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Research article

Formulation and analysis of an implicit non-standard finite difference scheme for the Black-Scholes option pricing model

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Abstract: This paper introduces a novel numerical scheme for estimating option prices under the Black-Scholes (B-S) model. The proposed method utilizes a non-standard finite difference (NSFD) approach that incorporates the powerful techniques of methods of sub-equation and exact finite difference (EFD). The proposed technique exhibits several positive characteristics: It preserves positivity by design, works with large step sizes, ensures dynamic consistency, and enhances stability. Notably, its implicit scheme and construction ensures that the fundamental properties of the solution are accurately captured. Finally, some numerical simulations are provided to demonstrate the effectiveness of the proposed implicit NSFD scheme.

Keywords: exact finite difference scheme; sub-equation; non-standard finite difference scheme; stability; simulation; numerical results; convergence analysis
Mathematics Subject Classification: 65M06, 65M12

1. Introduction

The financial sector employs a variety of derivative securities, such as options, futures, forwards, and swaps, to manage risk and generate profits. Among these, financial options grant the holder the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) on or before a specific date (expiration date). To acquire this right, the option holder pays a premium to the option writer. Options can be exercised in various ways, leading to different option styles. European options provide the holder with the right to exercise the option solely on the specific expiration date. In contrast, American options offer greater flexibility, allowing the holder to exercise the option at any point prior to or on the expiration date.

In their seminal work [1], Fischer Black and Myron Scholes developed the Black-Scholes partial differential equation (PDE) model and its corresponding solution, which laid the foundation for the theoretical analysis of option pricing. In this paper, we consider the following B-S model for determining the price of European call option V(s,τ):

$$-\frac{\partial V}{\partial \tau} = \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} + \tilde{r}s\frac{\partial V}{\partial s} - \tilde{r}V, \quad s > 0, \quad \tau \in (0, \mathcal{T}), \tag{1.1}$$

with the final condition

$$V(s, \mathcal{T}) = \max\{s - K, 0\}, \quad s \in [0, \infty), \tag{1.2}$$

where K and s represent the strike price and the current asset price, respectively. The current time is denoted by τ, and the maturity time is denoted by $\mathcal{T}$. $\tilde{r}$ denotes the interest rate, which is free of risk. When the volatility σ and risk-free interest rate $\tilde{r}$ are assumed to be constant, the B-S equation can be reduced to the well-known heat equation solvable analytically, as demonstrated in [2, 3].

It is well known that the B-S PDE (1.1) can be derived from the stochastic differential equation governing geometric Brownanian motion (GBM) for the underlying asset price, via an application of Itô's lemma and the no-arbitrage principle under the risk-neutral measure [1–3]. The Feynman-Kač theorem further establishes that the solution to (1.1) admits a probabilistic representation as the discounted expected payoff under the risk-neutral measure. The B-S model also has been widely studied in the context of financial mathematics and econophysics [4, 5]. In the present work, however, we focus entirely on the deterministic PDE framework and its numerical treatment.

A wide range of techniques have been developed for valuing options. Early methods included finite difference methods and lattice methods. Hull and White generalized lattice methods to option pricing in [6]. Researchers have utilized a diverse array of numerical techniques to price derivatives, ranging from traditional finite difference and finite element methods [7,8] to the method of lines [9] and stochastic Monte-Carlo simulations [2, 10]. The applicability of these techniques is often facilitated by the fact that the B-S equation with constant coefficients can be transformed into a diffusion equation. The binomial tree method, introduced by Cox et al. [11], is another popular technique for option pricing.

Numerical methods provide a deterministic approach to approximating option prices, avoiding the complexities of probabilistic and stochastic models. In the absence of analytical solutions, a thorough analysis of different numerical methods for option pricing can be found in [12]. By applying a logarithmic transformation to the B-S equation, researchers have developed a variety of numerical techniques [13–15]. Kadalbajoo et al. [16] and Mangal and Gupta [17] utilized B-spline-based collocation methods to address the numerical solution of the generalized B-S equation.

Standard finite difference (SFD) methods can be suffer from numerical instability [18], leading to solutions that may not accurately represent the underlying physical or financial processes. In financial applications, it is crucial to ensure that numerical solutions remain positive to avoid nonsensical results. Hence, it is good enough to develop a positive preserving scheme [19] that ignores the solution's irrational negative values and is stable in relation to the maximum norm. Nonstandard finite difference (NSFD) methods offer a flexible approach to constructing such schemes [18, 20]. The fundamental principles of NSFD methods are outlined below:

(1) The order of discrete derivatives must correspond to the order of derivatives in the differential equation.
(2) Accurate discrete representations of derivatives often necessitate the use of non-trivial denominator functions to capture the underlying continuous behavior.
(3) Nonlinear terms should generally be approximated by nonlocal discrete operators.
(4) To ensure consistency between the continuous and discrete models, any special conditions (e.g., boundary conditions, initial conditions, symmetries) that apply to the differential equation or its solutions must also be satisfied by the difference equation or its solutions.

Patidar [21] offered a comprehensive review of NSFD methods, highlighting their theoretical foundations and practical applications. Mickins and Methihas [15] developed explicit NSFD methods for the linear B-S equation with constant coefficients. Mickens [22] further advanced the application of NSFD methods by proposing an explicit NSFD scheme for the B-S equation, drawing inspiration from the Cauchy-Euler equation.

In this article, we will construct an implicit NSFD scheme for the B-S model (1.1) and (1.2) by employing sub-equation methods in conjunction with an exact finite difference (EFD) scheme, following the principles of the NSFD strategy. This paper presents the first implicit NSFD method for solving the B-S equation using sub-equations. The structure is as follows: Introduction to the B-S equation for European options and its time-forward transformation is presented in Section 2. Section 3 discusses full discretization of the B-S model, which includes the explanation of the sub-equation method and the concept of EFD schemes in Subsection 3.1 and derivation of the implicit NSFD scheme for the B-S equation in Subsection 3.2. Section 4 illustrates the stablity and convergence of the NSFD scheme developed for the B-S equation. Some numerical experiments and discussions are given in Section 5. Finally, conclusions are drawn in Section 6 at end of the paper.

2. Option pricing

Option pricing is the process of determining the appropriate price for buying or selling an option contract. This price is influenced by various factors, including the underlying asset's current market price, the price at which the option can be exercised (strike price), the time remaining before the option expires, the market's expectation of future price volatility (implied volatility), and whether the option grants the right to buy (call) or sell (put) the underlying asset. Option pricing models, such as the B-S model (1.1) and (1.2), utilize these factors to calculate a theoretical fair value for the option.

2.1. The European call option

European put and call options are handled similarly, but we concentrate on European call options in this work. The generalized B-S model for determining the price of a European call option, represented by V, is dependent on the underlying assets of the current market price s, and the time τ is

$$-\frac{\partial V}{\partial \tau} = \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} + s\tilde{r}\frac{\partial V}{\partial s} - \tilde{r}V, \quad s > 0, \tau \in (0, \mathcal{T}), \tag{2.1}$$

along with the final condition

$$V(s, \mathcal{T}) = \max(s - K, 0), s \in [0, \infty), \tag{2.2}$$

and the boundary conditions for the European call option are

$$V(0,\tau) = 0, \quad 0 \le \tau \le \mathcal{T},$$
$$V(s,\tau) \to s, \text{ as } s \to \infty.$$

The payoff function has a kink (non-smoothness) at $s = K$. This singularity causes serious difficulties for numerical schemes, because the PDE (2.1) requires $\partial^2 V/\partial s^2$ to exist, which fails at the strike price K. To address the non-smooth initial conditions near the singularity point, we approximate the function with a ninth-degree polynomial within a relatively small ε-neighborhood of the singularity point to make it smoother.

Define a function Ψ as

$$\Psi(s) = \begin{cases} 0, & s \le -\varepsilon, \\ e_1 + e_2 s + \cdots + e_{10} s^9, & -\varepsilon \le s \le \varepsilon, \\ s, & s \ge \varepsilon. \end{cases}$$

To determine the coefficients e_j for $j = 1, 2, \ldots 10$ for the given $\varepsilon > 0$, we impose the following boundary conditions on the function Ψ:

$$\Psi(-\varepsilon) = \Psi'(-\varepsilon) = \Psi''(-\varepsilon) = \Psi'''(-\varepsilon) = \Psi^4(-\varepsilon) = 0,$$
$$\Psi(\varepsilon) = \varepsilon, \Psi'(\varepsilon) = 1, \Psi''(\varepsilon) = \Psi'''(\varepsilon) = \Psi^4(\varepsilon) = 0.$$

Solving these conditions uniquely yields the coefficients:

$$e_1 = \frac{35}{256}\varepsilon,\ e_2 = \frac{1}{2},\ e_3 = \frac{35}{64\varepsilon},\ e_5 = -\frac{35}{128\varepsilon^3},$$
$$e_7 = \frac{7}{64\varepsilon^5},\ e_9 = \frac{5}{256\varepsilon^7},\ e_4 = e_6 = e_8 = e_{10} = 0.$$

The smoothness of the initial condition of the payoff function, in the case of at-the-money options, is visualized in Figure 1.

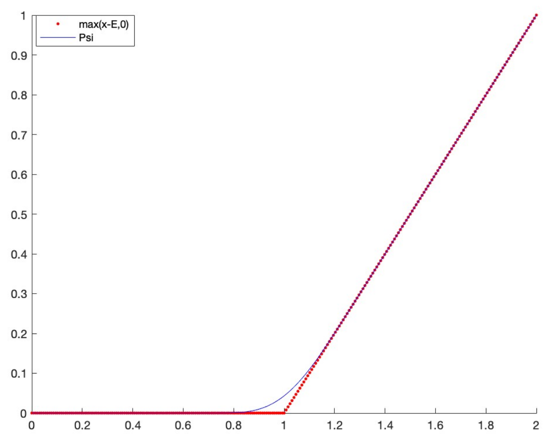


Figure 1. Comparison of the original non-smooth payoff function $\max(s - K, 0)$ and its smoothed approximation $\phi(s) = \Psi(s - K)$ near the strike price for a European call option.

Now define $\hat{V}$ as the solution to the same PDE (2.1) as V, but with the smoothed final condition. Since the PDE (2.1) is unchanged—it involves only the differential operator in s and τ, which is unaffected by the substitution at the terminal time—the governing equation for $\hat{V}$ takes the following form:

$$-\frac{\partial \hat{V}}{\partial \tau} = \frac{1}{2}s^2\sigma^2\frac{\partial^2 \hat{V}}{\partial^2 s} + \tilde{r}s\frac{\partial \hat{V}}{\partial s} - \tilde{r}\hat{V}, \quad s > 0, \quad \tau \in (0, T), \tag{2.3}$$

which is exactly Eq (2.1). The final condition (2.2) is reduced to

$$\hat{V}(s, \mathcal{T}) = \varphi(s), \quad s \in [0, \infty), \tag{2.4}$$

where $\varphi(s) = \Psi(s - K)$ is the smooth payoff function. This replaces the non-smooth $\max(s - K, 0)$ in the final condition. Since Ψ is constructed to be C^4 globally (by matching four derivatives at both junctions), $\varphi(s)$ is a fourth-order smooth function of s.

The following equation provides the error estimate caused by smoothing the final condition [23]:

$$|V - \hat{V}| \leq \hat{m}\|\varphi(s) - \max(s - K, 0)\|_{L^\infty}, \quad (s, \tau) \in [0, \infty) \times [0, \mathcal{T}], \tag{2.5}$$

where, $\hat{m}$ is independent to smoothing function φ and positive constant.

2.2. *The transformation of the equation backward to forward in time*

In order to transform the final value problem (2.3) and (2.4) into an initial value problem, a time reversal transformation $\tau = \mathcal{T} - t$ is applied. Also, for the application of numerical techniques, the unbounded domain of asset prices $[0, \infty)$ is truncated to a finite domain $[0, S_{\max}]$. The value of $S_{\max}$ should be chosen to be large enough to ensure that the truncated domain adequately captures the relevant dynamics of the problem. Thus, the model (2.3) and (2.4) is reduced to the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} = Ds^2\frac{\partial^2 u}{\partial^2 s} + rs\frac{\partial u}{\partial s} - ru, \quad s > 0, \quad t \in (0, \mathcal{T}), \tag{2.6}$$

where D and r are non-negative and $D = \frac{1}{2}\sigma^2$, along with the following initial-boundary conditions:

$$u(s, 0) = \varphi(s), \quad s \in [0, S_{\max}], \tag{2.7a}$$

$$u(0, t) = 0, \qquad t \in [0, \mathcal{T}], \tag{2.7b}$$

$$u(S_{\max}, t) = g(t) = S_{\max} - K\exp\left(-\int_0^t r(S_{\max}, y)dy\right), \quad t \in [0, \mathcal{T}], \tag{2.7c}$$

where, $u(s, t) = \hat{V}(s, \tau)$, $r = (\tilde{r})$. Note that the Eq (2.6) can be transformed to the classical heat equation by using some suitable transformation. Here, we assume that the initial-boundary data are Hölder continuous and satisfy compatibility conditions at the corner points of the domain so that the differential operator in Eq (2.6) satisfies the continuous minimum principle and the initial-boundary value problem (2.6) and (2.7) has a unique solution satisfying the following bounds [24]:

$$\left|\frac{\partial^{i+j}u(s, t)}{\partial s^i \partial t^j}\right| \leq C, \quad \forall (s, t) \in [0, S_{\max}] \times [0, T], \ 0 \leq i \leq 4, \ 0 \leq j \leq 3, \tag{2.8}$$

where C is a generic positive constant.

3. Full discretization of B-S equations

The full discretization of the B-S PDE can be achieved through the application of sub-equation methods. A sub-equation is a discrete approximation of a local portion of the differential equation. Below is a definition of a sub-equation [15]:

Definition 1. *A sub-equation of a differential equation with $n \geq 3$ terms is a PDE or ordinary differential equation (ODE) that is created by omitting either one or several terms from the original equation.*

Ehrhardt and Mickens [15] provided a comprehensive explanation of the sub-equation method in the derivation of NSFD schemes for PDEs. The core idea of NSFD technique is to construct EFD schemes for specific sub-equations of the whole PDE and then combine these individual EFD schemes to obtain a full NSFD scheme for the original equation.

In [18], Mickens provided a comprehensive exposition of EFD schemes. EFD schemes are considered superior to other numerical methods due to their inherent property of having zero truncation error, which leads to improved convergence and stability properties.

Equation (2.6) is composed of 10 individual sub-equations. However, in practical applications, we often employ only a select few of these sub-equations. Some examples of these sub-equations are listed below:

$$u_t = -ru, \tag{3.1}$$

$$Ds^2u_{ss} + rsu_s - ru = 0, \tag{3.2}$$

$$u_t = sru_s, \tag{3.3}$$

$$u_t = Ds^2u_{ss}, \tag{3.4}$$

$$rsu_s - r = 0, \tag{3.5}$$

$$u_t = Ds^2u_{ss}. \tag{3.6}$$

The sub-equation method is utilized to construct an NSFD scheme for Eq (2.6). In this article, we focus on the first two sub-equations. By employing their fundamental solutions, we derive EFD schemes for these sub-equations, which are subsequently combined to form the NSFD scheme for Eq (2.6).

3.1. Exact finite difference scheme for the sub-equations

Let us consider the computational domain $\bar{\Omega} = [0, S_{\max}] \times [0, T]$. Let $\bar{\Omega}_h$ and $\bar{\Omega}_k$ represent the space and the time directional partition.

$$\bar{\Omega}_h \equiv 0 = s_0 < s_1 < ... < s_{M-1} < s_M = S_{\max},$$

$$\bar{\Omega}_k \equiv 0 = t_0 < t_1 < ... < t_{N-1} < t_N = \mathcal{T},$$

where the mesh spacing in both directions are defined by $\delta s = \dfrac{s_M - s_0}{M}$, $\delta t = \dfrac{t_N - t_0}{N}$. Also, we define two more notations $\Omega_h \equiv \{s_i\}_{i=1}^{M-1}$ and $\Omega_k \equiv \{t_j\}_{j=1}^{N-1}$.

We consider the following time-independent Eq (3.3) as a sub-equation corresponding to which we have to find the EFD scheme:

$$Ds^2\frac{\partial^2 u}{\partial^2 s} + sr\frac{\partial u}{\partial s} - ru = 0. \tag{3.7}$$

It is important to note that Eq (3.7) is a Cauchy-Euler differential equation. As a result, its two linearly independent solutions are given by [25]

$$u^1(s) = s, \qquad u^2(s) = s^{-(r/D)}.$$

Let u_m^1 and u_m^2 be the discretization of these solutions, i.e.,

$$\begin{cases} u_m^1 \equiv u^1(s_m) = u^1(m\delta s), \\ u_m^2 \equiv u^2(s_m) = u^2(m\delta s), \qquad m = 0, 1, \ldots. \end{cases}$$

Thus, the EFD scheme associated with Eq (3.7) can be constructed following the approaches in [18,26] as

$$\det \begin{vmatrix} u_{m+1} & u_m & u_{m-1} \\ u_{m+1}^{(1)} & u_m^{(1)} & u_{m-1}^{(1)} \\ u_{m+1}^{(2)} & u_m^{(2)} & u_{m-1}^{(2)} \end{vmatrix} = 0.$$

Expanding the determinant for $m = 1, 2, \ldots, M-1$, we obtain the following EFD scheme for Eq (3.7):

$$A_{m-1}^{(1)}(\Delta^2 + 2\Delta + 1)u_{m-1} - A_{m-1}^{(2)}(\Delta + 1)u_{m-1} - A_{m-1}^{(3)}u_{m-1} = 0, \tag{3.8}$$

where Δ is the forward difference operator. The functions $A_m^{(1)}, A_m^{(2)}$, and $A_m^{(3)}$ are defined as follows:

$$\begin{aligned} A_m^{(1)} &= \left[\frac{m+1}{(m)^\alpha}\right] - \left[\frac{m}{(m+1)^\alpha}\right], \\ A_m^{(2)} &= \left[\frac{m+2}{(m)^\alpha}\right] - \left[\frac{m}{(m+2)^\alpha}\right], \\ A_m^{(3)} &= \left[\frac{m+2}{(m+1)^\alpha}\right] - \left[\frac{m+1}{(m+2)^\alpha}\right], \end{aligned}$$

where $\alpha = \dfrac{r}{D}$, and $A_m^{(2)} > A_m^{(1)} > A_m^{(3)} > 0$.

Equation (3.8) can be rewritten as:

$$A_{m-1}^{(1)}(\Delta^2 u_{m-1}) + (2A_{m-1}^{(1)} - A_{m-1}^{(2)})(\Delta u_{m-1}) + (A_{m-1}^{(1)} - A_{m-1}^{(2)} + A_{m-1}^{(3)})u_{m-1} = 0. \tag{3.9}$$

Equation (3.9) provides an EFD scheme for Eq (3.7). However, the parameters D and r are not explicitly present in Eq (3.9). As a result, Eq (3.9) does not share the same structural feature as the original Eq (3.7). However, this can be achieved in the following way:

(i) Divide all terms in Eq (3.9) by $(-A_{m-1}^{(1)} + A_{m-1}^{(2)} - A_{m-1}^{(3)})$ and multiply by r to get

$$\left[\frac{rA_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}}\right](\Delta^2 u_{m-1}) + r\left[\frac{2A_{m-1}^{(1)} - A_{m-1}^{(2)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}}\right](\Delta u_{m-1}) - ru_{m-1} = 0. \tag{3.10}$$

After algebraic manipulations, we obtain the following EFD scheme:

$$r(\delta s)^2\left[\frac{A_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}}\right]\frac{(u_{m+1} - 2u_m + u_{m-1})}{(\delta s)^2} + r\delta s\left[\frac{(2A_{m-1}^{(1)} - A_{m-1}^{(2)})}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)})}\right]\frac{(u_m - u_{m-1})}{\delta s} - ru_{m-1} = 0. \tag{3.11}$$

Note that under the limit $\delta s \to 0, m \to \infty, m\delta s = s$ (fixed),

$$\left(r(\delta s)^2\right)\left(\frac{A^{(1)}_{m-1}}{A^{(2)}_{m-1} - A^{(1)}_{m-1} - A^{(3)}_{m-1}}\right) \to Ds^2, \tag{3.12}$$

$$(r\delta s)\left[\frac{(2A^{(1)}_{m-1} - A^{(2)}_{m-1})}{A^{(2)}_{m-1} - A^{(1)}_{m-1} - A^{(3)}_{m-1})}\right] \to rs. \tag{3.13}$$

Now, the structural properties of Eq (3.7) are displayed in the final Eq (3.11).

Equation (3.1) is considered as the second sub-equation, which is given below:

$$\frac{\partial u}{\partial t} = -ru,$$

and its EFD [18] scheme is written as

$$\frac{u^k - u^{k-1}}{\psi(\delta t, r)} = -ru^k, \text{ where } \psi(\delta t, r) = \frac{e^{r\delta t} - 1}{r}. \tag{3.14}$$

$$\begin{aligned} u^k &= u(t_k), \\ t_k &= (\delta t)k, \qquad k = 0, 1, 2, \cdots, \\ \psi(\delta t, r) &= \delta t + O(\delta t^2). \end{aligned}$$

3.2. Non-standard finite difference scheme for the B-S PDEs

In the following sub-section, we present an NSFD method for the numerical solution of the B-S Eq (2.6). Mickens [15] outlined a sub-equation approach for the development of NSFD discretizations of PDEs. By combining the precise EFD schemes (3.11) and (3.14) in a novel way, we will introduce the NSFD numerical scheme.

Let $\bar{\Omega}^k_h$ represent the tensor product mesh of the discrete domains $\bar{\Omega}_h$ and $\bar{\Omega}_k$ with $\Omega^k_h = \Omega_h \times \Omega_k$, and $\partial\Omega^k_h = \bar{\Omega}^k_h \setminus \Omega^k_h$ represents the boundary points of the mesh $\bar{\Omega}^k_h$. Let M, and N denote the number of partitions of $\bar{\Omega}_h$ and $\bar{\Omega}_k$, respectively. Also, we consider that, at the point $(s_m, t_k) \in \bar{\Omega}^k_h$, v^k_m denotes the discrete approximation of the continuous solution $u(s, t)$.

Now, by combining the EFD schemes (3.11) and (3.14), the resulting NSFD scheme for the B-S model (2.6) and (2.7) is given as below:

$$\begin{cases} v^0_m = \varphi(s_m), & \forall s_m \in \bar{\Omega}_h, \\ \dfrac{v^k_m - v^{k-1}_m}{\psi(\delta t, r)} = \left[\dfrac{rA^{(1)}_{m-1}}{A^{(2)}_{m-1} - A^{(1)}_{m-1} - A^{(3)}_{m-1}}\right](v^k_{m+1} - 2v^k_m + v^k_{m-1}) & \\ +r\left[\dfrac{2A^{(1)}_{m-1} - A^{(2)}_{m-1}}{A^{(2)}_{m-1} - A^{(1)}_{m-1} - A^{(3)}_{m-1})}\right](v^k_m - v^k_{m-1}) - rv^k_{m-1}, & \forall (s_m, t_k) \in \Omega^k_h, \\ v^k_0 = 0, \quad v^k_M = g(t_k), & \forall t_k \in \bar{\Omega}_k. \end{cases} \tag{3.15}$$

By canceling common terms in the numerators and denominators and further simplifications of (3.15), it yields the following NSFD scheme:

$$\begin{cases} v_m^0 = \varphi(s_m), & \forall s_m \in \bar{\Omega}_h, \\ -\dfrac{1}{\psi_1(\delta t,r)} v_m^{k-1} = \left[\dfrac{A_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}}\right] v_{m+1}^k + \left[\dfrac{A_{m-1}^{(2)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}} - \dfrac{1}{\psi_1(\delta t, r)}\right] v_m^k & \\ + \left[\dfrac{A_{m-1}^{(2)} - A_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}} - 1\right] v_{m-1}^k, & \forall (s_m, t_k) \in \Omega_h^k, \\ v_0^k = 0, \quad v_M^k = g(t_k), & \forall t_k \in \bar{\Omega}_k, \end{cases} \tag{3.16}$$

where, $\psi_1(\delta t, r) = e^{r\delta t} - 1$. Thus, the final NSFD approximation scheme for the B-S model (2.6) and (2.7) takes the following form:

$$\begin{cases} v_m^0 = \varphi(s_m), & \forall s_m \in \bar{\Omega}_h, \\ B_{m-1}^l v_{m-1}^k + B_m^c v_m^k + B_{m+1}^r v_{m+1}^k = -\dfrac{1}{\psi_1(\delta t, r)} v_m^{k-1}, & \forall (s_m, t_k) \in \Omega_h^k, \\ v_0^k = 0, \quad v_M^k = g(t_k), & \forall t_k \in \bar{\Omega}_k, \end{cases} \tag{3.17}$$

where

$$B_{m-1}^l = \left[\frac{A_{m-1}^{(2)} - A_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}} - 1\right], \; B_m^c = \left[\frac{-A_{m-1}^{(2)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}} - \frac{1}{\psi_1(\delta t, r)}\right], \; B_{m+1}^r = \left[\frac{A_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}}\right].$$

It forms a tridiagonal system of $M - 1$ linear equations at each time level. Solving this system provides a numerical approximation to the solution of the original problem (2.6) and (2.7).

4. Stability and convergence analysis

In this section, we present the stability and consistency estimates, which establish the convergence of the proposed NSFD scheme (3.15). First, we will show that the NSFD scheme under conideration in this paper is stable through the discrete minimum principle.

Lemma 1. ***(M-matrix criterion)*** *For all step sizes δt, δs, the matrix associated with the above system (3.17) satisfies the M-matrix criterion.*

Proof. The associated matrix corresponding to (3.17) is tridiagonal. Also, from the discrete system (3.17), we observe the coefficient $B_{m-1}^l > 0$ and $B_{m+1}^r > 0$. Additionally, $B_{m-1}^l + B_m^c + B_{m+1}^r < 0$ for $m = 1, 2, \ldots, M - 1$. Thus, the coefficient matrix related to the problem (2.3) is a non-positive M-matrix. This implies that the above finite difference scheme (3.17) is stable. □

Lemma 2. ***(Discrete minimum principle)*** *Let $\mathcal{F}$ be any mesh function defined on $\bar{\Omega}_h^k$, and $\mathcal{L}$ is the discrete operator defined by the NSFD scheme (3.15) such that $\mathcal{F}(s_m, t_k) \geq 0 \, \forall (s_m, t_k) \in \partial\Omega_h^k$ and $\mathcal{L}\mathcal{F}(s_m, t_k) \leq 0 \, \forall (s_m, t_k) \in \Omega_h^k$, then $\mathcal{F}(s_m, t_k) \geq 0 \, \forall (s_m, t_k) \in \bar{\Omega}_h^k$.*

Proof. The proof immediately follows from the above Lemma 1. □

An immediate consequence of the discrete minimum principle is the following stability estimate for the proposed NSFD scheme (3.15).

Lemma 3. ***(Discrete stability estimate)*** *Let $\chi(s_m, t_k) = 0\,\forall (s_m, t_k) \in \partial\Omega_h^k$ for any mesh function χ defined on the mesh $\bar{\Omega}_h^k$, then we have the following discrete stability estimate:*

$$\chi(s_m, t_k) \leq \frac{1}{r}\max_{\Omega_h^k}|\mathcal{L}\chi(s_m, t_k)|, \qquad \forall (s_m, t_k) \in \bar{\Omega}_h^k.$$

Proof. We construct the following two barrier functions:

$$\Pi^{\pm}(s_m, t_k) = \frac{1}{r}\max_{\Omega_h^k}|\mathcal{L}\chi(s_m, t_k)| \pm \chi(s_m, t_k), \qquad \forall (s_m, t_k) \in \bar{\Omega}_h^k.$$

Here, $\Pi^{\pm}(s_m, t_k) \geq 0$, $\forall (s_m, t_k) \in \partial\Omega_h^k$, and on applying the discrete operator $\mathcal{L}$, one can notice that $\mathcal{L}\Pi^{\pm}(s_m, t_k) \leq 0\ \forall (s_m, t_k) \in \Omega_h^k$. Further, the discrete minimum principle (Lemma 2) implies that $\Pi^{\pm}(s_m, t_k) \geq 0\,\forall (s_m, t_k) \in \bar{\Omega}_h^k$, which establishes the desired discrete stability estimate for the proposed NSFD scheme (3.15). □

Finally, in the next theorem, we establish the convergence result for the proposed NSFD scheme.

Theorem 1. ***(Convergence estimate)*** *For each $(s_m, t_k) \in \bar{\Omega}_h^k$, the numerical approximation v obtained via the NSFD scheme (3.15) for the continuous solution u of the B-S model (2.6) and (2.7) satisfies the following error estimate:*

$$\|(u - v)(s_m, t_k)\|_{\bar{\Omega}_h^k} \leq C(\delta t + \delta s), \tag{4.1}$$

where C is a generic positive constant independent of mesh parameters δt, δs and $\|\bullet\|_{\bar{\Omega}_h^k}$ is the maximum norm.

Proof. To determine the above convergence estimate, first we find the truncation error associated with the proposed scheme. For that, we consider a mesh function $(u - v)(s_m, t_k)$ at the mesh point $(s_m, t_k) \in \Omega_h^k$. The associated local truncation error with the NSFD scheme (3.15) is given by

$$\begin{aligned}\mathcal{L}(u - v)(s_m, t_k) = &\left[\frac{\partial u}{\partial t} - \frac{u_m^k - u_m^{k-1}}{\psi(\delta t, r)}\right] - \left[Ds^2\frac{\partial^2 u}{\partial^2 s} - \left\{\frac{r(\delta s)^2 A_{m-1}^{(1)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)}}\right\}\frac{(u_{m+1}^k - 2u_m^k + u_{m-1}^k)}{(\delta s)^2}\right] \\ &- \left[rs\frac{\partial u}{\partial s} - r\delta s\left\{\frac{2A_{m-1}^{(1)} - A_{m-1}^{(2)}}{A_{m-1}^{(2)} - A_{m-1}^{(1)} - A_{m-1}^{(3)})}\right\}\frac{(u_m^k - u_{m-1}^k)}{\delta s}\right] + \left[ru - ru_{m-1}^k\right],\ \forall (s_m, t_k) \in \Omega_h^k.\end{aligned}$$

Using Taylor's series expansion, Eqs (3.12) and (3.13) and $\psi(\delta t, r) = \dfrac{e^{r\delta t} - 1}{r} = \delta t + O(\delta t^2)$, we get the following:

$$\mathcal{L}(u - v)(s_m, t_k) = \frac{\delta t}{2}(ru_t + u_{tt})(s_m, \eta_k) + \delta s\left(ru_s - \frac{s}{2}ru_{ss}\right)(\xi_m, t_k) + O(\delta t^2) + O(\delta s^2),$$

where $t_k < \eta_k < t_{k+1}$ and $s_{m-1} < \xi_m < s_{m+1}$. Thus, the principal term of the local truncation error is given by

$$|\mathcal{L}(u - v)(s_m, t_k)| \leq \left(\frac{\delta t}{2}(ru_t + u_{tt}) + \delta s\left(ru_s + \frac{s}{2}ru_{ss}\right)\right)(\xi_m, \eta_k).$$

Using the bounds on the continuous solution u and its derivatives given in Eq (2.8), the above truncation error estimates take the following form:

$$|\mathcal{L}(u - v)(s_m, t_k)| \leq C(\delta t + \delta s), \qquad \forall (s_m, t_k) \in \Omega_h^k.$$

Now, applying the discrete stability estimate (Lemma 3) for the mesh function $(u - v)(s_m, t_k)$, we get the desired convergence estimate, which completes the proof. □

5. Numerical results and discussions

This section is devoted to some numerical experiments consisting of two test problems, which verify our theoretical estimates and demonstrate the applicability of the proposed implicit NSFD scheme (3.15) for the discussed class of the B-S model (2.6) and (2.7). A closed-form solution to the B-S model problem (1.1) and (1.2) is given by

$$V(s,\tau) = sN(d_1) - K\exp(-\tilde{r}(\mathcal{T}-\tau))N(d_2), \tag{5.1}$$

where

$$d_1 = \frac{\ln s - \ln K + (\tilde{r} + \frac{1}{2}\sigma^2)(\mathcal{T}-\tau)}{\sigma\sqrt{\mathcal{T}-\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\mathcal{T}-\tau},$$

and $N(d_2)$ is the cumulative distribution function of the standard normal random variable.

Equation (3.8) is equivalent to the time-independent sub-equation, and serves as the foundation for the formulation. The reason for considering the particular ODE is that it is an ideal form for usage in both practical application and mathematical analysis of the B-S equation [27]. To compare the numerical results obtained with the proposed NSFD scheme and the SFD scheme, the following algorithm has been used for the discretization of the problem (2.6) and (2.7), as a standard classical finite difference algorithm:

$$\begin{cases} v_m^0 = \varphi(s_m) & \forall s_m \in \bar{\Omega}_h, \\ \dfrac{v_m^k - v_m^{k-1}}{\delta t} = D(m\delta s)^2\left[\dfrac{v_{m+1}^k - 2v_m^k + v_{m-1}^k}{(\delta s)^2}\right] + r(m\delta s)\left[\dfrac{v_m^k - v_{m-1}^k}{\delta s}\right] - rv_m^k, & \forall (s_m,t_k) \in \Omega_h^k, \\ v_0^k = 0, \qquad v_M^k = g(t_k), & \forall t_k \in \bar{\Omega}_k. \end{cases} \tag{5.2}$$

Note that in comparison to the above SFD scheme, the proposed implicit NSFD scheme (3.15) is independent of the step sizes δt and δs explicitly and depends on non-trivial denominator functions of them to capture the underlying continuous behavior of the original problem (2.6) and (2.7).

If u and U are the exact and numerical solutions respectively, then the maximum nodal error $E^{M,N}$, and corresponding order of convergence $p^{M,N}$ are computed as follows:

$$E^{M,N} = \max_{\substack{0\le m\le M\\ 0\le k\le N}} \left|u^{M,N}(s_m,t_k) - U^{M,N}(s_m,t_k)\right|,$$

$$p^{M,N} = \log_2 \frac{E^{M,N}}{E^{2M,2N}}.$$

Example 1. *In this example, we take into account the B-S model problem (2.6) and (2.7) for the pricing of a European call option with* $r = 0.04$, $\sigma = 0.4$, $\mathcal{T} = 1$, $K = 1$, $\varepsilon = 10^{-6}$, *and* $S_{\max} = 8$.

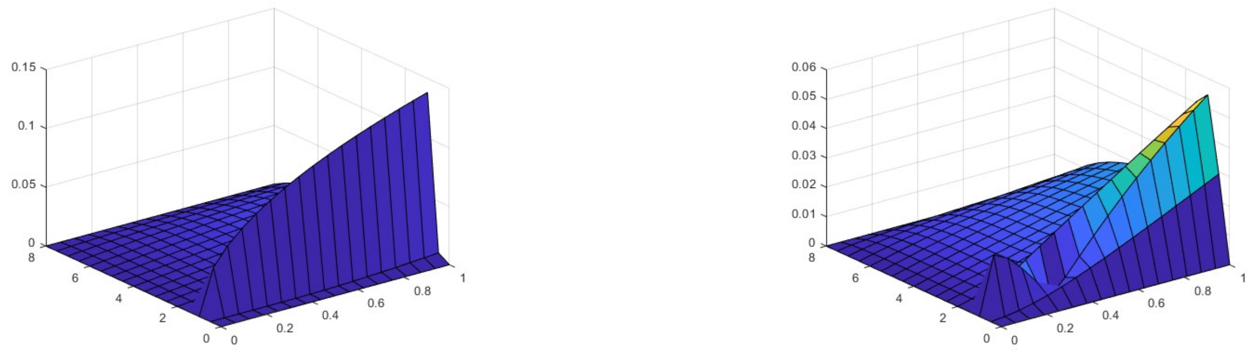
Table 1 represents the maximum nodal errors $E^{M,N}$ and corresponding order of convergence $p^{M,N}$ for varying values of M and N. The 3D plots of the analytical solution and numerical solution obtained by the NSFD scheme are shown in Figures 2a and 2b, respectively. The error profiles corresponding to the SFD and NSFD schemes are illustrated in Figures 3a and 3b, respectively, for the case $M = N = 16$.

Table 1. Maximum nodal errors $E^{M,N}$ and corresponding order of convergence $p^{M,N}$ for different values of M and N for Example 1 with NSFD and SFD schemes.

	$(M, N) \rightarrow$	$(8, 8)$	$(16, 16)$	$(32, 32)$	$(64, 64)$	$(128, 128)$	$(256, 256)$	$(512, 512)$	$(1024, 1024)$
NSFD	$E^{M,N}$	0.073604	0.054479	0.034852	0.019405	0.010242	0.005265	0.002669	0.001344
	$p^{M,N}$		0.4340	0.6444	0.8448	0.9219	0.9600	0.9796	0.9897
SFD	$E^{M,N}$	0.175782	0.138140	0.061714	0.026865	0.0128447	0.006296	0.003118	0.001552
	$p^{M,N}$		0.3476	1.1624	1.1998	1.0645	1.0285	1.0135	1.0066

(a) **(b)**

Figure 2. Analytical solution (a) and NSFD numerical solution (b) 3D plots for Example 1 with $M = N = 128$.



(a) **(b)**

Figure 3. Maximum absolute error profiles of the SFD (a) and NSFD (b) schemes for Example 1 with $M = N = 16$.

Example 2. *In this example, we consider the B-S model (2.6) and (2.7) of a European call option with* $\sigma = 0.4$, $r = 0.02$, $\mathcal{T} = 1$, $K = 1$, $S_{\max} = 8$, *and* $\varepsilon = 10^{-6}$.

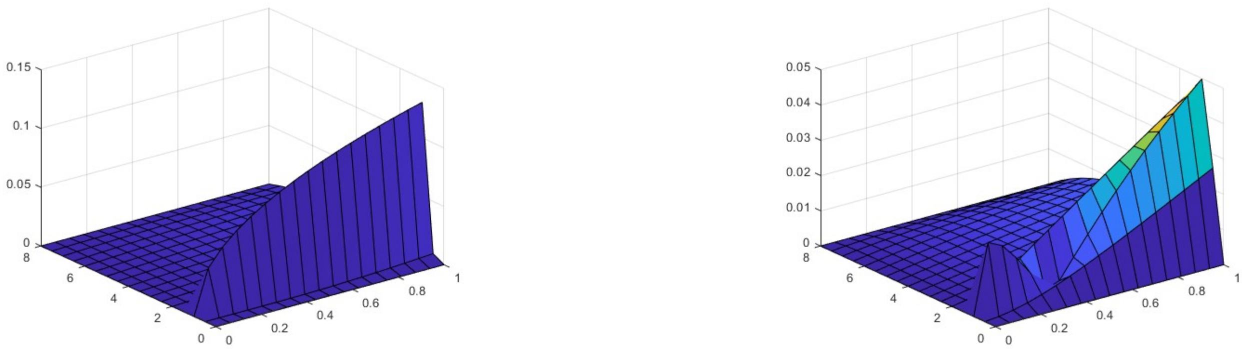
The 3D graphs of the analytical solution and computed solution obtained with the NSFD scheme are shown in Figures 4a and 4b, respectively. The maximum nodal errors and corresponding order of convergence are displayed in Table 2. The 3D error plots for the SFD and NSFD schemes are shown in Figures 5a and 5b, respectively, for $M = N = 16$.

(a) **(b)**

Figure 4. Analytical solution (a) and NSFD numerical solution (b) 3D plots for Example 2 with $M = N = 128$.

Table 2. Maximum nodal errors $E^{M,N}$ and corresponding order of convergence $p^{M,N}$ for different values of M and N for Example 2 with NSFD and SFD schemes.

	$(M, N) \rightarrow$	$(8, 8)$	$(16, 16)$	$(32, 32)$	$(64, 64)$	$(128, 128)$	$(256, 256)$	$(512, 512)$	$(1024, 1024)$
NSFD	$E^{M,N}$	0.064184	0.049882	0.032605	0.018293	0.009689	0.004989	0.002532	0.001275
	$p^{M,N}$		0.3636	0.6134	0.8337	0.9168	0.9575	0.9784	0.9891
SFD	$E^{M,N}$	0.167044	0.129748	0.0567805	0.0245497	0.011695	0.005723	0.0028322	0.0014089
	$p^{M,N}$		0.3645	1.1922	1.2096	1.0697	1.0311	1.0148	1.0072



(a) **(b)**

Figure 5. Maximum absolute error profiles of the SFD (a) and NSFD (b) schemes for Example 2 with $M = N = 16$.

To examine the applicability of the NSFD scheme for the European option pricing model, we consider two test problems, Examples 1 and 2. The computed absolute maximum pointwise errors $E^{M,N}$ and the corresponding rate of convergence $p^{M,N}$ for varying values of M and N are tabulated in Tables 1 and 2 for the NSFD and SFD schemes. Both schemes are first-order convergent; however, numerical results displayed in both tables demonstrate that the NSFD scheme provides better accuracy than the SFD scheme for a small number of mesh intervals in both directions. This is because the denominator functions are non-trivial functions of step sizes δt and δs in the discrete representations of derivatives in the NSFD scheme, which preserve the continuous behavior of the underlying B-S model. The comparison of error profiles for $M = N = 16$ in both Examples 1 and 2 further demonstrates that the proposed NSFD scheme achieves higher accuracy than the SFD scheme on a coarse mesh. Figures 2 and 4 demonstrate that the analytical and NSFD numerical solutions agree for both examples. Thus, tabulated numerical results and displayed 3D plots verify the effectiveness of the proposed implicit NSFD scheme to solve the European B-S option pricing model.

6. Conclusions

In this paper, we develop and analyze an implicit NSFD scheme for the valuation of European call options. To construct the proposed implicit NSFD scheme, we employ Micken's EFD approach on selected subequations of the underlying B-S equation. We derive EFD schemes for two sub-equations of the original B-S equations. One of these sub-equations is time-independent and exhibits a Cauchy-Euler form. By analytically solving this Cauchy-Euler equation, we obtain an exact solution. This exact solution is then leveraged to construct an EFD scheme for the respective subequation. Finally, merging the EFD schemes for both subequations results in the proposed implicit NSFD scheme. A key advantage of this sub-equation approach is that the resulting NSFD scheme preserves crucial dynamical properties like monotonicity and stability, ensuring the numerical solution accurately reflects the qualitative behavior of the exact solution. By targeting specific parts of the equation, we can design difference operators that more closely approximate the underlying differential operators.

Further, to assess the stability and convergence properties of the constructed NSFD scheme, we employ finite difference analysis tools. These tools include truncation error analysis to quantify the local discretization error, the discrete minimum principle to ensure qualitative behavior preservation, and the judicious selection of discrete barrier functions to establish stability bounds and convergence. We demonstrate that the proposed NSFD scheme is unconditionally stable and first-order convergent in both directions. To validate the theoretical estimates and demonstrate the practical efficacy of the proposed NSFD scheme for the B-S model, we apply the scheme to two numerical test problems. The resulting errors, presented in tabular form, confirm the stability and convergence properties of the NSFD scheme. Also, note that our proposed implicit NSFD scheme for the B-S model gives more accurate results with a small number of mesh points in comparison to the SFD scheme (5.2) with the same order of convergence. In summary, the proposed NSFD scheme integrates the unconditional stability of implicit methods with the structure-preserving properties of nonstandard discretization, offering enhanced robustness for larger step sizes and demonstrating improved overall reliability compared to standard implicit schemes and earlier explicit NSFD approaches. In future research, we aim to modify the proposed NSFD scheme to handle generalized B-S equations with non-constant coefficients and derivation of NSFD-type schemes on non-uniform grids. Additionally, we will explore the construction of NSFD schemes for spatially 2D PDEs and nonlinear B-S equations, with a particular focus on equations involving nonlinear convective terms.

Author contributions

Deepak Singh contributed to the conceptualization, methodology, software, investigation, writing–original draft; Vikas Gupta contributed to the conceptualization, methodology, formal analysis, investigation, supervision, validation, writing–review & editing; Mohammad Sajid contributed to the conceptualization, methodology, funding acquisition, validation, writing–review & editing. All authors confirm that they have read and approved the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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