1.5) has at most one coexistence equilibrium (u_0, v_0). If $g(u_{cc}) = 0$, then $g(u) \geq 0$, and $G(u)$ is increasing in $(0, 1)$ with a stationary point u_{cc}. If $G(u_{cc}) = 0$ simultaneously, the system has a unique coexistence equilibrium (u_{cc}, v_{cc}). If $g(u_{cc}) < 0$, $g(0) \leq 0$, and $g(1) \leq 0$, then $g(u) < 0$, and $G(u)$ is strictly decreasing in $(0, 1)$, so system (1.5) has at most one coexistence equilibrium (u_0, v_0). Similar to the model without anti-predator behavior, the following conclusions can be obtained through the geometric analysis of cubic functions:

Theorem 2.5. *If $g(u_{c_c}) < 0$, $g(0) > 0$, and $g(1) > 0$, then $g(u)$ has two zero points $u_{c_1} < u_{c_2}$ in $(0, 1)$; correspondingly, $G(u)$ has a maximum point u_{c_1} and a minimum point u_{c_2}. The equilibrium of system (1.5) are as follows:*

(1) *When $G(0) > 0$ and $G(1) < 0$, the system has only one coexistence equilibrium (u_*, v_*), $u_* \in (u_{c1}, u_{c_2})$.*

(2) *When* $G(0) < 0$ *and* $G(1) > 0$,

a) *If* $G(u_{c_1}) < 0$ *(or* $G(u_{c_2}) > 0$*) , the system has only one coexistence equilibrium.*

b) *If* $G(u_{c_1}) > 0$ *and* $G(u_{c_2}) < 0$*, the system has three coexistence equilibria* (u_i, v_i)*,* $i = 1, 2, 3$*, and* $0 < u_c < u_2 < u_3 < 1$.

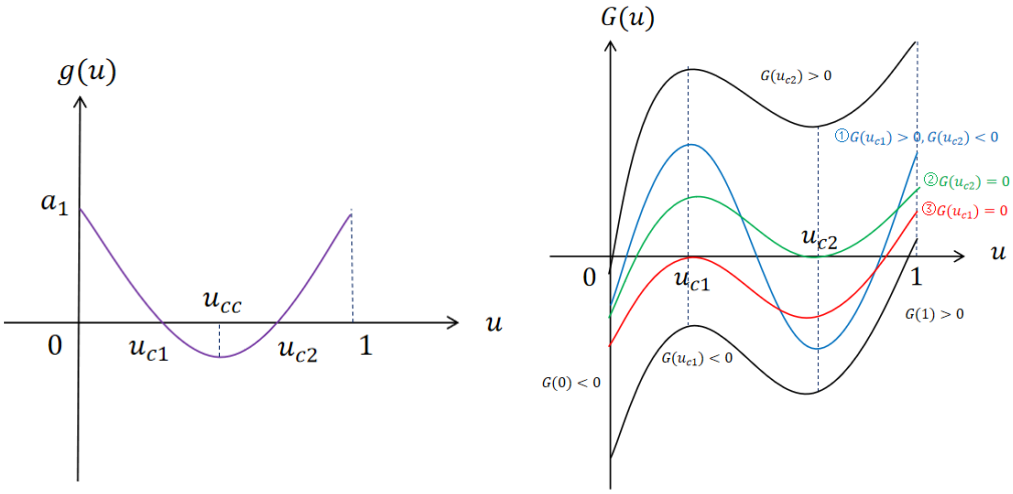
c) *If* $G(u_{c_1}) = 0$*, the system has equilibria* (u_{c_1}, v_{c_1}) *and* (u_3, v_3) *with* $u_3 \in (u_{c_2}, 1)$.

d) *If* $G(u_{c_2}) = 0$*, the system has equilibria* (u_{c_2}, v_{c_2}) *and* (u_1, v_1) *with* $u_1 \in (0, u_{c_1})$.

The above analysis is visualized in Figure 4, and these cases are summarized in Table 3.

Table 3. Existence of coexistence equilibrium of system (1.5) when $-\frac{1}{3} < a_2 < 0$, $g(u_{cc}) < 0$ and $g(0), g(1) > 0$.

Condition		Number of coexistence equilibrium
$G(0) > 0$, $G(1) < 0$		1
$G(0) > 0$, $G(1) > 0$	$G(u_{c1}) < 0$ (or $G(u_{c2}) > 0$)	1
	$G(u_{c1}) > 0$ and $G(u_{c2}) < 0$	3
	$G(u_{c1}) = 0$	2
	$G(u_{c2}) = 0$	2



(a) Image and zero points of $g(u)$ when $g(u_{cc}) < 0$, $g(0) > 0$, and $g(1) > 0$ (b) Corresponding image of $G(u)$ when $G(0) < 0$ and $G(1) > 0$

Figure 4. Existence of the coexistence equilibrium point of the system under the conditions of Theorem 2.5.

Theorem 2.6. *If* $g(u_{cc}) < 0$*,* $g(0) > 0$*, and* $g(1) < 0$*, then* $g(u)$ *has only one zero point* u_{c1} *in* $(0, 1)$*:*

(1) *If* $G(0) < 0$*,* $G(1) < 0$*, and* $G(u_{c1}) > 0$*, the system has two coexistence equilibria:* (u_c, v_c) *and* (u_2, v_2) *with* $0 < u_c < u_{c1} < u_2 < 1$.

(2) *If* $G(0) > 0$*,* $G(1) < 0$*, and* $G(u_{c1}) > 0$ *the system has one coexistence equilibrium* (u_1, v_1)*,* $u_{c1} < u_c < 1$.

(3) *If* $G(0) < 0$, $G(1) < 0$, *and* $G(u_{c1}) = 0$, *the system has only one coexistence equilibrium* (u_{c1}, v_{c1}).

Theorem 2.7. *If* $g(u_{cc}) < 0$, $g(0) < 0$, *and* $g(1) > 0$, *then* $g(u)$ *has only one zero point* u_{c_2} *in* $(0, 1)$, *corresponding to the minimum point* u_{c_2} *of* $G(u)$*:*

(1) *If* $G(0) > 0$, *and* $G(u_{c1}) > 0$, *system* (1.5) *has no coexistence equilibrium.*
(2) *If* $G(0) > 0$, $G(1) < 0$, *and* $G(u_{c_1}) < 0$ *the system has one coexistence equilibrium* (u_1, v_1) *with* $0 < u_c < u_{c1}$.
(3) *If* $G(0) > 0$, $G(1) > 0$, *and* $G(u_{c_1}) < 0$, *the system has two coexistence equilibria:* (u_c, v_c) *and* (u_2, v_2) *with* $0 < u_c < u_{c_1} < u_2 < 1$.
(4) *If* $G(0) > 0$, $G(1) > 0$, *and* $G(u_{c_1}) = 0$, *the system has only one coexistence equilibrium* (u_{c_1}, v_{c_1}).

The results are further summarized in Table 4.

Table 4. Existence of coexistence equilibrium of system (1.5) when $g(u_{cc}) < 0$ and $g(0)$ and $g(1)$ have opposite signs.

Condition			Number of coexistence equilibrium
$g(0) > 0$, $g(1) < 0$ and $G(1) < 0$	$G(0) < 0$	$G(u_{c_1}) > 0$	2
		$G(u_{c_1}) = 0$	1
	$G(0) > 0$	$G(u_{c_1}) > 0$	1
$g(0) < 0$, $g(1) > 0$ and $G(0) > 0$	$G(1) < 0$	$G(u_{c_1}) < 0$	1
	$G(1) > 0$	$G(u_{c_1}) < 0$	2
		$G(u_{c_1}) = 0$	1

2.2. Stability of boundary equilibrium

Let $f(u, v) = \begin{pmatrix} u(1-u) - \frac{\eta uv}{u+R} \\ \frac{\beta(u+K)v}{u+R} - \delta v - Auv - v^2 \end{pmatrix}$. To facilitate the discussion on the stability of the system's equilibrium, the Jacobian matrix of the system is first calculated as follows:

$$Df(u, v) = \begin{pmatrix} 1 - 2u - \frac{\eta Rv}{(u+R)^2} & -\frac{\eta u}{u+R} \\ \frac{\beta(R-K)v}{(u+R)^2} - Av & -\delta - 2v - Au + \frac{\beta(u+K)}{u+R} \end{pmatrix}. \tag{2.6}$$

Its determinant and trace are given by

$$\begin{aligned} \det Df(u, v) = &\left(1 - 2u - \frac{\eta Rv}{(u+R)^2}\right)\left(-\delta - 2v - Au + \frac{\beta(u+K)}{u+R}\right) \\ &+ \frac{\eta\beta(R-K)uv}{(u+R)^3} - \frac{A\eta uv}{u+R}, \end{aligned} \tag{2.7}$$

$$\operatorname{tr} Df(u, v) = 1 - \delta - 2u - 2v - Au + \frac{\beta(u+K)}{u+R} - \frac{\eta Rv}{(u+R)^2}. \tag{2.8}$$

Theorem 2.8. *The stability of the boundary equilibria* E_0, E_1, *and* E_2 *is as follows:*

(i) The extinction equilibrium $E_0 = (0, 0)$ is always unstable.

(ii) Let $\delta_1 = \frac{\beta(1+K)}{1+R}$. For the predator extinction equilibrium $E_1 = (1, 0)$, if $\delta > \delta_1 - A$, it is a stable node; if $\delta < \delta_1 - A$, it is an unstable saddle point.

(iii) Assume that the natural mortality rate of predators is sufficiently low ($\delta < \frac{\beta K}{R} - A$) such that the prey extinction equilibrium $E_2 = (0, v_0)$ exists, where $v_0 = \frac{\beta K}{R} - \delta$. Let $R_0 = \eta v_0$. If $R < R_0$, E_2 is a stable node; if $R > R_0$, E_2 is an unstable saddle point.

Proof. We analyze the local stability by computing the Jacobian matrix $Df(u, v)$ at each equilibrium point.

(i) For $E_0 = (0, 0)$, the Jacobian matrix is

$$Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta K}{R} - \delta \end{pmatrix}. \tag{2.9}$$

There exists an eigenvalue $\lambda_1 = 1 > 0$, implying that E_0 is always unstable.

(ii) For $E_1 = (1, 0)$, the Jacobian matrix is given by

$$Df(1,0) = \begin{pmatrix} -1 & -\frac{\eta}{1+R} \\ 0 & -\delta + \frac{\beta(1+K)}{1+R} - A \end{pmatrix}. \tag{2.10}$$

The eigenvalues are $\lambda_1 = -1 < 0$ and $\lambda_2 = \delta_1 - \delta - A$. Thus, E_1 is stable if $\lambda_2 < 0$ (i.e., $\delta > \delta_1 - A$) and unstable if $\lambda_2 > 0$ (i.e., $\delta < \delta_1 - A$).

(iii) For $E_2 = (0, v_0)$, the Jacobian matrix is

$$Df(0,v_0) = \begin{pmatrix} 1 - \frac{\eta v_0}{R} & 0 \\ \frac{\beta(R-K)v_0}{R^2} - Av_0 & -v_0 \end{pmatrix}. \tag{2.11}$$

The eigenvalues are $\lambda_1 = 1 - \frac{\eta v_0}{R}$ and $\lambda_2 = -v_0$. Since E_2 exists, we have $v_0 > 0$, so $\lambda_2 < 0$. The stability is determined by λ_1:

$$\lambda_1 = \frac{R - \eta v_0}{R} = \frac{R - R_0}{R}.$$

Therefore, if $R < R_0$, then $\lambda_1 < 0$, and E_2 is stable. If $R > R_0$, then $\lambda_1 > 0$, and E_2 is unstable.

□

2.3. Stability of coexistence equilibrium

To analyze the classification and stability of coexistence equilibrium, it should be noted that when $G(u) = 0$ and $u \neq 0$, from Eq (2.2), we have

$$v = \frac{(1-u)(u+R)}{\eta},$$

$$\delta = \frac{\beta(u+K)}{u+R} - \frac{(u+R)(1-u)}{\eta} + Au.$$

Substituting into the expression of $g(u)$, it can be seen that for the coexistence equilibrium (u_*, v_*), we have

$$g(u_*) = R(-1 - A\eta + R) + \frac{\beta\eta(R-K)}{u_* + R} + (-1 - 3A\eta + 3R)u_* + 2u_*^2.$$

Next, analyze the Jacobian matrix at the coexistence equilibrium (u_*, v_*):

$$Df(u_*, v_*) = \begin{pmatrix} 1 - 2u_* - \frac{\eta R v_*}{(u_*+R)^2} & -\frac{\eta u_*}{u_*+R} \\ \frac{\beta(R-K)v_*}{(u_*+R)^2} - Av_* & -\delta - 2v_* - Au_* + \frac{\beta(u_*+K)}{u_*+R} \end{pmatrix}.$$

From the equilibrium condition (2.2), we know that

$$1 - 2u_* - \frac{\eta R v_*}{(u_*+R)^2} = 1 - 2u_* - \frac{R}{u_*+R}(1-u_*) = \frac{u_*(1-R-2u_*)}{u_*+R},$$
$$-\frac{\eta u_*}{u_*+R} = -\frac{u_*(1-u_*)}{v_*}, \quad -\delta - 2v_* - Au_* + \frac{\beta(u_*+K)}{u_*+R} = -v_*.$$

Therefore,

$$\begin{aligned} \det Df(u_*v_*) &= \frac{u_*(1-R-2u_*)}{u_*+R}\frac{(1-u_*)(u_*+R)}{\eta} + \frac{\beta(R-K)}{(u_*+R)^2}u_*(1-u_*) - Au_*(1-u_*) \\ &= u_*(u_*-1)\frac{\beta\eta(K-R) - (u_*+R)^2(2u_*+2R-A\eta-1)}{\eta(u_*+R)^2} \\ &= -\frac{u_*(1-u_*)}{\eta(u_*+R)}g(u_*) - \frac{2A\eta u_* + R(u_*+R)}{\eta(u_*+R)}. \end{aligned} \tag{2.12}$$

Furthermore, since

$$\frac{u_*(1-u_*)}{\eta(u_*+R)} > 0,$$
$$\frac{2A\eta u_* + R(u_*+R)}{\eta(u_*+R)} > 0,$$

the following conclusion for the stability of the coexistence equilibrium can be drawn:

Theorem 2.9. *When the coexistence equilibrium (u_*, v_*) of system (1.5) exists, if $g(u_*) > -\frac{2A\eta u_*+R(u_*+R)}{u_*(1-u_*)}$ holds further, then (u_*, v_*) is an unstable saddle point; if $g(u_*) < -\frac{2A\eta u_*+R(u_*+R)}{u_*(1-u_*)}$, the equilibrium is an anti-saddle point.*

In the case of an anti-saddle point, further judgment needs to be made by combining the trace of the Jacobian matrix at this point:

$$\begin{aligned} \operatorname{tr} Df(u_*, v_*) &= 2Au_* + 2\frac{\beta(u_*+K)}{u_*+R} - \frac{-2u_*^2 + u_*(1-R-\beta) - \beta K}{u_*+R} - \frac{(u_*+R)(1-u_*)}{\eta} \\ &= \frac{1}{\eta(R+u_*)}\{-R^2 + 3\beta\eta(K+u_*) + u_*\Big[-2R - u_* + (R+u_*)^2 \\ &\quad + \eta(-1+R+2AR+2(1+A)u_*)]\}. \end{aligned} \tag{2.13}$$

If $\operatorname{tr} Df(u_*, v_*) < 0$, the coexistence equilibrium (u_*, v_*) is asymptotically stable; if $\operatorname{tr} Df(u_*, v_*) > 0$, it is unstable; if $\operatorname{tr} Df(u_*, v_*) = 0$, further analysis is required.

To distinguish whether the anti-saddle point is a node or a focus, the following needs to be considered:

$$\Delta = [\operatorname{tr} Df(u_*, v_*)]^2 - 4\det Df(u_*v_*).$$

When $\Delta \geq 0$, the equilibrium is a node. When $\Delta < 0$, the equilibrium is a focus.

In addition, for the case where $g(u_*) = -\frac{2A\eta u_*+R(u_*+R)}{u_*(1-u_*)}$, the equilibrium is unstable only when $\operatorname{tr} Df(u_*, v_*) > 0$; other cases are degenerate and difficult to analyze.

3. Bifurcation analysis of boundary equilibria

This section analyzes the possible bifurcation phenomena of system (1.5).

3.1. Bifurcation at $E_1 = (1, 0)$

Theorem 3.1. *At $E_1 = (1, 0)$, with the natural mortality rate of predators δ as the bifurcation parameter, as δ changes around $\delta_1 = \frac{\beta(1+K)}{1+R} - A$, a transcritical bifurcation occurs in the system. The specific bifurcation phenomena are as follows:*

In the case where $\delta < \frac{\beta(1+K)}{1+R} - A$, E_1 is unstable and isolated from other equilibria.

At the bifurcation point $\delta = \frac{\beta(1+K)}{1+R} - A$, E_1 becomes semi-stable.

Finally, the condition $\delta > \frac{\beta(1+K)}{1+R} - A$ renders E_1 a stable predator extinction equilibrium, accompanied by the appearance of a new unstable coexistence equilibrium E_*.

Proof. For the predator extinction equilibrium $E_1 = (1, 0)$, we have

$$Df(1, 0; \delta) = \begin{pmatrix} -1 & -\frac{\eta}{1+R} \\ 0 & -\delta + \frac{\beta(1+K)}{1+R} - A \end{pmatrix}. \tag{3.1}$$

Then $Df(1, 0; \delta_1) = \begin{pmatrix} -1 & -\frac{\eta}{1+R} \\ 0 & 0 \end{pmatrix}$, $\delta_1 = \frac{\beta(1+K)}{1+R} - A$.

$Df(1, 0; \delta_1)$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0$. For $\lambda_2 = 0$, there exist a left eigenvector w and a right eigenvector v, where

$$w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} v_c \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{-\eta}{K+1} \\ 1 \end{pmatrix}.$$

Furthermore, since

$$f_\delta(1, 0; \delta_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Df_\delta(1, 0; \delta_1) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned} D^2 f\,(1, 0; \delta_1)\,(v, v) &= \left[\sum_{j_1, j_2 = 1}^{n} \frac{\partial^2 f_i}{\partial \xi_{j_1} \partial \xi_{j_2}} v_{j_1} v_{j_2} \right]^\top \\ &= \left(\begin{matrix} \frac{\partial^2 f_1}{\partial x^2} v_c^2 + \frac{\partial^2 f_1}{\partial x \partial y} v_c + \frac{\partial^2 f_1}{\partial y \partial x} v_c + \frac{\partial^2 f_1}{\partial y^2} \cdot 1 \\ \frac{\partial^2 f_2}{\partial x^2} v_c^2 + \frac{\partial^2 f_2}{\partial x \partial y} v_c + \frac{\partial^2 f_2}{\partial y \partial x} v_c + \frac{\partial^2 f_2}{\partial y^2} \cdot 1 \end{matrix} \right) \Bigg|_{(1,0;\delta_1).} \\ &= \begin{pmatrix} -2v_c^2 - \frac{\eta R}{(1+R)^2} v_c + \frac{-\eta R}{(1+R)^2} v_c \\ \frac{\beta(R-K)}{(1+R)^2} v_c + \frac{\beta(R-K)}{(1+R)^2} v_c - 2 \end{pmatrix}. \end{aligned}$$

According to Sotomayor's theorem (Cf. P330 in [33]), since the following conditions hold:

(1) $w^\top f_\delta\,(1, 0; \delta_1) = 0$,

(2) $w^\top \left[Df_\delta\,(1, 0; \delta_1)\, v \right] = -1$,

(3) $w^\top \left(D^2 f\,(1, 0, \delta_1)\,(v, v) \right) = -\frac{2\beta(R-K)}{(1+R)^2} \cdot \frac{\eta}{R+1} - 2 = -2\left(\frac{\eta\beta(R-K)}{(1+R)^3} - 1 \right) \neq 0$.

We can know that as δ changes around $\delta_1 = \frac{\beta(1+K)}{1+R} - A$ at E_1, a transcritical bifurcation occurs in the system. When $\delta < \frac{\beta(1+K)}{1+R} - A$, the equilibrium E_1 is unstable, and there are no other equilibrium near E_1 in the system. When $\delta = \frac{\beta(1+K)}{1+R} - A$, E_1 is semi-stable. When $\delta > \frac{\beta(1+K)}{1+R} - A$, the system has a new unstable internal equilibrium E_*, while E_1 becomes stable.

□

It can be seen that when anti-predator behavior exists in the system, the survival pressure on predators increases significantly. Under the same mortality rate, it is more likely for predators to go extinct while the pest population still survives.

3.2. Bifurcation at $E_2 = (0, v_0)$

When $\delta < \delta_0 = \frac{\beta K}{R}$, system (1.5) has the equilibrium $E_2 = (0, v_0)$. Assuming the natural mortality rate of predators δ is a constant, with $R = R(a)$ as the bifurcation parameter, by controlling the additional food input amount a to change R, it can be proved that a bifurcation phenomenon also occurs near the equilibrium E_2 when R changes around $R_0 = \eta v_0$.

Theorem 3.2. *Under the condition $\delta < \frac{\beta K}{R}$, by controlling the parameter a to make R change around $R_0 = \eta v_0$, a saddle-node bifurcation occurs near the equilibrium E_2 of system* (1.5).

Proof. Consider

$$Df(0, v_0; R) = \begin{pmatrix} 1 - \frac{\eta v_0}{R} & 0 \\ \frac{\beta(R-K)v_0}{R^2} - Av_0 & -v_0 \end{pmatrix}. \tag{3.2}$$

When $R = R_0$, we have

$$Df(0, v_0; R_0) = \begin{pmatrix} 0 & 0 \\ \frac{\beta(R_0-K)}{\eta R_0} - A\frac{R_0}{\eta} & -v_0 \end{pmatrix}. \tag{3.3}$$

The eigenvalues of $Df(0, v_0; R_0)$ are $\lambda_1 = 0$ and $\lambda_2 = v_0 > 0$. For $\lambda_1 = 0$, consider its left eigenvector w and right eigenvector v

$$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} v_c \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{R_0^2}{\beta(R_0-K)-AR_0^2} \\ 1 \end{pmatrix}.$$

Finally

$$f_R(0, v_0; R_0) = \begin{pmatrix} R_0^{-1} \\ \frac{-\beta K v_0}{R_0^2} \end{pmatrix}, \quad Df_R(0, v_0; R_0) = \begin{pmatrix} R_0^{-1} & 0 \\ \frac{v_0\beta}{R_0^2} - \frac{2(-K+R_0)v_0\beta}{R_0^3} - \frac{A}{\eta} & 0 \end{pmatrix}.$$

Substituting the right eigenvector v for calculation, we get:

$$\begin{aligned} D^2 f(0, v_0; R_0)(v, v) &= \left[\sum_{j_1, j_2=1}^{n} \frac{\partial^2 f_i}{\partial \xi_{j_1} \partial \xi_{j_2}} v_{j_1} v_{j_2} \right]^\top \\ &= \left(\begin{matrix} \frac{\partial^2 f_1}{\partial x^2} v_c^2 + \frac{\partial^2 f_1}{\partial x \partial y} v_c + \frac{\partial^2 f_1}{\partial y \partial x} v_c + \frac{\partial^2 f_1}{\partial y^2} \cdot 1 \\ \frac{\partial^2 f_2}{\partial x^2} v_c^2 + \frac{\partial^2 f_2}{\partial x \partial y} v_c + \frac{\partial^2 f_2}{\partial y \partial x} v_c + \frac{\partial^2 f_2}{\partial y^2} \cdot 1 \end{matrix} \right) \Bigg|_{(0, v_0; R_0).} \\ &= \begin{pmatrix} -2v_c^2 - \frac{\eta R}{(1+R)^2} v_c + \frac{-\eta R}{(1+R)^2} v_c \\ \frac{\beta(R-K)}{(1+R)^2} v_c + \frac{\beta(R-K)}{(1+R)^2} v_c - 2 \end{pmatrix}. \end{aligned}$$

By using of Sotomayor Theorem (Cf. P330 in [33]), we verify the following conditions:

(1) $w^\top f_R(0, v_0; R_0) = R_0^{-1} > 0$.

(2) $w^\top D^2 f(0, v_0; R_0)(v, v) = -2v_c^2 - \frac{\eta R}{(1+R)^2} v_c + \frac{-\eta R}{(1+R)^2} v_c \neq 0$.

At this time, a saddle-node bifurcation occurs in the system at E_2 as R changes. When $R > R_0$, two internal equilibria split from E_2. □

4. Bifurcation analysis of coexistence equilibria

4.1. The analysis of the Hopf bifurcation

As shown in **Theorem 2.2 (1) and 2.9**, when

$$\begin{aligned} &a_2 \geq 0, \quad a_1 < 0, \quad a_0 > 0, \\ &G(1) < 0, \quad g(1) = 3 + 2a_2 + a_1 \leq 0, \end{aligned}$$

there exists a unique coexistence equilibrium (u_0, v_0). Moreover, if

$$\begin{aligned} &g(u_0) < -\frac{2A\eta u_0 + R(u_0 + R)}{u_0(1 - u_0)}, \\ &Au_0 + 2\frac{\beta(u_0 + K)}{u_0 + R} - \frac{-2u_0^2 + u_0(1 - R - \beta) - \beta K}{u_0 + R} - \frac{(u_0 + R)(1 - u_0)}{\eta} = 0, \end{aligned}$$

the equilibrium (u_0, v_0) satisfies

$$\operatorname{Det} J(u_0, v_0) > 0,$$

and

$$\operatorname{Tr} J(u_0, v_0) = 0,$$

and the Jacobian matrix $J(u_0, v_0)$ has a pair of conjugate pure imaginary eigenvalues

$$\lambda_{1,2} = \pm \mathrm{i} \sqrt{\operatorname{Det} J(u_0, v_0)} = \pm \mathrm{i} \sqrt{-\frac{u_0(1 - u_0)}{\eta(u_0 + R)} g(u_0) - \frac{2A\eta u_0 + R(u_0 + R)}{\eta(u_0 + R)}}.$$

Which means system (1.5) may undergo Hopf bifurcation on the point $E^* = (u_0, v_0)$.

Since

$$\delta = \frac{\beta(u_0 + K)}{u_0 + R} - \frac{(u_0 + R)(1 - u_0)}{\eta} + Au_0,$$

Now, to we choose the natural mortality rate of predator δ as the bifurcate parameter, and consider the bifurcation parametric curve

$$H(\delta) : \frac{\beta(u_0 + K)}{u_0 + R} - \frac{-2u_0^2 + u_0(1 - R - \beta) - \beta K}{u_0 + R} - \delta = 0.$$

Denote the critical value of Hopf bifurcation parameter

$$\delta^* := \frac{\beta(u_0 + K)}{u_0 + R} - \frac{-2u_0^2 + u_0(1 - R - \beta) - \beta K}{u_0 + R},$$

and let $\delta = \delta^* + \varepsilon$, where $|\varepsilon| << 1$ is a pertubation of δ^*.

The Jacobian matrix $J(x_0, y_0; \varepsilon)$ has a pair of conjugate complex eigenvalues

$$\lambda_{1,2}(\varepsilon) = \operatorname{Re} \lambda_{1,2}(\varepsilon) \pm \mathrm{i} \operatorname{Im} \lambda_{1,2}(\varepsilon),$$

where

$$\begin{aligned} \operatorname{Re}\lambda_{1,2}(\varepsilon) &= \frac{1}{2}\operatorname{Tr} Df(x_0, y_0; \varepsilon), \\ \operatorname{Im}\lambda_{1,2}(\varepsilon) &= \frac{1}{2}\sqrt{4\operatorname{Det} Df(x_0, y_0; \varepsilon) - [\operatorname{Tr} Df(x_0, y_0; \varepsilon)]^2}, \end{aligned} \tag{4.1}$$

and $\operatorname{Re}\lambda_{1,2}(0) = 0$, $\operatorname{Im}\lambda_{1,2}(0) = \pm \mathrm{i}\sqrt{\operatorname{Det} Df(x_0, y_0; \varepsilon)}\big|_{r=r^*} = x_0 y_0\left[\eta(\frac{2K}{A_0}x_0 - 1 - \frac{K}{A_0}) + \beta - \theta\right]$.

Now, to verify Hopf bifurcation conditions, we calculate by (4.1):

$$\begin{aligned} &\textbf{(H1.)} \quad \frac{\mathrm{d}\operatorname{Re}\lambda_{1,2}(\epsilon)}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{2} > 0, \\ &\textbf{(H2.)}\ [\operatorname{Im}(\lambda_{1,2}(0))]^2 = \operatorname{Det} Df(x_0, y_0; 0) > 0, \end{aligned} \tag{4.2}$$

Therefore, it can be determined by (4.2) that system (1.5) undergoes Hopf bifurcation at the coexistence equilibrium $E^* = (x_0, y_0)$ as the parameter δ changes around δ^*. Next, we will discuss **the first Lyapunov coefficient** to determine the direction of bifurcation and the stability of the bifurcated limit cycle.

Step 1: Transform the equilibrium to the origin.

Let $U = u - u_c$, $V = v - v_c$, $\delta = \delta_0$,

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = (U + u_1)(1 - U - u_1) - \frac{\eta(U+u_1)(V+v_1)}{(U+u_1)+R}, \\ \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\beta(U+u_1+K)(V+v_1)}{(U+u_1)+R} - A(U + u_1)(V + v_1) - \delta(V + v_1) - (V + v_1)^2, \\ \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0. \end{cases} \tag{4.3}$$

Step 2: Perform Taylor expansion on the right-hand function of system (4.3)**.**

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = (1 - 2u_c - \dfrac{\eta R v_c}{(u_c + R)^2})U - \dfrac{\eta u_c}{u_c + R}V + [-1 + \dfrac{\eta R v_c}{(R + u_c)^3}]U^2 \\ \qquad - \dfrac{\eta R}{(R + u1)^2}UV + \dfrac{\eta R}{(u_c + R)^3}U^2V - \dfrac{\eta R v_c}{(u_c + R)^4}U^3 + O(4), \\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = [-Av_c + \dfrac{\beta(R - K)v_c}{(u_c + R)^2}]U + [-Au_c - \delta - 2v_c + \dfrac{\beta(u_c + K)}{u_c + R}]V + \dfrac{\beta(K - R)v_c}{(u_c + R)^3}U^2 - V^2 \\ \qquad + [-A + \dfrac{\beta(R - K)}{(u_c + R)^2}]UV + \dfrac{\beta(K - R)}{(u_c + R)^3}U^2V + \dfrac{\beta(R - K)v_c}{(u_c + R)^4}U^3 + O(4), \\ \dfrac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0. \end{cases} \tag{4.4}$$

Step 3: Transform to normal form.

System (4.4) can be written as

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = J_1U + J_2V + F_1(U, V), \\ \frac{\mathrm{d}V}{\mathrm{d}t} = J_3U + J_4V + F_2(U, V), \\ \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0, \end{cases} \tag{4.5}$$

where

$$J_1 = J_1(u_c, v_c) = 1 - 2u_c - \frac{\eta R v_c}{(u_c + R)^2},$$
$$J_2 = J_2(u_c, v_c) = -\frac{\eta u_c}{u_c + R},$$
$$J_3 = J_3(u_c, v_c) = -Av_c + \frac{\beta(R - K)v_c}{(u_c + R)^2},$$
$$J_4 = J_4(u_c, v_c) = -Au_c - \delta - 2v_c + \frac{\beta(u_c + K)}{u_c + R},$$

then $\operatorname{tr} Df(u_c, v_c) = J_1 + J_4$, $\det Df(u_c, v_c) = J_1 J_4 - J_2 J_3$.

Let

$$F_1(U, V) = [-1 + \frac{\eta R v_c}{(R + u_c)^3}]U^2 - \frac{\eta R}{(R + u1)^2}UV + \frac{\eta R}{(u_c + R)^3}U^2V - \frac{\eta R v_c}{(u_c + R)^4}U^3 + O(4),$$
$$F_2(U, V) = +\frac{\beta(K - R)v_c}{(u_c + R)^3}U^2 - V^2 + [-A + \frac{\beta(R - K)}{(u_c + R)^2}]UV + \frac{\beta(K - R)}{(u_c + R)^3}U^2V + \frac{\beta(R - K)v_c}{(u_c + R)^4}U^3 + O(4).$$

From the branching condition $\delta = \delta_0$, and the equilibrium condition, we have $J_1(u_c, v_c) = -J_4(u_c, v_c) = v_c$, and system (4.5) is equivalent to the following system:

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = v_c U + J_2 V + F_1(U, V), \\ \frac{\mathrm{d}V}{\mathrm{d}t} = J_3 U - v_c V + F_2(U, V), \\ \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0. \end{cases} \tag{4.6}$$

Consider a invertible matrix

$$T = \begin{pmatrix} \frac{\sqrt{-J_2 J_3 - v_c^2}}{J_3} & \frac{v_c}{J_3} \\ 0 & 1 \end{pmatrix},$$

and its inverse matrix

$$T^{-1} = \begin{pmatrix} \frac{J_3}{\sqrt{-J_2 J_3 - v_c^2}} & -\frac{v_c}{\sqrt{-J_2 J_3 - v_c^2}} \\ 0 & 1 \end{pmatrix},$$

and make a non-degenerate linear transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = T \begin{pmatrix} U \\ V \end{pmatrix},$$

we get

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t} = -\sqrt{\det Df(u_c, v_c; 0)}Y + G^1(X, Y), \\ \frac{\mathrm{d}Y}{\mathrm{d}t} = \sqrt{\det Df(u_c, v_c; 0)}X + G^2(X, Y), \\ \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0. \end{cases} \tag{4.7}$$

Let $Q = \det Df(u_c, v_c; 0)$, where

$$\begin{aligned} G^1(X,Y) = &\left(-1+\frac{\eta R v_c}{(u_c+R)^3}\right)\frac{J_3}{Q}X^2 + \left[\left(-1+\frac{\eta R v_c}{(u_c+R)^3}\right)\frac{v_c^2}{Q} + \frac{\eta R}{(u_c+R)^2}\cdot\frac{v_c}{Q}\right]Y^2 \\ &+\left(-1+\frac{\eta R v_c}{(u_c+R)^3}\right)\left(-\frac{2v_c J_3}{Q}\right)XY \\ &-\frac{\eta R v_c}{(u_c+R)^4}\cdot\frac{J_3^3}{Q\sqrt{Q}}X^3 + \left[\frac{\eta R}{(u_c+R)^3}\cdot\frac{v_c^2}{Q} + \frac{\eta R v_c}{(u_c+R)^4}\cdot\frac{v_c^3}{Q\sqrt{Q}}\right]Y^3 \\ &+\left[\frac{\eta R}{(u_c+R)^3}\cdot\frac{J_3^2}{Q} + \frac{3\eta R v_c}{(u_c+R)^4}\cdot\frac{J_3^2 v_c}{Q\sqrt{Q}}\right]X^2Y \\ &-\left[\frac{2\eta R}{(u_c+R)^3}\cdot\frac{v_c J_3}{Q} - \frac{3\eta R v_c}{(u_c+R)^4}\cdot\frac{J_3 v_c^2}{Q\sqrt{Q}}\right]XY^2 + O(4). \end{aligned}$$

$$\begin{aligned} G^2(X,Y) = &\frac{J_3^2(K-R)v_c\beta}{Q(R+u_c)^3}X^2 + \frac{J_3^3(-K+R)v_c\beta}{Q^{3/2}(R+u_c)^4}X^3 - \frac{J_3^2(K-R)\left(\sqrt{Q}(R+u_c)+3v_c^2\right)\beta}{Q^{3/2}(R+u_c)^4}X^2Y \\ &-\frac{J_3(K-R)v_c\left(2\sqrt{Q}(R+u_c)+3v_c^2\right)\beta}{Q^{3/2}(R+u_c)^4}XY^2 - \frac{(K-R)v_c^2\left(\sqrt{Q}(R+u_c)+v_c^2\right)\beta}{Q^{3/2}(R+u_c)^4}Y^3 \\ &+\frac{J_3\left(-A\sqrt{Q}-\frac{(K-R)\left(\sqrt{Q}(R+u_c)-2v_c^2\right)\beta}{(R+u_c)^3}\right)}{Q}XY \\ &+\left(-1+\frac{(K-R)v_c^3\beta}{Q(R+u_c)^3} - \frac{v_c\left(A(R+u_c)^2+(K-R)\beta\right)}{\sqrt{Q}(R+u_c)^2}\right)Y^2 + O(4). \end{aligned}$$

Step 4: Calculate the first Lyapunov coefficient when $\varepsilon = 0$.

The first Lyapunov coefficient is

$$\begin{aligned} l(\delta_0) = &\frac{1}{16}\left[G^1_{XXX} + G^1_{XYY} + G^2_{XXY} + G^2_{YYY}\right] \\ &+\frac{1}{16\sqrt{Q}}\left[G^1_{XY}\left(G^1_{XX}+G^1_{YY}\right) - G^2_{XY}\left(G^2_{XX}+G^2_{YY}\right)\right. \\ &\left.-G^1_{XX}G^2_{XX} + G^1_{YY}G^2_{YY}\right]|_{X=Y=0}, \end{aligned} \tag{4.8}$$

where

$$G^1_{XX}\big|_{X=Y=0} = 2\left(-1+\frac{\eta R v_c}{(u_c+R)^3}\right)\frac{J_3}{Q},$$
$$G^1_{YY}\big|_{X=Y=0} = 2\left[\left(-1+\frac{\eta R v_c}{(u_c+R)^3}\right)\frac{v_c^2}{Q}+\frac{\eta R}{(u_c+R)^2}\cdot\frac{v_c}{Q}\right],$$
$$G^1_{XXX}\big|_{X=Y=0} = -6\frac{\eta R v_c}{(u_c+R)^4}\cdot\frac{J_3^3}{Q\sqrt{Q}},$$
$$G^1_{YYY}\big|_{X=Y=0} = 6\left[\frac{\eta R}{(u_c+R)^3}\cdot\frac{v_c^2}{Q}+\frac{\eta R v_c}{(u_c+R)^4}\cdot\frac{v_c^3}{Q\sqrt{Q}}\right],$$
$$G^1_{XY}\big|_{X=Y=0} = \left[-1+\frac{\eta R v_c}{(u_c+R)^3}\right]\left(-\frac{2v_c J_3}{Q}\right).$$

$$G^2_{XX}\big|_{X=Y=0} = \frac{2J_3^2(K-R)v_c\beta}{Q(R+u_c)^3}+\frac{6J_3^3(-K+R)v_c X\beta}{Q^{3/2}(R+u_c)^4}-\frac{2J_3^2(K-R)\left(\sqrt{Q}(R+u_c)+3v_c^2\right)Y\beta}{Q^{3/2}(R+u_c)^4},$$
$$G^2_{YY}\big|_{X=Y=0} = -\frac{2J_3(K-R)v_c\left(2\sqrt{Q}(R+u_c)+3v_c^2\right)X\beta}{Q^{3/2}(R+u_c)^4}-\frac{6(K-R)v_c^2\left(\sqrt{Q}(R+u_c)+v_c^2\right)Y\beta}{Q^{3/2}(R+u_c)^4}$$
$$+2\left(-1+\frac{(K-R)v_c^3\beta}{Q(R+u_c)^3}-\frac{v_c\left(A(R+u_c)^2+(K-R)\beta\right)}{\sqrt{Q}(R+u_c)^2}\right),$$
$$G^2_{XXX}\big|_{X=Y=0} = \frac{6J_3^3(-K+R)v_c\beta}{Q^{3/2}(R+u_c)^4},$$
$$G^2_{YYY}\big|_{X=Y=0} = -\frac{6(K-R)v_c^2\left(\sqrt{Q}(R+u_c)+v_c^2\right)\beta}{Q^{3/2}(R+u_c)^4},$$
$$G^2_{XY}\big|_{X=Y=0} = -\frac{2J_3^2(K-R)\left(\sqrt{Q}(R+u_c)+3v_c^2\right)X\beta}{Q^{3/2}(R+u_c)^4}-\frac{2J_3(K-R)v_c\left(2\sqrt{Q}(R+u_c)+3v_c^2\right)Y\beta}{Q^{3/2}(R+u_c)^4}$$
$$+\frac{J_3\left(-A\sqrt{Q}-\frac{(K-R)\left(\sqrt{Q}(R+u_c)-2v_c^2\right)\beta}{(R+u_c)^3}\right)}{Q}.$$

Theorem 4.1. *If $l(\delta_0)\neq 0$, system* (1.5) *will undergo a Hopf bifurcation at E^*.*

Moreover, if $l(\delta_0)<0$, there exists a stable limit cycle in the small neighborhood of E^* when $\delta<\delta_0$; and there exists an unstable limit cycle in the small neighborhood of E^* when $\delta>\delta_0$ if $l(\delta_0)>0$.

4.2. The analysis of Bogdanov-Takens bifurcation

As shown in **Theorem 2.2(1)**, there exists a unique non-hyperbolic equilibrium $E_c=(x_c,y_c)$, if

$$\begin{aligned}\det Df(u_*v_*) &= \frac{u_*(1-R-2u_*)}{u_*+R}\frac{(1-u_*)(u_*+R)}{\eta}+\frac{\beta(R-K)}{(u_*+R)^2}u_*(1-u_*)-Au_*(1-u_*)\\ &= u_*(u_*-1)\frac{\beta\eta(K-R)-(u_*+R)^2(2u_*+2R-A\eta-1)}{\eta(u_*+R)^2}\\ &= -\frac{u_*(1-u_*)}{\eta(u_*+R)}g(u_*)-\frac{2A\eta u_*+R(u_*+R)}{\eta(u_*+R)}=0.\end{aligned}\tag{4.9}$$

Moreover, if

$$\begin{aligned} \operatorname{tr} Df(u_*, v_*) &= 2Au_* + 2\frac{\beta(u_* + K)}{u_* + R} - \frac{-2u_*^2 + u_*(1 - R - \beta) - \beta K}{u_* + R} - \frac{(u_* + R)(1 - u_*)}{\eta} \\ &= 3Au_* + \delta + \frac{1 - u_*}{\eta v_*}[\beta u_* + 2\beta K + 2u_*^2 - u_*(1 - R - \beta)] = 0, \end{aligned} \tag{4.10}$$

the Jacobian matrix $J(x_c, y_c)$ has a pair of zero eigenvalues.

Choose β and r as bifurcation parameters. Since

$$\begin{aligned} &- u_*(1 - u_*)g(u_*) - 2A\eta u_* - R(u_* + R) = 0, \\ &3Au_* + \delta + \frac{1 - u_*}{\eta v_*}[\beta u_* + 2\beta K + 2u_*^2 - u_*(1 - R - \beta)] = 0, \\ &1 - u_* - \frac{\eta v_*}{u_* + R} = 0, \\ &\frac{\beta(u_* + K)}{u_* + R} - \delta - Au_* - v_* = 0. \end{aligned}$$

We get the critical value of Bogdanov-Takens bifurcation parameters as

$$\begin{aligned} \delta_b &= \frac{\beta(u_* + K)(1 - u_*)}{\eta v_*} - Au_* - v_*, \\ R_b &= \frac{\eta}{1 - \eta u_*}\left[\beta(u_* + K) + \frac{\eta v_*^2}{u_* - 1} + \beta(u_* + 2K) + 2u_*^2 - u_*(1 - \beta) - u_* v_* g(u_*)\right]. \end{aligned}$$

Consider a pertubation $\varepsilon = (\varepsilon_1, \varepsilon_2)$ for the parameters (β_0, R), let

$$\delta = \delta_b + \varepsilon_1, R = R_b + \varepsilon_2, \tag{4.11}$$

and take

$$U = u - u_c, V = v - v_c,$$

we get the pertubation system as follows:

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = (U + u_c)\left(1 - U - u_c - \frac{(V + v_c)\eta}{R + U + u_c + \varepsilon_1}\right), \\ \frac{\mathrm{d}V}{\mathrm{d}t} = (V + v_c)\left(-A(U + u_c) - V - v_c - \delta + \frac{(K + U + u_c)\beta}{R + U + u_c + \varepsilon_1} - \varepsilon_2\right). \end{cases} \tag{4.12}$$

Step 1: Perform the Taylor expansion on the right-hand function of the translated system (4.12) **at the point (U, V) = (0, 0).**

We can get:

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} =u_c\left(1-u_c-\dfrac{v_c\eta}{R+u_c+\varepsilon_1}\right)-\dfrac{u_c\eta}{R+u_c+\varepsilon_1}V+U^2\left(-1-\dfrac{u_cv_c\eta}{(R+u_c+\varepsilon_1)^3}+\dfrac{v_c\eta}{(R+u_c+\varepsilon_1)^2}\right)\\ \qquad +\dfrac{(R+\varepsilon_1)\eta}{(R+u_c+\varepsilon_1)^2}UV+U\left(1-\dfrac{v_c\eta}{R+u_c+\varepsilon_1}+u_c\left(-2+\dfrac{v_c\eta}{(R+u_c+\varepsilon_1)^2}\right)\right)+O(3),\\ \dfrac{\mathrm{d}V}{\mathrm{d}t} =v_c\left(-Au_c-v_c-\delta+\dfrac{K\beta}{R+u_c+\varepsilon_1}+\dfrac{u_c\beta}{R+u_c+\varepsilon_1}-\varepsilon_2\right)+\\ \qquad V\left(-Au_c-2v_c-\delta+\dfrac{K\beta}{R+u_c+\varepsilon_1}+\dfrac{u_c\beta}{R+u_c+\varepsilon_1}-\varepsilon_2\right)+\\ \qquad v_c\left(-A+\dfrac{\beta(-K+R+\varepsilon_1)}{(R+u_c+\varepsilon_1)^2}\right)U+UV\left(-A+\dfrac{\beta(-K+R+\varepsilon_1)}{(R+u_c+\varepsilon_1)^2}\right)+\\ \qquad -V^2+\dfrac{v_c\beta(K-R-\varepsilon_1)}{(R+u_c+\varepsilon_1)^3}U^2+O(3),\\ \dot{\varepsilon_1}=0,\\ \dot{\varepsilon_2}=0. \end{cases} \tag{4.13}$$

Simplifying, we obtain:

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = p_{00}(\varepsilon)+\alpha(\varepsilon)U+\beta(\varepsilon)V+p_{20}(\varepsilon)U^2+p_{11}(\varepsilon)UV+P_1^*(U,V,\varepsilon),\\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = q_{00}(\varepsilon)+\gamma(\varepsilon)U+\lambda(\varepsilon)V+q_{20}(\varepsilon)U^2+q_{11}(\varepsilon)UV+q_{02}(\varepsilon)V^2+Q_1^*(U,V,\varepsilon), \end{cases} \tag{4.14}$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2)$, and

$$\begin{aligned} p_{00}(\varepsilon) &= u_c\left(1-u_c-\frac{v_c\eta}{R+u_c+\varepsilon_1}\right),\\ \alpha(\varepsilon) &= 1-\frac{v_c\eta}{R+u_c+\varepsilon_1}+u_c\left(-2+\frac{v_c\eta}{(R+u_c+\varepsilon_1)^2}\right),\\ \beta(\varepsilon) &= -\frac{u_c\eta}{R+u_c+\varepsilon_1},\\ p_{20}(\varepsilon) &= -1-\frac{u_cv_c\eta}{(R+u_c+\varepsilon_1)^3}+\frac{v_c\eta}{(R+u_c+\varepsilon_1)^2},\\ p_{11}(\varepsilon) &= \frac{(R+\varepsilon_1)\eta}{(R+u_c+\varepsilon_1)^2},\\ q_{00}(\varepsilon) &= v_c\left(-Au_c-v_c-\delta+\frac{K\beta}{R+u_c+\varepsilon_1}+\frac{u_c\beta}{R+u_c+\varepsilon_1}-\varepsilon_2\right), \end{aligned}$$

$$
\begin{aligned}
&\gamma(\varepsilon) = v_c\left(-A + \frac{\beta(-K+R+\varepsilon_1)}{(R+u_c+\varepsilon_1)^2}\right),\\
&\lambda(\varepsilon) = -Au_c - 2v_c - \delta + \frac{K\beta}{R+u_c+\varepsilon_1} + \frac{u_c\beta}{R+u_c+\varepsilon_1} - \varepsilon_2,\\
&q_{20}(\varepsilon) = \frac{v_c\beta(K-R-\varepsilon_1)}{(R+u_c+\varepsilon_1)^3},\\
&q_{11}(\varepsilon) = (-A + \frac{\beta(-K+R+\varepsilon_1)}{(R+u_c+\varepsilon_1)^2},\\
&q_{02}(\varepsilon) = -1,\\
&\delta = \delta_b + \varepsilon_1, R = R_b + \varepsilon_2,
\end{aligned}
$$

and $P_1^*(U,V,\varepsilon)$ and $Q_1^*(U,V,\varepsilon)$ are terms of at least order three in (U,V).

Step 2: Reduction to a nonlinear oscillator.

Let $X = U$, $Y = \frac{\mathrm{d}U}{\mathrm{d}t}$ in (4.14), then system (4.14) is transformed into

$$
\begin{cases}
\dot{X} = Y,\\
\dot{Y} = n_{00}(\varepsilon) + n_{10}(\varepsilon)X + n_{01}(\varepsilon)Y + n_{20}(\varepsilon)X^2 + n_{11}(\varepsilon)XY + n_{02}(\varepsilon)Y^2 + Q_2^*(X,Y,\varepsilon),
\end{cases}
\tag{4.15}
$$

where

$$
\begin{aligned}
&n_{00}(\varepsilon) = -\lambda(\varepsilon)p_{00}(\varepsilon),\\
&n_{10}(\varepsilon) = \beta(\varepsilon)\gamma(\varepsilon) - \alpha(\varepsilon)\lambda(\varepsilon),\\
&n_{01}(\varepsilon) = \alpha(\varepsilon) + \lambda(\varepsilon) - \frac{p_{00}(\varepsilon)p_{11}(\varepsilon)}{\beta(\varepsilon)},\\
&n_{20}(\varepsilon) = -\lambda(\varepsilon)p_{20}(\varepsilon) + \gamma(\varepsilon)p_{11}(\varepsilon),\\
&n_{11}(\varepsilon) = 2p_{20}(\varepsilon) + \frac{p_{00}(\varepsilon)p_{11}^2(\varepsilon) - \alpha(\varepsilon)p_{11}(\varepsilon)}{\beta^2(\varepsilon)},\\
&n_{02}(\varepsilon) = \frac{p_{11}(\varepsilon)}{\beta(\varepsilon),}
\end{aligned}
$$

and $Q_2^*(X,Y,\varepsilon)$ are terms of at least order three in (X,Y).

Step 3: Parameter-dependent transformation.

Since

$$
g_{11}(0) = 1 + \frac{3K}{A_0} - \frac{4Kx_c}{A_0} \neq 0,
$$

by a parameter-dependent shift of coordinates in the X-direction $X_2 = X + g_{01}/g_{11}$, $Y_2 = Y$, we arrive at

$$
\begin{cases}
\dfrac{\mathrm{d}X_2}{\mathrm{d}t} = Y_2,\\
\dfrac{\mathrm{d}Y_2}{\mathrm{d}t} = h_{00}(\varepsilon) + h_{10}(\varepsilon)X_2 + h_{20}(\varepsilon)X_2^2 + h_{11}(\varepsilon)X_2Y_2 + h_{02}(\varepsilon)Y_2^2 + O\left(|X_2,Y_2|^3\right),
\end{cases}
\tag{4.16}
$$

$$
\begin{aligned}
h_{00}(\varepsilon) &= x_c\left(\varepsilon_2-\varepsilon_1 x_c\right) y_c \\
&+\frac{\left[-A_0 x_c+2K\left(1+x_c^2\right)\right]\left(-\varepsilon_2+\varepsilon_1 x_c-\eta y_c\right)\left(-\varepsilon_2+x_c+\varepsilon_1 x_c-\frac{Kx_c}{A_0}-\frac{2Kx_c^2}{A_0}-\eta y\right)^2}{A_0 x_c\left(1+\frac{3K}{A_0}-\frac{4Kx_c}{A_0}\right)^2} \\
&-\frac{\left(-\varepsilon_2+x_c+\varepsilon_1 x_c-\frac{Kx_c}{A_0}-\frac{2Kx_c^2}{A_0}-\eta y_c\right)\left(-\varepsilon_2 y_c+\varepsilon_1 x_c y_c-\frac{\left(\varepsilon_c-\varepsilon_1 x_c+\eta y\right)\left[3Kx_c^2+A_0(1-2x_c+y_c)\right]}{A_0}\right)}{1+\frac{3K}{A_0}-\frac{4Kx}{A_0}}, \\
h_{10}(\varepsilon) &= -\varepsilon_2 y_c+\varepsilon_1 x_c y_c \\
&-\frac{2\left[-A_0 x_c+2K\left(1+x_c\right)^2\right]\left(-\varepsilon_2+\varepsilon_1 x_c-\eta y_c\left(-\varepsilon_2+x_c+\varepsilon_1 x_c-\frac{Kx_c}{A_0}-\frac{2Kx_c^2}{A_0}-\eta y_c\right)\right.}{A_0 x_c\left(1+\frac{3K}{A_0}-\frac{4Kx_c}{A_0}\right)} \\
&-\frac{\left(\varepsilon_2-\varepsilon_1 x_c+\eta y\right)\left[3Kx_c^2+A_0\left(1-2x_c+y_c\right)\right]}{A_0}, \\
h_{20}(\varepsilon) &= \left[-1+\frac{2K\left(1+x_c^2\right)}{A_0 x_c}\right]\left(\varepsilon_1 x_c-\varepsilon_2-\eta y_c\right), \\
h_{11}(\varepsilon) &= 1+\frac{3K}{A_0}-\frac{4Kx_0}{A_0}, \\
h_{02}(\varepsilon) &= \frac{1}{x_0}.
\end{aligned}
$$

Step 4: Time reparameterization.

Now, we introduce a new time variable τ with $\mathrm{d}t=(1-n_{02}(\varepsilon)X)\,\mathrm{d}\tau$ and rewrite τ as t for simplicity. Then system (4.15) becomes

$$
\begin{cases}
\dot{X} = Y\left(1-n_{02}(\varepsilon)X\right), \\
\dot{Y} = \left(1-n_{02}(\varepsilon)X\right)\left[n_{00}(\varepsilon)+n_{10}(\varepsilon)X\right. \\
\qquad +n_{01}(\varepsilon)Y+n_{20}(\varepsilon)X^2+n_{11}(\varepsilon)XY \\
\qquad \left.+n_{02}(\varepsilon)Y^2+Q_2^*(X,Y,\varepsilon)\right].
\end{cases}
\tag{4.17}
$$

Letting $X_1=X$ and $Y_1=Y\left(1-n_{02}(\varepsilon)X\right)$ transforms system (4.17) into

$$
\begin{cases}
\dot{w}_5 = Y_1, \\
\dot{w}_6 = \zeta_{00}(\varepsilon)+\zeta_{10}(\varepsilon)X_1+\zeta_{01}(\varepsilon)Y_1+\zeta_{20}(\varepsilon)X_1^2 \\
\qquad +\zeta_{11}(\varepsilon)X_1Y_1+Q_3^*(X_1,Y_1,\varepsilon),
\end{cases}
\tag{4.18}
$$

where

$$
\begin{aligned}
\zeta_{00}(\varepsilon) &= n_{00}(\varepsilon), \\
\zeta_{10}(\varepsilon) &= n_{10}(\varepsilon)-2n_{00}(\varepsilon)n_{02}(\varepsilon), \\
\zeta_{01}(\varepsilon) &= n_{01}(\varepsilon), \\
\zeta_{20}(\varepsilon) &= n_{20}(\varepsilon)-2n_{02}(\varepsilon)n_{10}(\varepsilon)+n_{00}(\varepsilon)n_{02}(\varepsilon)^2, \\
\zeta_{11}(\varepsilon) &= -n_{01}(\varepsilon)n_{02}(\varepsilon)+n_{11}(\varepsilon),
\end{aligned}
$$

and $Q_3(X_1, Y_1, \varepsilon)$ are terms of at least order three in (X_1, Y_1).

Note that

$$\zeta_{20}(0) = \frac{Rv_c\eta\left(-A - \frac{(-K+R)u_c\eta}{(R+u_c)^3}\right)}{(R+u_c)^2} + \left(Au_c + 2v_c + \delta + \frac{Ku_c\eta}{(R+u_c)^2} + \frac{u_c^2\eta}{(R+u_c)^2}\right)\left(-1 - \frac{u_cv_c\eta}{(R+u_c)^3} + \frac{v_c\eta}{(R+u_c)^2}\right) + \frac{R^2\left(Au_c + 2v_c + \delta + \frac{Ku_c\eta}{(R+u_c)^2} + \frac{u_c^2\eta}{(R+u_c)^2}\right)\left(1 - u_c - \frac{v_c\eta}{R+u_c}\right)}{u_c(R+u_c)^2} + \frac{2R\left(-\frac{u_cv_c\eta\left(-A - \frac{(-K+R)u_c\eta}{(R+u_c)^3}\right)}{R+u_c} - \left(-Au_c - 2v_c - \delta - \frac{Ku_c\eta}{(R+u_c)^2} - \frac{u_c^2\eta}{(R+u_c)^2}\right)\left(1 - \frac{v_c\eta}{R+u_c} + u_c\left(-2 + \frac{v_c\eta}{(R+u_c)^2}\right)\right)\right)}{u_c(R+u_c)}.$$

Thus $\zeta_{20}(\varepsilon) < 0$ when ε_1 and ε_2 are sufficiently small. With the following coordinate changes,

$$X_2 = X_1, \quad Y_2 = \frac{Y_1}{\sqrt{-\zeta_{20}(\varepsilon)}}, \quad \tau = \sqrt{-\zeta_{20}(\varepsilon)}t,$$

system (4.18) can be transformed into

$$\begin{cases} \dot{w}_7 = Y_2, \\ \dot{w}_8 = \vartheta_{00}(\varepsilon) + \vartheta_{10}(\varepsilon)X_1 + \vartheta_{01}(\varepsilon)Y_2 \\ \qquad - X_2^2 + \vartheta_{11}(\varepsilon)X_2Y_2 + Q_4^*(X_2, Y_2, \varepsilon), \end{cases} \tag{4.19}$$

where

$$\vartheta_{00}(\varepsilon) = -\frac{\zeta_{00}(\varepsilon)}{\zeta_{20}(\varepsilon)}, \quad \vartheta_{10}(\varepsilon) = -\frac{\zeta_{10}(\varepsilon)}{\zeta_{20}(\varepsilon)},$$
$$\vartheta_{01}(\varepsilon) = \frac{\zeta_{01}(\varepsilon)}{\sqrt{-\zeta_{20}(\varepsilon)}}, \quad \vartheta_{11}(\varepsilon) = \frac{\zeta_{11}(\varepsilon)}{\sqrt{-\zeta_{20}(\varepsilon)}}.$$

$Q_4^*(X_2, Y_2, \varepsilon)$ are terms of at least order three in (X_2, Y_2). With a further transformation

$$X_3 = X_2 - \frac{\vartheta_{10}(\varepsilon)}{2}, \quad Y3 = Y_2,$$

system (4.19) can be rewritten as

$$\begin{cases} \dot{w}_9 = Y_3, \\ \dot{w}_{10} = m_{00}(\varepsilon) + m_{01}(\varepsilon)Y_3 - X_3^2 \\ \qquad + m_{11}(\varepsilon)X_3Y3 + Q_5^*(X_3, Y_3, \varepsilon), \end{cases} \tag{4.20}$$

where

$$m_{00}(\varepsilon) = \vartheta_{00}(\varepsilon) + \frac{1}{4}\vartheta_{10}(\varepsilon)^2,$$
$$m_{01}(\varepsilon) = \vartheta_{01}(\varepsilon) + \frac{1}{2}\vartheta_{10}(\varepsilon)\vartheta_{11}(\varepsilon),$$
$$m_{11}(\varepsilon) = \vartheta_{11}(\varepsilon),$$

and $Q_5^*(X_3, Y3, \varepsilon)$ are terms of at least order three in (X_3, Y_3).

From $m_{11}(0) = \frac{2+2h-\sqrt{a}}{h\sqrt{a}} \neq 0$, we have $m_{11}(\varepsilon) \neq 0$ when ε_1 and ε_2 are small. Employing the change of variables

$$x = m_{11}(\varepsilon)^2 X_3, \quad y = -m_{11}(\varepsilon)^3 Y_3, \quad \tau = -\frac{1}{m_{11}(\varepsilon)} t,$$

we obtain the universal unfolding of system (4.20),

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu_1(\varepsilon) + \mu_2(\varepsilon) y + x^2 + xy + Q_6^*(x, y, \varepsilon), \end{cases}$$

where

$$\mu_1 = -m_{00}(\varepsilon) m_{11}(\varepsilon)^4,$$
$$\mu_2 = -m_{01}(\varepsilon) m_{11}(\varepsilon),$$

and $Q_6^*(x, y, \varepsilon)$ are terms of at least order three in (x, y).

$$\left| \frac{\partial(\mu_1, \mu_2)}{\partial(\varepsilon_1, \varepsilon_2)} \right|_{\varepsilon_1 = \varepsilon_2 = 0} \neq 0.$$

Therefore, from the Bogdanov-Takens bifurcation theorem [33, 34], system (4.14) undergoes a Bogdanov-Takens bifurcation of codimension two when the parameters ε_1 and ε_2 vary in a small neighborhood of $(0, 0)$. The local representations of the bifurcation curves in a small neighborhood of $(0, 0)$ are as follows:

(i) The saddle-node bifurcation curve $SN = \{(\varepsilon_1, \varepsilon_2) : \mu_1(\varepsilon_1, \varepsilon_2) = 0, \mu_2(\varepsilon_1, \varepsilon_2) \neq 0\}$;

(ii) the Hopf-bifurcation curve $H = \left\{(\varepsilon_1, \varepsilon_2) : \mu_1(\varepsilon_1, \varepsilon_2) < 0, \mu_2(\varepsilon_1, \varepsilon_2) = -\sqrt{-\mu_1(\varepsilon_1, \varepsilon_2)}\right\}$;

(iii) the homoclinic curve $HL = \left\{(\varepsilon_1, \varepsilon_2) : \mu_1(\varepsilon_1, \varepsilon_2) < 0, \mu_2(\varepsilon_1, \varepsilon_2) = -\frac{5}{7}\sqrt{-\mu_1(\varepsilon_1, \varepsilon_2)}\right\}$.

5. Numerical investigations and biological implications

For the simulations, we have used PhasePlane app in MATLAB. Among the parameters, $\bar{A}$ represents the anti-predator behavior of prey against the predator population, $\frac{1}{\alpha}$ is the quality of the additional food provided to the predator, ξ is the quantity of additional food provided to the predator, γ is the carrying capacity of the pest, and c is the natural mortality rate of predators.

5.1. Hopf bifurcation

Fix $A = 0.001$, $K = 15$, $\xi = 0.45$ in the model (1.3); we first set $\beta = 0.9$, $c = 0.0702$, $\alpha = 0.5$. As shown in Figure 5, for $\delta = 0.315$, the unique coexistence equilibrium E_1 is a stable focus, indicating a stable coexistence state.

Increasing δ to 0.31552 destabilizes E_1 and generates a stable limit cycle, corresponding to a supercritical Hopf bifurcation.

The system is pushed away from an unstable equilibrium, but then pulled toward a stable limit cycle. At this point, the sizes of populations will exhibit sustained and periodic fluctuations. In this case, coexistence is possible, but it is dynamic and oscillatory rather than static.

The stable limit cycle represents the cyclic variation of population size along a closed orbit. This implies that lower zooplankton mortality drives periodic population oscillations.

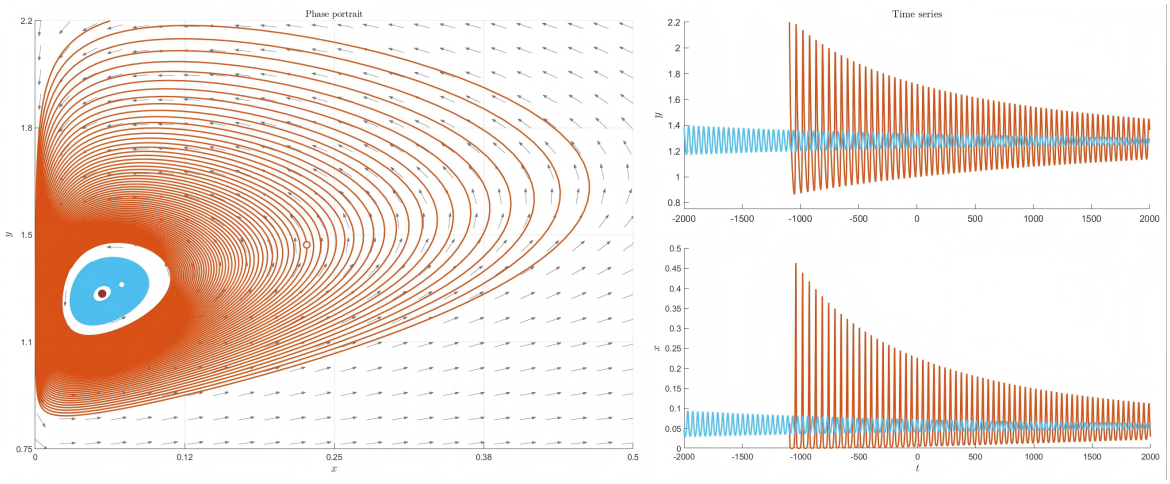


Figure 5. $\delta = 0.315$, there exists a unique stable focus, where the phytoplankton and zooplankton can coexist stably for a long time.

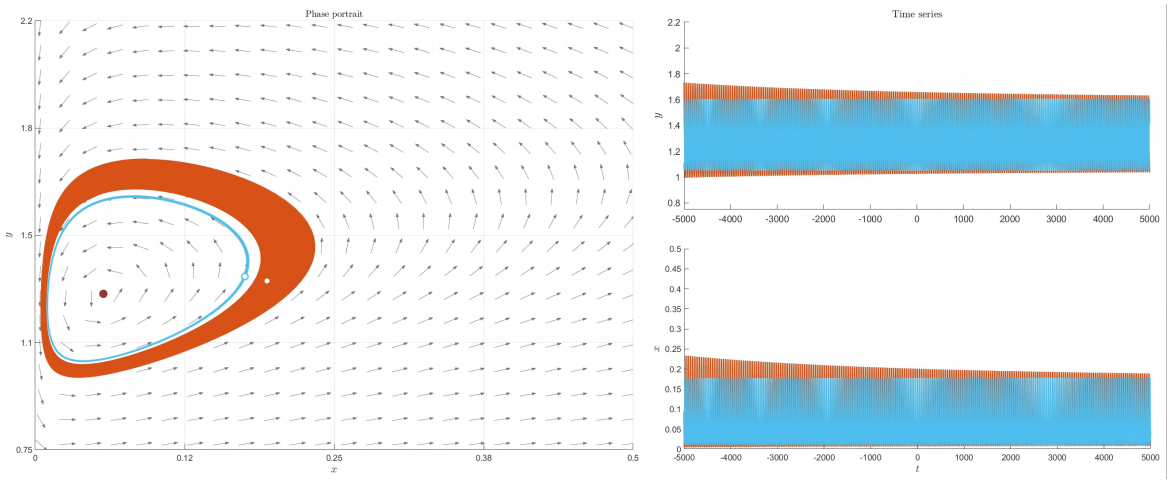


Figure 6. $\delta = 0.31552$, a stable limit cycle means that the population oscillates periodically and steadily.

In the case of $\beta = 0.9$, $c = 0.0702$, $\alpha = 0.5$, there exists a unique coexistence equilibrium E_1. If $\delta = 0.31552$, E_1 is an unstable focus. The low mortality rate of planktonic animals makes it difficult for the environment to support the population. Under such conditions, as shown in Figure 6, the coexistence equilibrium is unstable, and any state near the equilibrium will tend toward extinction over time.

This equilibrium is almost impossible to observe in reality. It is a state of "blade edge". Any small environmental fluctuations, random events, or subtle differences in initial conditions can cause the system to deviate from coexistence.

When δ increases to 0.31552, then the focus E_1 changes to unstable, and there is a stable limit cycle in its neighborhood. System (1.5) will undergo a **supercritical Hopf bifurcation**.

The stable limit cycle implies the existence of a critical population size threshold. If the population size remains below the threshold (within the loop), coexistence is possible. If the population size exceeds this threshold (outside the ring) for some reason, it may lead to catastrophic ecological consequences, such as population collapse or species extinction.

The emergence of limit cycle demonstrates the natural laws of population fluctuations. The limit cycle represents the periodic oscillation of a population, and a stable limit cycle represents the ability of an ecosystem to maintain long-term oscillatory equilibrium. If the limit cycle (periodic orbit) is unstable, small disturbances may cause the population to collapse or shift to other states (such as extinction).

From a biological perspective, the occurrence of Hopf bifurcation indicates the transition of the system from a stable equilibrium to sustained oscillatory dynamics. This phenomenon corresponds to periodic fluctuations in predator and prey populations, which are widely observed in natural ecosystems. In the context of the present model, such oscillations may arise when the intensity of additional food supply or the strength of anti-predator behavior crosses a critical threshold. For instance, excessive additional food may initially promote predator growth, but the resulting increase in intraspecific competition can destabilize the system, leading to persistent population cycles. Similarly, strong anti-predator behavior reduces predation efficiency, introducing a time-delay-like effect that can also induce oscillations.

These results suggest that both food supplementation and behavioral defense, while beneficial in isolation, may generate unintended dynamic consequences when their effects exceed certain levels.

5.2. Bogdanov-Takens bifurcation

Keep $A = 0.001$, $K = 15$, $\xi = 0.45$, then choose $\beta = 0.45$, and $c = 0.069358$. Let the bifurcation parameters δ and α vary within a small neighborhood of their critical values. The bifurcation curves **H**, **HL**, and **SN** obtained from the above calculations are codimension-1 bifurcation curves, and divide the small neighborhood of $(0, 0)$ in the (μ_1, μ_2)-plane into four sections. We will focus on the different topological phase portraits in the area I-II in Figure 7.

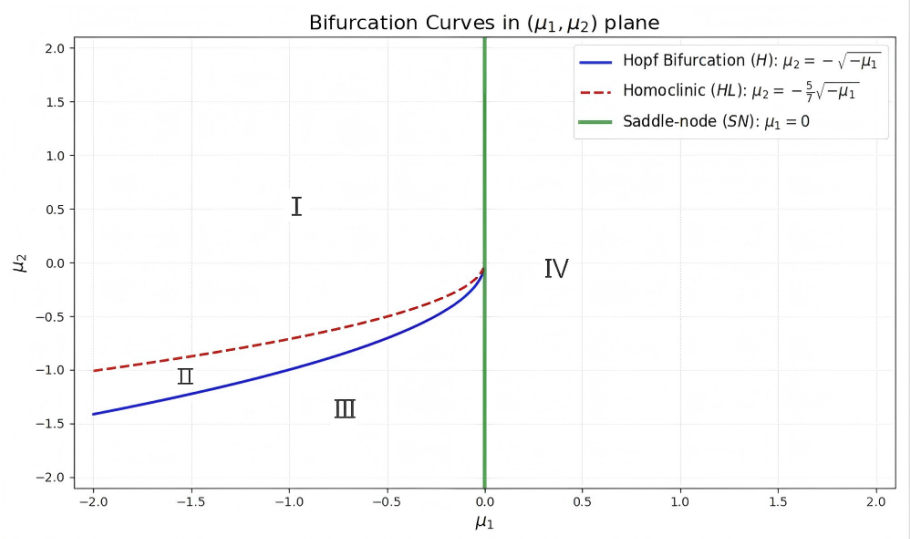


Figure 7. The Bogdanov-Takens bifurcation diagram.

As shown in Figure 8, when the parameter $\delta = 0.2701$, the equilibrium is a stable focus. When δ decreases to 0.27008 in Figure 9, the focus changes from stable to unstable, and a stable limit cycle emerges around it. The parameters will change from the area III to II in Figure 7.

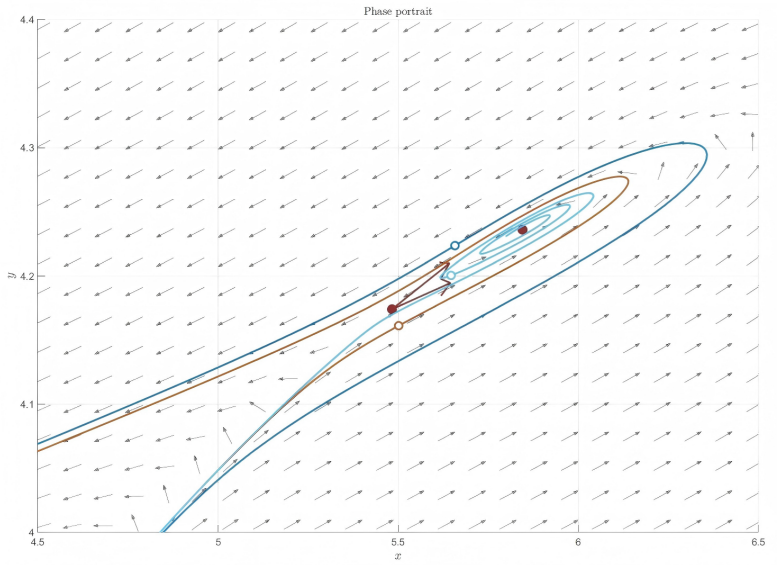


Figure 8. $\delta = 0.2701$, the equilibrium E is a stable focus.

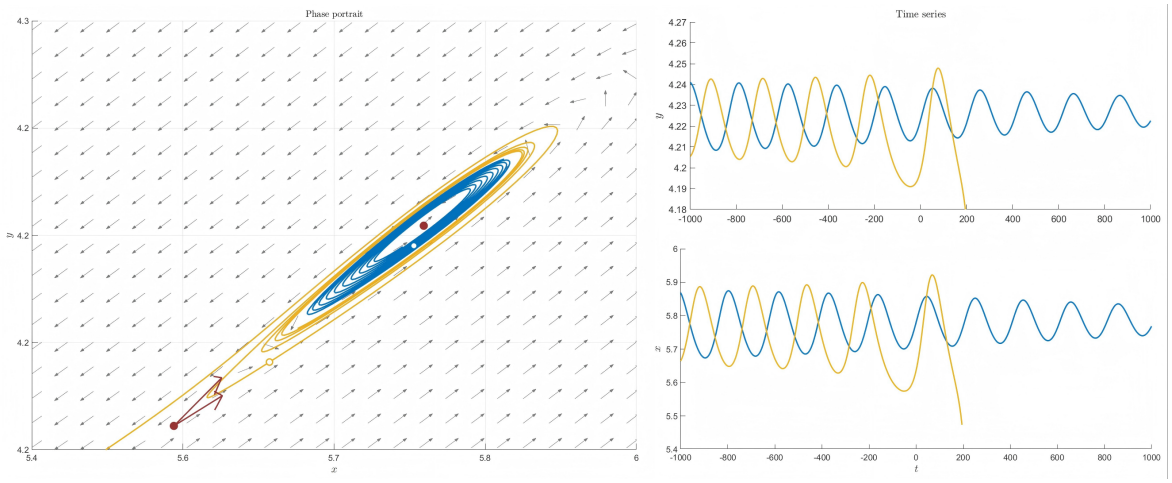


Figure 9. $\delta = 0.27008$, the focus changes from stable to unstable, and a stable limit cycle emerges around it.

When $\alpha = 0.212$ and $\delta = 0.27009$ in Figure 10, the parameters satisfy the condition of the curve **HL** in Figure 7. The system exhibits a homoclinic loop, as shown in the next figure, where the stable manifold and unstable manifold of a saddle point intersect.

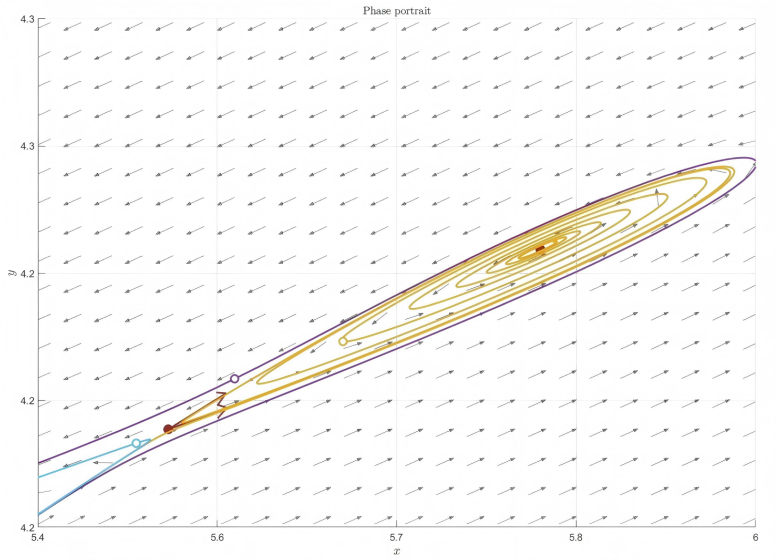


Figure 10. $\alpha = 0.212$ and $\delta = 0.27009$; the system exhibits a homoclinic orbit.

As δ decreases from 0.27009 to 0.27007 in Figure 11, the homoclinic orbit breaks, and an unstable focus is generated, as shown in the figure. The parameters change from the curve **HL** to the area I in Figure 7.

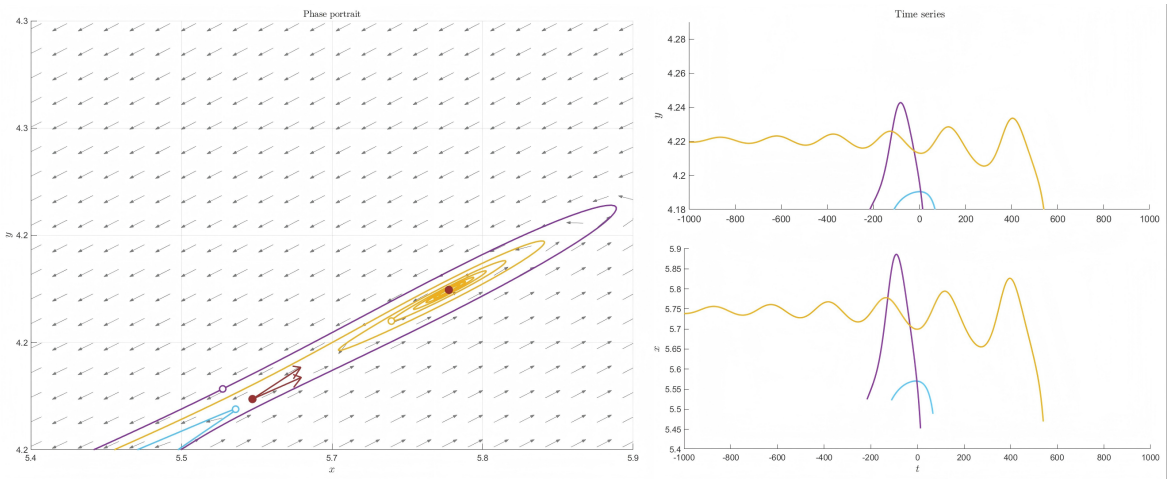


Figure 11. $\alpha = 0.212$ and $\delta = 0.27007$; the homoclinic orbit breaks, giving rise to an unstable focus F.

As α increases from 0.212 to 0.213, the parameters change from the curve **HL** to the area III in Figure 7. The homoclinic loop breaks, giving rise to a stable focus, as illustrated in Figure 12.

The Bogdanov-Takens bifurcation represents a higher-order degenerate bifurcation, indicating the presence of multiple coexisting dynamical behaviors in a small parameter region. Ecologically, this implies that the system is at a critical tipping point, where slight variations in environmental or biological parameters may lead to qualitatively different outcomes.

In predator-prey systems, such a bifurcation may correspond to transitions among stable coexistence, population oscillations, and even extinction scenarios. In particular, changes in additional

food input or the intensity of anti-predator behavior may shift the system across this critical threshold, resulting in sudden and potentially irreversible changes in population dynamics.

This highlights the importance of carefully regulating external interventions, such as supplementary feeding in biological control programs, as inappropriate parameter choices may destabilize the ecosystem rather than enhance its resilience.

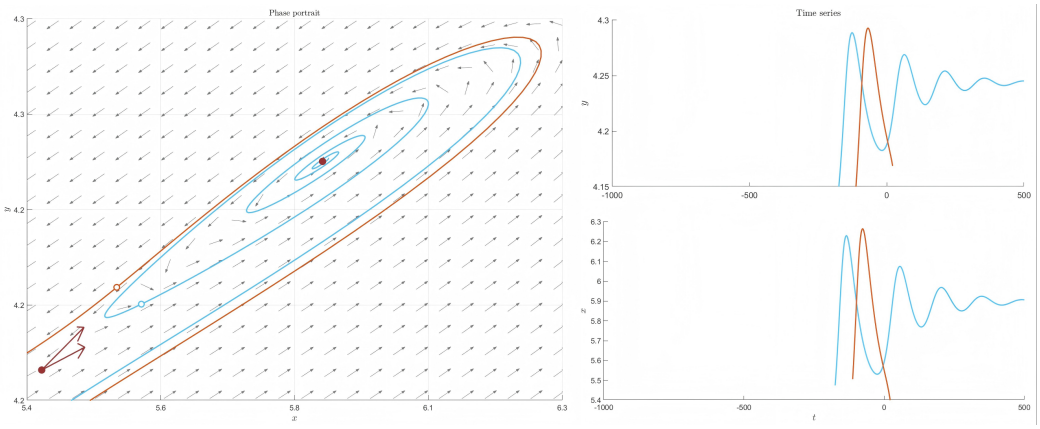


Figure 12. $\alpha = 0.213$ and $\delta = 0.27009$; the homoclinic orbit breaks, giving rise to a stable focus F.

5.3. Comparison with the case of no anti-predator behavior

In this section, we will compare the impact of anti-predator behavior on the system dynamics. Assume $A = 0$, and assume that the quality parameter of the supplementary food supplied satisfies $\frac{1}{\alpha} = 10$, i.e., $\alpha = 0.1$, the predator assimilation rate is $\beta = 0.35$, and the intraspecific competition coefficient is $c = 0.069$. We then investigate the impact of environmental factor-induced changes in the natural mortality rate of predators on the model. Phase portraits of the model are plotted for natural mortality rates of $\delta = 0.1200$ and $\delta = 0.1243$, as illustrated in Figures 13 and 14.

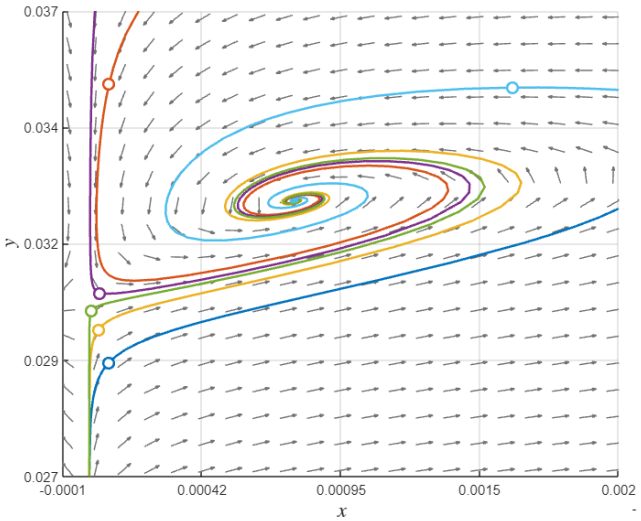


Figure 13. $\delta = 0.1200$.

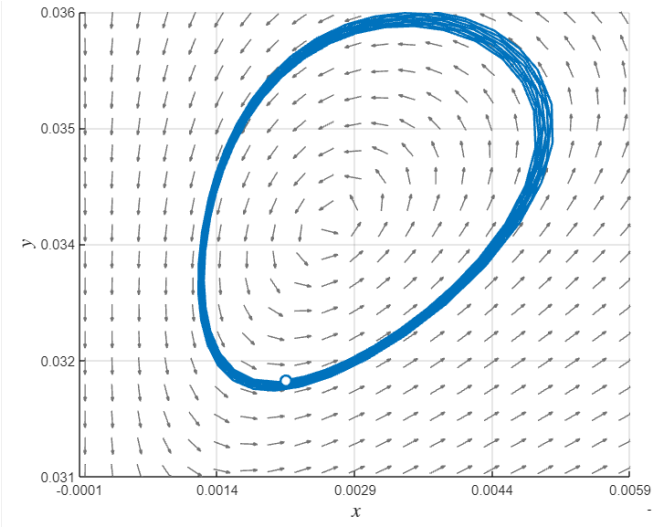


Figure 14. $\delta = 0.1243$.

It should be noted that when $\delta = 0.1200$, the system sustains a prey-extinction equilibrium and a coexistence equilibrium, which is a stable focus. When the predator mortality rate δ rises to $\delta = 0.1243$, the prey-extinction equilibrium point ceases, the coexistence equilibrium becomes unstable, and a stable limit cycle will occur (see Figures 13 and 14). The Hopf bifurcation induced by the predator mortality rate δ still exists.

With the above parameters unchanged, when $A = 0.45$, a stable coexistence equilibrium still exists at a predator mortality rate of 0.1200, but there is no equilibrium leading to prey extinction (see Figure 15). When the predator mortality rate δ reaches 0.1230, the stable coexistence equilibrium is disrupted, giving rise to periodic coexistence (see Figure 16). In pest control models, such a phenomenon indicates that the pest population cannot be completely eradicated.

It can thus be concluded that the anti-predation behavior of the prey significantly enhances its survival competitiveness, increases the survival pressure on the predator, and accelerates the decline of the predator population—both through anti-predation behavior and intraspecific competition. Consequently, the possibility of long-term coexistence between the two species is reduced.

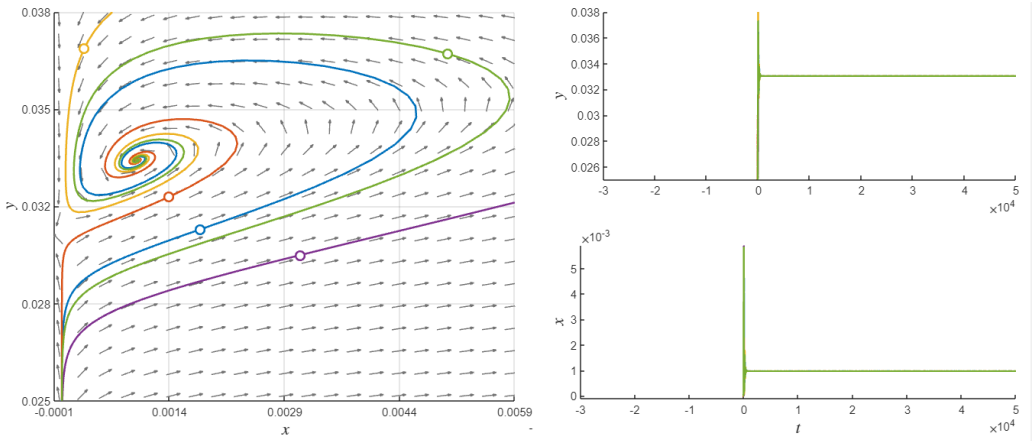


Figure 15. $\delta = 0.1200$.

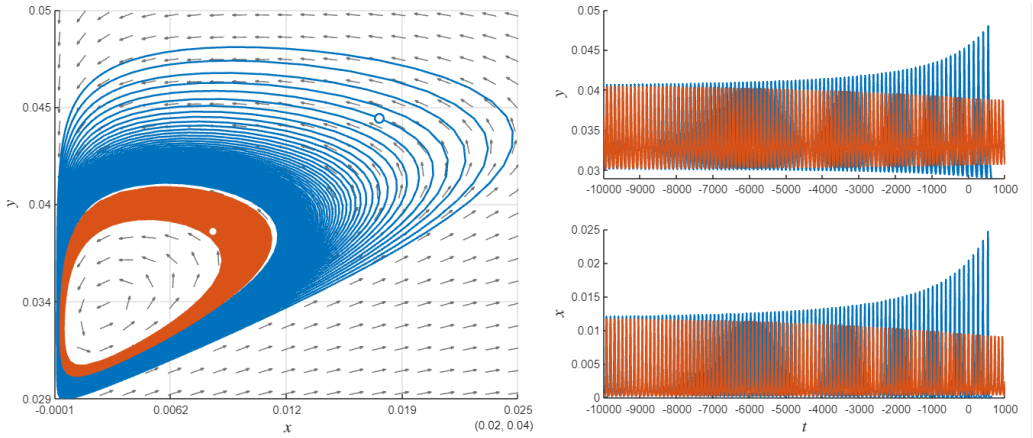


Figure 16. $\delta = 0.1230$.

6. Conclusions

In this paper, we have proposed and analyzed a predator-prey model with additional food inducing competition among predators incorporating anti-predator behavior.

We have discussed the existence and local stability of equilibria. It is shown that the system undergoes Hopf bifurcation when the bifurcation parameter passes through critical values, leading to the appearance of periodic solutions. Furthermore, we have investigated the Bogdanov-Takens bifurcation. By using the normal form theory and center manifold theorem, we have derived the approximate expressions for the saddle-node bifurcation curve, the Hopf bifurcation curve, and the homoclinic bifurcation curve.

Finally, numerical simulations are presented to verify the correctness of our theoretical analysis. The results indicate that the dynamical behaviors of the system are sensitive to the quality and quantity of additional food as well as the intensity of anti-predator behavior. The system exhibits complex phenomena including stable equilibria, limit cycles, and homoclinic loops. Our study suggests that providing additional food and considering anti-predator behavior are effective strategies for biological control, as they can significantly influence the stability and coexistence of the predator-prey system.

Author contributions

Feiyu Shi: Formal analysis, Conceptualization, Methodology, Writing–review and editing, Writing–original draft, Resources, Software; Yulong Li: Formal analysis, Writing–review and editing, Writing–original draft, Software; Deng Zhao: Supervision, Writing–review and editing, Funding acquisition, Validation, Project administration. All the authors have agreed and given their consent for the publication of this research paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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