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Research article

Pfaff reduction for a terminating bivariate hypergeometric polynomial

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Abstract: This paper studies a terminating (“modified”) Appell function

$$F_1^*(\alpha,\beta,\beta,2\beta;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^m y^n}{m!n!},$$

defined for integers $\alpha \geq 1$ and $\beta \leq -1$, together with the associated terminating Gauss function

$${}_2F_1^*(\alpha,\beta;2\beta;z)=\sum_{k=0}^{-\beta}\frac{(\alpha)_k(\beta)_k}{(2\beta)_k}\frac{z^k}{k!}.$$

The classical Pfaff-type reduction for the non-terminating Appell function

$$F_1(\alpha,\beta,\beta',\beta+\beta';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\beta+\beta')_{m+n}}\frac{x^m y^n}{m!n!}=\frac{1}{(1-y)^{\alpha}}\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \beta+\beta'\end{matrix};\frac{x-y}{1-y}\right),$$

is recalled as background. The paper argues that, for the modified terminating case with $\beta'=\beta\leq -1$ and $\gamma=2\beta$, the direct Pfaff reduction fails and must be replaced by a corrected identity that involves an explicit additional term $V^{(\alpha,\beta)}(x,y)$. A derivation of an explicit closed form for the correction term is given; it is first computed in low cases (notably $\alpha=1,2,3$), and then stated and proven in general by an induction on α. The final formula exhibits a structured binomial/Pascal-type pattern in its coefficients and yields several corollaries, including simplified boundary cases (for example $\beta=-1$) and an open extension problem for unequal negative integers $(\beta,\ \beta')$ is stated.

Keywords: hypergeometric series of one and two variables; summation formulae and transformations; terminating hypergeometric series of one and two variables; Pfaff transformation; binomial sums; integer sequences; Pascal’s triangle; differential equation; partial differential equation
Mathematics Subject Classification: 05A10, 05A19, 33C05, 33C50

1. Introduction

Hypergeometric functions in mathematical analysis, particularly in the context of two or more variables, provide deeper insights into complex phenomena, thereby enabling more sophisticated models and solutions. They serve as powerful tools to solve partial differential equations (PDEs), integrals, and boundary value problems that arise in fields such as quantum mechanics, statistical mechanics, and fluid dynamics. Additionally, these functions exhibit intricate relationships with other mathematical objects, such as Lie groups, algebraic varieties, and integrable systems, thus further expanding their applicability. The study of multivariable hypergeometric functions has gained significant attention due to their ability to provide analytical solutions in problems that involve multiple independent variables, thus offering a more generalized approach than their univariate counterparts. The expansion of hypergeometric functions into higher dimensions opens up new possibilities to solve complex systems, thus making them indispensable in contemporary mathematical and physical theories. For instance, and not limited to, see Y. Brychkov's work in [1].

As an example, hypergeometric functions of two variables are solutions of the following system of PDEs:

$$x(1-x)r + y(1-x)s + (\gamma - (\alpha+\beta+1)x)p - \beta yq - \alpha\beta z = 0, \tag{1.1}$$

$$y(1-y)t + x(1-y)s + (\gamma - (\alpha+\beta'+1)y)q - \beta' xq - \alpha\beta' z = 0, \tag{1.2}$$

in which x and y are independent variables, z is the unknown function of x and y, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, and $t = \frac{\partial^2 z}{\partial y^2}$. Monge's well-known notation for partial derivatives has been investigated by many writers.

Appell (1880) introduced this system of PDEs in connection with the hypergeometric series in two variables:

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!}, \mid x \mid < 1, \mid y \mid < 1, \tag{1.3}$$

which is a solution of (1.1).

This Appell function (1.3), of two variables x, y, can be found in any mathematical book that deals with hypergeometric bivariate series; for example (and not limited to), in Gradshteyn, Ryzhik, Tables of Integrals, Series and Products, page 1018 [2]. For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed either in terms of hypergeometric functions of a single variable or in terms of elementary functions as shown in page 1019 of the above mentioned reference:

$$F_1(\alpha,\beta,\beta',\beta+\beta';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\beta+\beta')_{m+n}} \frac{x^m y^n}{m!n!} = \frac{1}{(1-y)^{\alpha}}\, {}_2F_1\left(\begin{matrix}\alpha,\beta \\ \beta+\beta'\end{matrix}; \frac{x-y}{1-y}\right). \tag{1.4}$$

For more information about all the hypergeometric functions of two variables, we refer the reader to Kimura's book [3], starting from page 40.

In the case where the parameter α is a positive integer and the other parameters β, β' such that $\beta' = \beta$ are negative integers, we define the following:

- the modified hypergeometric function ${}_2F_1^*$ by [4–9]

$$ {}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};z\right)=\sum_{k=0}^{-\beta}\frac{(\alpha)_k(\beta)_k}{(2\beta)_k k!}z^k,\ \alpha\in\{1,2,3,...\},\ \beta\in\{...,-3,-2,-1\}, \tag{1.5}$$

- the modified Appell function F_1^* of two variables by

$$F_1^*(\alpha,\beta,\beta,2\beta;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^m y^n,\ \alpha\in\{1,2,3,...\},\ \beta\in\{...,-3,-2,-1\}. \tag{1.6}$$

These two definitions are well-defined, as both are two terminating series and double series, respectively, since the summations are only for $m,n=0,..,-\beta$, and the fact that 2β is also a negative integer does not make any harm.

The most important question that we may ask is as follows: does F_1^* fulfil the Eq (1.4)? (i.e., does the following equation still holds true?):

$$F_1^*(\alpha,\beta,\beta,2\beta;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^m y^n}{m!n!}=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right)? \tag{1.7}$$

We prove that (1.7) is no longer true, and, in the case where α is a positive integer and β is a positive integer, we explicitly give, the correction term $V^{(\alpha,\beta)}(x,y)$ such that

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^m y^n=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right)+V^{(\alpha,\beta)}(x,y). \tag{1.8}$$

First, let us prove that (1.7) is no longer true.

Remark 1.1. As we deal with terminating sums, we only impose that y is NOT equal to 1. The domain of validity of the corrected identity is reduced to the singular loci such as $y=1$, with an extension by continuity when $x=1$; since the sums terminate, convergence is not an issue.

Proposition 1.2. *For any positive integer α and any negative integer β, we have the following:*

$$F_1^*(\alpha,\beta,\beta,2\beta;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^m y^n}{m!n!}\neq\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right). \tag{1.9}$$

Proof. The left hand side of (1.7),

$$F_1^*(\alpha,\beta,\beta,2\beta;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^m y^n}{m!n!},$$

is a polynomial on x and y. The right hand side of (1.7) is given by $\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right)$. Let us denote by $A_k=\frac{(\alpha)_k(\beta)_k}{k!(2\beta)_k}$ and by $\beta=-n,\ n\in\mathbb{N}$; then,

$$\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right)=\frac{1}{(1-y)^\alpha}\left(1+A_1\frac{x-y}{1-y}+...+A_n\left(\frac{x-y}{1-y}\right)^n\right).$$

Then, the following hold:

- when $x = 1$, using the extension by continuity, we get

$$\frac{1}{(1-y)^{\alpha}}\, {}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};1\right) = \frac{1}{(1-y)^{\alpha}}\left(1 + A_1 + ... + A_n\right),$$

which is a rational function;
- when $x \neq 1$, we get

$$\begin{aligned}\frac{1}{(1-y)^{\alpha}}\, {}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};1\right) &= \frac{1}{(1-y)^{\alpha}}\left(1 + A_1\frac{x-y}{1-y} + ... + A_n\left(\frac{x-y}{1-y}\right)^n\right)\\ &= \frac{1}{(1-y)^{\alpha+n}}\left((1-y)^n + A_1(1-y)^{n-1}(x-y) + ... + A_n(x-y)^n\right),\end{aligned}$$

which is also a rational function and CANNOT be a polynomial. Indeed, the degree of y in the numerator is less than n, whereas the degree of y in the denominator is $\alpha + n > n,\ \alpha \geq 1$.

□

Therefore, we have to give the correction term $V^{(\alpha,\beta)}(x,y)$ such that

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^m y^n}{m!n!} = \frac{1}{(1-y)^{\alpha}}\, {}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right) + V^{(\alpha,\beta)}(x,y). \tag{1.10}$$

This paper will be organized as follows. First, we show how we found our first main result when α is equal to 1, which will play an important role for our second main result for α being equal to 2. Then, with this later result, we give our third main result when α is equal to 3. These three steps will lead us to deduce our final main result for any α in $\{1, 2, 3, ...\}$ by induction. The paper concludes with several lemmas and corollaries, linking thereby some double series to classical integer sequences.

2. First main result for α equal 1

When $\alpha = 1$, the correction term $V^{(1,\beta)}(x,y)$ is given in the following theorem.

Theorem 2.1. *We have the following result:*

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = \frac{1}{(1-y)}\, {}_2F_1^*\left(\begin{array}{c}1,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right) + V^{(1,\beta)}(x,y), \tag{2.1}$$

where $V^{(1,\beta)}(x,y) = -\dfrac{x^{-\beta}y^{-\beta+1}}{(1-y)}\, {}_2F_1^*\left(\begin{array}{c}1,\beta\\ 2\beta\end{array};\dfrac{x-y}{x(1-y)}\right)$*; then, we can write the following:*

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = \frac{1}{(1-y)}\, {}_2F_1^*\left(\begin{array}{c}1,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right) - \frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\, {}_2F_1^*\left(\begin{array}{c}1,\beta\\ 2\beta\end{array};\frac{x-y}{x(1-y)}\right).$$

Proof. Let us start from the following:

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = \frac{1}{(1-y)}\, {}_2F_1^*\left(\begin{array}{c}1,\beta\\ 2\beta\end{array};\frac{x-y}{1-y}\right) + V^{(1,\beta)}(x,y),$$

in such a way that we do NOT know $V^{(1,\beta)}(x,y)$. Then, the proof is carried out by first using the change of variables $x \leftarrow \frac{1}{x},\ y \leftarrow \frac{1}{y}$ in (2.1), which is written as follows

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{1}{x^m}\frac{1}{y^n} = \frac{1}{(1-\frac{1}{y})}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{\frac{1}{x}-\frac{1}{y}}{1-\frac{1}{y}}\right) + V^{(1,\beta)}\left(\frac{1}{x},\frac{1}{y}\right). \tag{2.2}$$

Second, the proof is carried out using the following three steps:

- the first step comes from the left hand side of (2.2) and it is given by

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{1}{x^m}\frac{1}{y^n} = (xy)^{\beta}\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n;$$

- the second step comes from the right hand side of (2.2) and it is given by

$$\frac{1}{(1-\frac{1}{y})}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{\frac{1}{x}-\frac{1}{y}}{1-\frac{1}{y}}\right) = -\frac{y}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right);$$

- if we combine the first and second step, then we get

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = \frac{x^{-\beta}y^{-\beta+1}}{(y-1)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right) + (xy)^{-\beta}V^{(1,\beta)}\left(\frac{1}{x},\frac{1}{y}\right),$$

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = \frac{1}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) + V^{(1,\beta)}(x,y),$$

  and an identification between these two equations give

$$V^{(1,\beta)}(x,y) = -\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right),$$

  provided that

$$(xy)^{-\beta}V^{(1,\beta)}\left(\frac{1}{x},\frac{1}{y}\right) = \frac{1}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right),$$

  which can be easily proven.

□

Remark 2.2. Using the change of variables $x \leftarrow \frac{1}{x},\ y \leftarrow \frac{1}{y}$ in $\frac{1}{(1-y)^{\alpha}}\ {}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)$ and by multiplying by $(xy)^{-\beta}$, we find $(xy)^{-\beta}\frac{(-1)^{\alpha}y^{\alpha}}{(1-y)^{\alpha}}\ {}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$. In the sequel, we adopt the following notation:

$$W^{(\alpha,\beta)}(x,y) = \frac{(-1)^{\alpha}x^{-\beta}y^{\alpha-\beta}}{(1-y)^{\alpha}}\ {}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right). \tag{2.3}$$

Corollary 2.3. *We have the following interesting particular cases:*

$$\bullet \sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^{m+n} = \sum_{m=0}^{-2\beta}x^m = \frac{x^{-2\beta+1}-1}{(x-1)},$$

$$\bullet \sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}y^{n} = \frac{(-2\beta+1)}{(-\beta+1)}\sum_{m=0}^{-\beta}y^m,$$

$$\bullet \sum_{n=0}^{-\beta}\frac{(\beta)_n}{(2\beta)_n}y^n = \frac{1}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{y}{y-1}\right) - \frac{y^{-2\beta+1}}{(1-y)^{-\beta+1}}\frac{(\beta)_{-\beta}}{(2\beta)_{-\beta}}.$$

Proof. • Immediate consequence of (2.1) with $x = y$.

• For $x = 0$, (2.1) can be written as follows

$$\sum_{n=0}^{-\beta}\binom{0+n}{0}\frac{(\beta)_0(\beta)_n}{(2\beta)_{0+n}}x^0y^n = \frac{1}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{0-y}{1-y}\right) + \frac{y^{-2\beta+1}}{(y-1)^{-\beta+1}}\frac{(\beta)_{-\beta}}{(2\beta)_{-\beta}}.$$

This is equivalent to

$$\sum_{n=0}^{-\beta}\frac{(\beta)_n}{(2\beta)_n}y^n = \frac{1}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{y}{y-1}\right) + \frac{y^{-2\beta+1}}{(y-1)^{-\beta+1}}\frac{(\beta)_{-\beta}}{(2\beta)_{-\beta}},$$

and this is given in [8], Lemma 1, Eq 2.5 therein.

□

Corollary 2.4. *Additionally, the previous theorem can be written as the following:*

$$\begin{aligned}\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(1)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^my^n}{m!n!} &= \frac{1}{(1-x)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-x}\right) + \frac{x^{-\beta}y^{1-\beta}}{(y-1)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\ &\quad + \frac{(\beta)_{-\beta}(x-y)^{1-2\beta}}{(2\beta)_{-\beta}(1-y)^{1-\beta}(x-1)^{1-\beta}}.\end{aligned}$$

Proof. Using the result given in [8], page 20149,

$${}_2F_1^*\left(\begin{matrix}1,-N\\-2N\end{matrix};z\right) = \frac{1}{1-z}\, {}_2F_1^*\left(\begin{matrix}1,-N\\-2N\end{matrix};\frac{z}{1-z}\right) + \frac{(-N)_N z^{2N+1}}{(-2N)_N(z-1)^N}.$$

If we substitute z by $\frac{x-y}{(1-y)}$, multiply by $\frac{1}{1-y}$, and put it in the above Theorem, we get the desired result. □

The first terms are written as follows:

$$\bullet 1 + \frac{1}{2}(x+y) + xy = -\frac{1}{2(x-1)}\left(2 + 1\frac{x-y}{(x-1)}\right) + \frac{xy^2}{2(y-1)}\left(2 + 1\frac{x-y}{x(1-y)}\right) + \frac{(x-y)^3}{2(y-1)^2(x-1)^2},$$

$$\bullet 1+\frac{1}{2}(x+y)+\frac{1}{6}(x^2+y^2)+\frac{2}{3}xy+\frac{1}{2}(xy^2+x^2y)+x^2y^2$$
$$=-\frac{1}{2(x-1)}\left(6+3\frac{x-y}{(x-1)}+1(\frac{x-y}{(x-1)})^2\right)+\frac{x^2y^3}{2(y-1)}\left(6+3\frac{x-y}{x(1-y)}+1(\frac{x-y}{x(1-y)})^2\right)-\frac{(x-y)^5}{6(y-1)^3(x-1)^3},$$

and the colored coefficients can be found in Pascal's triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & \ddots & & \ddots & & 20 & & 15 & & 6 & & 1;
\end{array}$$

Remark 2.5. Now, we are going to find $V^{(\alpha,\beta)}(x,y),\ \alpha=2,3,4,...$ such that

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(\alpha)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^my^n=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{array}{c}\alpha,\beta\\2\beta\end{array};\frac{x-y}{1-y}\right)+V^{(\alpha,\beta)}(x,y).$$

We will prove that the new expression $W^{(\alpha,\beta)}(x,y)$ given in (2.3) above will play an essential role in finding $V^{(\alpha,\beta)}(x,y),\ \alpha=2,3,4,....$

3. The case when α is equal to 2

The aim of this paragraph is to give the explicit expression of the correction term $V^{(2,\beta)}(x,y)$ such that the following equation holds

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(2)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^my^n=\frac{1}{(1-y)^2}\,{}_2F_1^*\left(\begin{array}{c}2,\beta\\2\beta\end{array};\frac{x-y}{1-y}\right)+V^{(2,\beta)}(x,y).$$

For this matter, we need a relation between the contiguous functions of F_1. In fact, we have the following proposition given in [3], p 54.

Proposition 3.1. *We have the following:*

$$\alpha F_1^*(\alpha+1,\beta,\beta',\gamma;x,y)=\alpha F_1^*(\alpha,\beta,\beta',\gamma;x,y)+x\frac{d}{dx}F_1^*(\alpha,\beta,\beta',\gamma;x,y)+y\frac{d}{dy}F_1^*(\alpha,\beta,\beta',\gamma;x,y).\quad(3.1)$$

The proof of this proposition is given in [3]. Indeed, let us denote by

$$F_1^*(\alpha+1,\beta,\beta',\gamma;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}A_{m,n}x^my^n,$$

where

$$A_{m,n} = \frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}};$$

clearly, we have

$$\begin{aligned} x\frac{\partial F_1^*}{\partial x} &= \sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta} mA_{m,n}x^m y^n, \\ y\frac{\partial F_1^*}{\partial y} &= \sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta} nA_{m,n}x^m y^n, \\ F_1^*(\alpha+1,\beta,\beta',\gamma;x,y) &= \sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta} \frac{\alpha+m+n}{\alpha}A_{m,n}x^m y^n, \end{aligned}$$

which gives (3.1).

In order to give the explicit expression of the correction term $V^{(2,\beta)}(x,y)$, we need the following steps:

- with $\alpha = 1$ and $\beta = \beta'$ in (3.1), we obtain

$$F_1^*(2,\beta,\beta,2\beta;x,y) = F_1^*(1,\beta,\beta,2\beta;x,y) + x\frac{d}{dx}F_1^*(1,\beta,\beta,2\beta;x,y) + y\frac{d}{dy}F_1(1,\beta,\beta,2\beta;x,y);$$

- now, we use the result of our first main theorem (2.1):

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = \frac{1}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) + V^{(1,\beta)}(x,y),$$

$$where \quad V^{(1,\beta)}(x,y) = -\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Remark 3.2. Let us denote by

$$(2F1)^{(\alpha,\beta)} = \frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right);$$

then,

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\binom{m+n}{m}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^m y^n = (2F1)^{(1,\beta)} + V^{(1,\beta)}(x,y).$$

- By combining these previous three steps, we obtain the following:

$$\begin{aligned} F_1(2,\beta,\beta,2\beta;x,y) &= (2F1)^{(1,\beta)}(x,y) + V^{(1,\beta)}(x,y) + x\frac{d}{dx}\left((2F1)^{(1,\beta)} + V^{(1,\beta)}(x,y)\right) \\ &\quad + y\frac{d}{dy}\left((2F1)^{(1,\beta)} + V^{(1,\beta)}(x,y)\right) \end{aligned}$$

$$
\begin{aligned}
&= (2F1)^{(1,\beta)}(x,y) - \frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&+x\frac{d}{dx}\left((2F1)^{(1,\beta)}(x,y) - \frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\right)\\
&+y\frac{d}{dy}\left((2F1)^{(1,\beta)}(x,y) - \frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\right)\\
&= (2F1)^{(1,\beta)}(x,y) - \frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&+x\Big[\frac{1}{2(y-1)^2}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{1-y}\right)+\frac{\beta x^{-\beta-1}y^{1-\beta}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{\beta x^{-\beta-2}y^{2-\beta}}{(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]+y\Big[\frac{1}{(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)\\
&+\frac{x-1}{2(1-y)^3}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{1-y}\right)-\frac{(1-\beta)\beta x^{-\beta}y^{-\beta}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{\beta x^{-\beta}y^{1-\beta}}{(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\frac{(x-1)\beta x^{-\beta-1}y^{1-\beta}}{2(1-y)^3}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big].
\end{aligned}
$$

It is easy to prove that the terms with $\frac{x-y}{1-y}$ give the following:

$$
\begin{aligned}
&(2F1)^{(1,\beta)}+\frac{x}{2(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{1-y}\right)+\frac{y}{(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)\\
&+\frac{y(x-1)}{2(1-y)^3}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{1-y}\right)=\frac{1}{(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)=(2F1)^{(2,\beta)};
\end{aligned}
$$

additionally, it is, also, easy to prove that the remaining terms, with $\frac{x-y}{x(1-y)}$, give the following

$$
\begin{aligned}
&\frac{x^{-\beta}y^{-\beta+1}}{(y-1)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+x\Big[-\frac{\beta x^{-\beta-1}y^{1-\beta}}{(y-1)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{\beta x^{-\beta-2}y^{2-\beta}}{(y-1)^2}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]+y\Big[\frac{(1-\beta)\beta x^{-\beta}y^{-\beta}}{(y-1)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{\beta x^{-\beta}y^{1-\beta}}{(y-1)^2}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+\frac{(x-1)\beta x^{-\beta-1}y^{1-\beta}}{2(y-1)^3}\ {}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]\\
&=-\frac{x^{-\beta}y^{-\beta+2}}{(y-1)^2}\ {}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+(2\beta-2)\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\ {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&=-W^{2,\beta}(x,y)-(2\beta-2)_1W^{1,\beta}(x,y)=V^{(2,\beta)}(x,y).
\end{aligned}
$$

Now, we can state our **second main result for** $\alpha = 2$

Theorem 3.3. *We have the following result:*

$$
\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(2)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^my^n=\frac{1}{(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)+V^{(2,\beta)}(x,y),
$$

$$\text{where}\quad V^{(2,\beta)}(x,y) = -W^{2,\beta}(x,y) - (2\beta-2)_1 W^{1,\beta}(x,y)$$
$$= -\frac{x^{-\beta}y^{-\beta+2}}{(y-1)^2}\, {}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right) - (2-2\beta)\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Corollary 3.4. *We have the following interesting particular cases:*

- $$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(2)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}m!n!}x^{m+n} = \left(\sum_{k=0}^{-2\beta+1}x^k\right)' = \left(\frac{x^{2-2\beta}-1}{x-1}\right)',$$
- $$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(2)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}m!n!}y^n = \frac{1-y^{-\beta+2}}{(1-y)^2}\, {}_2F_1^*(\begin{matrix}2,\beta\\2\beta\end{matrix};1) - (2-2\beta)\frac{y^{-\beta+1}}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};1\right)$$
$$= -\frac{2(1-2\beta)\left((\beta-1)y^{2-\beta}-(\beta-2)y^{1-\beta}-1\right)}{(2-\beta)(1-y)^2},$$
- $$\sum_{n=0}^{-\beta}\frac{(2)_n(\beta)_n}{(2\beta)_n n!}y^n = {}_2F_1^*(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{y}{y-1}) - \frac{y^{-2\beta+2}}{(1-y)^{-\beta+2}}\frac{(1-\beta)(\beta)_{-\beta}}{(2\beta)_{-\beta}} - (2-2\beta)\frac{y^{-2\beta+1}}{(1-y)^{-\beta+1}}\frac{(\beta)_{-\beta}}{(2\beta)_{-\beta}}.$$

Proof. • For $x=y$, the above Theorem can be written as:

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(2)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}m!n!}x^{m+n} = \frac{1}{(1-x)^2} - \frac{x^{2-2\beta}}{(x-1)^2} - (2-2\beta)\frac{x^{1-2\beta}}{(1-x)}$$
$$= \frac{1-x^{2-2\beta}+(1-x)(2\beta-2)x^{1-2\beta}}{(1-x)^2} = \left(\sum_{k=0}^{-2\beta+1}x^k\right)' = \left(\frac{x^{2-2\beta}-1}{x-1}\right)'.$$

- For $x=1$, the above Theorem can be written as

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(2)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}y^n = \frac{1-y^{-\beta+2}}{(1-y)^2}\, {}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};1\right) - (2-2\beta)\frac{y^{-\beta+1}}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};1\right);$$

  by taking into account that ${}_2F_1^*(\begin{matrix}1,\beta\\2\beta\end{matrix};1) = \frac{(-2\beta+1)}{(-\beta+1)}$, and ${}_2F_1^*(\begin{matrix}2,\beta\\2\beta\end{matrix};1) = \frac{2(-2\beta+1)}{(-\beta+2)}$, we get the desired result.
- For $x=0$, the above Theorem can be written as follows:

$$\sum_{n=0}^{-\beta}\frac{(2)_n(\beta)_n}{(2\beta)_n n!}y^n = \frac{1}{(1-y)^2}\, {}_2F_1^*(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{0-y}{1-y}) - \frac{x^{-\beta}y^{-\beta+2}}{(y-1)^2}\, {}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{0-y}{x(1-y)}\right)$$
$$-(2-2\beta)\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\, {}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{0-y}{x(1-y)}\right)$$
$$= {}_2F_1^*(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{y}{y-1}) - \frac{y^{-2\beta+2}}{(1-y)^{-\beta+2}}\frac{(1-\beta)(\beta)_{-\beta}}{(2\beta)_{-\beta}} + (2\beta-2)\frac{y^{-2\beta+1}}{(1-y)^{-\beta+1}}\frac{(\beta)_{-\beta}}{(2\beta)_{-\beta}}.$$

□

4. The case when α is equal 3

In this paragraph, we follow the same steps as done in the case when $\alpha = 2$ to obtain the explicit expression of the correction term $V^{(3,\beta)}(x,y)$ such that the following equality holds:

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(3)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}}x^my^n = \frac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)+V^{(3,\beta)}(x,y).$$

For this matter, we follow the following steps.

- With $\alpha = 2$ and $\beta = \beta'$ in (3.1), we obtain the following:

$$2F_1^*(3,\beta,\beta,2\beta;x,y) = 2F_1^*(2,\beta,\beta,2\beta;x,y) + x\frac{d}{dx}F_1^*(2,\beta,\beta,2\beta;x,y) + y\frac{d}{dy}F_1^*(2,\beta,\beta,2\beta;x,y).$$

- Now, we use the result of our second main theorem (3.3) as follows:

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(2)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^my^n = (2F1)^{(2,\beta)}(x,y) + V^{(2,\beta)}(x,y),$$

$$where\quad V^{(2,\beta)}(x,y) = -\frac{x^{-\beta}y^{-\beta+2}}{(y-1)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-(2-2\beta)\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

- By Combining these previous three steps, we obtain the following

$$\begin{aligned}
&2F_1^*(3,\beta,\beta,2\beta;x,y) = 2\,(2F1)^{(2,\beta)}(x,y) + 2V^{(2,\beta)}(x,y) + x\frac{d}{dx}\Big((2F1)^{(2,\beta)} + 2V^{(2,\beta)}(x,y)\Big)\\
&+y\frac{d}{dy}\Big((2F1)^{(2,\beta)} + V^{(2,\beta)}(x,y)\Big) = \frac{1}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{(1-y)}\right)\\
&-\frac{x^{-\beta}y^{2-\beta}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-(2-2\beta)\frac{x^{-\beta}y^{1-\beta}}{1-y}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&+x\Bigg(\frac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{(1-y)}\right)+\frac{x^{-\beta-1}\beta y^{2-\beta}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{x^{-2-\beta}y^{3-\beta}}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)-\frac{2\beta(\beta-1)x^{-1-\beta}y^{1-\beta}}{1-y}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&+\frac{(\beta-1)x^{-2-\beta}y^{2-\beta}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Bigg)+y\Bigg(\frac{2}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{(1-y)}\right)\\
&+\frac{x-1}{(1-y)^4}\,{}_2F_1^*\left(\begin{matrix}3,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{(1-y)}\right)-\frac{x^{-\beta}y^{1-\beta}(2-\beta)}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{2\,x^{-\beta}y^{2-\beta}}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\frac{(x-1)x^{-1-\beta}y^{2-\beta}}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&-\frac{2(\beta-1)^2x^{-\beta}y^{-\beta}}{1-y}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+\frac{2(\beta-1)x^{-\beta}y^{1-\beta}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\\
&+\frac{(\beta-1)(x-1)x^{-1-\beta}y^{1-\beta}}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}2,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Bigg).
\end{aligned}$$

- The terms with $\dfrac{x-y}{1-y}$ give the following:

$$\frac{1}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{(1-y)}\right)+x\frac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{(1-y)}\right)$$

$$+y\Big(\frac{2}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{(1-y)}\right)+\frac{x-1}{(1-y)^4}\,{}_2F_1^*(\begin{matrix}3,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{(1-y)}\Big),$$

which is, exactly, $\dfrac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\dfrac{x-y}{(1-y)}\right)$.

- The remaining terms with $\dfrac{x-y}{x(1-y)}$ give the following:

$$=-\frac{x^{-\beta}y^{-\beta+3}}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+\frac{(2\beta-3)_1}{1!}\frac{x^{-\beta}y^{-\beta+2}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\frac{(2\beta-3)_2}{2!}\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$=-\binom{3-2\beta}{0}\frac{x^{-\beta}y^{-\beta+3}}{(y-1)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\binom{3-2\beta}{1}\frac{x^{-\beta}y^{-\beta+2}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\binom{3-2\beta}{2}\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Now, we can state our **third main result for $\alpha = 3$**

Theorem 4.1. *We have the following result:*

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(3)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^my^n=\frac{1}{(1-y)^3}\,{}_2F_1^*(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{1-y})-\binom{3-2\beta}{0}\frac{x^{-\beta}y^{-\beta+3}}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\binom{3-2\beta}{1}\frac{x^{-\beta}y^{-\beta+2}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\binom{3-2\beta}{2}\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Corollary 4.2. *We have the following interesting particular cases:*

- $\displaystyle\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(3)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}m!n!}x^{m+n}=\frac{1}{2}\left(\sum_{k=0}^{-2\beta+2}x^k\right)'=\frac{1}{2}\left(\frac{x^{2-2\beta}-1}{x-1}\right)''$,
- $\displaystyle\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(3)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}m!n!}y^n=\frac{1-y^{-\beta+3}}{(1-y)^3}\,{}_2F_1^*(\begin{matrix}3,\beta\\2\beta\end{matrix};1)+\frac{(2\beta-3)}{1!}\frac{y^{-\beta+2}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};1\right)$

$$-\frac{(2\beta-3)(2\beta-2)}{2!}\frac{y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};1\right)$$
$$=\frac{(1-2\beta)(3-2\beta)}{(3-\beta)(2-\beta)}\frac{\left(y^{3-\beta}(2-\beta)(1-\beta)-2y^{2-\beta}(3-\beta)(1-\beta)+y^{1-\beta}(3-\beta)(2-\beta)-2\right)}{(-1+y)^3};$$

- $$\sum_{n=0}^{-\beta}\frac{(3)_n(\beta)_n}{(2\beta)_n n!}y^n = {}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{y}{y-1}\right)-\left(\frac{y^2}{(1-y)^2}+\frac{(3-2\beta)y}{(1-y)}+\frac{(3-2\beta)(2-2\beta)y}{2!(1-y)}\right)\frac{(-1)^\beta y^{1-2\beta}(\beta)_{-\beta}}{(1-y)^{1-\beta}(2\beta)_{-\beta}}.$$

Proof. • For $x=y$, the above Theorem can be written as follows:

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(3)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}m!n!}x^{m+n} = \frac{1}{(1-x)^2}-\frac{x^{2-2\beta}}{(x-1)^2}+(2\beta-2)\frac{x^{1-2\beta}}{(1-x)}$$
$$=\frac{1-x^{2-2\beta}+(1-x)(2\beta-2)x^{1-2\beta}}{(1-x)^2}=\left(\sum_{k=0}^{-2\beta+1}x^k\right)'=\left(\frac{x^{2-2\beta}-1}{x-1}\right)'.$$

- For $x=1$, the above Theorem can be written as follows:

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(3)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}y^n = \frac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};1\right)=-\frac{y^{-\beta+2}}{(y-1)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};1\right)$$
$$+(2\beta-2)\frac{y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};1\right);$$

by taking into account that ${}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};1\right)=\frac{(-2\beta+1)}{(-\beta+1)}$, and ${}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};1\right)=\frac{2(-2\beta+1)}{(-\beta+2)}$, we get the desired result.

- For $x=0$, the above Theorem can be written as follows:

$$\frac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{0-y}{1-y}\right)-\frac{x^{-\beta}y^{-\beta+2}}{(y-1)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{0-y}{x(1-y)}\right)+(2\beta-2)\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{0-y}{x(1-y)}\right)$$
$$={}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{y}{y-1}\right)-\frac{y^{-2\beta+2}}{(1-y)^{-\beta+2}}\frac{(1-\beta)(\beta)_{-\beta}}{(2\beta)_{-\beta}}+(2\beta-2)\frac{y^{-2\beta+1}}{(1-y)^{-\beta+1}}\frac{(\beta)_{-\beta}}{(2\beta)_{-\beta}}.$$

□

5. The case when α is any integer greater or equal 1

Let us summarize. In paragraphs 2, 3, and 4, we found the following:

- for $\alpha=1$, we found

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(1)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^m y^n=\frac{1}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)-\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right);$$

- for $\alpha=2$, we found

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(2)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^m y^n=\frac{1}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)-\frac{x^{-\beta}y^{-\beta+2}}{(y-1)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\binom{2-2\beta}{1}\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right);$$

- for $\alpha = 3$, we found

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(3)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^m y^n = \frac{1}{(1-y)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \frac{x^{-\beta}y^{-\beta+3}}{(y-1)^3}\,{}_2F_1^*\left(\begin{matrix}3,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\binom{3-2\beta}{1}\frac{x^{-\beta}y^{-\beta+2}}{(1-y)^2}\,{}_2F_1^*\left(\begin{matrix}2,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right) - \binom{3-2\beta}{2}\frac{x^{-\beta}y^{-\beta+1}}{(1-y)}\,{}_2F_1^*\left(\begin{matrix}1,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Then, by induction, we can deduce the following theorem.

Theorem 5.1. *For any α integer greater or equal* 1*, we have the following result*

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^m y^n$$
$$= \frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \sum_{k=1}^{\alpha}\frac{\binom{\alpha-2\beta}{\alpha-k}x^{-\beta}y^{-\beta+k}}{(1-y)^k}\,{}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right). \tag{5.1}$$

Proof. The proof will be done by induction on $\alpha \geq 1$.
- The case $\alpha = 1$ is done in paragraph 1.
- We suppose that (5.1) is true for any $k,\ 1 \leq k \leq \alpha$ and we prove it for $\alpha + 1$.
- We use (1.3), and we take $\beta' = \beta$ and $\gamma = 2\beta$, we obtain the following:

$$\alpha F_1^*(\alpha+1,\beta,\beta,2\beta;x,y) = \alpha F_1^*(\alpha,\beta,\beta,2\beta;x,y) + x\frac{d}{dx}F_1^*(\alpha,\beta,\beta,2\beta;x,y) + y\frac{d}{dy}F_1^*(\alpha,\beta,\beta,2\beta;x,y).$$

- These two later facts give the following

$$\alpha F_1^*(\alpha+1,\beta,\beta,2\beta;x,y) = \alpha\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \alpha\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^k}\,{}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$+x\frac{d}{dx}\left(\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^k}\,{}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\right)$$
$$+y\frac{d}{dy}\left(\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^k}\,{}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\right),$$

which is equivalent to the following:

$$\alpha F_1^*(\alpha+1,\beta,\beta,2\beta;x,y) = \alpha\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \alpha\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^k}\,{}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$+x\left(\frac{\alpha}{2(1-y)^{\alpha+1}}\,{}_2F_1^*\left(\begin{matrix}\alpha+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{1-y}\right) - \sum_{k=1}^{\alpha}\left[\beta\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k}}{(1-y)^k}\,{}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)\right.\right.$$
$$\left.\left.+\frac{k}{2}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-2}y^{-\beta+k+1}}{(1-y)^{k+1}}\,{}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\right]\right)$$
$$+y\left(\frac{\alpha}{(1-y)^{\alpha+1}}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right) - \frac{\alpha(x-1)}{2(1-y)^{\alpha+1}}\,{}_2F_1^*\left(\begin{matrix}\alpha+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{1-y}\right)\right.$$

$$-\sum_{k=1}^{\alpha}\Big[(k-b)\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k-1}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+k\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k-1}}{(1-y)^{k+1}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\frac{k(x-1)}{2}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k-1}}{(1-y)^{k+2}}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]\Big).$$

Lemma 5.2. *The terms which are outside the three summations give the following:*

$$\frac{\alpha}{(1-y)^{\alpha+1}}\ {}_2F_1^*(\begin{matrix}\alpha+1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}).$$

Proof. The terms which are outside the three summations give the following:

$$\frac{\alpha}{(1-y)^{\alpha+1}}\ {}_2F_1^*(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y})+\frac{\alpha(x-y)}{2(1-y)^{\alpha+2}}\ {}_2F_1^*\left(\begin{matrix}\alpha+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right),$$

which is, exactly,

$$\frac{\alpha}{(1-y)^{\alpha+1}}\ {}_2F_1^*(\begin{matrix}\alpha+1,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}).$$

□

Lemma 5.3. *The terms with the three summations give the following:*

$$-\alpha\sum_{k=1}^{\alpha+1}\binom{\alpha+1-2\beta}{\alpha+1-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Proof.

$$-\alpha\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+x\Big(-\sum_{k=1}^{\alpha}\Big[\beta\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$+\frac{k}{2}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-2}y^{-\beta+k+1}}{(1-y)^{k+1}}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]\Big)$$
$$+y\Big(-\sum_{k=1}^{\alpha}\Big[(k-b)\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k-1}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+k\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k-1}}{(1-y)^{k+1}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\frac{k(x-1)}{2}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k-1}}{(1-y)^{k+2}}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]\Big).$$

This is equal to

$$-\alpha\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\sum_{k=1}^{\alpha}\Big[\beta\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$+\frac{k}{2}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k+1}}{(1-y)^{k+1}}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Big]$$
$$-\sum_{k=1}^{\alpha}\Big[(k-b)\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^{k}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+k\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^{k+1}}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$

$$-\frac{k(x-1)}{2}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k}}{(1-y)^{k+2}}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Bigg].$$

This is equal to $\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^k}\Big[$

$$-\alpha\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\beta\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\frac{ky}{2(1-y)}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-(k-\beta)\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)-\frac{k}{1-y}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)+\frac{k(x-1)}{2x(1-y)^2}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right)\Bigg].$$

$$-\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta}y^{-\beta+k}\left((\alpha-2\beta)(1-y)+k\right)}{(1-y)^{k+1}}(k+1-y)\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right)$$
$$-\sum_{k=1}^{\alpha}\binom{\alpha-2\beta}{\alpha-k}\frac{x^{-\beta-1}y^{-\beta+k+1}}{(1-y)^{k+2}}\frac{(x-y)}{2}\ {}_2F_1^*\left(\begin{matrix}k+1,\beta+1\\2\beta+1\end{matrix};\frac{x-y}{x(1-y)}\right).$$

This gives, exactly,

$$-\alpha\sum_{k=1}^{\alpha+1}\binom{\alpha+1-2\beta}{\alpha+1-k}\frac{x^{-\beta}y^{-\beta+k}}{(1-y)^k}\ {}_2F_1^*\left(\begin{matrix}k,\beta\\2\beta\end{matrix};\frac{x-y}{x(1-y)}\right).$$

□

□

We conclude this paper by giving the following corollary, which comes from the boundary conditions.

Corollary 5.4. *For any α integer greater than or equal to* 1*, we have the following result:*

$$xy\sum_{k=1}^{\alpha}\binom{\alpha+2}{\alpha-k}\left(\frac{y}{(1-y)}\right)^k\left(1+\frac{k}{2}\frac{x-y}{x(1-y)}\right)=\frac{1}{(1-y)^\alpha}\left(1+\frac{\alpha}{2}\frac{x-y}{(1-y)}\right)-\left(1+\frac{\alpha(x+y)}{2}+\frac{\alpha(\alpha+1)}{2}xy\right).\tag{5.2}$$

Proof. Let us take (5.1) with $\beta=-1$; then we obtain the following:

$$\sum_{m=0}^{1}\sum_{n=0}^{1}\frac{(\alpha)_{m+n}(-1)_m(-1)_n}{(-2)_{m+n}n!m!}x^my^n=\frac{1}{(1-y)^\alpha}\ {}_2F_1^*\left(\begin{matrix}\alpha,-1\\-2\end{matrix};\frac{x-y}{1-y}\right)-\sum_{k=1}^{\alpha}\frac{\binom{\alpha+2}{\alpha-k}x^1y^{1+k}}{(1-y)^k}\ {}_2F_1^*\left(\begin{matrix}k,-1\\-2\end{matrix};\frac{x-y}{x(1-y)}\right).$$

Taking

- $\sum_{m=0}^{1}\sum_{n=0}^{1}\frac{(\alpha)_{m+n}(-1)_m(-1)_n}{(-2)_{m+n}n!m!}x^my^n=1+\frac{\alpha(x+y)}{2}+\frac{\alpha(\alpha+1)}{2}xy,$
- ${}_2F_1^*\left(\begin{matrix}\gamma,-1\\-2\end{matrix};Z\right)=1+\frac{\gamma}{2}Z$

into account, we get the desired result. □

Conclusion and open problem. In the case when α is a positive integer and β is a negative integer, the equation

$$F_1^*(\alpha,\beta,\beta,2\beta;x,y)=\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(2\beta)_{m+n}}\frac{x^my^n}{m!n!}=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right),$$

is no longer true, and we give an explicit expression of the correction term $V^{(\alpha,\beta)}(x,y)$ such that

$$\sum_{m=0}^{-\beta}\sum_{n=0}^{-\beta}(\alpha)_{m+n}\frac{(\beta)_m(\beta)_n}{(2\beta)_{m+n}n!m!}x^my^n=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\2\beta\end{matrix};\frac{x-y}{1-y}\right)+V^{(\alpha,\beta)}(x,y). \tag{5.3}$$

This result is very important because of the following reasons: first, it has four variables α, β, x, and y; second, all researchers know how difficult is to manipulate double summations with two variables; and, third, it has many applications in various applied mathematics .

This leads us to state the following open problem. In the case when α is a positive integer and β_1, β_2 are negative integers, the equation

$$F_1^*(\alpha,\beta,\beta',\beta+\beta';x,y)=\sum_{m=0}^{-\beta'}\sum_{n=0}^{-\beta}\frac{(\alpha)_{m+n}(\beta')_m(\beta)_n}{(\beta+\beta')_{m+n}}\frac{x^my^n}{m!n!}=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\\beta+\beta'\end{matrix};\frac{x-y}{1-y}\right),$$

is no longer true. Find an explicit expression of the correction term $V^{(\alpha,\beta,\beta')}(x,y)$ such that the following result holds true:

$$\sum_{m=0}^{-\beta'}\sum_{n=0}^{-\beta}(\alpha)_{m+n}\frac{(\beta')_m(\beta)_n}{(\beta+\beta')_{m+n}n!m!}x^my^n=\frac{1}{(1-y)^\alpha}\,{}_2F_1^*\left(\begin{matrix}\alpha,\beta\\\beta+\beta'\end{matrix};\frac{x-y}{1-y}\right)+V^{(\alpha,\beta,\beta')}(x,y). \tag{5.4}$$

Use of Generative-AI tools declaration

The author declares the Artificial Intelligence (AI) tools are not used in the creation of this article.

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Conflict of interest

The author declares that there are no conflicts of interest.

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Appendix 1.

Here we provide here the Maple instructions for our main theorem in order to reassure the reader of the accuracy and correctness of our result:

> restart;
> $G1 := (b, x, y) \to sum(sum((pochhammer(1, n+m).(pochhammer(b, m) * pochhammer(b, n)/(pochhammer(2b, n+m) * factorial(m) * factorial(n)))) * x^m * y^n, n = 0..-b), m = 0..-b);$
> $F11 := (b, x, y) \to (1-y)^{-1} * hypergeom([1, b], [2*b], (x-y)/(1-y));$
> $F12 := (b, x, y) \to x^{-b} * y^{-b+1} * (y-1)^{-1} * hypergeom([1, b], [2*b], (x-y)/(x*(1-y)));$
> $simplify(G1(-3, x, y) - F11(-3, x, y) - F12(-3, x, y));$

The answer will be 0.

Appendix 2.

Here we provide here the Maple instructions for our Corollary(2.4) in order to reassure the reader of the accuracy and correctness of our result:

> restart;
> $G1 := (b, x, y) \to sum(sum((pochhammer(1, n+m).(pochhammer(b, m) * pochhammer(b, n)/(pochhammer(2b, n+m) * factorial(m) * factorial(n)))) * x^m * y^n, n = 0..-b), m = 0..-b);$
> $K1x := (b, x, y) \to 1/(1-x) * hypergeom([1, b], [2*b], (x-y)/(x-1));$
> $F12 := (b, x, y) \to x^{-b} * y^{-b+1} * (y-1)^{-1} * hypergeom([1, b], [2*b], (x-y)/(x*(1-y)));$

```
> K1xy := (b, x, y) → pochhammer(b, −b)/pochhammer(2b, −b) ∗ (x − y)^(1−2∗b)/(1 − y)^(−b+1)/(x − 1)^(1−b);
> simplify(G1(−3, x, y) − K1x(−3, x, y) − K1xy(−3, x, y) − F12(−3, x, y));
```

The answer will be 0.

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