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Research article

Analytical solutions to some one-dimensional compressible fluid equations

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Abstract: In this paper, we construct some self-similar analytical solutions for some one-dimensional compressible fluid equations. For the isentropic Euler equations with heat conduction, we present an analytical solution with the temperature decaying in time at the rate of $O((1+t)^{-1})$ for the free boundary problem, provide an analytical solution for the Cauchy problem, and investigate its blowup and decay phenomena. We also construct some analytical solutions for the isothermal Euler equations with heat conduction. Moreover, we give an exact solution to the isentropic Euler equations with time-dependent damping and construct an analytical solution to the Navier-Stokes equations with density-dependent viscosity for $\gamma = 3$, respectively, where γ is the adiabatic exponent.

Keywords: Euler equations; Navier-Stokes equations; self-similar; analytical solutions
Mathematics Subject Classification: 35C06, 35Q30, 35Q31

1. Introduction

In the last several decades, the analytical solutions to the Euler equations and the Navier-Stokes equations for compressible fluids have attracted many researchers' interest. Constructing special analytical solutions can help one better understand the nonlinear phenomena of the system under consideration. For instance, the blowup and expanding phenomena of the radially symmetric Euler equations for a compressible fluid were exhibited in [1,2] by providing some analytical solutions. In [3], Gugat and Ulbrich studied the flow reversal phenomena in the pipeline networks modeled by the one-dimensional isothermal Euler equations with gravity and friction by constructing some product solutions. The concentration and cavitation phenomena were investigated for the isentropic Euler system with the logarithmic equation of state in [4] by presenting some special analytical solutions. For the Navier-Stokes equations, Guo and Xin [5] studied the formation of a vacuum and the spreading rate of the free boundary for the free boundary problem in the spherical symmetry case. We can refer to [6–8] and the references therein for more results about the analytical solutions to the Euler equations and the Navier-Stokes equations for compressible fluids.

In this paper, we are concerned with the analytical solutions to three systems of compressible fluid equations. The first one is the following one-dimensional Euler equations with heat conduction (see [9] for instance):

$$\begin{cases} \rho(x,t)_t + [\rho(x,t)u(x,t)]_x = 0, \\ u(x,t)_t + u(x,t)u(x,t)_x = -\frac{1}{\rho(x,t)}p(x,t)_x, \\ T(x,t)_t + u(x,t)T(x,t)_x = kT(x,t)_{xx}, \end{cases} \tag{1.1}$$

where the fluid density $\rho(x,t)$, the fluid velocity $u(x,t)$, and the fluid temperature $T(x,t)$ are the unknown variables and $p(x,t) = A\rho^\gamma$ denotes the pressure function of density with $A > 0$ and $\gamma \geq 1$ being two constants. When $\gamma = 1$, the fluid is called isothermal; when $\gamma > 1$, the fluid is called isentropic. $k > 0$ is the heat conduction coefficient. In [9], Barna and Matyas found an exact solution for (1.1) with $\gamma = 3$ by using the self-similar Ansatz of the form

$$V(x,t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right), \tag{1.2}$$

where $V(x,t)$ is an arbitrary variable of (1.1) and α is the rate of decay of the magnitude $V(x,t)$, whereas β represents the rate of spread (or contraction if $\beta < 0$) of the space distribution as time grows up. In this paper, we use another self-similar Ansatz to construct analytical solutions to (1.1):

$$\rho(x,t) = \frac{f\left(\frac{x}{a(t)}\right)}{a(t)}, \;\; u(x,t) = \frac{a'(t)}{a(t)}x, \tag{1.3}$$

where $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$. In fact, (1.3) is the one-dimensional version of Lemma 3 in [10], which has been widely used to construct analytical solutions to the Navier-Stokes equations and the Euler equations for compressible fluids; see [4,5,11] for instance. Although some analytical solutions to the Euler equations were provided in [11–13] for the isentropic case, there was no information about the fluid temperature in these works. In this paper, we will present some analytical solutions to (1.1) by using the Ansatz (1.3); see the next section.

The second system that we study in this paper is the following one-dimensional Euler equations with time-dependent damping (see [14]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + [p(\rho)]_x = -\frac{\mu}{(1+t)^\lambda}\rho u, \end{cases} \tag{1.4}$$

where the fluid density ρ and the fluid velocity u are the unknown variables, the function $p(\rho) = A\rho^\gamma$ with $A > 0$ and $\gamma \geq 1$ is the pressure, and $\mu > 0$ and $\lambda \in \mathbb{R}$ denote the damping coefficients. System (1.4) can be used to describe the motion of compressible flow through a porous medium. The term $-\frac{\mu}{(1+t)^\lambda}\rho u$ represents the damping effect, which is called under-damping if $\lambda > 0$ and over-damping if $\lambda < 0$ (see [15,16]), where the large-time behavior of the multi-dimensional model of (1.4) was studied for an under-damping case and over-damping case, respectively. Furthermore, the damping is called critically over-damping* for $\lambda = -1$; see [17]. The global existence and blowup of smooth

*Here, the phrase "critically over-damping" and later "critically under-damping" are both mathematical definitions in the context of damping.

solutions to (1.4) with non-vacuum initial data for some values of λ and μ were investigated in [18,19]; see [20,21] for the multi-dimensional case. For the analytical solutions to (1.4), Pan [14] constructed a solution on a half line with one-side physical vacuum and proved its stability for $0 < \lambda < 1, \mu > 0$ or $\lambda = 1, \mu > 2$. For the N-dimensional radially symmetric case, Dong, Lou, and Zhang [11] constructed some analytical blowup solutions for $\lambda > 0$. Moreover, the self-similar analytical solution to the free boundary problem of spherical Euler equations was presented in [12] for $\lambda > 1$; see [13] for the cylindrically symmetric case with $\lambda \geq 1$. In this paper, we will provide an exact solution to (1.4) for $\lambda = 1$ by using the following self-similar Ansatz:

$$\rho(x,t) = (1+t)^{-\alpha} h\left(\frac{x}{(1+t)^{\beta}}\right), \;\; u(x,t) = (1+t)^{-\delta} g\left(\frac{x}{(1+t)^{\beta}}\right), \tag{1.5}$$

which is similar to (1.2); see Section 3. For $\lambda = 1$, the damping is called critically under-damping. The analytical solutions for this case play an important role in investigating the critical behavior of the system.

The third system that we consider in this paper is the following one-dimensional Navier-Stokes equations with density-dependent viscosity (see [8]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + [p(\rho)]_x - (\mu(\rho) u_x)_x = 0, \end{cases} \tag{1.6}$$

where the unknown variables are the fluid density ρ and the fluid velocity u, $p(\rho) = A\rho^{\gamma}$ is the pressure, and $\mu(\rho) = B\rho$ is the viscosity coefficient with $B > 0$ being a constant. In [8], Dong and Zhang constructed an exact solution to (1.6) for $\gamma = 1$ by using a non-self-similar Ansatz and investigated the large-time asymptotic behavior of the density according to various A. For the multi-dimensional case, Guo and Xin [5] provided some spherical analytical solutions to the free boundary problem of Navier-Stokes equations with density-dependent viscosity; see [22] for the corresponding improved results and [23] for the cylindrically symmetric case. In this paper, we will present an analytical solution to (1.6) with $\gamma = 3$ on the half line $[0, +\infty)$ by using the Ansatz (1.5). The choice of $[0, +\infty)$ ensures the positivity of the density, and the class of solutions we construct does not involve solutions for which the density is positive on the entire line; see Section 4.

We choose the above three systems as our research object because they represent progressive physical complexities added to the standard Euler equations for a compressible fluid (heat conduction, damping, and viscosity). The self-similar transformations such as (1.2) (or (1.5)) and (1.3) are mathematically necessary to decouple the partial differential equations into integrable ordinary differential equations by separating temporal and spatial variables, for other Ansatzes; see [24] and references therein. Compared to [9], using the Ansatz (1.3) rather than (1.2), we obtain some new information about the behavior of the temperature for the system (1.1); see Remark 2.1 of the next section. Let us mention that the equations of state for the pressure used in [4,25,26] are different from (1.4). While in [11–13], the function $a(t)$ in the Ansatz (1.3) satisfies an ordinary differential equation, which does not have an analytical solution, so we present an exact solution for (1.4) by using the Ansatz (1.5) and discuss some qualitative behavior of the density depending on μ. For the system (1.6), the analytical solutions constructed in [8] are non-self-similar, whereas the analytical solutions in this paper are self-similar. We obtain an analytical solution for (1.6) that does not feature emergence of vacuum in finite time, whereas analytical solutions to the physical vacuum boundary problem for the multi-dimensional Navier-Stokes equations with a certain symmetry were constructed in [5,22,23,27].

While this paper focuses on classical equations for compressible fluids, we note that related analytical techniques have been extended to more complex frameworks, including the two-phase flow models [28], the fractional models for incompressible fluid flow [29], hydromagnetic free convection flow [30], Maxwell fluid flow [31], and magnetohydrodynamic free convection flow [32].

2. Analytical solutions to the system (1.1)

First, we consider (1.1) on the moving interval $[-a(t), a(t)]$ $(a(t) > 0)$, where $\pm a(t)$ is the free boundary of fluid. We supplement (1.1) with the following initial and boundary data:

$$\rho(x,0) = \rho_0(x), \;\; u(x,0) = u_0(x), \;\; T(x,0) = T_0(x), \;\; x \in [-a_0, a_0], \tag{2.1}$$

$$\rho(-a(t),t) = \rho(a(t),t) = 0, \tag{2.2}$$

where $[-a_0, a_0]$ is the interval occupied by the fluid initially. The problems (1.1), (2.1), and (2.2) describe the motion of the one-dimensional fluid on an interval surrounded by vacuum. The first result of this section is as follows.

Theorem 2.1. *If $1 < \gamma \le 2$ and $T_0(x) = b_0 x$ with b_0 being a constant, there is an analytical solution to the problems (1.1), (2.1), and (2.2) of the form*

$$\rho(x,t) = \frac{\left[\max\left\{0, \frac{\gamma-1}{2\gamma}\left(1 - \frac{x^2}{a(t)^2}\right)\right\}\right]^{\frac{1}{\gamma-1}}}{a(t)}, \;\; u(x,t) = \frac{a'(t)}{a(t)}x, \;\; T(x,t) = \frac{a_0 b_0}{a(t)}x, \tag{2.3}$$

where $a(t)$ satisfies

$$a''(t) = \frac{A}{a(t)^\gamma}, \;\; a(0) = a_0 > 0, \;\; a'(0) = a_1 \tag{2.4}$$

with a_1 being the initial slope of the free boundary. Moreover, if $F(0) = \int_{-a_0}^{a_0} \rho_0 u_0 x dx > 0$, then, the interval $[-a(t), a(t)]$ occupied by the fluid grows linearly in time, and the absolute value of the fluid temperature $|T(x,t)|$ decays to zero at the rate of $O((1+t)^{-1})$.

Outline of the proof of Theorem 2.1. To prove Theorem 2.1, we first obtain the analytical solution (2.3) and reduce the original partial differential equations to the ordinary differential equation (2.4) by using the Ansatz method, then we prove the free boundary $a(t)$ expands outward linearly in time by using the averaged quantity method.

Proof. It is not difficult to verify that the Ansatz (1.3) satisfies the first equation of (1.1). When $p(x,t) = A\rho^\gamma$ with $\gamma > 1$, we plug (1.3) into the second equation of (1.1) to have

$$\frac{a''(t)}{a(t)}x + \frac{A\gamma}{\gamma-1} \cdot \frac{\left[f^{\gamma-1}(\frac{x}{a(t)})\right]_x'}{a(t)^{\gamma-1}} = 0, \tag{2.5}$$

which becomes

$$a''(t)s + \frac{A\gamma}{\gamma-1} \cdot \frac{[f^{\gamma-1}(s)]_s'}{a(t)^\gamma} = 0 \tag{2.6}$$

by letting $s = \frac{x}{a(t)}$. We further require

$$-s = \frac{\gamma}{\gamma - 1} \cdot [f^{\gamma-1}(s)]'_s, \quad f(1) = 0. \tag{2.7}$$

Then we get

$$f(s) = \left[\frac{\gamma - 1}{2\gamma}(1 - s^2)\right]^{\frac{1}{\gamma-1}}, \quad s \in [-1, 1], \tag{2.8}$$

which together with (1.3) leads to

$$\rho(x,t) = \frac{\left[\frac{\gamma-1}{2\gamma}\left(1 - \frac{x^2}{a(t)^2}\right)\right]^{\frac{1}{\gamma-1}}}{a(t)}, \quad x \in [-a(t), a(t)]. \tag{2.9}$$

Moreover, by (2.6) and (2.7), $a(t)$ satisfies (2.4).

For (2.4), we can prove it has a local C^2-solution by using a fixed-point theorem as in [5]; here, we omit the details. By $a(0) = a_0 > 0$ and continuity, $a(t) > 0$ on a short time interval $[0, t^*)$. Set

$$T^* = \sup\{t^* | a(t) > 0, \ \forall t \in [0, t^*)\}. \tag{$*$}$$

We can prove $T^* = +\infty$ by contradiction. Indeed, if $T^* < +\infty$, then $\lim_{t \to T^{*-}} a(t) = 0$ by continuity. For any fixed $t \in (0, T^*)$, we multiply (2.4) by $a'(t)$ and integrate it over $[0, t]$ to have

$$\frac{1}{2}a'(t)^2 + \frac{A}{(\gamma-1)a(t)^{\gamma-1}} = \frac{1}{2}a_1^2 + \frac{A}{(\gamma-1)a_0^{\gamma-1}}. \tag{$**$}$$

Letting $t \to T^{*-}$ in ($**$) to have $\lim_{t \to T^{*-}} a'(t)^2 = -\infty$ is a contradiction. Similarly, we can also prove $a(t) \nrightarrow 0$ as $t \to +\infty$. Consequently, there exists a positive constant M such that $a(t) \geq M$.

To obtain a formula for $T(x,t)$, we assume that $T(x,t) = b(t)x$, where $b(t) \in C^1$ will be determined later. Substituting $T(x,t) = b(t)x$ and $u(x,t) = \frac{a'(t)}{a(t)}x$ into the third equation of (1.1), we get

$$b'(t)x + \frac{a'(t)}{a(t)}b(t)x = 0, \tag{2.10}$$

which can be solved as

$$b(t) = \frac{a_0 b_0}{a(t)}, \tag{2.11}$$

where b_0 is a constant satisfying $T_0(x) = b_0 x$. So, we obtain $T(x,t) = \frac{a_0 b_0}{a(t)}x$.

To investigate the spreading rate of the fluid and the decay rate of the temperature, we need to study the large-time behavior of $a(t)$ through the ordinary differential equation (2.4) combining with the following averaged quantities:

$$I(t) = \int_{-a(t)}^{a(t)} \rho x^2 dx, \tag{2.12}$$

$$m(t) = \int_{-a(t)}^{a(t)} \rho dx, \tag{2.13}$$

$$F(t)=\int_{-a(t)}^{a(t)}\rho uxdx, \tag{2.14}$$

$$E(t)=\frac{1}{2}\int_{-a(t)}^{a(t)}\rho u^2dx+\frac{A}{\gamma-1}\int_{-a(t)}^{a(t)}\rho^\gamma dx. \tag{2.15}$$

We use the first equation of (1.1) and the condition (2.2) to obtain

$$I'(t)=\int_{-a(t)}^{a(t)}\rho_t x^2dx=-\int_{-a(t)}^{a(t)}(\rho u)_x x^2dx=2\int_{-a(t)}^{a(t)}\rho uxdx=2F(t), \tag{2.16}$$

$$m'(t)=\int_{-a(t)}^{a(t)}\rho_t dx=-\int_{-a(t)}^{a(t)}(\rho u)_x dx=0, \tag{2.17}$$

which implies that

$$m(t)=m(0). \tag{2.18}$$

Multiplying $(1.1)_1$ and $(1.1)_2$ by u and ρ, respectively, the sum of the two resultant equations is

$$(\rho u)_t+(\rho u^2)_x+[p(\rho)]_x=0. \tag{2.19}$$

By virtue of (2.19) and the condition (2.2), we get

$$F'(t)=\int_{-a(t)}^{a(t)}(\rho u)_t xdx=-\int_{-a(t)}^{a(t)}(\rho u^2)_x xdx-A\int_{-a(t)}^{a(t)}(\rho^\gamma)_x xdx=\int_{-a(t)}^{a(t)}\rho u^2dx+A\int_{-a(t)}^{a(t)}\rho^\gamma dx. \tag{2.20}$$

Multiplying $(1.1)_1$ and (2.19) by $-\frac{1}{2}u^2$ and u, respectively, the sum of the two resultant equations is

$$\frac{1}{2}(\rho u^2)_t+\frac{1}{2}(\rho u^3)_x+A(\rho^\gamma)_x u=0. \tag{2.21}$$

We integrate (2.21) over $[-a(t),a(t)]$ with respect to x to have

$$\frac{1}{2}\cdot\frac{d}{dt}\int_{-a(t)}^{a(t)}\rho u^2dx+\frac{1}{2}\int_{-a(t)}^{a(t)}(\rho u^3)_x dx+A\int_{-a(t)}^{a(t)}(\rho^\gamma)_x udx=0. \tag{2.22}$$

With the aid of the condition (2.2) and the first equation of (1.1), one has

$$\frac{1}{2}\int_{-a(t)}^{a(t)}(\rho u^3)_x dx=0, \tag{2.23}$$

$$\begin{aligned}A\int_{-a(t)}^{a(t)}(\rho^\gamma)_x udx&=A\gamma\int_{-a(t)}^{a(t)}\rho^{\gamma-1}\rho_x udx=A\gamma\int_{-a(t)}^{a(t)}\rho^{\gamma-2}\rho_x(\rho u)dx\\&=\frac{A\gamma}{\gamma-1}\int_{-a(t)}^{a(t)}(\rho^{\gamma-1})_x(\rho u)dx=-\frac{A\gamma}{\gamma-1}\int_{-a(t)}^{a(t)}\rho^{\gamma-1}(\rho u)_x dx\\&=\frac{A\gamma}{\gamma-1}\int_{-a(t)}^{a(t)}\rho^{\gamma-1}\rho_t dx=\frac{A}{\gamma-1}\cdot\frac{d}{dt}\int_{-a(t)}^{a(t)}\rho^\gamma dx.\end{aligned} \tag{2.24}$$

Combining (2.15) and (2.22)–(2.24), we get

$$E'(t) = 0, \tag{2.25}$$

which implies that

$$E(t) = E(0). \tag{2.26}$$

It follows from (2.16), (2.20), (2.15), and (2.26) that

$$\begin{aligned} I''(t) = 2F'(t) &= 2\int_{-a(t)}^{a(t)} \rho u^2 dx + 2A\int_{-a(t)}^{a(t)} \rho^\gamma dx \\ &\geq 2(\gamma-1)E(t) = 2(\gamma-1)E(0). \end{aligned} \tag{2.27}$$

Integrating (2.27) over $[0, t]$ twice and using (2.16), we obtain

$$I(t) \geq (\gamma-1)E(0)t^2 + 2F(0)t + I(0). \tag{2.28}$$

By (2.12), (2.13), and (2.18),

$$I(t) \leq a(t)^2 \int_{-a(t)}^{a(t)} \rho dx = m(t)a(t)^2 = m(0)a(t)^2, \tag{2.29}$$

which together with (2.28) implies

$$a(t)^2 \geq \frac{(\gamma-1)E(0)t^2 + 2F(0)t + I(0)}{m(0)}. \tag{2.30}$$

So, there exists a constant $C_1 > 0$ such that

$$a(t) \geq C_1(1+t). \tag{2.31}$$

We integrate (2.4) over $[0, t]$ to have

$$a'(t) = a_1 + A\int_0^t \frac{ds}{a(s)^\gamma}, \tag{2.32}$$

which together with (2.31) leads to

$$a'(t) \leq a_1 + A\int_0^{+\infty} \frac{dt}{[C_1(1+t)]^\gamma}. \tag{2.33}$$

Note that the integral $\int_0^{+\infty} \frac{dt}{[C_1(1+t)]^\gamma}$ is convergent due to $\gamma > 1$, and there exists a constant $C_2 > 0$ such that

$$a(t) \leq C_2(1+t). \tag{2.34}$$

By (2.31), (2.34), and (2.3), we can see that the interval $[-a(t), a(t)]$ occupied by the fluid grows linearly in time, and the absolute value of the fluid temperature $|T(x,t)|$ decays to zero at the rate of $O((1+t)^{-1})$. We have completed the proof of Theorem 2.1.

Figure 1 shows the plot of $\rho(x,t)$ for the solution in Theorem 2.1.

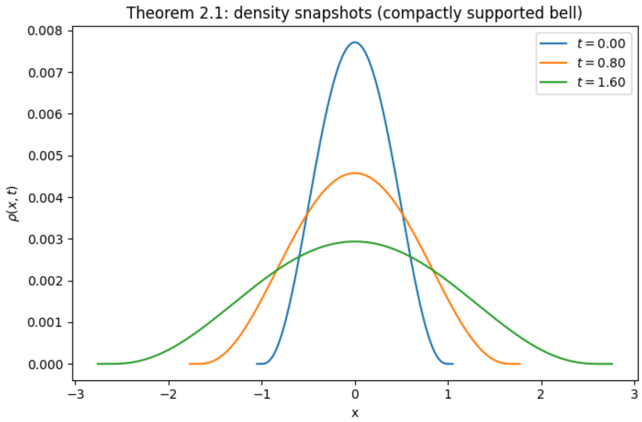

Figure 1. The plot of density of Theorem 2.1.

Next, we study the system (1.1) with the following initial data:

$$\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \quad T(x,0) = T_0(x). \tag{2.35}$$

For the isentropic case, we have the following result.

Theorem 2.2. *If $\gamma > 1$ and $T_0(x) = b_0 x$ with b_0 being a constant, there is an analytical solution to the problems (1.1) and (2.35) of the form*

$$\rho(x,t) = \frac{\left[\frac{(\gamma-1)x^2}{2\gamma a(t)^2} + 1\right]^{\frac{1}{\gamma-1}}}{a(t)}, \quad u(x,t) = \frac{a'(t)}{a(t)}x, \quad T(x,t) = \frac{a_0 b_0}{a(t)}x, \tag{2.36}$$

where $a(t)$ satisfies

$$a''(t) = -\frac{A}{a(t)^\gamma}, \quad a(0) = a_0 > 0, \quad a'(0) = a_1. \tag{2.37}$$

Moreover, if $a_1 \geq \sqrt{\frac{2A}{(\gamma-1)a_0^{\gamma-1}}}$, then

$$\lim_{t\to+\infty} a(t) = +\infty, \quad \lim_{t\to+\infty} \rho(x,t) = 0, \quad \lim_{t\to+\infty} T(x,t) = 0; \tag{2.38}$$

if $a_1 < \sqrt{\frac{2A}{(\gamma-1)a_0^{\gamma-1}}}$, then there exists a finite time $t_0 > 0$ such that

$$\lim_{t\to t_0^-} a(t) = 0, \quad \lim_{t\to t_0^-} \rho(x,t) = +\infty, \quad \lim_{t\to t_0^-} T(x,t) = \infty \ (x \neq 0). \tag{2.39}$$

Proof. Similar to the proof of Theorem 2.1, after we obtain (2.5) and (2.6), we require

$$s = \frac{\gamma}{\gamma-1} \cdot [f^{\gamma-1}(s)]'_s, \quad f(0) = 1. \tag{2.40}$$

Then we get

$$f(s) = \left[\frac{\gamma-1}{2\gamma}s^2 + 1\right]^{\frac{1}{\gamma-1}}, \tag{2.41}$$

which together with (1.3) leads to

$$\rho(x,t) = \frac{\left[\frac{(\gamma-1)x^2}{2\gamma a(t)^2}+1\right]^{\frac{1}{\gamma-1}}}{a(t)}. \tag{2.42}$$

By (2.6) and (2.40), we get (2.37). Similar to how we treated (2.10) and (2.11), we can obtain $T(x,t) = \frac{a_0 b_0}{a(t)}x$.

We multiply (2.37) by $a'(t)$ and integrate it over $[0,t]$ to have

$$\frac{1}{2}a'(t)^2 - \frac{A}{(\gamma-1)a(t)^{\gamma-1}} = \frac{1}{2}a_1^2 - \frac{A}{(\gamma-1)a_0^{\gamma-1}}. \tag{2.43}$$

Also, we integrate (2.37) over $[0,t]$ to get

$$a'(t) = a_1 - A\int_0^t \frac{ds}{a(s)^\gamma}. \tag{2.44}$$

From (2.37), we can see that $a(t)$ is strictly concave and $a'(t)$ is strictly decreasing. If there exists $t_1 > 0$ such that $a'(t_1) \le 0$, then, there exists a finite time $t_0 > 0$ such that $\lim_{t\to t_0^-} a(t) = 0$, and we have $\lim_{t\to t_0^-}\rho(x,t) = +\infty$, $\lim_{t\to t_0^-} T(x,t) = \infty$ $(x \neq 0)$ by (2.36). If $a'(t) > 0$ for all $t \in [0,+\infty)$, then (2.44) implies $\lim_{t\to+\infty} a(t) = +\infty$, which leads to $\lim_{t\to+\infty}\rho(x,t) = 0$, $\lim_{t\to+\infty} T(x,t) = 0$ by (2.36). The escape velocity[†] that characterizes these two cases is the initial velocity that leads to $\lim_{t\to+\infty} a'(t) = 0$, in which case $a(t)$ tends to $+\infty$ according to (2.44). By (2.43), we know that the escape velocity is $\sqrt{\frac{2A}{(\gamma-1)a_0^{\gamma-1}}}$.

Figure 2 shows the plot of $\rho(x,t)$ for the solution in Theorem 2.1 (collapsing case).

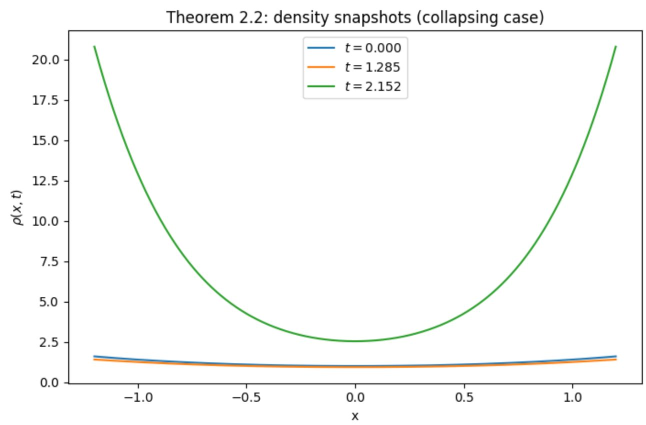

Figure 2. The plot of density of Theorem 2.2.

For the isothermal case, we have the following two results.

Theorem 2.3. *If $\gamma = 1$ and $T_0(x) = b_0 x$ with b_0 being a constant, there is an analytical solution to the problems (1.1) and (2.35) of the form*

$$\rho(x,t) = \frac{e^{1-\frac{x^2}{2a(t)^2}}}{a(t)},\quad u(x,t) = \frac{a'(t)}{a(t)}x,\quad T(x,t) = \frac{a_0 b_0}{a(t)}x, \tag{2.45}$$

[†]In this paper, the escape velocity is the critical initial expansion rate required to overcome the pressure gradients that would otherwise cause the fluid to collapse back on itself.

where $a(t)$ satisfies

$$a''(t) = \frac{A}{a(t)}, \quad a(0) = a_0 > 0, \quad a'(0) = a_1. \tag{2.46}$$

Moreover, there exists a constant $M > 0$ such that $\rho(x,t) \leq M$ and

$$\lim_{t\to+\infty} \rho(x,t) = \lim_{t\to+\infty} T(x,t) = 0. \tag{2.47}$$

Proof. It is easy to verify that $(1.1)_1$ has solutions of the form

$$\rho(x,t) = \frac{e^{f(\frac{x}{a(t)})}}{a(t)}, \quad u(x,t) = \frac{a'(t)}{a(t)}x, \tag{2.48}$$

where $f \in C^1$ and $a(t) > 0 \in C^1$. When $\gamma = 1$, we plug (2.48) into $(1.1)_2$ to have

$$a''(t)s + \frac{Af'(s)}{a(t)} = 0, \tag{2.49}$$

where $s = \frac{x}{a(t)}$. Let

$$-s = f'(s), \quad f(0) = 1, \tag{2.50}$$

which can be solved as

$$f(s) = 1 - \frac{s^2}{2}. \tag{2.51}$$

By (2.48) and (2.51), we get $\rho(x,t) = \frac{e^{1-\frac{x^2}{2a(t)^2}}}{a(t)}$. In view of (2.49) and (2.50), we obtain (2.46). The proof of $T(x,t) = \frac{a_0 b_0}{a(t)}x$ is the same as in the proof of Theorem 2.1.

We multiply (2.46) by $a'(t)$ and integrate it over $[0,t]$ to get

$$\frac{1}{2}a'(t)^2 - A\ln a(t) = \frac{1}{2}a_1^2 - A\ln a_0, \tag{2.52}$$

which leads to

$$-A\ln a(t) \leq \frac{1}{2}a_1^2 - A\ln a_0. \tag{2.53}$$

Consequently,

$$a(t) \geq a_0 e^{-\frac{a_1^2}{2A}} > 0. \tag{2.54}$$

From (2.45) and (2.54), we find that there exists a constant $M > 0$ such that $\rho(x,t) \leq M$.

Integrating (2.46) over $[0,t]$, we obtain

$$a'(t) = a_1 + A\int_0^t \frac{ds}{a(s)}. \tag{2.55}$$

We claim that

$$\lim_{t\to+\infty} a(t) = +\infty. \tag{2.56}$$

Otherwise, there exists a constant $M_1 > 0$ such that

$$a(t) \leq M_1, \tag{2.57}$$

which together with (2.55) leads to

$$a'(t) \geq a_1 + A\int_0^t \frac{ds}{M_1} = a_1 + \frac{A}{M_1}t \to +\infty, \quad t \to +\infty. \tag{2.58}$$

This is a contradiction. By virtue of (2.45) and (2.56), (2.47) holds true.

Theorem 2.4. *If $\gamma = 1$ and $T_0(x) = b_0 x$ with b_0 being a constant, there is an analytical solution to the problems (1.1) and (2.35) of the form*

$$\rho(x,t) = \frac{e^{1+\frac{x^2}{2a(t)^2}}}{a(t)}, \quad u(x,t) = \frac{a'(t)}{a(t)}x, \quad T(x,t) = \frac{a_0 b_0}{a(t)}x, \tag{2.59}$$

where $a(t)$ satisfies

$$a''(t) = -\frac{A}{a(t)}, \quad a(0) = a_0 > 0, \quad a'(0) = a_1. \tag{2.60}$$

Moreover, there exists a finite time $t_0 > 0$ such that (2.39) holds.

Proof. Unlike in the proof of Theorem 2.3, now we require

$$s = f'(s), \quad f(0) = 1, \tag{2.61}$$

which can be solved as

$$f(s) = 1 + \frac{s^2}{2}. \tag{2.62}$$

By (2.48) and (2.62), we get $\rho(x,t) = \frac{e^{1+\frac{x^2}{2a(t)^2}}}{a(t)}$. In view of (2.49) and (2.61), we obtain (2.60).

Integrating (2.60) over $[0,t]$ yields

$$a'(t) = a_1 - A\int_0^t \frac{ds}{a(s)}. \tag{2.63}$$

We claim that there exists $t_1 > 0$ such that $a'(t_1) \leq 0$. Otherwise, we have $a'(t) > 0$ for all $t \in [0,+\infty)$, which together with (2.63) implies that $\lim_{t\to+\infty} a(t) = +\infty$. We multiply (2.60) by $a'(t)$ and integrate it over $[0,t]$ to get

$$\frac{1}{2}a'(t)^2 + A\ln a(t) = \frac{1}{2}a_1^2 + A\ln a_0, \tag{2.64}$$

which leads to

$$A\ln a(t) \leq \frac{1}{2}a_1^2 + A\ln a_0. \tag{2.65}$$

By (2.65),

$$a(t) \leq a_0 e^{\frac{a_1^2}{2A}}, \tag{2.66}$$

which contradicts to $\lim_{t\to+\infty} a(t) = +\infty$.

By (2.60), $a(t)$ is concave, and $a'(t)$ is decreasing; since $a'(t_1) \leq 0$ for some $t_1 > 0$, this implies the existence of a finite time $t_0 > 0$ for which (2.39) holds.

Remark 2.1. *Compared to (2.11) in [9], which describes the fluid temperature as a quickly oscillating and decaying function or a slowly oscillating and slowly decaying function, the absolute value of* *the fluid temperature $|T(x,t)|$ constructed in Theorem 2.1 of this paper decays to zero at the rate of $O((1+t)^{-1})$. Moreover, depending on the initial data, the blowup or decay of the analytical solution can occur, as Theorem 2.2 reveals. Some qualitative behavior of the density and temperature are analyzed in Theorems 2.3 and 2.4 for the isothermal fluid.*

Remark 2.2. *Different from the isentropic case in Theorem 2.2, the solution constructed for the isothermal case in Theorem 2.4 will blow up in finite time for all $a_1 \in \mathbb{R}$.*

Remark 2.3. *In Theorem 2.1, we restrict $\gamma \le 2$ to ensure that ρ remains C^1-smooth at $x = -a(t)$ and $x = a(t)$. In fact, by $\gamma \le 2$, the power $\frac{1}{\gamma-1}$ in (2.9) is not smaller than 1, which implies that the derivative of ρ with respect to x is continuous at $x = -a(t)$ and $x = a(t)$.*

Remark 2.4. *We should remark that the solution (2.3) satisfies the physical vacuum boundary condition (see [33–35] for instance). Indeed, by (2.9),*

$$p'(\rho) = A\gamma\rho^{\gamma-1} = \frac{A(\gamma-1)(a(t)+x)(a(t)-x)}{2a(t)^{\gamma+1}}, \quad x \in [-a(t), a(t)], \tag{2.67}$$

which together with (2.31) and (2.34) implies that the sound speed $c = \sqrt{p'(\rho)}$ is $C^{1/2}$-Hölder continuous (with respect to x) across the vacuum boundary. For the well-posedness theory of the physical vacuum boundary problem for the Euler equations, we can see [33,34] and the references cited therein.

3. An exact solution to the system (1.4)

In this section, we will use the Ansatz (1.5) with $h \ge 0$ to present an exact solution to the system (1.4) for $\lambda = 1$. We impose the following initial conditions:

$$\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x). \tag{3.1}$$

The main result of this section is stated as follows.

Theorem 3.1. *If $\gamma = 3$, there exists an exact solution to the problems (1.4) and (3.1) for $\lambda = 1$ of the form*

$$\rho(x,t) = \sqrt{\frac{1}{6A}\left(\frac{1}{2} - \mu\right)\frac{x^2}{1+t} + \frac{\rho(0,0)^2}{1+t}}, \quad u(x,t) = \frac{x}{2(1+t)}, \tag{3.2}$$

where $\rho(0,0)$ is the initial density at the point $x = 0$.

Proof. We plug (1.5) into $(1.4)_1$ to have

$$-\alpha(1+t)^{-\alpha-1}h(\eta) - \beta(1+t)^{-\alpha-1}\eta h'(\eta) + (1+t)^{-\alpha-\beta-\delta}[h(\eta)g(\eta)]' = 0, \tag{3.3}$$

where $\eta = \frac{x}{(1+t)^\beta}$. Let $\beta = \delta = \frac{1}{2}$, then, (3.3) becomes

$$-\alpha h(\eta) - \frac{1}{2}\eta h'(\eta) + [h(\eta)g(\eta)]' = 0. \tag{3.4}$$

We set $\alpha = \frac{1}{2}$. So, (3.4) can be solved as

$$g(\eta) = \frac{1}{2}\eta. \tag{3.5}$$

In view of $(1.4)_1$, we can rewrite $(1.4)_2$ as

$$u_t + uu_x + \frac{1}{\rho}[p(\rho)]_x = -\frac{\mu}{(1+t)^{\lambda}}u \tag{3.6}$$

for $\rho > 0$. When $\gamma = 3$ and $\lambda = 1$, we substitute (3.5) and

$$\rho(x,t) = (1+t)^{-\frac{1}{2}} h\left(\frac{x}{(1+t)^{\frac{1}{2}}}\right), \quad u(x,t) = (1+t)^{-\frac{1}{2}} g\left(\frac{x}{(1+t)^{\frac{1}{2}}}\right) \tag{3.7}$$

into (3.6) to get

$$\left(\frac{\mu}{2} - \frac{1}{4}\right)\eta = -3Ah(\eta)h'(\eta) = -\frac{3A}{2}[h^2(\eta)]', \tag{3.8}$$

which can be solved as

$$h(\eta) = \sqrt{\frac{1}{6A}\left(\frac{1}{2} - \mu\right)\eta^2 + h(0)^2}. \tag{3.9}$$

By (1.5), we know that $h(0) = \rho(0,0)$, so (3.9) becomes

$$h(\eta) = \sqrt{\frac{1}{6A}\left(\frac{1}{2} - \mu\right)\eta^2 + \rho(0,0)^2}. \tag{3.10}$$

Combining the above results, we obtain (3.2).

Figure 3 shows the plot of $\rho(x,t)$ for the solution in Theorem 3.1.

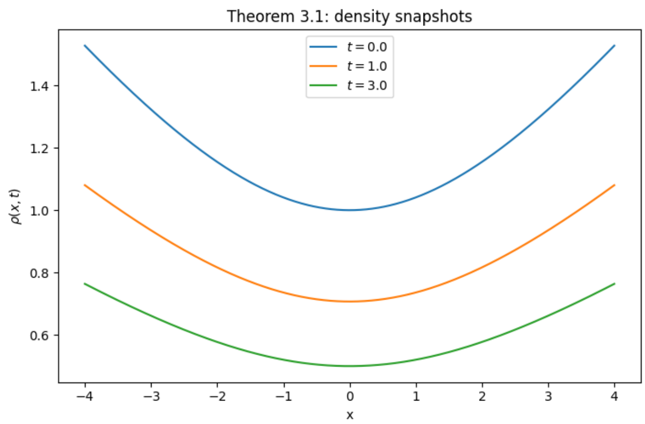

Figure 3. The plot of density of Theorem 3.1.

Remark 3.1. *For the solution constructed in Theorem 3.1, the qualitative behavior of the density varies depending on the values of* $\rho(0,0)$ *and* μ*:*

Case (i). *If* $\rho(0,0) = 0$ *and* $\mu < \frac{1}{2}$*, we have* $\rho(x,t) = \sqrt{\frac{1}{6A}\left(\frac{1}{2} - \mu\right)}\frac{|x|}{(1+t)^{\frac{1}{2}}}$*, which means that the density increases linearly according to the distance of the fluid from the symmetry center* $x = 0$ *for any fixed time* t*, and it decays in time to zero at the rate of* $O((1+t)^{-\frac{1}{2}})$ *for any fixed point* $x \neq 0$*;*

Case (ii). *If* $\rho(0,0) > 0$ *and* $\mu \leq \frac{1}{2}$*, then, the density decays in time to zero at the rate of* $O((1+t)^{-\frac{1}{2}})$ *for any fixed point* $x \in \mathbb{R}$*;*

Case (iii). *If $\rho(0,0) > 0$ and $\mu > \frac{1}{2}$, then the density $\rho(x,t)$ is only well-defined in the interval*

$$\left(-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0),\ \sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)\right). \tag{3.11}$$

In this case, we can assume that there is vacuum outside of $\left(-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0),\ \sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)\right)$, so, (3.2) is a solution for the vacuum boundary problem corresponding to the system (1.4) in $\left(-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0),\ \sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)\right)$.

Remark 3.2. *In this section and the next, we choose $\gamma = 3$, which is not entirely for algebraic convenience. In fact, $\gamma = 3$ is structurally special in one-dimensional fluid dynamics (often related to one-dimensional shallow water equations) because it allows the nonlinear convective terms and the pressure gradient to scale compatibly, enabling exact integration. For instance, an exact solution was available in [9] for one-dimensional Euler equations with heat conduction for a compressible fluid when the pressure $p(\rho) \sim \rho^3$.*

Remark 3.3. *For Case (iii) in Remark 3.1, the total energy $E(t)$ is finite and dissipative for the explicit solution (3.2) (note $\gamma = 3$):*

$$\begin{aligned}
E(t) =& \frac{1}{2}\int_{-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)}^{\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)} \rho u^2 dx + \frac{A}{2}\int_{-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)}^{\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)} \rho^3 dx \\
=& \frac{1}{2}\int_{-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)}^{\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)} \sqrt{\frac{1}{6A}\left(\frac{1}{2}-\mu\right)\frac{x^2}{1+t} + \frac{\rho(0,0)^2}{1+t}} \cdot \frac{x^2}{4(1+t)^2} dx \\
&+ \frac{A}{2}\int_{-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)}^{\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)} \left[\sqrt{\frac{1}{6A}\left(\frac{1}{2}-\mu\right)\frac{x^2}{1+t} + \frac{\rho(0,0)^2}{1+t}}\right]^3 dx \\
<& \frac{1}{2}\int_{-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)}^{\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)} \sqrt{\frac{1}{6A}\left(\frac{1}{2}-\mu\right)x^2 + \rho(0,0)^2} \cdot \frac{x^2}{4} dx \\
&+ \frac{A}{2}\int_{-\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)}^{\sqrt{\frac{6A}{\mu-\frac{1}{2}}}\rho(0,0)} \left[\sqrt{\frac{1}{6A}\left(\frac{1}{2}-\mu\right)x^2 + \rho(0,0)^2}\right]^3 dx \\
=& E(0), \quad \forall t > 0.
\end{aligned} \tag{3.12}$$

4. An analytical solution to the system (1.6)

In this section, we will provide an analytical solution to the system (1.6) for $\gamma = 3$ on the half line $[0, +\infty)$ by using the Ansatz (1.5). We state the result as follows.

Theorem 4.1. *If $\gamma = 3$ and $\mu(\rho) = B\rho$ with $B > 0$ being a constant, there exists an analytical solution to the problems (1.6) and (3.1) of the form*

$$\rho(x,t) = (1+t)^{-\frac{1}{2}} h\left(\frac{x}{(1+t)^{\frac{1}{2}}}\right), \quad u(x,t) = \frac{x}{2(1+t)}, \tag{4.1}$$

where $h(\eta)$ satisfies

$$-\frac{1}{4}\eta + 3Ah(\eta)h'(\eta) = \frac{B}{2}h^{-1}(\eta)h'(\eta) \tag{4.2}$$

with $\eta = \frac{x}{(1+t)^{\frac{1}{2}}}$. Moreover, if $\rho(0,0) > 0$, then the density is positive for any $(x,t) \in [0,+\infty) \times [0,+\infty)$.

Proof. We demonstrate (4.1) by the same steps that were used to derive (3.3)–(3.5). When $\gamma = 3$ and $\mu(\rho) = B\rho$, by $(1.6)_1$, we can rewrite $(1.6)_2$ as

$$u_t + uu_x + 3A\rho\rho_x = B\rho^{-1}\rho_x u_x + Bu_{xx} \tag{4.3}$$

for $\rho > 0$. We plug (4.1) into (4.3) to get (4.2). By (4.1), we know that if $\rho(0,0) > 0$, then $h(0) = \rho(0,0) > 0$, so $h(\eta) > 0$ for small $\eta > 0$. Set

$$\eta_0 = \sup\{\eta^* | h(\eta) > 0, \ \forall \eta \in [0,\eta^*)\}.$$

We can prove $\eta_0 = +\infty$ by contradiction. In fact, if $\eta_0 < +\infty$, then $\lim_{\eta\to\eta_0^-} h(\eta) = 0$ by continuity. For any fixed $\eta \in (0,\eta_0)$, integrating (4.2) over $[0,\eta]$, we obtain

$$-\frac{1}{8}\eta^2 + \frac{3A}{2}h^2(\eta) - \frac{3A}{2}h^2(0) = \frac{B}{2}\ln h(\eta) - \frac{B}{2}\ln h(0), \tag{4.4}$$

that is,

$$12Ah^2(\eta) - 4B\ln h(\eta) = \eta^2 + 12Ah^2(0) - 4B\ln h(0). \tag{4.5}$$

Letting $\eta \to \eta_0^-$ in (4.5), we get

$$\eta_0^2 + 12Ah^2(0) - 4B\ln h(0) = +\infty, \tag{4.6}$$

which is a contradiction. Consequently, $h(\eta) > 0$ for any $\eta > 0$, which together with (4.1) means that the density is positive for any $(x,t) \in [0,+\infty) \times [0,+\infty)$.

Figure 4 shows the plot of $\rho(x,t)$ for the solution in Theorem 4.1.

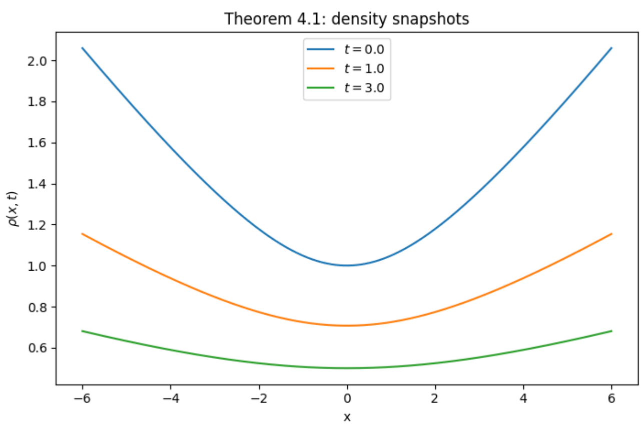

Figure 4. The plot of density of Theorem 4.1.

5. Conclusions

In this paper, we construct several analytical and self-similar solutions for different one-dimensional compressible fluid models, including Euler equations with heat conduction, Euler equations with time-dependent damping, and Navier-Stokes equations with density-dependent viscosity. The decay, blow-up behavior, and qualitative properties of the solutions are analyzed. For the Euler equations with heat conduction, we present some analytical solutions with density decaying in time or blowing up in finite time. While $u \propto x$ implies the velocity could grow unbounded in an infinite domain, our solutions for the free boundary problem are compactly supported within the interval $x \in [-a(t), a(t)]$. Therefore, the fluid velocity never reaches infinity; it is strictly bounded by the finite expansion speed of the boundary. This linear velocity profile is a well-known hallmark of expanding flows, such as Sedov-Taylor blast waves and Hubble-type expansions. In Section 3, we provide an exact solution for the Euler equations with time-dependent damping, and some qualitative properties of the density depending on the values of $\rho(0,0)$ and μ are exhibited. Such exact solutions were not available in the previous related works [11–14]. In Section 4, we construct a self-similar analytical solution with positive fluid density for the Navier-Stokes equations with density-dependent viscosity. This solution is different from the ones constructed in [5,7,10,22,23,27,35]. Our solutions provide some analytical insight into the systems under consideration and can be used to test numerical methods.

However, there are some defects in this paper. For instance, the temperature constructed in Section 2 decays linearly in time or blows up in a finite time, which seems to be restricted in physical applications. Could one construct analytical solutions with temperature oscillating within a certain range by using the Ansatz (1.3)? In addition, we take the adiabatic exponent $\gamma = 3$ in Sections 3 and 4. Could this be generalized to the case of other values of γ? These problems will be investigated in future papers.

Author contributions

Jianwei Dong, Manwai Yuen, Litao Zhang and Junhui Zhu: Formal analysis, Writing–original draft; Jianwei Dong and Manwai Yuen: Investigation, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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