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Research article

Advancing Boole's rule inequalities through fractal analysis and neural network modeling

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Abstract: The goal of this study is to improve some known results related to Boole's type inequalities that use five points (Boole's rule). We first prove an important auxiliary identity connected to these inequalities. Using this auxiliary identity, we develop new Boole's type inequalities by applying a differentiable convex function within the setting of local fractional calculus. In this work, we study different types of functions, including convex, bounded, and Lipschitz functions over fractal sets. Additionally, a feedforward Artificial Neural Network (ANN) was used to approximate the left-hand side and right-hand side of fractal Boole-type inequalities. The model takes two input values and passes them through hidden layers to produce two outputs as predictions. This type of ANN is widely used because it can learn complex relationships from data without needing any fixed formulas. In this work, we apply an ANN model for the first time to predict the bounds of inequalities in fractal dimensions, which is an important outcome of our study. The ReLU activation function was applied to help the model learn nonlinear patterns, while training was carried out using the Mean Squared Error (MSE) loss and the Adam optimizer for stable and efficient learning. The network was trained for 500 epochs, and its performance was evaluated using loss curves. Finally, 3-dimensional surface plots were created to compare the predicted and actual inequality values. We also present examples and applications to show the usefulness of our main results.

Keywords: bounds of Boole's type; fractal sets; generalized convex function; special functions; special means; Artificial Neural Network (ANN)
Mathematics Subject Classification: 26D15, 26A51, 68T07, 68T30

1. Introduction and preliminaries

Since convexity theory and integral inequalities are essential to mathematical evaluation especially in dealing with the solution of differential equations and the enhancement of computational approaches applied to real-world issues, they have been investigated widely. Due to their numerous applications in probability theory, physics, engineering and economics, integral inequalities are highly beneficial. Various types of inequalities use in the estimation of functions behaviour and in the understanding of mathematical connections. For instance, the solutions of differential and integral equations are estimated using Gronwall's inequality. In the fields of statistics and probability, Chebyshev's inequality is frequently used to compare weighted averages. Because it clarifies how convex functions react in relation to assumptions or integrals Jensen's inequality has relevance in convex analysis. Hölder's inequality is helpful in functional analysis and expands on a notion of absolute values. Numerical techniques employ the results of Milne and the Simpson's inequalities to enhance integration and approximation result. The Hermite-Hadamard inequality will be essential for business optimization and model development and it facilitates when calculating the mean value of convex functions. In addition to helping us learn more about mathematical functions, these inequalities are valuable tools for addressing practical issues. Those who have an interest in further investigations can consult publications like [1–4], because they offer comprehensive descriptions of the way these inequalities can be utilized in many fields. In computational integration, the Simpson's formulas are commonly and frequently used. Additionally, they are very helpful in determining the error in quadrature ways.

Definition 1.1. *[5] Assume that the function* $\Phi : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be convex if, for all* $\mathrm{r}_1, \mathrm{r}_2 \in \mathrm{I}$ *and* $\mathrm{s} \in [0,1]$*, the following inequality holds:*

$$\Phi(\mathrm{s}\mathrm{r}_1 + (1-\mathrm{s})\mathrm{r}_2) \leq \mathrm{s}\Phi(\mathrm{r}_1) + (1-\mathrm{s})\Phi(\mathrm{r}_2).$$

Theorem 1.1. *[6] Assume that* $\Phi : \mathrm{I} \to \mathbb{R}$ *is a convex function. Then the following celebrated result, known as the Hermite-Hadamard inequality, holds:*

$$\Phi\left(\frac{\mathrm{r}_1 + \mathrm{r}_2}{2}\right) \leq \frac{1}{\mathrm{r}_2 - \mathrm{r}_1} \int_{\mathrm{r}_1}^{\mathrm{r}_2} \Phi(\varkappa) d\varkappa \leq \frac{\Phi(\mathrm{r}_1) + \Phi(\mathrm{r}_2)}{2}, \tag{1.1}$$

for every $\mathrm{r}_1, \mathrm{r}_2 \in \mathrm{I}$ *such that* $\mathrm{r}_1 < \mathrm{r}_2$.

Several generalizations of the double inequality (1.1), involving significant convexities and functional characteristics have been proposed and are systematically presented in the monographs [6, 7]. The fractal theory, being a newly developed hot spot, has played a comparatively crucial role in numerous fields that are involved in theoretical mathematics and applied mathematics. It has been frequently used in solving local fractional wave equations, non-differentiable problems, nonlinear local fractional PDEs and so on. Occasionally, numerous scholars have investigated important integral inequalities in the field of fractals.

One of the most commonly used methods in this area is Simpson's rule. It is often used to estimate the error that comes from using quadrature formulas.

Theorem 1.2. *[8] Let* $\Phi : [\mathrm{r}_1, \mathrm{r}_2] \to \mathrm{R}$, *is a four time continuously differentiable mapping on* $(\mathrm{r}_1, \mathrm{r}_2)$ *and let* $\|\Phi^{(4)}\|_\infty = sup_{\varkappa \in (\mathrm{r}_1, \mathrm{r}_2)} |\Phi^{(4)}(\varkappa)| < \infty$*, then the below mentioned inequality holds:*

$$\left|\frac{1}{6}\left[\Phi(\mathrm{r}_1) + 4\Phi\left(\frac{\mathrm{r}_1 + \mathrm{r}_2}{2}\right) + \Phi(\mathrm{r}_2)\right] - \frac{1}{\mathrm{r}_2 - \mathrm{r}_1} \int_{\mathrm{r}_1}^{\mathrm{r}_2} \Phi(\varkappa) d\varkappa\right|$$

$$\leq \frac{1}{2880}\|\Phi^{(4)}\|_{\infty}(\mathrm{r}_2-\mathrm{r}_1)^4.$$

In mathematics, Boole's rule gives a formula to approximate the integral of a function Φ over the interval $[r_1, r_2]$ as shown below:

$$\int_{\mathrm{r}_1}^{\mathrm{r}_2}\Phi(\varkappa)d\varkappa \approx \frac{2h}{45}\Big[7\Phi(\varkappa_o)+32\Phi(\varkappa_1)+12\Phi(\varkappa_2)+32\Phi(\varkappa_3)+7\Phi(\varkappa_4)\Big], \tag{1.2}$$

where $h = \frac{\mathrm{r}_2-\mathrm{r}_1}{4}$ and $\varkappa_i$ values are equally spaced point on the interval. This formula uses different weights for the function values at these points to get a better estimate of the integral.

Recent work by Kim et al. [10, 11] on heterogeneous Stirling numbers and Bell polynomials provides useful tools for combinatorics and network applications. These studies show the importance of modern analytical and computational methods in mathematics. Inspired by these ideas, this paper studies new Boole's rule type inequalities using fractal analysis and validates the results with artificial neural networks. For instance, Butt et al. [9, 12] worked on fractal-fractional forms of Bullen-type inequalities and on a parameterized fractal-fractional version of Ostrowski- and Simpson-type inequalities. They also showed how these results can be used in fractal domains. Akkurt et al. [13], Sarikaya and Budak [14], and Tomar et al. [15] introduced some generalized Ostrowski-type integral inequalities based on generalized moments, generalized convexity, and generalized s-convexity, respectively. Erden and Sarikaya [16] developed generalized Pompeiu-type inequalities and discussed their applications using local fractional integrals. Erden et al. [17] and Sarikaya et al. [18] worked on generalized Gröss-type inequalities and presented related applications in fractal spaces. Iftikhar et al. [19, 20] established Newton-type integral inequalities concerning generalized convexity and generalized harmonic convex functions, respectively. Krnic and Vukovic [21] obtained multidimensional Hilbert-type inequalities involving local fractional continuous kernels and weight functions. Budak et al. [22] and Almutairi and Kilicman [23] proved Hermite-Hadamard-type inequalities based on generalized convexity and generalized s-convexity, respectively. Butt et al. [24] discussed Hadamard-Mercer-type inequalities and their applications in fractal settings. Furthermore, Luo et al. [25] and Almutairi and Kilicman [26] established Fejer-Hermite-Hadamard-type inequalities involving generalized h-convexity and (h-m)-convexity, respectively. More studies related to local fractional calculus can be found in [27–31] and the references cited therein. In addition, applications concerning fractal spaces are discussed in several works such as [32–36] and their related bibliographies.

We start by recalling Yang's definition of the local fractional integral and derivative, which is based on the set $\mathbb{R}^{\xi_0}$ of real line numbers. One of the important recent advances in fractional calculus is Yang's local fractional calculus, defined as follows:

For $0 < \xi_0 \leq 1$, the elements belonging to the ξ_0-type set are specified below:

$\mathbb{Z}^{\xi_o}$: The ξ_0-type set of integer numbers is used to define the corresponding set as follows: $\Big\{0^{\xi_o}, \pm 1^{\xi_o}, \pm 2^{\xi_o}, ..., \pm n^{\xi_o}, ...\Big\}$.

$\mathbb{Q}^{\xi_o}$: The ξ_o-type set of rational numbers is used to define the corresponding set as follows: $\Big\{\mathrm{m}^{\xi_o} = \left(\frac{\varkappa}{\varkappa_1}\right)^{\xi_o} : \varkappa, \varkappa_1 \in \mathbb{Z}, \varkappa_1 \neq 0\Big\}$.

$\mathbb{J}^{\xi_o}$: The ξ_o-type set of irrational numbers is used to define the corresponding set as follows: $\Big\{\mathrm{m}^{\xi_o} \neq \left(\frac{\varkappa}{\varkappa_1}\right)^{\xi_o} : \varkappa, \varkappa_1 \in \mathbb{Z}, \varkappa_1 \neq 0\Big\}$.

$\mathbb{R}^{\xi_o}$: The ξ_o-type set of real numbers is used to define the corresponding set as follows:
$\mathbb{R}^{\xi_o}=\mathbb{Q}^{\xi_o} \cup \mathbb{J}^{\xi_o}$.
The following below mentioned operations hold for $\varkappa^{\xi_o}, {\varkappa_1}^{\xi_o}$ and ${s_o}^{\xi_o}$ belong to the set R^{ξ_o} of real line numbers:
(i) $\varkappa^{\xi_o} + {\varkappa_1}^{\xi_o}$ and $\varkappa^{\xi_o}{\varkappa_1}^{\xi_o}$ belong to the set R^{ξ_o};
(ii)$\varkappa^{\xi_o} + {\varkappa_1}^{\xi_o} = {\varkappa_1}^{\xi_o} + \varkappa^{\xi_o} = (\varkappa + \varkappa_1)^{\xi_o} = (\varkappa + \varkappa_1)^{\xi_o}$;
(iii)$\varkappa^{\xi_o} + ({\varkappa_1}^{\xi_o} + s^{\xi_o}) = (\varkappa + \varkappa_1)^{\xi_o} + s^{\xi_o}$;
(iv)$\varkappa^{\xi_o}{\varkappa_1}^{\xi_o} = {\varkappa_1}^{\xi_o}\varkappa^{\xi_o} = (\varkappa\varkappa_1)^{\xi_o} = (\varkappa_1\varkappa)^{\xi_o}$;
(v)$\varkappa^{\xi_o}({\varkappa_1}^{\xi_o}s^{\xi_o}) = (\varkappa^{\xi_o}{\varkappa_1}^{\xi_o})s^{\xi_o}$;
(vi)$\varkappa^{\xi_o}({\varkappa_1}^{\xi_o} + s^{\xi_o}) = \varkappa^{\xi_o}{\varkappa_1}^{\xi_o} + \varkappa^{\xi_o}s^{\xi_o}$;
(vii)$\varkappa^{\xi_o} + 0^{\xi_o} = 0^{\xi_o} + \varkappa^{\xi_o} = \varkappa^{\xi_o}$ and $\varkappa^{\xi_o}1^{\xi_o} = 1^{\xi_o}\varkappa^{\xi_o} = \varkappa^{\xi_o}$;
(viii)$\varkappa^{\xi_o} = {\varkappa_1}^{\xi_o}$ just in case $\varkappa = \varkappa_1, \varkappa, \varkappa_1 \subset R$;
(ix)$\varkappa^{\xi_o} \geq \varkappa_1$ just in case $\varkappa \geq \varkappa_1, \varkappa, \varkappa_1 \subset R$;
(x)$(\varkappa^{\xi_o})^{\mathfrak{q}}=(\varkappa^{\mathfrak{q}})^{\xi_o}$, $\mathfrak{q} > 0$ and $\varkappa > 0$.

In [36], Yang expanded the notion of convexity on fractal sets as follows:

Definition 1.2. *[36] If for all* $r_1, r_2 \in I$ *and* $s \in [0, 1]$ *the inequality*

$$\Phi(sr_1 + (1 - s)r_2) \leq s^{\xi_o}\Phi(r_1) + (1 - s)^{\xi_o}\Phi(r_2)$$

holds, then the function $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}^{\xi_o}$ *is called a generalized convex function on* I.

Theorem 1.3. *[36] Suppose* $\Phi(\varkappa) \in {}_{r_1}I_{r_2}^{\xi_o}$ *is a generalized convex function defined on* $[r_1, r_2]$, *where* $r_1 < r_2$. *Then*

$$\Phi\Big(\frac{r_1 + r_2}{2}\Big) \leq \frac{\Gamma(1 + \xi_o)}{(r_2 - r_1)^{\xi_o}}\ {}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa) \leq \frac{\Phi(r_1) + \Phi(r_2)}{2^{\xi_o}}. \tag{1.3}$$

Remark 1.1. *In* (1.3), *if we choose* $\xi_o = 1$, *then it reduces to the classical Hermite–Hadamard inequality defined in* (1.1).

Definition 1.3. *[36] Consider a mapping* $\Phi : \mathbb{R} \to \mathbb{R}^{\xi_o}$ *given by* $\varkappa_1 \mapsto \Phi(\varkappa_1)$, *which is not differentiable. The mapping* Φ *is said to be local fractional continuous at the point* $\varkappa$ *if, for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|\Phi(\varkappa_1) - \Phi(\varkappa_0)| < \varepsilon^{\xi_o}. \tag{1.4}$$

This condition holds whenever $|\varkappa_1 - \varkappa_o| < \delta$, *where* $\Phi, \delta \in \mathbb{R}$. *If the function* $\Phi(\varkappa_1)$ *is locally fractional continuous on the interval* (r_1, r_2), *we denote this by* $\Phi(\varkappa) \in C_{\xi_o}(r_1, r_2)$.

Definition 1.4. *[36] The local fractional derivative of order* ξ_o *of the function* $\Phi(\varkappa)$ *at the point* $\varkappa = \varkappa_o$ *is defined by*

$$\Phi^{(\xi_o)}(\varkappa_o) = {}_{\varkappa_o}D_{\varkappa}^{\xi_o}\Phi(\varkappa) = \frac{d^{(\xi_o)}\Phi(\varkappa)}{d(\varkappa^{\xi_o})}\Big|_{\varkappa=\varkappa_o} = \lim_{\varkappa\to\varkappa_o}\frac{\Delta^{\xi_o}(\Phi(\varkappa) - \Phi(\varkappa_o))}{(\varkappa - \varkappa_o)^{\xi_o}}, \tag{1.5}$$

where $\Delta^{\xi_o}(\Phi(\varkappa) - \Phi(\varkappa_o)) \cong \Gamma(1 + \xi_o)(\Phi(\varkappa) - \Phi(\varkappa_o))$, *and* Γ *denotes the well-known gamma function.*

Let $\Phi^{(\xi_o)}(\varkappa_o) = D_\varkappa^{\xi_o}\Phi(\varkappa)$. *If for any* $\varkappa \in I \subseteq \mathbb{R}$, *the higher-order local fractional derivatives*

$$\Phi^{(k+1)\xi_o}(\varkappa) = \overbrace{D_\varkappa^{\xi_o}\cdots D_\varkappa^{\xi_o}\Phi(\varkappa)}^{k+1\ times}$$

exist, then we denote $\Phi \in D_{(k+1)\xi_o}(I)$ *for* $k = 0, 1, 2, 3, \ldots$.

Definition 1.5. *[36] The local fractional integral of order* ξ_o *of the function* Φ *over the interval* $[r_1, r_2]$ *is defined as:*

$$_{r_1}I_{r_2}^{(\xi_o)}\Phi = \frac{1}{\Gamma(\xi_o+1)}\int_{r_1}^{r_2}\Phi(\varkappa)(d\varkappa)^{\xi_o} := \frac{1}{\Gamma(\xi_o+1)}\lim_{\Delta\eta\to 0}\sum_{j=0}^{N-1}\Phi(\eta_j)(\Delta\eta_j)^{\xi_o}, \tag{1.6}$$

provided that this limit exists.

From this definition, it follows that

$$_{r_1}I_{r_2}^{(\xi_o)}\Phi = 0 \quad \text{if} \quad r_1 = r_2,$$

and

$$_{r_1}I_{r_2}^{(\xi_o)}\Phi = -_{r_2}I_{r_1}^{(\xi_o)}\Phi \quad \text{if} \quad r_1 < r_2.$$

Lemma 1.1. *We review some key properties of local fractional calculus (see [36]) that play a crucial role in deriving our main results. These properties include:*
(1) (The local fractional derivative of $\varkappa^{s\xi_o}$ *in the local fractional sense):*

$$\frac{d^{\xi_o}\varkappa^{s\xi_o}}{d\varkappa^{\xi_o}} = \frac{\Gamma(1+s\xi_o)}{\Gamma(1+(s-1)\xi_o)}\varkappa^{(s-1)\xi_o}. \tag{1.7}$$

(2) (Anti-differentiation is local fractional integration):
Suppose that $\Phi(\varkappa) = {\Phi_o}^{(\xi_o)}(\varkappa) \in C_{\xi_o}[r_1, r_2]$. *Then, we have*

$$_{r_1}I_{r_2}^{(\xi_o)}\Phi(\varkappa) = \Phi_o(r_2) - \Phi_o(r_1). \tag{1.8}$$

(3) (Fractional integration by parts applied locally):
Suppose that $\Phi(\varkappa), \Phi_o(\varkappa) \in D_{\xi_o}[r_1, r_2]$ *and* $\Phi^{(\xi_o)}(\varkappa), {\Phi_o}^{(\xi_o)}(\varkappa) \in C_{\xi_o}[r_1, r_2]$. *Then we have*

$$_{r_1}I_{r_2}^{(\xi_o)}\Phi(\varkappa){\Phi_o}^{(\xi_o)}(\varkappa) = \Phi(\varkappa)\Phi_o(\varkappa)|_{r_1}^{r_2} - {}_{r_1}I_{r_2}^{(\xi_o)}\Phi^{(\xi_o)}(\varkappa)\Phi_o(\varkappa). \tag{1.9}$$

Lemma 1.2. *The definite local fractional integral forms of* $\varkappa^{s\xi_o}$ *are as given below:*

$$\frac{1}{\Gamma(1+\xi_o)}\int_{r_1}^{r_2}\varkappa^{s\xi_o}(d\varkappa)^{\xi_o} = \frac{\Gamma(1+s\xi_o)}{\Gamma(1+(s+1)\xi_o)}({r_2}^{(s+1)\xi_o} - {r_1}^{(s+1)\xi_o}), s \in R. \tag{1.10}$$

As presented in [36], Yang established the generalized Hölder's inequality for the fractal domain as follows.

Lemma 1.3. *[36] Let Φ and Φ_1 be functions defined on the interval* $[\mathrm{r}_1, \mathrm{r}_2]$ *with* $\Phi, \Phi_1 \in C_{\xi_o}[\mathrm{r}_1, \mathrm{r}_2]$. *Assume that* $|\Phi|^p$ *and* $|\Phi_1|^q$ *are locally fractional integrable on* $[\mathrm{r}_1, \mathrm{r}_2]$, *where* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, the following integral inequality holds:*

$$\frac{1}{\Gamma(1+\xi_o)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\Phi(\varkappa)\Phi_1(\varkappa)|(d\varkappa)^{\xi_o}$$
$$\leq \left(\frac{1}{\Gamma(1+\xi_o)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\Phi(\varkappa)|^{\mathrm{p}}(d\varkappa)^{\xi_o}\right)^{\frac{1}{\mathrm{p}}} \left(\frac{1}{\Gamma(1+\xi_o)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\Phi_1(\varkappa)|^{\mathrm{q_o}}(d\varkappa)^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}. \tag{1.11}$$

In 2021, Luo et al. introduced a revised version of Hölder-Yang's integral inequality within the framework of fractal sets, providing an improved error estimate compared to the original. It can be stated as follows.

In [37], Yu et al. gave the generalized power mean inequality in fractal domain as follows.

Lemma 1.4. *[37] Assume* $q \geq 1$ *and let* Φ, Φ_1 *be functions defined on the interval* $[\mathrm{r}_1, \mathrm{r}_2]$ *with* $\Phi, \Phi_1 \in C_{\xi_o}[\mathrm{r}_1, \mathrm{r}_2]$. *If the functions* $|\Phi|$ *and* $|\Phi| \cdot |\Phi_1|^q$ *are locally fractional integrable on* $[\mathrm{r}_1, \mathrm{r}_2]$, *then the following generalized power-mean integral inequality holds:*

$$\frac{1}{\Gamma(1+\xi_o)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\Phi(\varkappa)\Phi_1(\varkappa)|(d\varkappa)^{\xi_o}$$
$$\leq \left(\frac{1}{\Gamma(1+\xi_o)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\Phi(\varkappa)|(d\varkappa)^{\xi_o}\right)^{1-\frac{1}{\mathrm{q_o}}} \left(\frac{1}{\Gamma(1+\xi_o)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\Phi(\varkappa)||\Phi_1(\varkappa)|^{\mathrm{q_o}}(d\varkappa)^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}. \tag{1.12}$$

Recently Hussain et al. [38] introduced some new Boole's type inequalities through conformable fractional integrals. Mateen et al. [39] developed new version of Boole's formula type estimates in multiplicative sense. Anwar et al. [40] established some new Boole-type inequalities via modified convex functions. Moreover, the proposed framework is closely related to recent studies on degenerate special functions and probabilistic concepts. This indicates that the present results can be extended to stochastic models and inequalities involving expectations of random variables. T. Kim and D. S. Kim et al. [41] introduced identities involving expectations of various random variables and degenerate Stirling numbers, showing fundamental relation between probabilistic concepts and degenerate combinatorial frameworks. In another related work, D. S. Kim and T. Kim et al. [42] studied degenerate harmonic and hyperharmonic numbers and introduced several associated polynomials and numbers.

This research generalizes existing Boole-type inequalities previously established in classical, fractional and multiplicative settings by introducing and developing them within a fractal framework. It is not presently available in the literature a comprehensive treatment of inequalities of Boole's-type to problems involving convexity theory. This gap in the literature obstructs both the advancement of the theoretical framework and the practical applicability of these inequalities to real-world problems, including the resolution of special means.

The structure of this document is described in the paragraph that follows: The problem statement, literature review and introduction are presented in Section 1. The derivation of Boole's-type inequalities in the context of local fractional calculus is covered in detail in Section 2. Boole's-type inequalities for bounded and Lipschitzian functions in fractal spaces are covered in Sections 3 and 4, respectively. We provide examples and graphical representations of the results in Section 5. In

Section 6, we present the numerical analysis of our new results. We also show how these results relate to modern fields such as artificial intelligence and neural networks. Several uses of the results are examined in Section 7. The work is finally concluded in Section 8, which also offers ideas for future research directions.

2. Main results

First, in the context of local fractional calculus, we develop an significant identity for generalized local fractional integrals. A number of Boole-type inequalities for generalized convex functions are then derived.

Lemma 2.1. *Suppose* $\Phi : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}^{\xi_o}$ *is a mapping with* $\Phi \in \mathrm{D}_{\xi_o}(\mathrm{I}^o)$ *and let* $\mathrm{r}_1, \mathrm{r}_2 \in \mathrm{I}^o$*, where* $\mathrm{r}_1 < \mathrm{r}_2$*. If* $\Phi^{(\xi_o)}$ *belongs to* $\mathrm{C}_{\xi_o}[\mathrm{r}_1, \mathrm{r}_2]$*, then the following equality is satisfied:*

$$\begin{aligned}
&\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(\mathrm{r}_1) + 32^{\xi_o}\Phi\left(\frac{3\mathrm{r}_1 + \mathrm{r}_2}{4}\right) + 12^{\xi_o}\Phi\left(\frac{\mathrm{r}_1 + \mathrm{r}_2}{2}\right) + 32^{\xi_o}\Phi\left(\frac{\mathrm{r}_1 + 3\mathrm{r}_2}{4}\right) + 7^{\xi_o}\Phi(\mathrm{r}_2)\right] \\
&- \frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}}\,{}_{\mathrm{r}_1}\mathrm{I}_{\mathrm{r}_2}^{\xi_o}\Phi(\varkappa) \\
&= \frac{(\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}}{2^{\xi_o}} \\
&\times\left[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(\mathrm{s} - \frac{7}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1) - \Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1 + (1-\mathrm{s})\mathrm{r}_2)](d\mathrm{s})^{\xi_o}\right. \\
&+ \frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\mathrm{s} - \frac{39}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1) - \Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1 + (1-\mathrm{s})\mathrm{r}_2)](d\mathrm{s})^{\xi_o} \\
&+ \frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\mathrm{s} - \frac{51}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1) - \Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1 + (1-\mathrm{s})\mathrm{r}_2)](d\mathrm{s})^{\xi_o} \\
&\left.+ \frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left(\mathrm{s} - \frac{83}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1) - \Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1 + (1-\mathrm{s})\mathrm{r}_2)](d\mathrm{s})^{\xi_o}\right].
\end{aligned} \tag{2.1}$$

Proof. By using the rules of local fractional integration by parts, we have

$$\begin{aligned}
\mathrm{I}_1 &= (\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(\mathrm{s} - \frac{7}{90}\right)^{\xi_o}\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1)(d\mathrm{s})^{\xi_o} \\
&= \left[\left(\mathrm{s} - \frac{7}{90}\right)^{\xi_o}\Phi(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1)\Big|_0^{\frac{1}{4}}\right. \\
&\left.- \frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Gamma(1+\xi_o)\Phi(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1)(d\mathrm{s})^{\xi_o}\right], \\
&= \left[\left(\frac{31}{180}\right)^{\xi_o}\Phi\left(\frac{3\mathrm{r}_1 + \mathrm{r}_2}{4}\right) + \left(\frac{7}{90}\right)^{\xi_o}\Phi(\mathrm{r}_1)\right] \\
&- \frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Gamma(1+\xi_o)\Phi(\mathrm{s}\mathrm{r}_2 + (1-\mathrm{s})\mathrm{r}_1)(d\mathrm{s})^{\xi_o},
\end{aligned} \tag{2.2}$$

$$
\begin{aligned}
I_2 &= (r_2 - r_1)^{\xi_o} \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(s - \frac{39}{90}\right)^{\xi_o} \Phi^{(\xi_o)}(sr_2 + (1-s)r_1)(ds)^{\xi_o} \\
&= \Bigg[\left(s - \frac{39}{90}\right)^{\xi_o} \Phi(sr_2 + (1-s)r_1)\Big|_{\frac{1}{4}}^{\frac{1}{2}} \\
&- \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{1}{4}}^{\frac{1}{2}} \Gamma(1+\xi_o)\Phi(sr_2 + (1-s)r_1)(ds)^{\xi_o}\Bigg], \\
&= \left[\left(\frac{12}{180}\right)^{\xi_o} \Phi\left(\frac{r_1 + r_2}{2}\right) + \left(\frac{33}{180}\right)^{\xi_o} \Phi\left(\frac{3r_1 + r_2}{4}\right)\right] \\
&- \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{1}{4}}^{\frac{1}{2}} \Gamma(1+\xi_o)\Phi(sr_2 + (1-s)r_1)(ds)^{\xi_o},
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
I_3 &= (r_2 - r_1)^{\xi_o} \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{1}{2}}^{\frac{3}{4}} \left(s - \frac{51}{90}\right)^{\xi_o} \Phi^{(\xi_o)}(sr_2 + (1-s)r_1)(ds)^{\xi_o} \\
&= \Bigg[\left(s - \frac{51}{90}\right)^{\xi_o} \Phi(sr_2 + (1-s)r_1)\Big|_{\frac{1}{2}}^{\frac{3}{4}} \\
&- \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{1}{2}}^{\frac{3}{4}} \Gamma(1+\xi_o)\Phi(sr_2 + (1-s)r_1)(ds)^{\xi_o}\Bigg], \\
&= \left[\left(\frac{33}{180}\right)^{\xi_o} \Phi\left(\frac{r_1 + 3r_2}{4}\right) + \left(\frac{12}{180}\right)^{\xi_o} \Phi\left(\frac{r_1 + r_2}{2}\right)\right] \\
&- \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{1}{2}}^{\frac{3}{4}} \Gamma(1+\xi_o)\Phi(sr_2 + (1-s)r_1)(ds)^{\xi_o},
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
I_4 &= (r_2 - r_1)^{\xi_o} \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{3}{4}}^{1} \left(s - \frac{83}{90}\right)^{\xi_o} \Phi^{(\xi_o)}(sr_2 + (1-s)r_1)(ds)^{\xi_o} \\
&= \Bigg[\left(s - \frac{83}{90}\right)^{\xi_o} \Phi(sr_2 + (1-s)r_1)\Big|_{\frac{3}{4}}^{1} \\
&- \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{3}{4}}^{1} \Gamma(1+\xi_o)\Phi(sr_2 + (1-s)r_1)(ds)^{\xi_o}\Bigg], \\
&= \left[\left(\frac{31}{180}\right)^{\xi_o} \Phi\left(\frac{r_1 + 3r_2}{4}\right) + \left(\frac{7}{90}\right)^{\xi_o} \Phi(r_2)\right] \\
&- \frac{1}{\Gamma(1+\xi_o)} \int_{\frac{3}{4}}^{1} \Gamma(1+\xi_o)\Phi(sr_2 + (1-s)r_1)(ds)^{\xi_o}.
\end{aligned} \tag{2.5}
$$

Thus, we obtain the following equality by adding (2.2)–(2.5)

$$
\begin{aligned}
&I_1 + I_2 + I_3 + I_4 \\
&= \left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1) + 32^{\xi_o}\Phi\left(\frac{3r_1 + r_2}{4}\right) + 12^{\xi_o}\Phi\left(\frac{r_1 + r_2}{2}\right) + 32^{\xi_o}\Phi\left(\frac{r_1 + 3r_2}{4}\right) + 7^{\xi_o}\Phi(r_2)\right]
\end{aligned} \tag{2.6}
$$

$$-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}\frac{1}{\Gamma(1+\xi_o)}\int_{r_1}^{r_2}\Phi(\varkappa)(d\varkappa)^{\xi_o}.$$

In same manner, we have

$$\begin{aligned}
I_5+I_6+I_7+I_8 &= (r_2-r_1)^{\xi_o}\Big[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(\frac{7}{90}-s\right)^{\xi_o}\Phi^{(\xi_o)}(sr_1+(1-s)r_2)(ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{39}{90}-s\right)^{\xi_o}\Phi^{(\xi_o)}(sr_1+(1-s)r_2)(ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\frac{51}{90}-s\right)^{\xi_o}\Phi^{(\xi_o)}(sr_1+(1-s)r_2)(ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left(\frac{83}{90}-s\right)^{\xi_o}\Phi^{(\xi_o)}(sr_1+(1-s)r_2)(ds)^{\xi_o}\Big].
\end{aligned}\tag{2.7}$$

It implies that

$$\begin{aligned}
&I_5+I_6+I_7+I_8\\
&=\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]\\
&-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}\frac{1}{\Gamma(1+\xi_o)}\int_{r_1}^{r_2}\Phi(\varkappa)(d\varkappa)^{\xi_o}.
\end{aligned}\tag{2.8}$$

By combining Eqs (2.6) and (2.7) one we can get the required equality. □

Theorem 2.1. *Suppose* $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}^{\xi_o}$ *satisfies* $\Phi \in D_{\xi_o}(I^o)$ *with* $r_1, r_2 \in I^o$ *and* $r_1 < r_2$. *If* $\Phi^{(\xi_o)} \in C_{\xi_o}[r_1, r_2]$ *and the function* $|\Phi^{(\xi_o)}|$ *is generalized convex on the interval* $[r_1, r_2]$, *the following inequality follows:*

$$\begin{aligned}
&\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]\\
&-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}{}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa)\Bigg|\\
&\leq\left(\frac{239}{1620}\right)^{\xi_o}\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}\left(\left|\Phi^{(\xi_o)}(r_1)\right|+\left|\Phi^{(\xi_o)}(r_2)\right|\right).
\end{aligned}\tag{2.9}$$

Proof. In Lemma 2.1, by the properties of the modulus, we have

$$\begin{aligned}
&\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]\\
&-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}{}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa)\Bigg|\\
&\leq\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\\
&\times\Big[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left|\left(s-\frac{7}{90}\right)^{\xi_o}\right|\left[\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|+\left|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\right|\right](ds)^{\xi_o}
\end{aligned}$$

$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(s-\frac{39}{90}\right)^{\xi_o}\right|\left[\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|+\left|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\right|\right](ds)^{\xi_o}$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(s-\frac{51}{90}\right)^{\xi_o}\right|\left[\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|+\left|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\right|\right](ds)^{\xi_o}$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left|\left(s-\frac{83}{90}\right)^{\xi_o}\right|\left[\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|+\left|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\right|\right](ds)^{\xi_o}\Bigg].$$

By utilizing the generalized convexity of $|\Phi^{(\xi_o)}|$, we have

$$\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]$$
$$-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}{}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa)\Bigg|$$
$$\leq\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left|\left(s-\frac{7}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}$$
$$\times\left[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+s^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|\right](ds)^{\xi_o}$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(s-\frac{39}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}$$
$$\times\left[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+s^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|\right](ds)^{\xi_o}$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(s-\frac{51}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}$$
$$\times\left[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+s^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|\right](ds)^{\xi_o}$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left|\left(s-\frac{83}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}$$
$$\times\left[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+s^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|\right](ds)^{\xi_o}\Bigg]$$
$$=\left(\frac{239}{1620}\right)^{\xi_o}\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}\left(\left|\Phi^{(\xi_o)}(r_1)\right|+\left|\Phi^{(\xi_o)}(r_2)\right|\right).$$

This completes the proof of Theorem 2.1. □

Remark 2.1. *In Theorem 2.1 if we choose $\xi_o = 1$ then, we have the below mentioned inequality proved in [8] (Theorem 3):*

$$\left|\frac{1}{90}\left[7\Phi(r_1)+32\Phi\left(\frac{3r_1+r_2}{4}\right)+12\Phi\left(\frac{r_1+r_2}{2}\right)+32\Phi\left(\frac{r_1+3r_2}{4}\right)+7\Phi(r_2)\right]\right.$$
$$\left.-\frac{1}{r_2-r_1}\int_{r_1}^{r_2}\Phi(\varkappa)d\varkappa\right|$$
$$\leq\frac{239(r_2-r_1)}{6480}\left[\left|\Phi'(r_1)\right|+\left|\Phi'(r_2)\right|\right].$$

Theorem 2.2. *Suppose* $\Phi : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}^{\xi_0}$ *with* $\Phi \in \mathrm{D}_{\xi_o}(\mathrm{I}^o)$, $\mathrm{r}_1, \mathrm{r}_2 \in \mathrm{I}^o$, *and* $\mathrm{r}_1 < \mathrm{r}_2$. *Let* $\Phi^{(\xi_o)} \in \mathrm{C}_{\xi_o}[\mathrm{r}_1, \mathrm{r}_2]$. *If* $|\Phi^{(\xi_o)}|^{q_o}$ *is generalized convex on the interval* $[\mathrm{r}_1, \mathrm{r}_2]$ *for some* $q_o > 1$ *such that* $\frac{1}{p} + \frac{1}{q_o} = 1$, *the following inequality follows:*

$$\begin{aligned}
&\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(\mathrm{r}_1) + 32^{\xi_o}\Phi\left(\frac{3\mathrm{r}_1 + \mathrm{r}_2}{4}\right) + 12^{\xi_o}\Phi\left(\frac{\mathrm{r}_1 + \mathrm{r}_2}{2}\right) + 32^{\xi_o}\Phi\left(\frac{\mathrm{r}_1 + 3\mathrm{r}_2}{4}\right) + 7^{\xi_o}\Phi(\mathrm{r}_2)\right] \\
&- \frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}}\, {}_{\mathrm{r}_1}\mathrm{I}^{\xi_o}_{\mathrm{r}_2}\Phi(\varkappa)\Bigg| \\
&\le \frac{(\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\left(\mathrm{K}_1\left(\left(\frac{14}{45}\right)^{(\mathrm{p}+1)\xi_o} + \left(\frac{31}{45}\right)^{(\mathrm{p}+1)\xi_o}\right)\right)^{\frac{1}{\mathrm{p}}} \\
&\times\left\{\left(\left(\frac{1}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}} + \mathrm{K}_3|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\left(\frac{1}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}} + \mathrm{K}_3|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}}\right\} \\
&+ \left(\mathrm{K}_1\left(\left(\frac{11}{15}\right)^{(\mathrm{p}+1)\xi_o} + \left(\frac{4}{15}\right)^{(\mathrm{p}+1)\xi_o}\right)\right)^{\frac{1}{\mathrm{p}}} \\
&\times\left\{\left(\left(\frac{3}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}} + \mathrm{K}_4|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\left(\frac{3}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}} + \mathrm{K}_4|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}}\right\} \\
&+ \left(\mathrm{K}_1\left(\left(\frac{4}{15}\right)^{(\mathrm{p}+1)\xi_o} + \left(\frac{11}{15}\right)^{(\mathrm{p}+1)\xi_o}\right)\right)^{\frac{1}{\mathrm{p}}} \\
&\times\left\{\left(\left(\frac{5}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}} + \mathrm{K}_5|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\left(\frac{5}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}} + \mathrm{K}_5|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}}\right\} \\
&+ \left(\mathrm{K}_1\left(\left(\frac{14}{45}\right)^{(\mathrm{p}+1)\xi_o} + \left(\frac{31}{45}\right)^{(\mathrm{p}+1)\xi_o}\right)\right)^{\frac{1}{\mathrm{p}}} \\
&\times\left\{\left(\left(\frac{7}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}} + \mathrm{K}_6|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\left(\frac{7}{16}\right)^{\xi_o}\mathrm{K}_2|\Phi^{(\xi_o)}(\mathrm{r}_1)|^{\mathrm{q_o}} + \mathrm{K}_6|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}}\right)^{\frac{1}{\mathrm{q_o}}}\right\}\Bigg],
\end{aligned}$$

where

$$\mathrm{K}_1 = \frac{\Gamma(1+\mathrm{p}\xi_o)}{4^{(\mathrm{p}+1)\xi_o}\Gamma(1+(1+\mathrm{p})\xi_o)}, \qquad \mathrm{K}_2 = \frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}, \qquad \mathrm{K}_3 = \frac{4^{\xi_o}\Gamma(1+2\xi_o) - (\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)},$$

$$\mathrm{K}_4 = \frac{4^{\xi_o}\Gamma(1+2\xi_o) - 3^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}, \qquad \mathrm{K}_5 = \frac{4^{\xi_o}\Gamma(1+2\xi_o) - 5^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)},$$

and

$$\mathrm{K}_6 = \frac{4^{\xi_o}\Gamma(1+2\xi_o) - 7^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}.$$

Proof. Using Hölder's inequality on the absolute value of (2.1), we derive

$$\begin{aligned}
&\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(\mathrm{r}_1) + 32^{\xi_o}\Phi\left(\frac{3\mathrm{r}_1 + \mathrm{r}_2}{4}\right) + 12^{\xi_o}\Phi\left(\frac{\mathrm{r}_1 + \mathrm{r}_2}{2}\right) + 32^{\xi_o}\Phi\left(\frac{\mathrm{r}_1 + 3\mathrm{r}_2}{4}\right) + 7^{\xi_o}\Phi(\mathrm{r}_2)\right] \\
&- \frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}}\, {}_{\mathrm{r}_1}\mathrm{I}^{\xi_o}_{\mathrm{r}_2}\Phi(\varkappa)\Bigg| \\
&\le \frac{(\mathrm{r}_2 - \mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}
\end{aligned}$$

$$\times\Bigg[\Bigg\{\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Bigg|\Bigg(s-\frac{7}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Big|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Bigg|\Bigg(s-\frac{7}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Big|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}$$

$$+\Bigg\{\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Bigg|\Bigg(s-\frac{39}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Big|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Bigg|\Bigg(s-\frac{39}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Big|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}$$

$$+\Bigg\{\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Bigg|\Bigg(s-\frac{51}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Big|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Bigg|\Bigg(s-\frac{51}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Big|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}$$

$$+\Bigg\{\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Bigg|\Bigg(s-\frac{83}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Bigg|\Bigg(s-\frac{83}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\Big|^{q_o}(ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg].$$

Taking advantage of the generalized convexity of $|\Phi^{(\xi_o)}|^{q_o}$, we have

$$\Bigg|\Bigg(\frac{1}{90}\Bigg)^{\xi_o}\Bigg[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\Bigg(\frac{3r_1+r_2}{4}\Bigg)+12^{\xi_o}\Phi\Bigg(\frac{r_1+r_2}{2}\Bigg)+32^{\xi_o}\Phi\Bigg(\frac{r_1+3r_2}{4}\Bigg)+7^{\xi_o}\Phi(r_2)\Bigg]$$

$$-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}{}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa)\Bigg|$$

$$\leq\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Bigg|\Bigg(s-\frac{7}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}$$

$$\times\Bigg\{\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}[s^{\xi_o}\Big|\Phi^{(\xi_o)}(r_2)\Big|^{q_o}+(1-s)^{\xi_o}\Big|\Phi^{(\xi_o)}(r_1)\Big|^{q_o}](ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}[s^{\xi_o}\Big|\Phi^{(\xi_o)}(r_1)\Big|^{q_o}+(1-s)^{\xi_o}\Big|\Phi^{(\xi_o)}(r_2)\Big|^{q_o}](ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Bigg|\Bigg(s-\frac{39}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}$$

$$\times\Bigg\{\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}[s^{\xi_o}\Big|\Phi^{(\xi_o)}(r_2)\Big|^{q_o}+(1-s)^{\xi_o}\Big|\Phi^{(\xi_o)}(r_1)\Big|^{q_o}](ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}[s^{\xi_o}\Big|\Phi^{(\xi_o)}(r_1)\Big|^{q_o}+(1-s)^{\xi_o}\Big|\Phi^{(\xi_o)}(r_2)\Big|^{q_o}](ds)^{\xi_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}$$

$$+\Bigg(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Bigg|\Bigg(s-\frac{51}{90}\Bigg)^{\xi_o}\Bigg|^{p}(ds)^{\xi_o}\Bigg)^{\frac{1}{p}}$$

$$\times\left\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|^{q_o}+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|^{q_o}](ds)^{\xi_o}\right)^{\frac{1}{q_o}}\right.$$
$$\left.+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|^{q_o}+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|^{q_o}](ds)^{\xi_o}\right)^{\frac{1}{q_o}}\right\}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left|\left(s-\frac{83}{90}\right)^{\xi_o}\right|^{p}(ds)^{\xi_o}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|^{q_o}+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|^{q_o}](ds)^{\xi_o}\right)^{\frac{1}{q_o}}\right.$$
$$\left.\left.+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}[s^{\xi_o}\left|\Phi^{(\xi_o)}(r_1)\right|^{q_o}+(1-s)^{\xi_o}\left|\Phi^{(\xi_o)}(r_2)\right|^{q_o}](ds)^{\xi_o}\right)^{\frac{1}{q_o}}\right\}\right]$$
$$=\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\left[\left(K_1\left(\left(\frac{14}{45}\right)^{(p+1)\xi_o}+\left(\frac{31}{45}\right)^{(p+1)\xi_o}\right)\right)^{\frac{1}{p}}\right.$$
$$\times\left\{\left(\left(\frac{1}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_2)|^{q_o}+K_3|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}+\left(\left(\frac{1}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_1)|^{q_o}+K_3|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\right\}$$
$$+\left(K_1\left(\left(\frac{11}{15}\right)^{(p+1)\xi_o}+\left(\frac{4}{15}\right)^{(p+1)\xi_o}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left(\frac{3}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_2)|^{q_o}+K_4|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}+\left(\left(\frac{3}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_1)|^{q_o}+K_4|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\right\}$$
$$+\left(K_1\left(\left(\frac{4}{15}\right)^{(p+1)\xi_o}+\left(\frac{11}{15}\right)^{(p+1)\xi_o}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left(\frac{5}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_2)|^{q_o}+K_5|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}+\left(\left(\frac{5}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_1)|^{q_o}+K_5|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\right\}$$
$$+\left(K_1\left(\left(\frac{14}{45}\right)^{(p+1)\xi_o}+\left(\frac{31}{45}\right)^{(p+1)\xi_o}\right)\right)^{\frac{1}{p}}$$
$$\left.\times\left\{\left(\left(\frac{7}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_2)|^{q_o}+K_6|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}+\left(\left(\frac{7}{16}\right)^{\xi_o}K_2|\Phi^{(\xi_o)}(r_1)|^{q_o}+K_6|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\right\}\right].$$

This completes the proof of Theorem 2.2. □

Remark 2.2. *If we take $\xi_o = 1$ in Theorem 2.2, the resulting inequality corresponds to the one proven in [8] (Theorem 4):*

$$\left|\frac{1}{90}\left[7\Phi(r_1)+32\Phi\left(\frac{3r_1+r_2}{4}\right)+12\Phi\left(\frac{r_1+r_2}{2}\right)+32\Phi\left(\frac{r_1+3r_2}{4}\right)+7\Phi(r_2)\right]\right.$$
$$\left.-\frac{1}{r_2-r_1}\int_{r_1}^{r_2}\Phi(\varkappa)d\varkappa\right|$$
$$\leq\frac{(r_2-r_1)}{2}$$
$$\times\left[\left(\frac{(\frac{14}{45})^{p+1}+(\frac{31}{45})^{p+1}}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|\Phi'(r_2)|^{q_o}+7|\Phi'(r_1)|^{q_o}}{32}\right)^{\frac{1}{q_o}}+\left(\frac{|\Phi'(r_1)|^{q_o}+7|\Phi'(r_2)|^{q_o}}{32}\right)^{\frac{1}{q_o}}\right\}\right.$$

$$+\left(\frac{(\frac{4}{15})^{p+1}+(\frac{11}{15})^{p+1}}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|\Phi'(r_2)|^{q_o}+5|\Phi'(r_1)|^{q_o}}{32}\right)^{\frac{1}{q_o}}+\left(\frac{3|\Phi'(r_1)|^{q_o}+5|\Phi'(r_2)|^{q_o}}{32}\right)^{\frac{1}{q_o}}\right\}$$
$$+\left(\frac{(\frac{4}{15})^{p+1}+(\frac{11}{15})^{p+1}}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{5|\Phi'(r_2)|^{q_o}+3|\Phi'(r_1)|^{q_o}}{32}\right)^{\frac{1}{q_o}}+\left(\frac{5|\Phi'(r_1)|^{q_o}+3|\Phi'(r_2)|^{q_o}}{32}\right)^{\frac{1}{q_o}}\right\}$$
$$+\left(\frac{(\frac{14}{45})^{p+1}+(\frac{31}{45})^{p+1}}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{7|\Phi'(r_2)|^{q_o}+|\Phi'(r_1)|^{q_o}}{32}\right)^{\frac{1}{q_o}}+\left(\frac{7|\Phi'(r_1)|^{q_o}+|\Phi'(r_2)|^{q_o}}{32}\right)^{\frac{1}{q_o}}\right\}\Bigg].$$

Theorem 2.3. *Consider the mapping* $\Phi : I \subseteq R \to R^{\xi_o}$ *such that* $\Phi \in D_{\xi_o}(I^o)$*, with* $r_1, r_2 \in I^o$*,* $r_1 < r_2$*, and* $\Phi^{(\xi_o)} \in C_{\xi_o}[r_1, r_2]$*. If the function* $|\Phi^{(\xi_o)}|^{q_o}$ *is generalized convex on* $[r_1, r_2]$ *for* $q_o \geq 1$*, then the following inequality follows:*

$$\left|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]\right.$$
$$\left.-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}\,{}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa)\right| \leq \frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}$$
$$\times\left\{\left[\left(K_2\left(\frac{1157}{32400}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.\right.$$
$$\times\left\{\left(\left(K_7\left(\frac{85637}{5832000}\right)^{\xi_o}-K_2\left(\frac{22862}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right.\right.$$
$$\left.+\left(K_2\left(\frac{31680}{5832000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{42840}{5832000}\right)^{\xi_o}-K_7\left(\frac{85637}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}$$
$$+\left(\left(K_7\left(\frac{85637}{5832000}\right)^{\xi_o}-K_2\left(\frac{22862}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right.$$
$$\left.\left.\left.+\left(K_2\left(\frac{31680}{5832000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{42840}{5832000}\right)^{\xi_o}-K_7\left(\frac{85637}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\right\}\right]$$
$$+\left[\left(K_2\left(\frac{137}{3600}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.$$
$$\times\left\{\left(\left(K_2\left(\frac{2951}{108000}\right)^{\xi_o}-K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right.\right.$$
$$\left.+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{91}{1800}\right)^{\xi_o}-K_2\left(\frac{9761}{108000}\right)^{\xi_o}+K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}$$
$$+\left(\left(K_2\left(\frac{2951}{108000}\right)^{\xi_o}-K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right.$$
$$\left.\left.\left.+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{91}{1800}\right)^{\xi_o}-K_2\left(\frac{9761}{108000}\right)^{\xi_o}+K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\right\}\right]$$
$$+\left[\left(K_2\left(\frac{137}{3600}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.$$
$$\times\left\{\left(\left(K_7\left(\frac{39517}{216000}\right)^{\xi_o}-K_2\left(\frac{10421}{108000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right.\right.$$
$$\left.+\left(K_2\left(\frac{28811}{108000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{119}{1800}\right)^{\xi_o}-K_7\left(\frac{39517}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right)^{\frac{1}{q_o}}$$

$$+\left(\left(K_7\left(\frac{39517}{216000}\right)^{\xi_o}-K_2\left(\frac{10421}{108000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}$$
$$+\left(K_2\left(\frac{28811}{108000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{119}{1800}\right)^{\xi_o}-K_7\left(\frac{39517}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\Bigg\}\Bigg]$$
$$+\left[\left(K_2\left(\frac{1157}{32400}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.$$
$$\times\Bigg\{\left(\left(K_2\left(\frac{372421}{2916000}\right)^{\xi_o}-K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right.$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{1411}{16200}\right)^{\xi_o}-K_2\left(\frac{776251}{2916000}\right)^{\xi_o}+K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\Bigg)^{\frac{1}{q_o}}$$
$$+\left(\left(K_2\left(\frac{372421}{2916000}\right)^{\xi_o}-K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right.$$
$$\left.+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{1411}{16200}\right)^{\xi_o}-K_2\left(\frac{776251}{2916000}\right)^{\xi_o}+K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\right)^{\frac{1}{q_o}}\Bigg\}\Bigg]\Bigg\}.$$

Where

$$K_2=\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)},\qquad K_7=\frac{\Gamma(1+2\xi_o)}{\Gamma(1+3\xi_o)}.$$

Proof. Using power mean inequality on the absolute value of (2.1), we derive

$$\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]$$
$$-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}\,{}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa)\Bigg|$$
$$\le\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\left(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left|\left(s-\frac{7}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}\right)^{1-\frac{1}{q_o}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left|\left(s-\frac{7}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|^{q_o}(ds)^{\xi_o}\right)^{\frac{1}{q_o}}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left|\left(s-\frac{7}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\right|^{q_o}(ds)^{\xi_o}\right)^{\frac{1}{q_o}}\Bigg\}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(s-\frac{39}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}\right)^{1-\frac{1}{q_o}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(s-\frac{39}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|^{q_o}(ds)^{\xi_o}\right)^{\frac{1}{q_o}}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(s-\frac{39}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(sr_1+(1-s)r_2)\right|^{q_o}(ds)^{\xi_o}\right)^{\frac{1}{q_o}}\Bigg\}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(s-\frac{51}{90}\right)^{\xi_o}\right|(ds)^{\xi_o}\right)^{1-\frac{1}{q_o}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(s-\frac{51}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(sr_2+(1-s)r_1)\right|^{q_o}(ds)^{\xi_o}\right)^{\frac{1}{q_o}}$$

$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\right|^{\mathrm{q_o}}(d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}\Bigg\}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left|\left(\mathrm{s}-\frac{83}{90}\right)^{\xi_o}\right|(d\mathrm{s})^{\xi_o}\right)^{1-\frac{1}{\mathrm{q_o}}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left|\left(\mathrm{s}-\frac{83}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2+(1-\mathrm{s})\mathrm{r}_1)\right|^{\mathrm{q_o}}(d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left|\left(\mathrm{s}-\frac{83}{90}\right)^{\xi_o}\right|\left|\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\right|^{\mathrm{q_o}}(d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}\Bigg\}\Bigg].$$

Taking advantage of the generalized convexity of $|\Phi^{(\xi_o)}|^{\mathrm{q_o}}$, we have

$$\Bigg|\left(\frac{1}{90}\right)^{\xi_o}\left[7^{\xi_o}\Phi(\mathrm{r}_1)+32^{\xi_o}\Phi\left(\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}\right)+7^{\xi_o}\Phi(\mathrm{r}_2)\right]$$
$$-\frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}\,{}_{\mathrm{r}_1}\mathrm{I}_{\mathrm{r}_2}^{\xi_o}\Phi(\varkappa)\Bigg|$$
$$\leq\frac{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\left(\frac{1}{\Gamma(1+\xi_o)}\int_{0}^{\frac{1}{4}}\left|\left(\mathrm{s}-\frac{7}{90}\right)^{\xi_o}\right|(d\mathrm{s})^{\xi_o}\right)^{1-\frac{1}{\mathrm{q_o}}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{0}^{\frac{1}{4}}\left|\left(\mathrm{s}-\frac{7}{90}\right)^{\xi_o}\right|\left[\mathrm{s}^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_2)\right|^{\mathrm{q_o}}+(1-\mathrm{s})^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_1)\right|^{\mathrm{q_o}}\right](d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{0}^{\frac{1}{4}}\left|\left(\mathrm{s}-\frac{7}{90}\right)^{\xi_o}\right|\left[\mathrm{s}^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_1)\right|^{\mathrm{q_o}}+(1-\mathrm{s})^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_2)\right|^{\mathrm{q_o}}\right](d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}\Bigg\}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\mathrm{s}-\frac{39}{90}\right)^{\xi_o}\right|(d\mathrm{s})^{\xi_o}\right)^{1-\frac{1}{\mathrm{q_o}}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\mathrm{s}-\frac{39}{90}\right)^{\xi_o}\right|\left[\mathrm{s}^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_2)\right|^{\mathrm{q_o}}+(1-\mathrm{s})^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_1)\right|^{\mathrm{q_o}}\right](d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\left(\mathrm{s}-\frac{39}{90}\right)^{\xi_o}\right|\left[\mathrm{s}^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_1)\right|^{\mathrm{q_o}}+(1-\mathrm{s})^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_2)\right|^{\mathrm{q_o}}\right](d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}\Bigg\}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\right|(d\mathrm{s})^{\xi_o}\right)^{1-\frac{1}{\mathrm{q_o}}}$$
$$\times\Bigg\{\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\right|\left[\mathrm{s}^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_2)\right|^{\mathrm{q_o}}+(1-\mathrm{s})^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_1)\right|^{\mathrm{q_o}}\right](d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}$$
$$+\left(\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\right|\left[\mathrm{s}^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_1)\right|^{\mathrm{q_o}}+(1-\mathrm{s})^{\xi_o}\left|\Phi^{(\xi_o)}(\mathrm{r}_2)\right|^{\mathrm{q_o}}\right](d\mathrm{s})^{\xi_o}\right)^{\frac{1}{\mathrm{q_o}}}\Bigg\}$$
$$=\frac{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}$$
$$\times\Bigg\{\Bigg[\left(\mathrm{K}_2\left(\frac{1157}{32400}\right)^{\xi_o}\right)^{1-\frac{1}{\mathrm{q_o}}}$$
$$\times\Bigg\{\Bigg(\left(\mathrm{K}_7\left(\frac{85637}{5832000}\right)^{\xi_o}-\mathrm{K}_2\left(\frac{22862}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(\mathrm{r}_2)|^{\mathrm{q_o}}$$

$$
\begin{aligned}
&+\left(K_2\left(\frac{31680}{5832000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{42840}{5832000}\right)^{\xi_o}-K_7\left(\frac{85637}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\left(\left(K_7\left(\frac{85637}{5832000}\right)^{\xi_o}-K_2\left(\frac{22862}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\right.\\
&+\left(K_2\left(\frac{31680}{5832000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{42840}{5832000}\right)^{\xi_o}-K_7\left(\frac{85637}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg]\\
&+\left[\left(K_2\left(\frac{137}{3600}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.\\
&\times\Bigg\{\Bigg(\left(K_2\left(\frac{2951}{108000}\right)^{\xi_o}-K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\\
&+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{91}{1800}\right)^{\xi_o}-K_2\left(\frac{9761}{108000}\right)^{\xi_o}+K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\left(K_2\left(\frac{2951}{108000}\right)^{\xi_o}-K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\\
&+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{91}{1800}\right)^{\xi_o}-K_2\left(\frac{9761}{108000}\right)^{\xi_o}+K_7\left(\frac{4777}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg]\\
&+\left[\left(K_2\left(\frac{137}{3600}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.\\
&\times\Bigg\{\Bigg(\left(K_7\left(\frac{39517}{216000}\right)^{\xi_o}-K_2\left(\frac{10421}{108000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\\
&+\left(K_2\left(\frac{28811}{108000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{119}{1800}\right)^{\xi_o}-K_7\left(\frac{39517}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\left(K_7\left(\frac{39517}{216000}\right)^{\xi_o}-K_2\left(\frac{10421}{108000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\\
&+\left(K_2\left(\frac{28811}{108000}\right)^{\xi_o}-\frac{1}{\Gamma(1+\xi_o)}\left(\frac{119}{1800}\right)^{\xi_o}-K_7\left(\frac{39517}{216000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg]\\
&+\left[\left(K_2\left(\frac{1157}{32400}\right)^{\xi_o}\right)^{1-\frac{1}{q_o}}\right.\\
&\times\Bigg\{\Bigg(\left(K_2\left(\frac{372421}{2916000}\right)^{\xi_o}-K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\\
&+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{1411}{16200}\right)^{\xi_o}-K_2\left(\frac{776251}{2916000}\right)^{\xi_o}+K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\left(K_2\left(\frac{372421}{2916000}\right)^{\xi_o}-K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_1)|^{q_o}\\
&+\left(\frac{1}{\Gamma(1+\xi_o)}\left(\frac{1411}{16200}\right)^{\xi_o}-K_2\left(\frac{776251}{2916000}\right)^{\xi_o}+K_7\left(\frac{856217}{5832000}\right)^{\xi_o}\right)|\Phi^{(\xi_o)}(r_2)|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg].
\end{aligned}
$$

This completes the proof of Theorem 2.3. □

Remark 2.3. *Choosing $\xi_o = 1$ in Theorem 2.3 yields the inequality stated below, as proved in [8]*

(Theorem 5):

$$\begin{aligned}
&\left|\frac{1}{90}\left[7\Phi(r_1)+32\Phi\left(\frac{3r_1+r_2}{4}\right)+12\Phi\left(\frac{r_1+r_2}{2}\right)+32\Phi\left(\frac{r_1+3r_2}{4}\right)+7\Phi(r_2)\right]\right.\\
&\left.-\frac{1}{r_2-r_1}\int_{r_1}^{r_2}\Phi(\varkappa)d\varkappa\right|\\
&\leq\frac{(r_2-r_1)}{2880}\\
&\times\left[\left(\frac{1157}{45}\right)^{1-\frac{1}{q_o}}\left\{\left(\frac{25672|\Phi'(r_2)|^{q_o}+130523|\Phi'(r_1)|^{q_o}}{6075}\right)^{\frac{1}{q_o}}+\left(\frac{25672|\Phi'(r_1)|^{q_o}+130523|\Phi'(r_2)|^{q_o}}{6075}\right)^{\frac{1}{q_o}}\right\}\right.\\
&+\left(\frac{137}{5}\right)^{1-\frac{1}{q_o}}\left\{\left(\frac{2038|\Phi'(r_2)|^{q_o}+4127|\Phi'(r_1)|^{q_o}}{225}\right)^{\frac{1}{q_o}}+\left(\frac{2038|\Phi'(r_1)|^{q_o}+4127|\Phi'(r_2)|^{q_o}}{225}\right)^{\frac{1}{q_o}}\right\}\\
&+\left(\frac{137}{5}\right)^{1-\frac{1}{q_o}}\left\{\left(\frac{4127|\Phi'(r_2)|^{q_o}+2038|\Phi'(r_1)|^{q_o}}{225}\right)^{\frac{1}{q_o}}+\left(\frac{4127|\Phi'(r_1)|^{q_o}+2038|\Phi'(r_2)|^{q_o}}{225}\right)^{\frac{1}{q_o}}\right\}\\
&\left.+\left(\frac{1157}{45}\right)^{1-\frac{1}{q_o}}\left\{\left(\frac{130523|\Phi'(r_2)|^{q_o}+25672|\Phi'(r_1)|^{q_o}}{6075}\right)^{\frac{1}{q_o}}+\left(\frac{130523|\Phi'(r_1)|^{q_o}+25672|\Phi'(r_2)|^{q_o}}{6075}\right)^{\frac{1}{q_o}}\right\}\right].
\end{aligned}$$

3. Fractal Boole's type inequality for bounded functions

In this section, we provide Boole's type inequalities for bounded functions in fractal space.

Theorem 3.1. *Assume that the assumptions of Lemma 2.1 are valid. If there exist* $n, N \in \mathbb{R}^{\xi_o}$, *such that* $n^{\xi_o} \leq \Phi^{\xi_o}(s) \leq N^{\xi_o}$, *for* $s \in [r_1, r_2]$, *then we have the below mentioned Boole's type inequality:*

$$\begin{aligned}
&\left|\frac{1^{\xi_o}}{90^{\xi_o}}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]\right.\\
&\left.-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}{}_{r_2}I_{r_1}^{\xi_o}\Phi(\varkappa)\right|\\
&\leq\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\frac{239^{\xi_o}}{1620^{\xi_o}}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}(N-n)^{\xi_o}. \qquad (3.1)
\end{aligned}$$

Proof. By the help of Lemma 2.1, we have

$$\begin{aligned}
&\frac{1^{\xi_o}}{90^{\xi_o}}\left[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\left(\frac{3r_1+r_2}{4}\right)+12^{\xi_o}\Phi\left(\frac{r_1+r_2}{2}\right)+32^{\xi_o}\Phi\left(\frac{r_1+3r_2}{4}\right)+7^{\xi_o}\Phi(r_2)\right]\\
&-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}{}_{r_2}I_{r_1}^{\xi_o}\Phi(\varkappa)\\
&=\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\\
&\times\left[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(s-\frac{7}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}\right.\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(s-\frac{39}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left(s-\frac{51}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}
\end{aligned}$$

$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left(\mathrm{s}-\frac{83}{90}\right)^{\xi_o}[\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)](d\mathrm{s})^{\xi_o}\Bigg]. \tag{3.2}$$

We add and subtract $\frac{n^{\xi_o}+N^{\xi_o}}{2^{\xi_o}}$ to rewrite the expression in a symmetric midpoint form, which allows us to apply the generalized convexity property

$$\begin{aligned}
&=\frac{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(\mathrm{s}-\frac{7}{90}\right)^{\xi_o}\\
&\times\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)+\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\mathrm{s}-\frac{39}{90}\right)^{\xi_o}\\
&\times\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)+\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\\
&\times\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)+\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2}-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left(\mathrm{s}-\frac{83}{90}\right)^{\xi_o}\\
&\times\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)+\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2}-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2}\right)(d\mathrm{s})^{\xi_o}\Bigg],\\
&=\frac{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}\\
&\times\Bigg[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(\mathrm{s}-\frac{7}{90}\right)^{\xi_o}\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\left(\mathrm{s}-\frac{7}{90}\right)^{\xi_o}\left(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\mathrm{s}-\frac{39}{90}\right)^{\xi_o}\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\mathrm{s}-\frac{39}{90}\right)^{\xi_o}\left(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\mathrm{s}-\frac{51}{90}\right)^{\xi_o}\left(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{sr}_1+(1-\mathrm{s})\mathrm{r}_2)\right)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\left(\mathrm{s}-\frac{83}{90}\right)^{\xi_o}\left(\Phi^{(\xi_o)}(\mathrm{sr}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\right)(d\mathrm{s})^{\xi_o}
\end{aligned}$$

$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big(\mathrm{s}-\frac{83}{90}\Big)^{\xi_o}\Big(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\Big)(d\mathrm{s})^{\xi_o}\Big].$$

By using the properties of the modulus in (3.2), we obtain

$$\begin{aligned}
&\Big|\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}\Phi(\mathrm{r}_1)+32^{\xi_o}\Phi\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}\Big)+12^{\xi_o}\Phi\Big(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\Big)+32^{\xi_o}\Phi\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}\Big)+7^{\xi_o}\Phi(\mathrm{r}_2)\Big]\\
&-\frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{}_{\mathrm{r}_2}\mathrm{I}_{\mathrm{r}_1}^{\xi_o}\Phi(\varkappa)\Big|\\
&\le\frac{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}\\
&\times\Big[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Big|\Big(\mathrm{s}-\frac{7}{90}\Big)^{\xi_o}\Big|\Big|\Big(\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\Big)\Big|(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Big|\Big(\mathrm{s}-\frac{7}{90}\Big)^{\xi_o}\Big|\Big|\Big(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\Big)\Big|(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Big|\Big(\mathrm{s}-\frac{39}{90}\Big)^{\xi_o}\Big|\Big|\Big(\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\Big)\Big|(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Big|\Big(\mathrm{s}-\frac{39}{90}\Big)^{\xi_o}\Big|\Big|\Big(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\Big)\Big|(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Big|\Big(\mathrm{s}-\frac{51}{90}\Big)^{\xi_o}\Big|\Big|\Big(\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\Big)\Big|(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Big|\Big(\mathrm{s}-\frac{51}{90}\Big)^{\xi_o}\Big|\Big|\Big(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\Big)\Big|(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big|\Big(\mathrm{s}-\frac{83}{90}\Big)^{\xi_o}\Big|\Big|\Big(\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\Big)(d\mathrm{s})^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big|\Big(\mathrm{s}-\frac{83}{90}\Big)^{\xi_o}\Big|\Big|\Big(\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\Big)(d\mathrm{s})^{\xi_o}\Big],
\end{aligned}$$

from the suppositions $\mathrm{n}^{\xi_o}\le\Phi^{(\xi_o)}(\mathrm{s})\le\mathrm{N}^{\xi_o}$ for $\mathrm{s}\in[\mathrm{r}_1,\mathrm{r}_2]$, we get

$$\Big|\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_2+(1-\mathrm{s})\mathrm{r}_1)-\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}\Big|\le\frac{\mathrm{N}^{\xi_o}-\mathrm{n}^{\xi_o}}{2^{\xi_o}},\tag{3.3}$$

and

$$\Big|\frac{\mathrm{n}^{\xi_o}+\mathrm{N}^{\xi_o}}{2^{\xi_o}}-\Phi^{(\xi_o)}(\mathrm{s}\mathrm{r}_1+(1-\mathrm{s})\mathrm{r}_2)\Big|\le\frac{\mathrm{N}^{\xi_o}-\mathrm{n}^{\xi_o}}{2^{\xi_o}},\tag{3.4}$$

by the inequalities (3.3) and (3.4), we have

$$\begin{aligned}
&\Big|\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}\Phi(\mathrm{r}_1)+32^{\xi_o}\Phi\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}\Big)+12^{\xi_o}\Phi\Big(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\Big)+32^{\xi_o}\Phi\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}\Big)+7^{\xi_o}\Phi(\mathrm{r}_2)\Big]\\
&-\frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{}_{\mathrm{r}_2}\mathrm{I}_{\mathrm{r}_1}^{\xi_o}\Phi(\varkappa)\Big|
\end{aligned}$$

$$\leq \frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}(N-n)^{\xi_o}$$
$$\times\Big[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Big|\Big(s-\frac{7}{90}\Big)^{\xi_o}\Big|(ds)^{\xi_o}+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Big|\Big(s-\frac{39}{90}\Big)^{\xi_o}\Big|(ds)^{\xi_o}$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Big|\Big(s-\frac{51}{90}\Big)^{\xi_o}\Big|(ds)^{\xi_o}+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big|\Big(s-\frac{83}{90}\Big)^{\xi_o}\Big|(ds)^{\xi_o}\Big]$$
$$=\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\frac{239^{\xi_o}}{1620^{\xi_o}}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}(N-n)^{\xi_o}.$$

This completes the proof of Theorem 3.1. □

Remark 3.1. *If we take $\xi_o = 1$ in Theorem 3.1, then the inequality below, proved in [8] (Theorem 6), holds:*

$$\Big|\frac{1}{90}\Big[7\Phi(r_1)+32\Phi\Big(\frac{3r_1+r_2}{4}\Big)+12\Phi\Big(\frac{r_1+r_2}{2}\Big)+32\Phi\Big(\frac{r_1+3r_2}{4}\Big)+7\Phi(r_2)\Big]$$
$$-\frac{1}{r_2-r_1}\int_{r_1}^{r_2}\Phi(\varkappa)d\varkappa\Big|$$
$$\leq\frac{239(r_2-r_1)}{6480}(N-n).$$

4. Fractal Boole's type inequality for lipschitzian functions

In this section, we provide Boole's type inequalities for lipschitzian functions in fractal space.

Definition 4.1. *[43] A function $\Phi : I = [r_1, r_2] \subset \mathbb{R} \to \mathbb{R}^{\xi_o}$ is said to be Lipschitzian of order ξ_o, where $0 < \xi_o \leq 1$, if there exists a constant $L_{\xi_0} > 0$ such that*

$$|\Phi(r_1)-\Phi(r_2)| \leq L_{\xi_o}\left|r_1^{\xi_o}-r_2^{\xi_o}\right| \quad \textit{for all } r_1, r_2 \in I.$$

Theorem 4.1. *Assume that the conditions of Lemma 2.1 are valid. If $\Phi^{(\xi_o)}$ is a generalized* L-*Lipschitzian function on* $[r_1, r_2]$ *then, we have the below mentioned inequality:*

$$\Big|\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\Big(\frac{3r_1+r_2}{4}\Big)+12^{\xi_o}\Phi\Big(\frac{r_1+r_2}{2}\Big)+32^{\xi_o}\Phi\Big(\frac{r_1+3r_2}{4}\Big)+7^{\xi_o}\Phi(r_2)\Big]$$
$$-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}\,{}_{r_2}I_{r_1}^{\xi_o}\Phi(\varkappa)\Big|$$
$$\leq\frac{L^{\xi_o}(r_2-r_1)^{2\xi_o}}{2^{\xi_o}}\Big[\Big(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac{1}{4}}\Big|\Big(s-\frac{7}{90}\Big)^{\xi_o}\Big|+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{4}}^{\frac{1}{2}}\Big|\Big(s-\frac{39}{90}\Big)^{\xi_o}\Big|$$
$$+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{1}{2}}^{\frac{3}{4}}\Big|\Big(s-\frac{51}{90}\Big)^{\xi_o}\Big|+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac{3}{4}}^{1}\Big|\Big(s-\frac{83}{90}\Big)^{\xi_o}\Big|\Big)(2s-1)^{\xi_o}(ds)^{\xi_o}\Big].$$

Proof. With the help of Lemma (2.1), we have

$$\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\Big(\frac{3r_1+r_2}{4}\Big)+12^{\xi_o}\Phi\Big(\frac{r_1+r_2}{2}\Big)+32^{\xi_o}\Phi\Big(\frac{r_1+3r_2}{4}\Big)+7^{\xi_o}\Phi(r_2)\Big]$$

$$
\begin{aligned}
&- \frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}} {}_{r_2}I_{r_1}^{\xi_o}\Phi(\varkappa)\\
&= \frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\\
&\times\Big[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac14}\Big(s-\frac{7}{90}\Big)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac14}^{\frac12}\Big(s-\frac{39}{90}\Big)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac12}^{\frac34}\Big(s-\frac{51}{90}\Big)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac34}^{1}\Big(s-\frac{83}{90}\Big)^{\xi_o}[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)](ds)^{\xi_o}\Big].
\end{aligned}
$$

By using the properties of modulus, since Φ^{ξ_o} is L-Lipschitzian function, we obtain

$$
\begin{aligned}
&\Big|\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}\Phi(r_1)+32^{\xi_o}\Phi\Big(\frac{3r_1+r_2}{4}\Big)+12^{\xi_o}\Phi\Big(\frac{r_1+r_2}{2}\Big)+32^{\xi_o}\Phi\Big(\frac{r_1+3r_2}{4}\Big)+7^{\xi_o}\Phi(r_2)\Big]\\
&-\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}} {}_{r_2}I_{r_1}^{\xi_o}\Phi(\varkappa)\Big| \le \frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\\
&\times\Big[\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac14}\Big|\Big(s-\frac{7}{90}\Big)^{\xi_o}\Big|\Big|[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)]\Big|(ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac14}^{\frac12}\Big|\Big(s-\frac{39}{90}\Big)^{\xi_o}\Big|\Big|[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)]\Big|(ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac12}^{\frac34}\Big|\Big(s-\frac{51}{90}\Big)^{\xi_o}\Big|\Big|[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)]\Big|(ds)^{\xi_o}\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac34}^{1}\Big|\Big(s-\frac{83}{90}\Big)^{\xi_o}\Big|\Big|[\Phi^{(\xi_o)}(sr_2+(1-s)r_1)-\Phi^{(\xi_o)}(sr_1+(1-s)r_2)]\Big|(ds)^{\xi_o}\Big],
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{L^{\xi_o}(r_2-r_1)^{2\xi_o}}{2^{\xi_o}}\Big[\Big(\frac{1}{\Gamma(1+\xi_o)}\int_0^{\frac14}\Big|\Big(s-\frac{7}{90}\Big)^{\xi_o}\Big|+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac14}^{\frac12}\Big|\Big(s-\frac{39}{90}\Big)^{\xi_o}\Big|\\
&+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac12}^{\frac34}\Big|\Big(s-\frac{51}{90}\Big)^{\xi_o}\Big|+\frac{1}{\Gamma(1+\xi_o)}\int_{\frac34}^{1}\Big|\Big(s-\frac{83}{90}\Big)^{\xi_o}\Big|\Big)(2s-1)^{\xi_o}(ds)^{\xi_o}\Big].
\end{aligned}
$$

This completes the proof of Theorem 4.1. □

Remark 4.1. *If we take $\xi_o = 1$ in Theorem 4.1, the inequality below, as proved in [8] (Theorem 7), holds:*

$$
\begin{aligned}
&\Big|\frac{1}{90}\Big[7\Phi(r_1)+32\Phi\Big(\frac{3r_1+r_2}{4}\Big)+12\Phi\Big(\frac{r_1+r_2}{2}\Big)+32\Phi\Big(\frac{r_1+3r_2}{4}\Big)+7\Phi(r_2)\Big]\\
&-\frac{1}{r_2-r_1}\int_{r_1}^{r_2}\Phi(\varkappa)d\varkappa\Big| \le \frac{80627(r_2-r_1)}{4374000}L.
\end{aligned}
$$

5. Examples with graphical analysis

In this section, we provide examples with numbers and visual illustrations to express the newly introduced inequalities. The graphs clearly show how these inequalities behave numerically, and it could be easier to understand their purpose and accuracy.

Example 5.1. *Case 1: Let*

$$\Phi(\varkappa_o) = \frac{\Gamma(1+2\xi_o)}{\Gamma(1+3\xi_o)}\varkappa_o^{3\xi_o}, \quad \varkappa_o > 0.$$

If we get $\mathrm{r}_1 = 3$, $\mathrm{r}_2 = 5$, *and* $\xi_o \in (0, 1]$, *then*

$$|\Phi^{(\xi_o)}(\varkappa_o)| = \varkappa_o^{2\xi_o}$$

is convex for $\varkappa_o > 0$. *Provided these conditions hold, the LHS of Theorem 2.1 reduces to:*

$$\left|\frac{68^{\xi_o}\Gamma(1+2\xi_o)}{\Gamma(1+3\xi_o)} - \frac{272^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}{\Gamma(1+4\xi_o)}\right| = LHS.$$

From another perspective, since

$$|\Phi^{(\xi_o)}(\mathrm{r}_1)| = 3^{2\xi_o} \quad \textit{and} \quad |\Phi^{(\xi_o)}(\mathrm{r}_2)| = 5^{2\xi_o},$$

the RHS of Theorem 2.1 simplifies to:

$$\left\{\frac{239^{\xi_o}}{1620^{\xi_o}}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}\left[(3)^{2\xi_o} + (5)^{2\xi_o}\right]\right\} = RHS.$$

The Table 1 next depicts the variation of numerical values of Boole's inequality over fractal dimension $\xi_o \in [0, 1]$.

Table 1. Numerical value of the LHS and RHS for the Boole-type fractal inequality (Example 5.1), with $r_1 = 3$, $r_2 = 5$ and $\xi_o \in [0, 1]$.

ξ_0	LHS	RHS
0.1	0.164414	1.217459
0.2	0.374711	1.428703
0.3	0.617445	1.629397
0.4	0.871175	1.815803
0.5	1.104779	1.984842
0.6	1.275812	2.134103
0.7	1.328893	2.261849
0.8	1.194136	2.366988
0.9	0.785630	2.449028
1.0	0.000000	2.508025

Case 2: Let

$$\Phi(\varkappa_o) = \frac{\Gamma(1+2\xi_o)}{\Gamma(1+3\xi_o)}\varkappa_o^{3\xi_o}, \quad \varkappa_o > 0.$$

If we take $\xi_o = 1$, *then*

$$|\Phi^{(\xi_o)}(\varkappa_o)| = \varkappa_o^2,$$

is convex for $\varkappa_o > 0$. *Provided these conditions hold, the LHS of Theorem 2.1 becomes:*

$$\begin{aligned}&\left|\frac{1}{270}\left[7(\mathrm{r}_1)^3 + 32\left(\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}\right)^3 + 12\left(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\right)^3 + 32\left(\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}\right)^3 + 7(\mathrm{r}_2)^3\right]\right.\\ &\left.-\frac{1}{\mathrm{r}_2-\mathrm{r}_1}\int_{\mathrm{r}_1}^{\mathrm{r}_2}\frac{\varkappa_o^3}{3}d\varkappa_\mathrm{o}\right|\\ &= LHS.\end{aligned}$$

From another perspective, since

$$|\Phi^{(\xi_o)}(\mathrm{r}_1)| = \mathrm{r}_1{}^{2\xi_o} \quad and \quad |\Phi^{(\xi_o)}(\mathrm{r}_2)| = \mathrm{r}_2{}^{2\xi_o},$$

the RHS of Theorem 2.1 becomes:

$$\begin{aligned}&\left\{\frac{239(\mathrm{r}_2-\mathrm{r}_1)}{6480}\left[(\mathrm{r}_2)^2 + (\mathrm{r}_1)^2\right]\right\}\\ &= RHS.\end{aligned}$$

Figures 1, 2 and Table 1 clearly illustrate that RHS $\geq$ LHS.

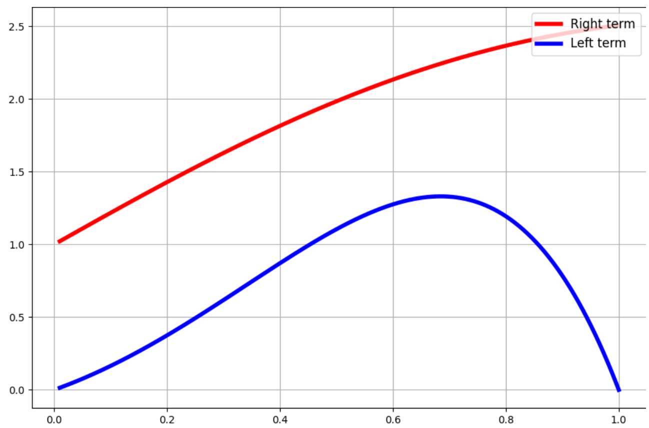

Figure 1. Case 1 considers the graphical analysis on the interval $[\mathrm{r}_1, \mathrm{r}_2] = [3, 5]$, where the parameter ξ_o varies within $[0.1, 1]$.

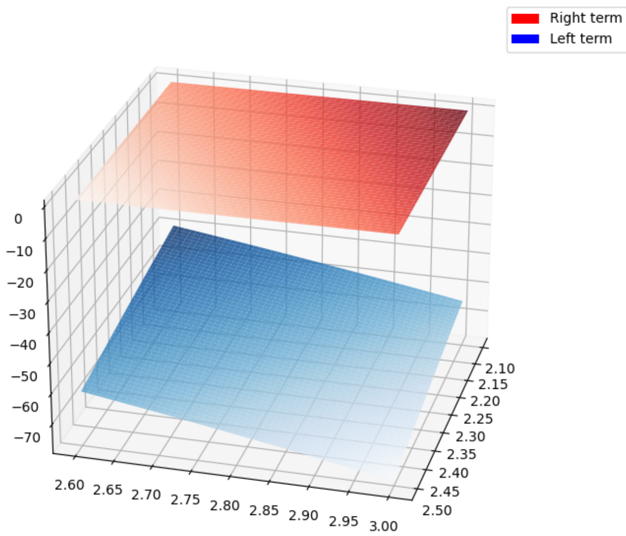

Figure 2. Analyze the graphical results for Case 2 is carried out for $\mathrm{r}_1 \in [2.1, 2.5]$ and $\mathrm{r}_2 \in [2.6, 3]$.

Example 5.2. *Case 1: Let*

$$\Phi(\varkappa_o)=\frac{\Gamma(1+2\xi_o)}{\Gamma(1+3\xi_o)}\varkappa_o^{3\xi_o},\quad \varkappa_o>0.$$

If we take $\xi_o=1$, $\mathrm{r}_1=1$, *and* $\mathrm{r}_2=2$, *then the derivative* $|\Phi'(\varkappa_o)|=\varkappa_o^2$ *is convex for* $\varkappa_o>0$. *Provided these conditions hold, the LHS of Theorem 2.2 becomes:*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\Phi(\mathrm{r}_1)+32\Phi\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}\Big)+12\Phi\Big(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\Big)+32\Phi\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}\Big)+7\Phi(\mathrm{r}_2)\right]\right.\\&\left.-\frac{1}{\mathrm{r}_2-\mathrm{r}_1}\int_{\mathrm{r}_1}^{\mathrm{r}_2}\Phi(\varkappa_o)d\varkappa_o\right|\\&=\left|\frac{675}{540}-\frac{1}{3}\int_1^2\varkappa_o^3d\varkappa_o\right|\\&=LHS.\end{aligned}$$

From another perspective, since

$$|\Phi'(\mathrm{r}_1)|^{q_o}=1^{2q_o}\quad and\quad |\Phi'(\mathrm{r}_2)|^{q_o}=2^{2q_o},\quad with\ q_o\in[1.1,10],$$

the RHS of Theorem 2.2 simplifies to:

$$\begin{aligned}&\frac{(\mathrm{r}_2-\mathrm{r}_1)}{2}\\&\times\Bigg[\Bigg(\frac{(\frac{14}{45})^{\frac{q_o}{q_o-1}+1}+(\frac{31}{45})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1}+1)}\Bigg)^{1-\frac{1}{q_o}}\Bigg\{\Bigg(\frac{(4)^{q_o}+7(1)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}+\Bigg(\frac{(1)^{q_o}+7(4)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}\Bigg\}\\&+\Bigg(\frac{(\frac{4}{15})^{\frac{q_o}{q_o-1}+1}+(\frac{11}{15})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1}+1)}\Bigg)^{1-\frac{1}{q_o}}\Bigg\{\Bigg(\frac{3(4)^{q_o}+5(1)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}+\Bigg(\frac{3(1)^{q_o}+5(4)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}\Bigg\}\\&+\Bigg(\frac{(\frac{4}{15})^{\frac{q_o}{q_o-1}+1}+(\frac{11}{15})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1}+1)}\Bigg)^{1-\frac{1}{q_o}}\Bigg\{\Bigg(\frac{5(4)^{q_o}+3(1)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}+\Bigg(\frac{5(1)^{q_o}+3(4)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}\Bigg\}\\&+\Bigg(\frac{(\frac{14}{45})^{\frac{q_o}{q_o-1}+1}+(\frac{31}{45})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1}+1)}\Bigg)^{1-\frac{1}{q_o}}\Bigg\{\Bigg(\frac{7(4)^{q_o}+(1)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}+\Bigg(\frac{7(1)^{q_o}+(4)^{q_o}}{32}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg].\\&=RHS.\end{aligned}$$

Case 2: Let

$$\Phi(\varkappa_o)=\frac{\Gamma(1+2\xi_o)}{\Gamma(1+3\xi_o)}\varkappa_o^{3\xi_o},\quad \varkappa_o>0.$$

If we take $\xi_o=1$, $q=2$, $\mathrm{r}_1\in[1,2]$, *and* $\mathrm{r}_2\in[3,4]$, *then the derivative* $|\Phi'(\varkappa_o)|={\varkappa_o}^2$ *is convex for* $\varkappa_o>0$. *Provided these conditions hold, the LHS of Theorem 2.2 becomes:*

$$\left|\frac{1}{270}\left[7(\mathrm{r}_1)^3+32\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}\Big)^3+12\Big(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\Big)^3+32\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}\Big)^3+7(\mathrm{r}_2)^3\right]\right.$$

$$
\begin{aligned}
&- \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} \frac{\varkappa_o^3}{3} d\varkappa_o \Big| \\
&= LHS.
\end{aligned}
$$

From another perspective, since

$$
|\Phi'(r_1)|^{q_o} = {r_1}^{2q_o} \quad \textit{and} \quad |\Phi'(r_2)|^{q_o} = {r_2}^{2q_o},
$$

the RHS of Theorem 2.2 reduces to:

$$
\begin{aligned}
&\frac{(r_2 - r_1)}{2} \\
&\times \Bigg[\left(\frac{(\frac{14}{45})^{\frac{q_o}{q_o-1}+1} + (\frac{31}{45})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1} + 1)} \right)^{1-\frac{1}{q_o}} \left\{ \left(\frac{({r_2}^2)^{q_o} + 7({r_1}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} + \left(\frac{({r_1}^2)^{q_o} + 7({r_2}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} \right\} \\
&+ \left(\frac{(\frac{4}{15})^{\frac{q_o}{q_o-1}+1} + (\frac{11}{15})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1} + 1)} \right)^{1-\frac{1}{q_o}} \left\{ \left(\frac{3({r_2}^2)^{q_o} + 5({r_1}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} + \left(\frac{3({r_1}^2)^{q_o} + 5({r_2}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} \right\} \\
&+ \left(\frac{(\frac{4}{15})^{\frac{q_o}{q_o-1}+1} + (\frac{11}{15})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1} + 1)} \right)^{1-\frac{1}{q_o}} \left\{ \left(\frac{5({r_2}^2)^{q_o} + 3({r_1}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} + \left(\frac{5({r_1}^2)^{q_o} + 3({r_2}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} \right\} \\
&+ \left(\frac{(\frac{14}{45})^{\frac{q_o}{q_o-1}+1} + (\frac{31}{45})^{\frac{q_o}{q_o-1}+1}}{4^{\frac{q_o}{q_o-1}+1}(\frac{q_o}{q_o-1} + 1)} \right)^{1-\frac{1}{q_o}} \left\{ \left(\frac{7({r_2}^2)^{q_o} + ({r_1}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} + \left(\frac{7({r_1}^2)^{q_o} + ({r_2}^2)^{q_o}}{32} \right)^{\frac{1}{q_o}} \right\} \Bigg]. \\
&= RHS.
\end{aligned}
$$

Figures 3 and 4 clearly demonstrate that RHS $\geq$ LHS.

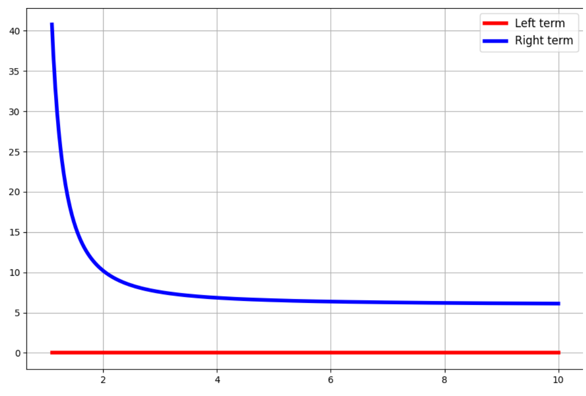

Figure 3. Analyze the graphical results for Case 1 is performed over the interval $[r_1, r_2] = [1, 2]$ with $q_o \in [1.1, 10]$.

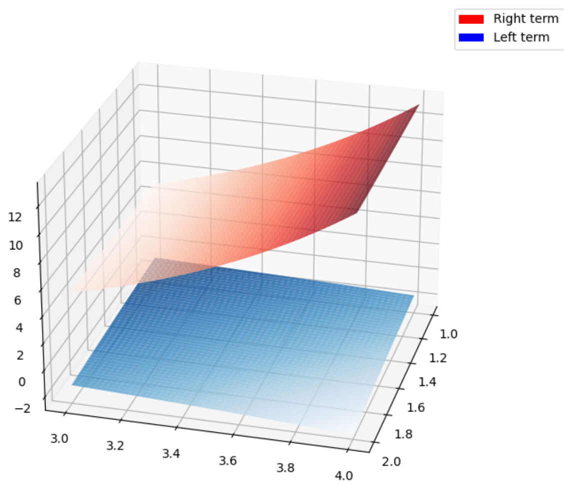

Figure 4. Analyze the graphical results for Case 2 is conducted with $q_o = 2$, $r_1 \in [1, 2]$, and $r_2 \in [3, 4]$.

Example 5.3. *Case 1: Let*

$$\Phi(\varkappa_o) = (2e^{\varkappa}_o)^{\xi_o}, \quad \varkappa_o > 0.$$

If we set $\xi_o = 1$*,* $\mathrm{r}_1 = 1$*, and* $\mathrm{r}_2 = 2$*, then the derivative*

$$|\Phi'(\varkappa_o)| = 2e^{\varkappa}_o,$$

is convex for $\varkappa_o > 0$*. Provided these conditions hold, the LHS of Theorem 2.3 becomes:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[14e + 64e^{\frac{5}{2}} + 24e^{\frac{3}{2}} + 64e^{\frac{7}{4}} + 14e^2\right] - 2(e^2 - e)\right| \\
&= 4.67254 \times 10^{-6} \\
&= LHS.
\end{aligned}$$

From another perspective, since

$$|\Phi'(\mathrm{r}_1)|^{q_o} = (2e^1)^{q_o}, \quad |\Phi'(\mathrm{r}_2)|^{q_o} = (2e^2)^{q_o}, \quad and \quad q_o \in [1.1, 10],$$

the RHS of Theorem 2.3 simplifies to:

$$\begin{aligned}
&= \frac{1}{2880} \\
&\times \left[\left(\frac{1157}{45}\right)^{1-\frac{1}{\mathrm{q_o}}}\left\{\left(\frac{25672(2e^2)^{\mathrm{q_o}} + 130523(2e^1)^{\mathrm{q_o}}}{6075}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\frac{25672(2e^1)^{\mathrm{q_o}} + 130523(2e^2)^{\mathrm{q_o}}}{6075}\right)^{\frac{1}{\mathrm{q_o}}}\right\}\right. \\
&+ \left(\frac{137}{5}\right)^{1-\frac{1}{\mathrm{q_o}}}\left\{\left(\frac{2038(2e^2)^{\mathrm{q_o}} + 4127(2e^1)^{\mathrm{q_o}}}{225}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\frac{2038(2e^1)^{\mathrm{q_o}} + 4127(2e^2)^{\mathrm{q_o}}}{225}\right)^{\frac{1}{\mathrm{q_o}}}\right\} \\
&+ \left(\frac{137}{5}\right)^{1-\frac{1}{\mathrm{q_o}}}\left\{\left(\frac{4127(2e^2)^{\mathrm{q_o}} + 2038(2e^1)^{\mathrm{q_o}}}{225}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\frac{4127(2e^1)^{\mathrm{q_o}} + 2038(2e^2)^{\mathrm{q_o}}}{225}\right)^{\frac{1}{\mathrm{q_o}}}\right\} \\
&\left.+ \left(\frac{1157}{45}\right)^{1-\frac{1}{\mathrm{q_o}}}\left\{\left(\frac{130523(2e^2)^{\mathrm{q_o}} + 25672(2e^1)^{\mathrm{q_o}}}{6075}\right)^{\frac{1}{\mathrm{q_o}}} + \left(\frac{130523(2e^1)^{\mathrm{q_o}} + 25672(2e^2)^{\mathrm{q_o}}}{6075}\right)^{\frac{1}{\mathrm{q_o}}}\right\}\right].
\end{aligned}$$

Case 2: Let

$$\Phi(\varkappa_o) = (2e^{\varkappa}_o)^{\xi_o}, \quad \varkappa_o > 0.$$

If we take $\xi_o = 1$*,* $q_o = 2$*,* $\mathrm{r}_1 \in [1, 2]$*, and* $\mathrm{r}_2 \in [3, 4]$*, then the derivative*

$$|\Phi'(\varkappa_o)| = 2e^{\varkappa_o},$$

is convex for $\varkappa_o > 0$*. Provided these conditions hold, the LHS of Theorem 2.3 becomes:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[14e^{\mathrm{r}_1} + 64e^{\frac{3\mathrm{r}_1+\mathrm{r}_2}{4}} + 24e^{\frac{\mathrm{r}_1+\mathrm{r}_2}{2}} + 64e^{\frac{\mathrm{r}_1+3\mathrm{r}_2}{4}} + 14e^{\mathrm{r}_2}\right] - \frac{2(e^{\mathrm{r}_2} - e^{\mathrm{r}_1})}{\mathrm{r}_2 - \mathrm{r}_1}\right| \\
&= LHS.
\end{aligned}$$

From another perspective, since

$$|\Phi'(\mathrm{r}_1)|^q = (2e^{\mathrm{r}_1})^2 \quad and \quad |\Phi'(\mathrm{r}_2)|^q = (2e^{\mathrm{r}_2})^2,$$

the RHS of Theorem 2.3 simplifies to:

$$
\begin{aligned}
&= \frac{1}{2880} \\
&\times \Bigg[\left(\frac{1157}{45}\right)^{\frac{1}{2}} \left\{ \left(\frac{25672(2e^{r_2})^2 + 130523(2e^{r_1})^2}{6075} \right)^{\frac{1}{2}} + \left(\frac{25672(2e^{r_1})^2 + 130523(2e^{r_2})^2}{6075} \right)^{\frac{1}{2}} \right\} \\
&+ \left(\frac{137}{5}\right)^{\frac{1}{2}} \left\{ \left(\frac{2038(2e^{r_2})^2 + 4127(2e^{r_1})^2}{225} \right)^{\frac{1}{2}} + \left(\frac{2038(2e^{r_1})^2 + 4127(2e^{r_2})^2}{225} \right)^{\frac{1}{2}} \right\} \\
&+ \left(\frac{137}{5}\right)^{\frac{1}{2}} \left\{ \left(\frac{4127(2e^{r_2})^2 + 2038(2e^{r_1})^2}{225} \right)^{\frac{1}{2}} + \left(\frac{4127(2e^{r_1})^2 + 2038(2e^{r_2})^2}{225} \right)^{\frac{1}{2}} \right\} \\
&+ \left(\frac{1157}{45}\right)^{\frac{1}{2}} \left\{ \left(\frac{130523(2e^{r_2})^2 + 25672(2e^{r_1})^2}{6075} \right)^{\frac{1}{2}} + \left(\frac{130523(2e^{r_1})^2 + 25672(2e^{r_2})^2}{6075} \right)^{\frac{1}{2}} \right\} \Bigg].
\end{aligned}
$$

Figures 5 and 6 clearly show that RHS $\geq$ LHS.

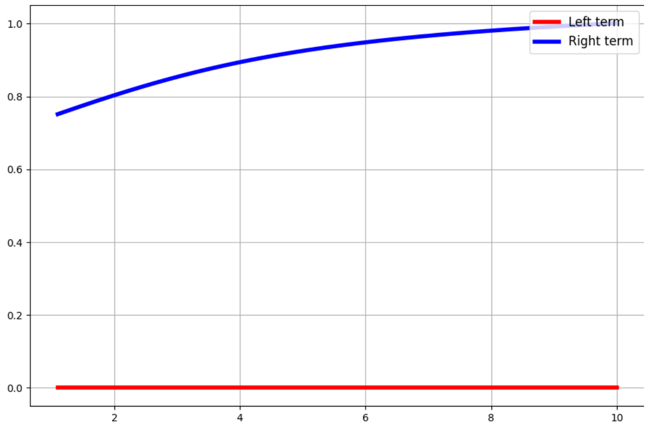

Figure 5. Analyze the graphical results for Case 1 over the interval $[r_1, r_2] = [1, 2]$ with $q_o \in [1.1, 10]$.

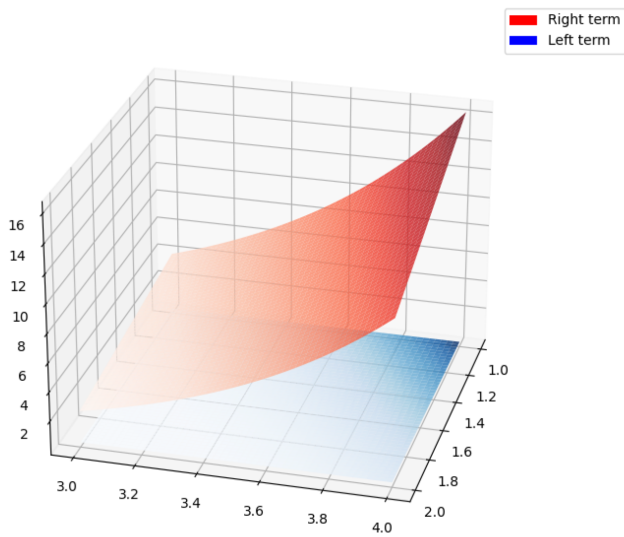

Figure 6. Analyze the graphical results for Case 2 with $q_o = 2$, $r_1 \in [1, 2]$, and $r_2 \in [3, 4]$.

6. Analysis with Artificial Neural Network (ANN)

The computational model employed in this study is a feedforward Artificial Neural Network (ANN), which represents one of the most fundamental and widely used architectures in deep learning [44]. This model consists of an interconnected structure of artificial neurons organized in layers: an input layer, one or more hidden layers, and an output layer [45]. The information in such a network flows strictly in one direction from inputs to outputs without feedback connections, which distinguishes it from recurrent or convolutional architectures. Each neuron processes incoming signals, applies a nonlinear transformation through an activation function, and passes the resulting signal to the next layer. In the present work, the ANN was designed with two input neurons (corresponding to the parameters r_1 and

r_2), two hidden layers of 64 neurons each, and an output layer containing two neurons corresponding to the predicted values of the left-hand side (LHS) and right-hand side (RHS) of the studied Boole's type inequalities on fractal domains. Such a model serves as a universal function approximator, capable of capturing complex nonlinear relationships between variables based on empirical data (see Figures 7 and 8).

The importance of a feedforward neural network lies in its universal approximation capability, as established by the Universal Approximation Theorem [46]. According to this theorem, a neural network with at least one hidden layer containing a finite number of neurons can approximate any continuous function on a compact subset of $\mathbb{R}^n$ with arbitrary accuracy, provided suitable parameters (weights and biases) are chosen. This makes ANNs powerful tools for modeling, regression, classification and prediction in both theoretical and applied contexts. In mathematics, they are quite helpful whenever we deal with problematic expressions, such as integrals that are difficult to work out or other complicated functions. They provide us with a more approximate answer, check whether our theoretical limits are correct using the numerical results, or represent relationships between the variables graphically for easy interpretation. Neural networks are especially effective because they learn from data rather than being limited by prefixed rules. This open them to capture ways in the data that are not obvious and when the connection is nonlinear or include a big number of input variables.

The basic building block of a neural network is the artificial neuron. Each neuron takes in one or more input values, multiplies each input $\varkappa_i$ with a corresponding weight w_i, adds a bias term b_o, and an activation function $\Phi(\cdot)$ to produce the neuron's output. Mathematically, the operation performed by a single neuron can be expressed as [47]:

$$z_o = \sum_{i=1}^{n} w_i \varkappa_i + b_o, \qquad y_o = \Phi(z_o).$$

Here, z_o represents the weighted sum of inputs and y_o is the activated output of the neuron. While training the network, the weights and biases are adapted such that the predicted outputs generated reduce the difference between computed and target values. This is achieved through an iterative update of parameters in which an optimization algorithm is employed, usually driven by gradient descent.

For a neural network with L layers Goodfellow et al. [48], the process of going from input to output can be written step by step. If $a_0^{(0)}$ is the input vector, then for each layer $l = 1, 2, \ldots, L$, we compute the new values using the output from the previous layer.

$$z_o{}^{(l)} = W^{(l)} a_o{}^{(l-1)} + b_o{}^{(l)}, \qquad a_o{}^{(l)} = \Phi^{(l)}(z_o{}^{(l)}),$$

where $W^{(l)}$ denotes the weight matrix, $b_o{}^{(l)}$ the bias vector, $z_o{}^{(l)}$ the linear combination of inputs, and $\Phi^{(l)}$ the activation function. The output of the last layer, $a_0{}^{(L)}$, is taken as the model's final prediction. In this study, the output layer has two neurons, which give the predicted values of the LHS and RHS.

$$\hat{y_o} = a_o{}^{(L)} = [\widehat{LHS}, \widehat{RHS}].$$

The activation function used in this model is the Rectified Linear Unit (ReLU), which is mathematically defined as [46]:

$$\Phi(x_o) = \max(0, x_o).$$

This function adds nonlinearity to the model, hence enabling it to learn complex and nonlinear mappings between input and output spaces. ReLU is efficient in terms of computation; it avoids the saturation problem inherent in traditional activation functions, such as (sigmoid) or (tanh) and it accelerates convergence during training. Piecewise linearity helps maintain sparse activation, meaning only a subset of neurons are active at any given time, hence improving the efficiency of the model.

The network was trained using Mean Squared Error (MSE) as the loss function. This measures how far the predicted values are from the true values by averaging the squared differences between them. The MSE is given by [49]:

$$\mathrm{MSE} = \frac{1}{\mathrm{n}}\sum_{i=1}^{\mathrm{n}}(\mathrm{y_{oi}} - \hat{\mathrm{y}}_{\mathrm{oi}})^2.$$

Here, $\mathrm{y_{oi}}$ is the true value and $\hat{\mathrm{y}}_{\mathrm{oi}}$ is the value predicted by the model. During training, we try to make this loss as small as possible so that the neural network gives more accurate results. MSE works well for regression problems, where the target values are continuous, because it punishes big errors more than small ones.

The network parameters were updated using the Adam (Adaptive Moment Estimation) optimizer, which is a very common choice in deep learning. Adam combines ideas from two other methods, Momentum and RMSProp. It gives each parameter its own learning rate by keeping running averages of the gradients and of their squares.

The update rules for Adam are given in [50] as:

$$\mathrm{m_s} = \beta_1 \mathrm{m_{s-1}} + (1-\beta_1)\mathrm{g_s}, \qquad \mathrm{v_s} = \beta_2 \mathrm{v_{s-1}} + (1-\beta_2)\mathrm{g_s^2},$$

$$\hat{\mathrm{m}}_{\mathrm{s}} = \frac{\mathrm{m_s}}{1-\beta_1^s}, \qquad \hat{\mathrm{v}}_{\mathrm{s}} = \frac{\mathrm{v_s}}{1-\beta_2^s},$$

$$\theta_{\mathrm{s}} = \theta_{\mathrm{s-1}} - \alpha\frac{\hat{\mathrm{m}}_{\mathrm{s}}}{\sqrt{\hat{v}_s} + \epsilon}.$$

Here, $\mathrm{g_s}$ is the gradient at step s, and θ_{s} are the model parameters at that step. The symbol α is the learning rate, which controls how big each update step is. The terms β_1 and β_2 are decay rates (usually 0.9 and 0.999), and ϵ is a very small number added to avoid dividing by zero. Because Adam adjusts the learning rate for each parameter on its own, it usually learns faster and more smoothly for many different types of problems. For more details about the ANN setup and training process, readers can see [51].

The model was trained for 500 epochs with a batch size of 32. A custom callback function was used to calculate and save the loss and accuracy at the end of each epoch. The accuracy was computed using the mean absolute error (MAE) between the predicted and true values, and then dividing it by the largest output value. The loss curve was checked to see if the model was learning and converging properly, while the accuracy curve showed how consistent the predictions were over time. Overall, the results showed that the model learned in a stable way and gave reliable predictions.

To clearly show how well the model was working, 3D surface plots were made to compare the predicted and actual values of the LHS and RHS functions. The two surfaces were colored in maroon and navy so they could be easily told apart. Additionally, loss and accuracy curves were plotted against epochs, with highlighted points representing the minimum loss and maximum accuracy. These visualizations validated the model's effectiveness in approximating the theoretical inequality relationships. The predictions from our ANN model are compared with the theoretical inequalities in Theorems 2.1–2.3 to illustrate and validate their accuracy.

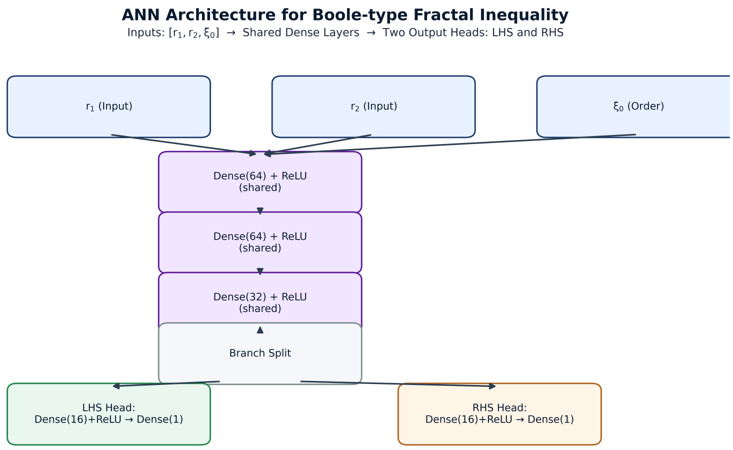

Figure 7. ANN architecture is used to predict the left-hand side (LHS) and right-hand side (RHS) of the Boole-type fractal inequality in Theorem 2.1. Inputs $[r_1, r_2, \xi_0]$ pass through a shared trunk (Dense 64-ReLU → Dense 64-ReLU → Dense 32-ReLU), then branch into two heads (each Dense 16-ReLU → Dense 1) producing the LHS and RHS outputs. Training uses Adam with MSE summed over both outputs.

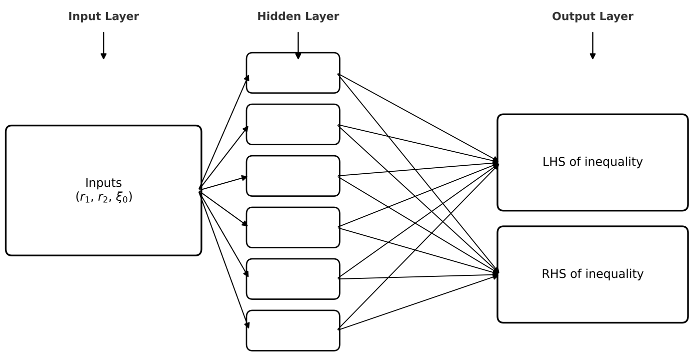

Figure 8. Schematic representation of the ANN architecture used to predict the Boole-type fractal inequality in Theorem 2.1. The inputs (r_1, r_2, ξ_0) pass through a single hidden layer of six neurons and output the predicted LHS and RHS of the inequality.

The Figures 9 and 10 shows the error histogram for the ANN model. It presents how often different errors happen in the training, validation and test sets. Most errors are close to zero, which means the model's predictions are mostly accurate. A few larger errors appear farther from zero, showing the parts where the model needs improvement.

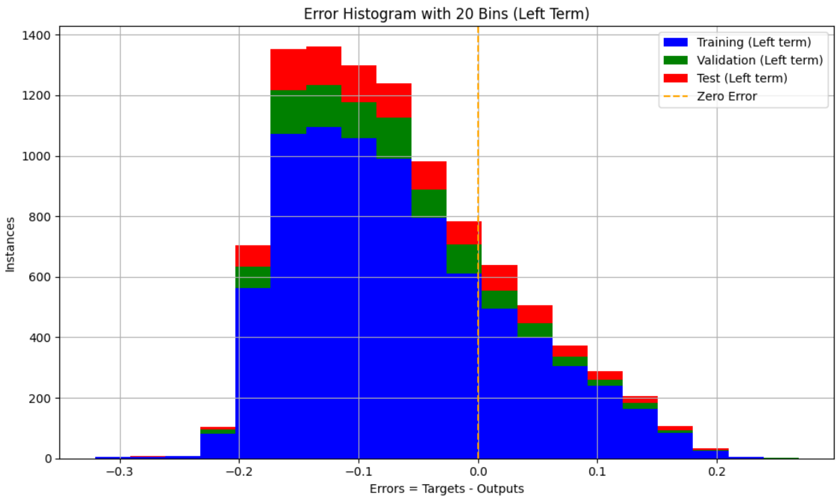

Figure 9. Error histogram of the ANN model for predicting the LHS of the fractal Boole-type inequality based on Example 5.1, which illustrates Theorem 2.1. The plot shows how much the model's predictions differ from the actual values in the training data.

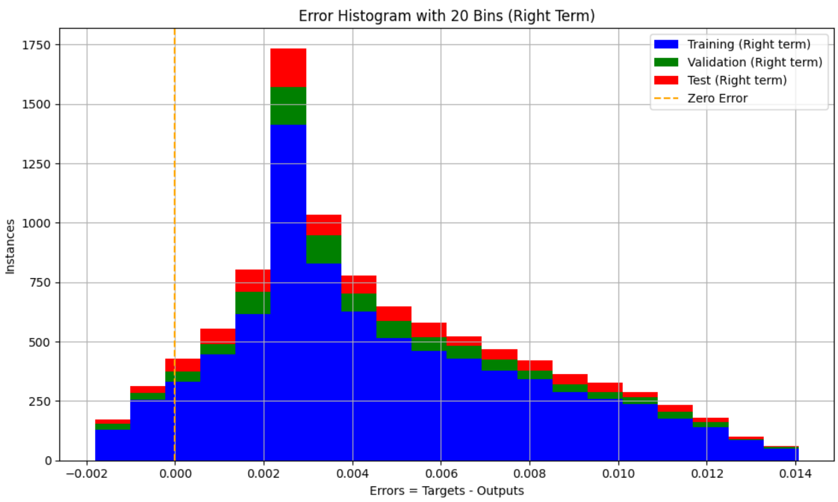

Figure 10. Error histogram of the ANN model for predicting RHS fractal Boole's-type inequality based on Example 5.1, which illustrates Theorem 2.1. The plot shows how much the model's predictions differ from the actual values in the training data.

As shown in Figures 11–13 the Artificial Neural Network (ANN) model illustrates the comparison between the predicted and exact values of the LHS and RHS of the considered Boole's type inequalities in Theorems 2.1–2.3. The model receives input parameters r_1 and r_2, which are processed through hidden layers with nonlinear activation functions to produce the outputs. The maroon and navy colored surfaces correspond to the predicted and exact values, respectively, enabling a clear visualization of the model's performance. The fact that the predicted and actual surfaces are very close to each other shows that the ANN has learned the relationship between the inputs and outputs very well.

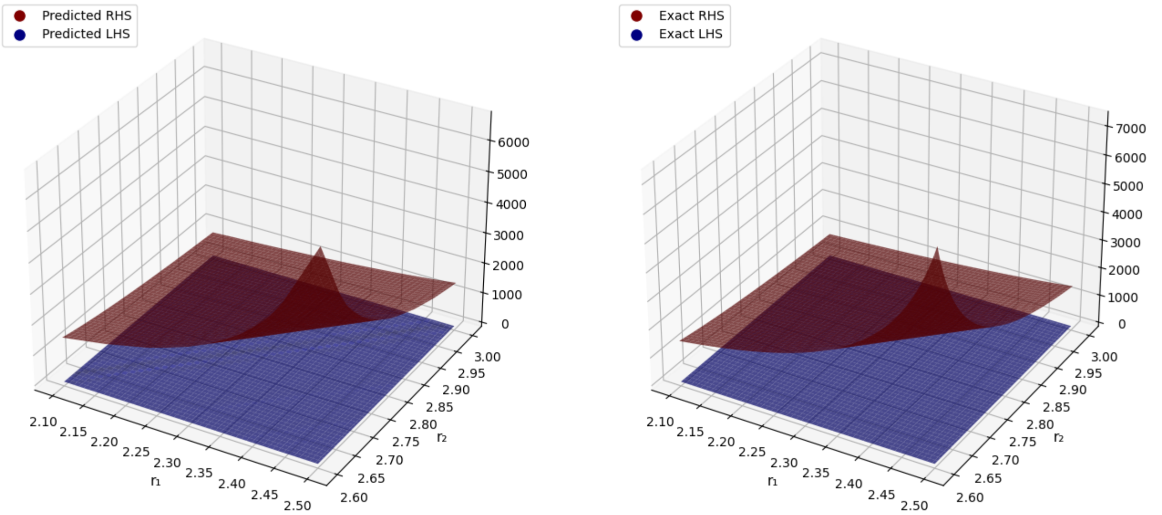

Figure 11. Predicted and actual 3D surfaces of the LHS and RHS for the fractal Boole-type inequalities in Example 5.1.

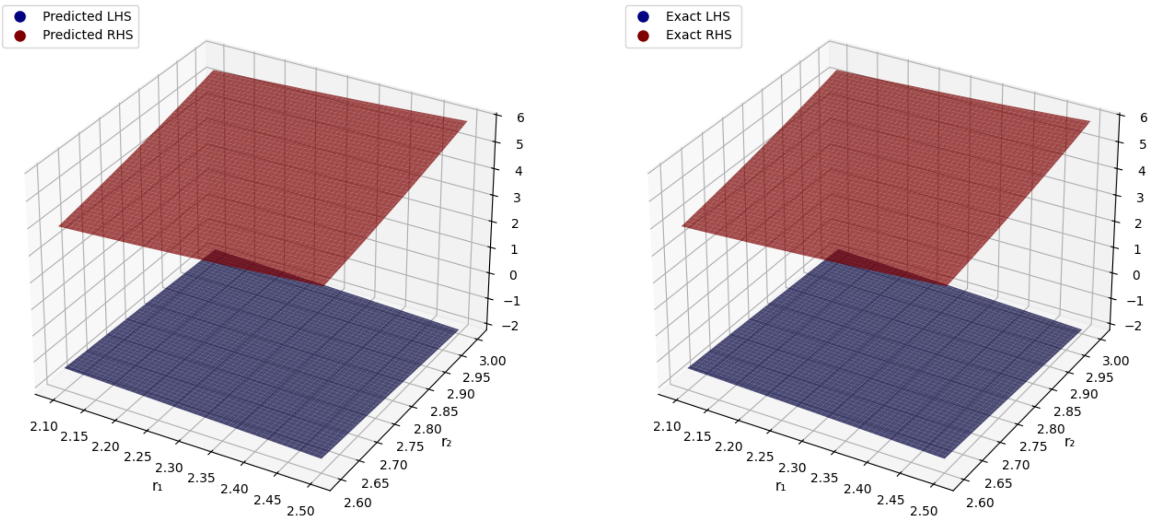

Figure 12. Predicted and actual 3D surfaces of the LHS and RHS for the fractal Boole-type inequalities in Example 5.2.

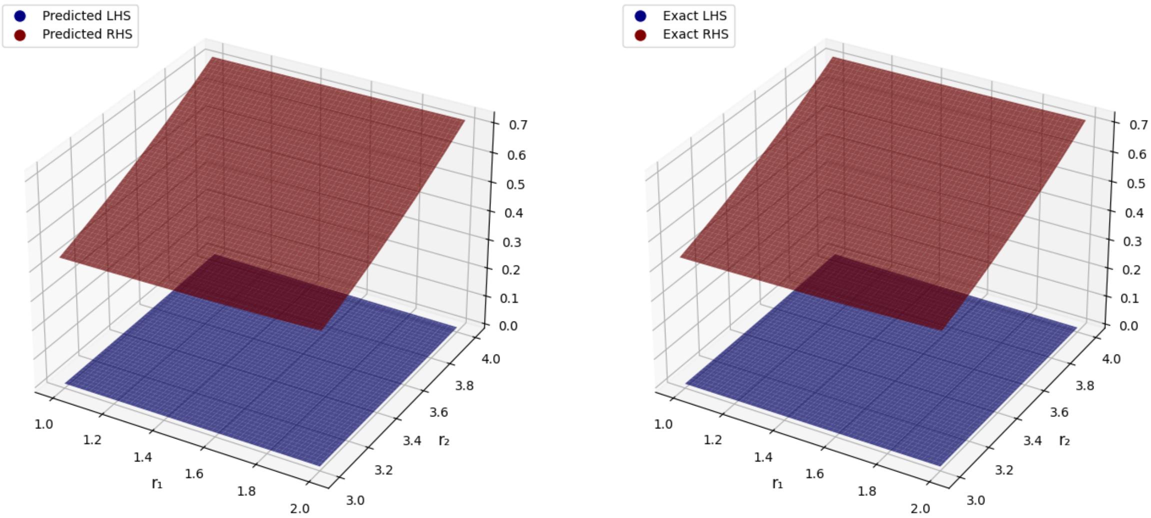

Figure 13. Predicted and actual 3D surfaces of the LHS and RHS for the fractal Boole-type inequalities in Example 5.3.

As shown in Figures 14–16, the training performance of the ANN is illustrated through the loss curves. The loss demonstrates a rapid decrease initially and gradually stabilizes as the training progresses, indicating that the model has effectively learned the underlying patterns in the data. These results confirm that the proposed neural network model has achieved stable convergence, demonstrating reliable predictive capability in estimating both sides of the inequality in examples.

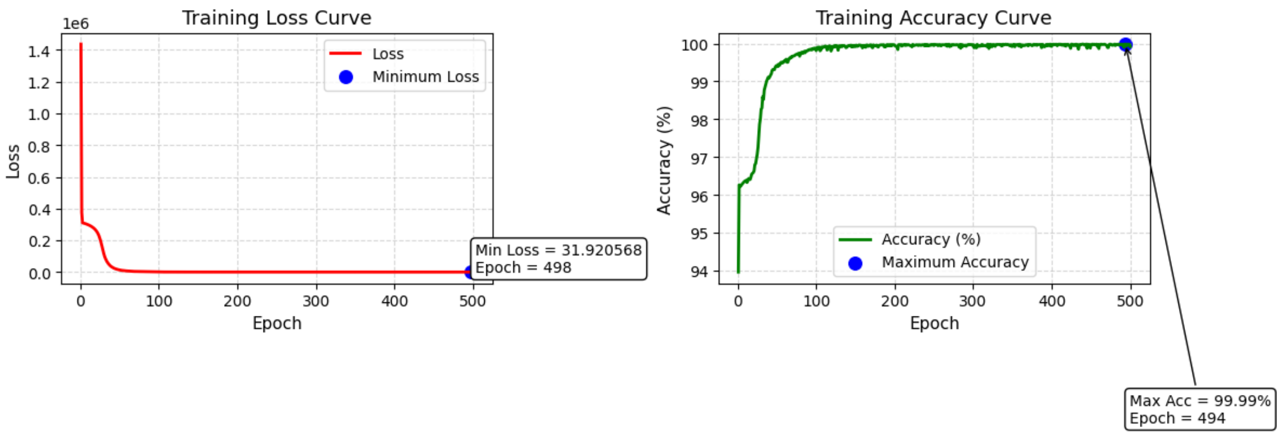

Figure 14. Training performance of the ANN model illustrating the loss and accuracy curves for the fractal Boole-type inequalities based on Example 5.1, which illustrates Theorem 2.1.

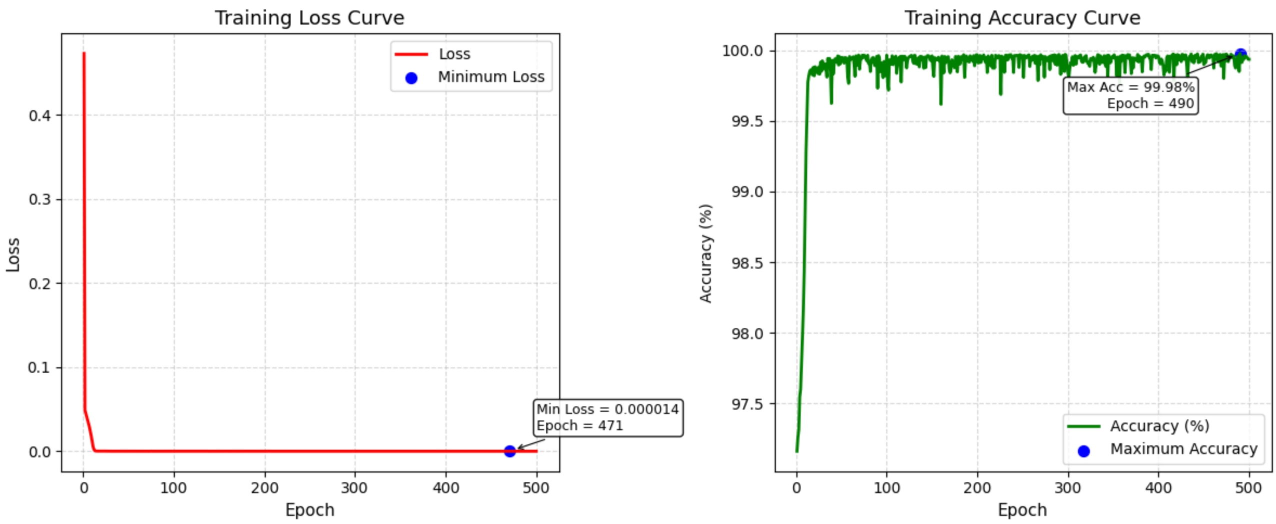

Figure 15. Training performance of the ANN model illustrating the loss and accuracy curves for the fractal Boole-type inequalities based on Example 5.2, which illustrates Theorem 2.2.

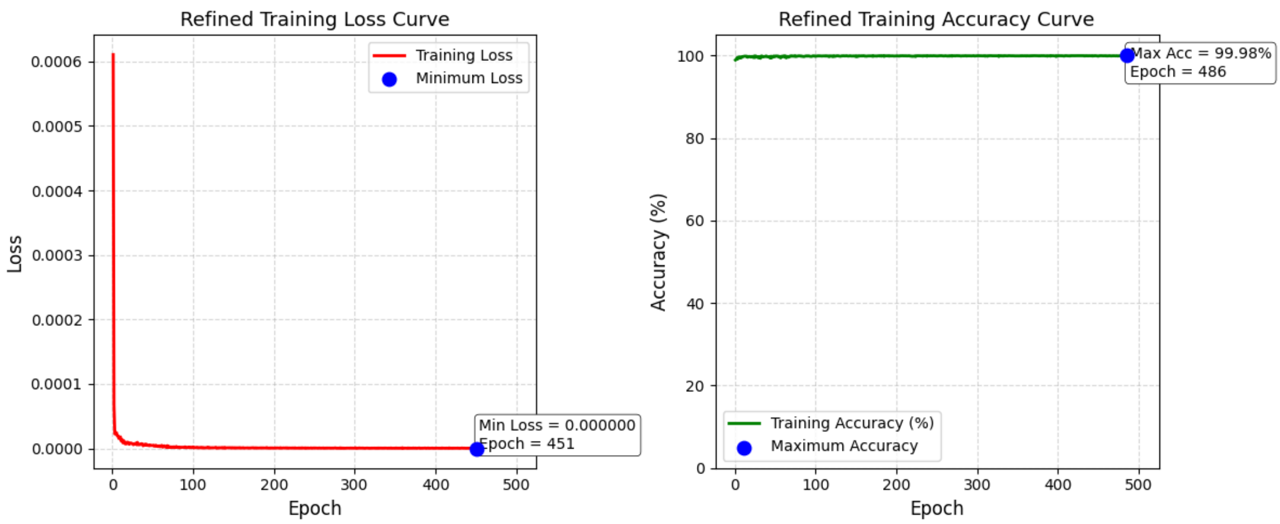

Figure 16. Training performance of the ANN model illustrating the loss and accuracy curves for the fractal Boole-type inequalities based on Example 5.3, which illustrates Theorem 2.3.

6.1. Numerical validation and error analysis

The performance of the neural network model is visualized through a graphical comparison between the exact analytical values and the predicted values generated by the trained neural network. This graphical comparison provides a clear visual representation of how closely the neural network predictions approximate the exact analytical solutions. To numerically validate the theoretical bounds established in the main Theorem 2.1, we implemented an Artificial Neural Network (ANN) model to approximate both sides of the derived inequality. The parameter $\xi_o \in (0, 1)$ was used as the input variable, while the LHS and RHS expressions involving Gamma functions were taken as output targets. Only in this case, the network was trained for 2000 epochs using a deep architecture consisting of three hidden layers with 128, 64, and 32 neurons respectively, and the hyperbolic tangent (tanh) activation function. An earlier configuration employing the ReLU activation function, with two hidden layers, 64 and 32 neurons and 500 epochs, resulted in noticeable discrepancies between the exact and predicted values. By increasing the network depth, extending the training duration and adopting the tanh activation function, which is more suitable for modeling smooth nonlinear behavior, the prediction accuracy improved significantly. The trained model was subsequently tested at selected values of ξ_o, and the predicted results were compared with the exact analytical values to evaluate the approximation accuracy.

To further check the accuracy of the trained Artificial Neural Network (ANN), predictions were generated on a dense grid of parameter values. In particular, 500 equally spaced values of $\xi_o \in [0.1, 0.8]$ were selected. These values were reshaped into column form to match the required input format of the neural network. For each selected value of ξ_o, the exact analytical expressions for the left-hand side (LHS) and right-hand side (RHS) of the derived inequality were computed using the Gamma function formulas obtained in the main theorems. These exact values were treated as reference values for comparison. The trained ANN model was then used to predict the corresponding LHS and RHS values over the same parameter range. To measure the accuracy of the predictions, the absolute error was calculated as

$$\text{Absolute Error}_{LHS} = \left|\text{LHS}_{\text{predicted}} - \text{LHS}_{\text{exact}}\right|,$$

$$\text{Absolute Error}_{RHS} = \left|\text{RHS}_{\text{predicted}} - \text{RHS}_{\text{exact}}\right|.$$

Absolute error was used to clearly show the size of the deviation without considering its sign. A comparison table was constructed containing the parameter values, exact results, predicted results, and their absolute errors. To identify the most accurate predictions, a total absolute error was defined as the sum of the LHS and RHS absolute errors, and the parameter values with the smallest total error were selected.

Finally, two graphical comparisons were presented (see Figures 17 and 18, Tables 2 and 3). The first plot compares the exact curves with the predicted values, demonstrating that the ANN results closely follow the analytical expressions. The second plot shows the absolute errors for both LHS and RHS as functions of ξ_o, illustrating that the prediction error remains small across the interval. These numerical results provide additional computational support for the theoretical findings.

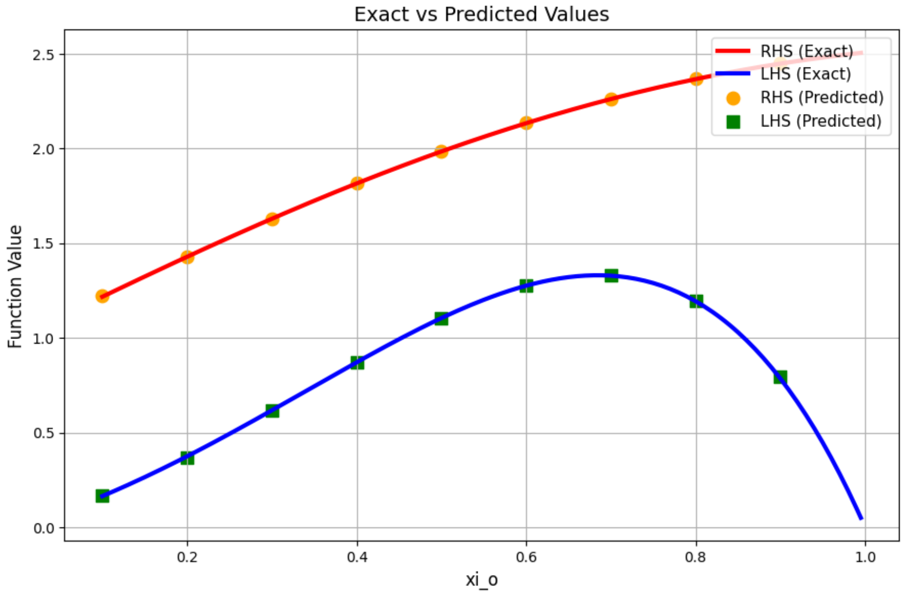

Figure 17. Exact vs predicted values of LHS and RHS inequality with respect to ξ_o, showing the accuracy of the neural network model based on Example 5.1, which represent Theorem 2.1.

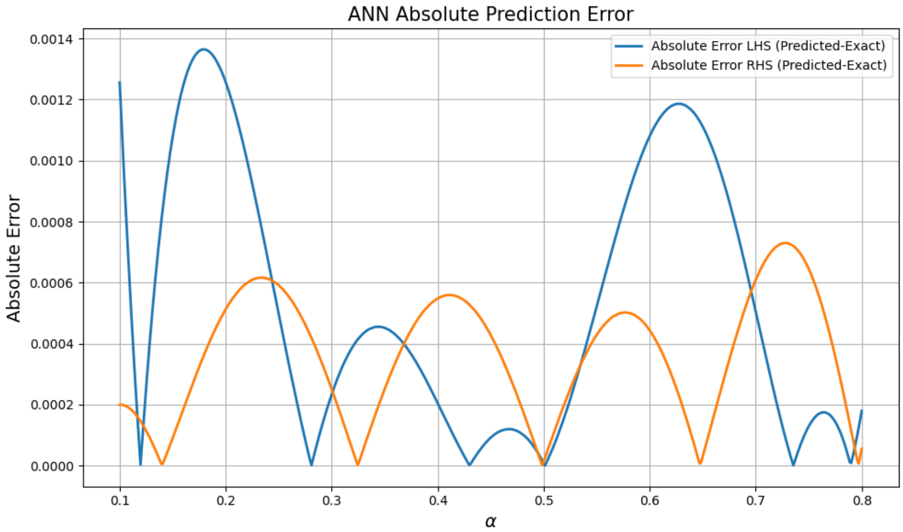

Figure 18. Absolute error comparison between exact and ANN-predicted values of LHS and RHS with respect to the fractal parameter α. The small magnitude of the errors confirms the high prediction accuracy of the trained neural network.

Table 2. Exact and ANN-Predicted Values for LHS and RHS.

ξ_o	LHS (Exact)	RHS (Exact)	LHS (Predicted)	RHS (Predicted)
0.498397	1.101398	1.982282	1.101379	1.982284
0.499800	1.104357	1.984522	1.104348	1.984510
0.501202	1.107305	1.986758	1.107305	1.986732
0.496994	1.098426	1.980038	1.098399	1.980054
0.502605	1.110240	1.988990	1.110250	1.988950
0.495591	1.095443	1.977790	1.095407	1.977821
0.504008	1.113163	1.991218	1.113184	1.991164
0.494188	1.092448	1.975539	1.092404	1.975583
0.505411	1.116073	1.993442	1.116105	1.993375
0.492786	1.089442	1.973283	1.089390	1.973342

Table 3. Absolute Errors between Exact and ANN-Predicted Values.

ξ_o	Abs Error (LHS)	Abs Error (RHS)
0.498397	1.843485×10^{-5}	0.000002
0.499800	9.376057×10^{-6}	0.000012
0.501202	3.206229×10^{-7}	0.000026
0.496994	2.746620×10^{-5}	0.000016
0.502605	1.043166×10^{-5}	0.000040
0.495591	3.600716×10^{-5}	0.000030
0.504008	2.085325×10^{-5}	0.000054
0.494188	4.383271×10^{-5}	0.000044
0.505411	3.183969×10^{-5}	0.000067
0.492786	5.167098×10^{-5}	0.000058

The ANN model is used only as a numerical tool to support and verify the theoretical result in Theorem 2.1. We compare the network's predicted values with the exact analytical values to check how close they are. In most cases, the ANN shows the same pattern as the theoretical results, but small differences appear, especially when ξ_o is small. This may happen because the data set is limited. The ANN does not replace the mathematical proofs, it only gives computational support. In the future, such models may help identify patterns that could lead to new inequalities, showing that machine learning can assist in exploring number patterns and discovering new results.

7. Applications

We introduce some innovative applications of our key results in this section.

7.1. Generalized special means

Within the framework of local fractional calculus, the following define the ξ_0-type special means and n-logarithmic means for $0 < r_1 < r_2$:

(1) The arithmetic mean in a generalized sense:

$$\mathrm{A}_{\xi_o}(\mathrm{r}_1,\mathrm{r}_2)=\frac{\mathrm{r_1}^{\xi_o}+\mathrm{r_2}^{\xi_o}}{2^{\xi_o}}.$$

(2) The n-logarithmic mean in a generalized sense:

$$\mathrm{L_n}\xi_o(\mathrm{r}_1,\mathrm{r}_2)=\Big[\frac{\Gamma(1+\mathrm{n}\xi_o)}{\Gamma(1+(\mathrm{n}+1)\xi_o)}\frac{(\mathrm{r}_2)^{(\mathrm{n}+1)\xi_o}-(\mathrm{r}_1)^{(\mathrm{n}+1)\xi_o}}{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}\Big]^{\frac{1}{\mathrm{n}}},$$

where $\mathrm{n}\in\mathbb{Z}\setminus\{-1,0\}$, $\mathrm{r}_1,\mathrm{r}_2\in\mathrm{R}$, $\mathrm{r}_1=\mathrm{r}_2$.

Proposition 7.1. *Given* $0<\mathrm{r}_1<\mathrm{r}_2$, $\xi_o\in(0,1]$, *and* $n\geq 1$, *and assuming all the requirements of Theorem 2.1 are fulfilled, it follows that:*

$$\begin{aligned}&\Big|\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}\mathrm{r_1}^{(\mathrm{u}_o+1)\xi_o}+32^{\xi_o}\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{2}\Big)^{(\mathrm{u}_o+1)\xi_o}+12^{\xi_o}(\mathrm{A}_{\xi_o}(\mathrm{r}_1,\mathrm{r}_2))^{(\mathrm{u}_o+1)}\\&+32^{\xi_o}\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{2}\Big)^{(\mathrm{u}_o+1)\xi_o}+7^{\xi_o}\mathrm{r_2}^{(\mathrm{u}_o+1)\xi_o}\Big]-\frac{\Gamma(1+\xi_o)\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\mathrm{L}^{(\mathrm{u}_o+1)}_{(\mathrm{u}_o+1)\xi_o}(\mathrm{r}_1,\mathrm{r}_2)\Big|\\&\leq\Big(\frac{239}{1620}\Big)^{\xi_o}\frac{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{2^{\xi_o}}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}[(\mathrm{r}_1)^{\mathrm{u}_o\xi_o}+(\mathrm{r}_2)^{\mathrm{u}_o\xi_o}].\end{aligned}$$

Proof. Consider the function $\Phi:\mathrm{I}=(0,\infty)\to\mathbb{R}^{\xi_0}$ defined by

$$\Phi(\varkappa)=\frac{\Gamma(1+\mathrm{u}_o\xi_0)}{\Gamma(1+(\mathrm{u}_o+1)\xi_0)}\varkappa^{(\mathrm{u}_o+1)\xi_0},$$

whose derivative

$$\Phi^{(\xi_o)}(\varkappa)=\varkappa^{\mathrm{u}_o\xi_0},$$

is a non-negative generalized convex function on the interval $(0,+\infty)$. Hence, we have

$$\begin{aligned}&\Phi(\mathrm{r}_1)=\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\mathrm{r_1}^{(\mathrm{u}_o+1)\xi_o},\quad|\Phi^{(\xi_o)}(\mathrm{r}_1)|=(\mathrm{r}_1)^{\mathrm{u}_o\xi_o}\\&\Phi(\mathrm{r}_2)=\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\mathrm{r_2}^{(\mathrm{u}_o+1)\xi_o},\quad|\Phi^{(\xi_o)}(\mathrm{r}_2)|=(\mathrm{r}_2)^{\mathrm{u}_o\xi_o}\\&\Phi\Big(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\Big)=\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\Big(\frac{\mathrm{r}_1+\mathrm{r}_2}{2}\Big)^{(\mathrm{u}_o+1)\xi_o}=\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}(\mathrm{A}_{\xi_o}(\mathrm{r}_1,\mathrm{r}_2))^{(\mathrm{u}_o+1)}\\&\Phi\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{2}\Big)=\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\Big(\frac{3\mathrm{r}_1+\mathrm{r}_2}{2}\Big)^{(\mathrm{u}_o+1)\xi_o}\\&\Phi\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{2}\Big)=\frac{\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\Big(\frac{\mathrm{r}_1+3\mathrm{r}_2}{2}\Big)^{(\mathrm{u}_o+1)\xi_o}\end{aligned}$$

and

$$\frac{\Gamma(1+\xi_o)}{(\mathrm{r}_2-\mathrm{r}_1)^{\xi_o}}{}_{\mathrm{r}_1}\mathrm{I}^{\xi_o}_{\mathrm{r}_2}\Phi(\varkappa)=\frac{\Gamma(1+\xi_o)\Gamma(1+\mathrm{u}_o\xi_o)}{\Gamma(1+(\mathrm{u}_o+1)\xi_o)}\mathrm{L}^{(\mathrm{u}_o+1)}_{(\mathrm{u}_o+1)\xi_o}(\mathrm{r}_1,\mathrm{r}_2).$$

By utilizing the aforementioned values and the definition of special means in Theorem 2.1, the desired result can be obtained. □

Proposition 7.2. *Given* $0 < r_1 < r_2$, $\xi_o \in (0,1]$, *and* $n \geq 1$, *and assuming all the requirements of Theorem 2.2 are fulfilled, it follows that:*

$$\begin{aligned}
&\Bigg|\frac{\Gamma(1+u_0\xi_o)}{\Gamma(1+(u_0+1)\xi_o)}\frac{1^{\xi_o}}{90^{\xi_o}}\Big[7^{\xi_o}{r_1}^{(u_0+1)\xi_o}+32^{\xi_o}\Big(\frac{3r_1+r_2}{2}\Big)^{(u_0+1)\xi_o}+12^{\xi_o}(A_{\xi_o}(r_1,r_2))^{(u_0+1)}\\
&+32^{\xi_o}\Big(\frac{r_1+3r_2}{2}\Big)^{(u_0+1)\xi_o}+7^{\xi_o}{r_2}^{(u_0+1)\xi_o}\Big]-\frac{\Gamma(1+\xi_o)\Gamma(1+u_0\xi_o)}{\Gamma(1+(u_0+1)\xi_o)}L^{(u_0+1)}_{(u_0+1)\xi_o}(r_1,r_2)\Bigg|\\
&\leq\frac{(r_2-r_1)^{\xi_o}}{2^{\xi_o}}\Bigg[\Big(\frac{\Gamma(1+p\xi_o)}{4^{(p+1)\xi_o}\Gamma(1+(1+p)\xi_o)}\Big(\Big(\frac{14}{45}\Big)^{(p+1)\xi_o}+\Big(\frac{31}{45}\Big)^{(p+1)\xi_o}\Big)\Big)^{\frac{1}{p}}\\
&\times\Bigg\{\Bigg(\Big(\frac{1}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_2)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_1)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\Big(\frac{1}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_1)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_2)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\\
&+\Big(\frac{\Gamma(1+p\xi_o)}{4^{(p+1)\xi_o}\Gamma(1+(1+p)\xi_o)}\Big(\Big(\frac{11}{15}\Big)^{(p+1)\xi_o}+\Big(\frac{4}{15}\Big)^{(p+1)\xi_o}\Big)\Big)^{\frac{1}{p}}\\
&\times\Bigg\{\Bigg(\Big(\frac{3}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_2)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-3^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_1)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\Big(\frac{3}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_1)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-3^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_2)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\\
&+\Big(\frac{\Gamma(1+p\xi_o)}{4^{(p+1)\xi_o}\Gamma(1+(1+p)\xi_o)}\Big(\Big(\frac{4}{15}\Big)^{(p+1)\xi_o}+\Big(\frac{11}{15}\Big)^{(p+1)\xi_o}\Big)\Big)^{\frac{1}{p}}\\
&\times\Bigg\{\Bigg(\Big(\frac{5}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_2)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-5^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_1)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\Big(\frac{5}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_1)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-5^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_2)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\\
&+\Big(\frac{\Gamma(1+p\xi_o)}{4^{(p+1)\xi_o}\Gamma(1+(1+p)\xi_o)}\Big(\Big(\frac{14}{45}\Big)^{(p+1)\xi_o}+\Big(\frac{31}{45}\Big)^{(p+1)\xi_o}\Big)\Big)^{\frac{1}{p}}\\
&\times\Bigg\{\Bigg(\Big(\frac{7}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_2)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-7^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_1)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\\
&+\Bigg(\Big(\frac{7}{16}\Big)^{\xi_o}\frac{\Gamma(1+\xi_o)}{\Gamma(1+2\xi_o)}|(r_1)^{u_0\xi_o}|^{q_o}+\Big(\frac{4^{\xi_o}\Gamma(1+2\xi_o)-7^{\xi_o}(\Gamma(1+\xi_o))^2}{16^{\xi_o}\Gamma(1+\xi_o)\Gamma(1+2\xi_o)}\Big)|(r_2)^{u_0\xi_o}|^{q_o}\Bigg)^{\frac{1}{q_o}}\Bigg\}\Bigg].
\end{aligned}$$

Proof. Consider the function $\Phi : I = (0,\infty) \to \mathbb{R}^{\xi_0}$ defined by

$$\Phi(\varkappa) = \frac{\Gamma(1+u_0\xi_0)}{\Gamma(1+(u_0+1)\xi_0)}\varkappa^{(u_0+1)\xi_0},$$

whose derivative

$$\Phi^{(\xi_o)}(\varkappa) = \varkappa^{u_0\xi_0},$$

is a non-negative generalized convex function on the interval $(0,+\infty)$. Therefore, we have

$$\Phi(r_1) = \frac{\Gamma(1+u_0\xi_o)}{\Gamma(1+(u_0+1)\xi_o)}{r_1}^{(u_0+1)\xi_o}, \quad |\Phi^{(\xi_o)}(r_1)| = (r_1)^{u_0\xi_o}$$

$$\Phi(r_2) = \frac{\Gamma(1+u_o\xi_o)}{\Gamma(1+(u_o+1)\xi_o)} r_2^{(u_o+1)\xi_o}, \quad |\Phi^{(\xi_o)}(r_2)| = (r_2)^{u_o\xi_o}$$

$$\Phi\Big(\frac{r_1+r_2}{2}\Big) = \frac{\Gamma(1+u_o\xi_o)}{\Gamma(1+(u_o+1)\xi_o)}\Big(\frac{r_1+r_2}{2}\Big)^{(u_o+1)\xi_o} = \frac{\Gamma(1+u_o\xi_o)}{\Gamma(1+(u_o+1)\xi_o)}(A_{\xi_o}(r_1,r_2))^{(u_o+1)}$$

$$\Phi\Big(\frac{3r_1+r_2}{2}\Big) = \frac{\Gamma(1+u_o\xi_o)}{\Gamma(1+(u_o+1)\xi_o)}\Big(\frac{3r_1+r_2}{2}\Big)^{(u_o+1)\xi_o}$$

$$\Phi\Big(\frac{r_1+3r_2}{2}\Big) = \frac{\Gamma(1+u_o\xi_o)}{\Gamma(1+(u_o+1)\xi_o)}\Big(\frac{r_1+3r_2}{2}\Big)^{(u_o+1)\xi_o}$$

and

$$\frac{\Gamma(1+\xi_o)}{(r_2-r_1)^{\xi_o}}\, {}_{r_1}I_{r_2}^{\xi_o}\Phi(\varkappa) = \frac{\Gamma(1+\xi_o)\Gamma(1+u_o\xi_o)}{\Gamma(1+(u_o+1)\xi_o)} L_{(u_o+1)\xi_o}^{(u_o+1)}(r_1,r_2).$$

By applying the above values and the definition of special means from Theorem 2.2, we obtain the desired result. □

8. Conclusions

In this study, we presented new Boole's type inequalities for local fractional calculus that applied to differentiable convex functions. We attained this through establishing novel inequalities by using the helpful fundamental identity that we first established. Furthermore, we investigated function of several types, such as convex, bounded and Lipschitz functions and used this method to produce novel findings over fractal sets. We also discussed applications of these fractal Boole's type inequalities to special means. Future research in local fractional integrals, convexity theory and problems involving higher-order derivatives can use the outcomes of this study. These outcomes may also be helpful for solving many real-world problems in mathematics. This study clearly shows that the Artificial Neural Network (ANN) can successfully estimate very complex mathematical relationships between the given input variables and the outputs of the proposed inequalities. The model successfully learned the main behaviour of the functions and produced exact values for both sides of the inequality, even with multiple hidden layers and nonlinear activation functions. The strong relation between the predicted and exact outcomes shows that the ANN is a reliable and effective tool for analyzing and testing the proposed mathematical framework. These results show that the model can be used in future for other areas of mathematics and other researchers can use our outcomes as a reference for their own work. Overall, this study provides a strong foundation for future work on integral inequalities using ANNs and researchers can further extend and apply this approach to other areas of mathematics.

Author contributions

Conceptualization: M.A., S.I.B., and Y.S.; methodology: M.M., and S.I.B.; software: M.A; validation: M.M., S.I.B., M.A., and Y.S.; formal analysis: Y.S., and M.M.; investigation: M.A., and S.I.B.; writing—original draft preparation: M.M.; writing—review and editing: M.A., S.I.B. and M.M.; visualization: Y.S.; supervision: S.I.B.; project administration: Y.S.; funding acquisition: Y.S. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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