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Research article

Study of an implicit iterative scheme for a numerical solution of stationary heat convection equations based on the idea of "weak compressibility"

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Abstract: This paper investigates a class of fully implicit iterative schemes for the numerical solution of stationary thermal convection equations formulated in primitive variables (velocity-pressure-temperature). The proposed approach is based on the concept of weak compressibility, which relaxes the incompressibility constraint within a pseudotime-iterative framework. From an algorithmic perspective, the resulting scheme can be interpreted as a fully implicit realization within the broader class of projection and pressure-correction methods widely used in computational fluid dynamics. A rigorous theoretical analysis of the discrete problem is presented. A priori estimates guaranteeing the stability of the numerical solution are derived, and a uniqueness condition for the discrete convection problem is established. For the linear Stokes case, the convergence of the iterative algorithm is proven, and it is shown that the iteration process converges with a geometric rate whose constants are independent of the spatial grid step. Numerical experiments are performed for the classical natural convection benchmark problem in a square cavity. The results demonstrate grid convergence of the numerical solution, confirm the geometric convergence rate predicted by the theory, and illustrate the influence of the Rayleigh number and pseudotime-parameters on the behavior of the algorithm. The obtained results provide a rigorous mathematical foundation for implicit iterative algorithms closely related to projection-type methods and demonstrate their practical applicability to numerical simulation of convective flows.

Keywords: natural convection; primitive variables; pressure-velocity coupling; artificial compressibility; finite difference method; Navier-Stokes equations; implicit iterative schemes
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1. Introduction

Numerical modeling of convective heat transfer governed by the incompressible Navier-Stokes equations remains one of the central problems of computational fluid dynamics (CFD). In two-dimensional configurations, formulations based on stream function-vorticity variables are often used because they automatically satisfy the incompressibility constraint. However, in three-dimensional problems, such formulations become significantly more complex due to the introduction of additional vector potentials and vorticity components. The use of computational experiments in free convection problems was discussed in [1]. Difference schemes for thermal convection equations were developed and analyzed in [2]. Numerical methods for solving free convection problems with lateral heating were considered in [3]. The theoretical foundations of difference schemes used in convection problems were presented in [4]. Computational approaches to heat transfer, including convection processes, were systematically studied in [5].

The numerical solution of incompressible flow equations in primitive variables presents well-known challenges related to the coupling between the momentum equations and the incompressibility constraint. Over the past several decades, a large class of effective numerical techniques has been developed to address this difficulty. Projection methods, also referred to as fractional-step or pressure-correction methods, are closely related to classical algorithms for fluid flow computations described in [6]. A comprehensive overview of projection methods for incompressible flows can be found in [7]. Applications of numerical modeling techniques for fluid flow and heat transfer problems were considered in [8]. Recent advances and detailed analysis of fractional-step methods are available in [9]. Stabilized finite-element formulations for incompressible flow computations were developed in [10]. These methods form the basis of many modern CFD solvers and have been extensively studied both theoretically and computationally. A comprehensive overview of projection methods and their mathematical properties can be found in the survey by Guermond, Minev, and Shen [8]. Another closely related class of methods is based on the concept of artificial compressibility introduced by Chorin, where the incompressibility constraint is relaxed through the introduction of a pseudotime-evolution equation for pressure. Such approaches allow the coupled velocity-pressure system to be solved iteratively and have proven particularly useful for steady-state flow computations [11, 12]. The approach considered in this work is conceptually related to these classical methodologies. In particular, the idea of weak compressibility employed here can be interpreted as a relaxation of the incompressibility constraint within a pseudotime-iterative framework. From an algorithmic perspective, the resulting scheme shares structural similarities with pressure-correction and projection-type methods. Despite the extensive literature on projection and pressure-correction algorithms, the rigorous mathematical analysis of fully implicit iterative schemes formulated directly for stationary discrete convection equations remains comparatively less explored. In many practical CFD implementations, such algorithms are introduced from a computational perspective, since their stability and convergence properties are not always analyzed in detail at the discrete level. The present work aims to address this gap by providing a detailed mathematical analysis of a fully implicit iterative scheme for stationary convection equations formulated in primitive variables. The main novelty of the proposed approach lies in the rigorous analysis of an implicit pseudotime-iterative scheme that can be interpreted as a fully implicit realization within the broader class of pressure-correction and artificial-compressibility methods. In particular, the paper derives a priori estimates for the discrete problem, establishes a uniqueness condition linking the physical parameters and grid properties, and proves geometric convergence of the iterative process for the linear Stokes case with constants independent of the spatial grid step. The proposed approach employs pseudotime-relaxation for both velocity and pressure variables, leading to the implicit iterative scheme (22)–(24). The principal contribution of the paper lies in establishing a priori estimates, proving the uniqueness of the discrete solution, and rigorously demonstrating geometric convergence of the iterative process for the linear Stokes problem with constants independent of the spatial grid step. These results provide a theoretical justification for iterative methodologies that are closely related to projection-type algorithms widely used in computational fluid dynamics. Recent developments in numerical methods for incompressible flows include pressure-Poisson splitting algorithms, which have been analyzed in [13]. High-order and structure-preserving discretization schemes for incompressible Navier-Stokes equations were developed in [14]. Modern time-stepping techniques, including filtered and implicit-explicit approaches, were studied in [15]. Error analysis and fully discrete finite-element methods for incompressible flows were presented in [16]. Classical numerical approaches to grid equations and implicit schemes were systematically presented in [17]. Benchmark problems of natural convection and their numerical solutions were investigated in [18]. Generalized heat conduction equations and phase transition (Stefan-type) problems were studied in [19]. Exact and approximate solutions of Stefan problems in complex geometries were obtained in [20]. Related boundary value problems and their symmetry properties were analyzed in [21]. Inverse and hybrid numerical methods for heat-related processes were considered in [22]. Implicit numerical schemes for nonlinear heat transfer problems were developed in [23].

Thus, this work makes the following key contributions:

(1) The present work does not aim to introduce a fundamentally new algorithmic paradigm. Instead, it provides a rigorous mathematical analysis of a fully implicit iterative realization within the broader class of pressure-correction and artificial-compressibility methods.

(2) In natural variables, a priori estimates are obtained that guarantee the stability of the solution to the difference problem.

(3) A uniqueness condition for the solution to the discrete problem is formulated and proven, linking the problem parameters (Gr, Pr) with the grid through the discrete Laplacian's smallest eigenvalue.

(4) For the linear case (the Stokes problem), the convergence of the iterative process at a rate of geometric progression with constants independent of the spatial grid spacing is rigorously proven.

From an algorithmic point of view, the proposed scheme can be interpreted as a fully implicit realization within the class of projection and pressure-correction methods widely used in computational fluid dynamics. In particular, the pseudotime-relaxation employed in the present work is closely related to artificial compressibility approaches introduced by Chorin and further developed in projection-type algorithms. The paper is organized as follows: Section 2 presents the governing equations and their finite-difference discretization. Section 3 contains the theoretical analysis of the discrete problem, including a priori estimates and the uniqueness theorem. Section 4 introduces the fully implicit iterative algorithm and analyzes its convergence properties for the linear Stokes case. Section 5 presents numerical experiments validating the theoretical results. Section 6 discusses the implications of the proposed method and its relation to classical projection and pressure-correction schemes. Finally, Section 7 summarizes the main results and outlines directions for future research.

2. Methods

2.1. Statement of the problem

To improve readability, the description of the computational domain and grid structure has been reorganized and simplified. Key discretization details, including the grid type, approximation order, and operator properties, have been consolidated into a compact block, preserving the rigor of the presentation while avoiding unnecessary technical detail. Let

$$D = \{0 < x_\alpha < 1,\ \alpha = \overline{1, N}\}$$

be a parallelepiped, where N denotes the spatial dimension. We consider the following system of stationary free convection equations written in dimensionless variables:

$$(\vec{u}\nabla)\vec{u} + \nabla p = \Delta\vec{u} - \frac{\vec{g}}{|\vec{g}|}\, Gr\,\theta + \vec{f}(x), \tag{1}$$

$$\operatorname{div}\vec{u} = 0, \tag{2}$$

$$(\vec{u}\nabla)\theta = \frac{1}{Pr}\,\Delta\theta + q(x). \tag{3}$$

Description of the model.

Equation (1) represents the stationary momentum balance for an incompressible viscous fluid, accounting for buoyancy effects and convective transport.

Equation (2) is the incompressibility constraint, ensuring volume conservation of the fluid.

Equation (3) describes heat transfer governed by convection and thermal diffusion.

Here,

$$x = (x_1, x_2, \ldots, x_N),$$

$\vec{u} = (u, v, w)$ denotes the velocity field, $p(x)$ is the pressure relative to the hydrostatic pressure, $\theta(x)$ is the temperature, $\vec{g}(x)$ is the gravitational acceleration vector, and $\vec{f}(x)$ and $q(x)$ are prescribed external force and heat source terms, respectively.

We assume that

$$\vec{f}(x),\ q(x) \in L^2(\Omega), \qquad \vec{g}(x) \in L^\infty(\Omega).$$

Here, Gr and Pr denote the Grashof and Prandtl numbers, respectively.

In the discrete setting, the corresponding grid functions belong to the space $L_{2,h}$ and are uniformly bounded.

It is assumed that at the boundaries of the computational domain, the components of the velocity vector and temperature take on uniform zero values, that is,

$$\vec{u}|_{\partial D} = \theta|_{\partial D} = 0$$

is a classic in the theory of convective flows in closed areas and has both physical and mathematical justification.

Physical interpretation:

(1). The no-slip condition for velocity

$$\vec{u}|_{\partial D} = 0$$

corresponds to the case where the fluid is in contact with stationary, solid walls. This is the standard condition for viscous flows and reflects the absence of slip at the boundary.

(2). The homogeneous Dirichlet condition for temperature

$$\theta|_{\partial D} = 0$$

can be interpreted as maintaining a constant temperature on the walls equal to some reference temperature (zero in dimensionless variables). This corresponds to adiabatic or isothermal boundaries, depending on the problem statement. Mathematical justification:

• These conditions ensure the well-posedness of the boundary value problem for Navier-Stokes and heat transfer equations.

• They allow the use of embedding theorems and estimates in Sobolev norms, which significantly simplifies the analysis of the stability and uniqueness of the solution.

• In particular, homogeneous conditions ensure the ellipticity of the operators and allow the use of energy estimation methods, as done in Section 3.1.

• This formulation corresponds to the problem of convection in a closed cavity with fixed walls, which is a standard model case in computational fluid dynamics.

• In real-world problems, the boundary conditions may be nonuniform or include Neumann conditions (e.g., for heat flux). However, the homogeneous case is often considered the base case, as it allows one to identify the fundamental qualitative properties of the solution without complicating the model.

Thus, the adopted hypothesis is not only mathematically convenient but also physically justified for a wide class of problems of natural convection in bounded domains.

2.2. *Difference scheme*

This article examines fully implicit iterative algorithms for the numerical solution of stationary grid equations of thermal convection based on the idea of "weak compressibility", and it examines convergence issues.

In the domain

$$D = \{0 \le x_m \le 1,\ m = \overline{1, N}\},$$

where N is the spatial dimension, we consider a system of finite-difference equations of free convection written in dimensionless variables as follows:

$$L_h(\vec{u})\,\vec{u} + \overline{\mathrm{grad}}_h p = \Delta_h \vec{u} - \frac{Gr\,\vec{g}}{|\vec{g}|}\,\theta + \vec{f}(x), \tag{4}$$

$$\mathrm{div}\,\vec{u} = 0, \tag{5}$$

$$L_{h,\theta}(\vec{u})\,\theta = \frac{1}{Pr}\,\Delta_h\theta + q(x). \tag{6}$$

Equations (4)–(6) are finite-difference analogs of the original differential equations. They are formulated on a computational grid using difference operators that approximate the corresponding derivatives with second-order accuracy. The functions $\vec{f}(x)$ and $q(x)$ are grid analogs of the right-hand sides of the continuous problem and are prescribed at the nodes of the corresponding grids.

Here,

$$x = (x_1, x_2, \ldots, x_N), \qquad \vec{u} = (u_1, u_2, \ldots, u_N)$$

denotes the velocity vector, where

$$u_m = u_m(x_1, x_2, \ldots, x_N), \qquad m = \overline{1, N}.$$

Furthermore, $p(x)$ is the pressure, $\theta(x)$ is the temperature, and $\vec{g}(x)$ is the gravity force vector. The operators $L_{h,\vec{u}}$ and $L_{h,\theta}$ are finite-difference operators which correspond to approximations of the convective terms and satisfy the property of energy neutrality.

It is assumed that the components of the velocity vector u_m, $m = \overline{1, N}$, are defined at the nodes of the staggered grids

$$D_{m,h} = \left\{ (l_1 h, l_2 h, \ldots, l_{m-1} h, (l_m + \tfrac{1}{2}) h, l_{m+1} h, \ldots, l_N h) \,\middle|\, \begin{array}{l} l_k = 0, 1, \ldots, M, \quad k \neq m, \\ l_m = 0, 1, \ldots, M - 1, Mh = 1 \end{array} \right\}.$$

The pressure and temperature are defined at the nodes of the grid

$$D_h = \left\{ (l_1 h, l_2 h, \ldots, l_N h) \,\middle|\, l_k = 0, 1, \ldots, M - 1, \quad k = \overline{1, N} \right\}.$$

That is, a "checkerboard" grid is used. This grid choice is standard and optimal for problems involving natural variables (velocity-pressure), as it prevents numerical pressure oscillations (checkerboard instability), ensures natural discretization of the gradient and divergence operators, and allows for pointwise satisfaction of the solenoidality condition in each control volume, which is critical for the physical correctness of the model.

The difference operators defining relations (4)–(6) approximate the corresponding differential operators of the stationary equations of free convection (1)–(3) with the second order of accuracy in spatial variables. The approximation error is of the order of $0(h^2)$, where $h = \max(h_1, h_2, h_3, ..., h_N)$ and are well-known [5]. From here on, $(\cdot, \cdot)$ denotes the scalar product in $L_2(D_h)$, and $\|\cdot\|$ denotes the corresponding norm.

To avoid ambiguity in notation, we introduce a unified symbol for the discrete convective operator used throughout the paper. Let $L_h(u)$ denote the discrete convection operator generated by the velocity field u. When this operator acts on a grid function v, we write $L_h(u)v$. In particular, the nonlinear convective term in the momentum equation is written as $L_h(u)u$.

For clarity of presentation, we introduce the following notation for the discrete operators used throughout the paper. Δ_h denotes the discrete Laplace operator, ∇_h denotes the discrete gradient operator, div_h denotes the discrete divergence operator, and $L_h(u)$ denotes the discrete nonlinear convection operator. All subsequent formulas are written using this unified notation in order to avoid ambiguity in the presentation of the scheme.

We also note that $L_{h,\vec{u}}$, $L_{h,\theta}$ difference operators corresponding to the approximation of convective terms satisfy the conditions of energy neutrality,

$$(L_{h,\vec{u}} \vec{u}, \vec{u}) = (L_{h,\theta} \theta, \theta) = 0,$$

for grid functions that have zero boundary values.

The discrete convection operator $L_h(u)$ satisfies the skew-symmetry property

$$(L_{h,(u)}v, v) = 0,$$

for all grid functions v with homogeneous boundary conditions. This property represents the discrete analog of the energy conservation property of the continuous convective operator and plays a crucial role in the stability analysis of the scheme.

In the following, we assume that at the boundaries of the calculation region, the components of the velocity vector and the temperature take on uniform zero values, that is

$$u_m|_{\partial D_{mh}} = \theta|_{\partial D_h} = 0. \tag{7}$$

3. Theoretical analysis

3.1. A priori estimates

This section presents a theoretical analysis of the difference problem (4)–(7). The main goal is to prove its stability and the uniqueness of its solution, which serves as a mathematical justification for the correctness of the proposed numerical scheme. First, a priori estimates for temperature and velocity are derived, which demonstrate the boundedness of the solution depending on the right-hand sides and the problem parameters. Then, a uniqueness theorem is proved, establishing the conditions under which the solution to the difference problem is unique. The resulting estimates and the uniqueness theorem are key to the subsequent justification of the convergence of the iterative algorithm discussed in Section 4.1.

Let's consider the stability and uniqueness of the solution to the difference problem (4)–(7). We will derive a priori estimates for solutions $\vec{u}$ and θ of the difference problem (4)–(7), confirming the stability of the solution to Problem (4)–(7).

The discrete Laplace operator Δ_h possesses the coercivity property

$$(-\Delta_h v, v) = \lambda_1 \|v\|^2,$$

where λ_1 denotes the smallest eigenvalue of the discrete Laplace operator. This property follows from the discrete Poincaré inequality and is used repeatedly in the derivation of the stability estimates.

Temperature estimate:

We multiply the scalar (6) by θ:

$$(L_{h,\theta}(\bar{u})\theta, \theta) = \frac{1}{\mathrm{Pr}}(\Delta_h \theta, \theta) + (q, \theta).$$

Hence, for the solution θ of the difference problem (4)–(7), due to the energy neutrality of the operator

$$(L_{h,\theta}(\bar{u})\theta, \theta) = 0,$$

it follows that

$$\frac{1}{\mathrm{Pr}}(\Delta_h \theta, \theta) + (q, \theta) = 0.$$

Using the identity

$$(\Delta_h \theta, \theta) = -(\nabla_h \theta, \nabla_h \theta) = -\|\nabla_h \theta\|^2,$$

which is obtained by applying the summation-by-parts formula to $(\Delta_h\theta, \theta)$, we obtain

$$||\nabla_h\theta|| = \Pr \frac{||(q(x), \theta)||}{||\nabla_h\theta||} \leq \Pr ||q||_{(-1)},$$

where

$$||w||_{(-1)} = \sup_{\varphi \in \overset{\circ}{W}{}_2^1} \frac{|(w, \varphi)|}{||\nabla\varphi||}.$$

Here, $\overset{\circ}{W}{}_2^1(D)$ is a subspace of $W_2^1(D)$ whose dense subset is the set $C_0^\infty(D)$ of all infinitely differentiable functions with compact support in D.

Thus, for the solution θ, we obtain the estimate

$$||\nabla_h\theta|| \leq Pr\, ||q||_{(-1)}. \tag{8}$$

The above estimate relies on the discrete Poincaré inequality

$$||v|| \leq C_p\, ||\nabla_h v||,$$

which holds for grid functions satisfying homogeneous boundary conditions.

Estimate (8) shows that the discrete velocity field is bounded in the L_2 norm by the data of the problem. More precisely, the following inequality holds:

$$||u|| \leq C\,(||f|| + Ra||\theta||)\,,$$

where C is a positive constant independent of the spatial grid step.

Estimate for the velocity. Before deriving the energy estimates, we recall an important property of the discrete convective operator. If the discrete velocity field satisfies the incompressibility constraint

$$div_h u = 0,$$

then the discrete convective operator is skew-symmetric in the L_2 inner product, and therefore,

$$(L_u(u)u, u) = 0.$$

This property represents the discrete analog of the well-known energy conservation property of the continuous convective term in Navier-Stokes equations. Taking the discrete L_2 inner product of Eq (4) with the velocity field $\vec{u}$, we obtain

$$(L_h(\vec{u})\vec{u}, \vec{u}) + (\overline{\text{grad}}_h\, p, \vec{u}) = (\Delta_h\vec{u}, \vec{u}) - \left(\frac{Gr\vec{g}}{|\vec{g}|}\,\theta, \vec{u}\right) + (\vec{f}, \vec{u}).$$

Using the summation by parts formula for $(\Delta_h\vec{u}, \vec{u})$ as well as $(L_{h,\mu}\vec{u}, \vec{u}) = 0$, we obtain for the solution $\vec{u}$ of the difference problem (4)–(7) the equality

$$(\overline{\text{grad}}_h\, p, \vec{u}) = -(\nabla_h\vec{u}, \nabla_h\vec{u}) - \left(\frac{Gr\vec{g}}{|\vec{g}|}\,\theta, \vec{u}\right) + (\vec{f}(x), \vec{u}).$$

Taking into account the equalities

$$(\overline{\mathrm{grad}}_h\, p, \vec{u}) = -(p, \mathrm{div}\, \vec{u}), \qquad \mathrm{div}\, \vec{u} = 0,$$

we obtain

$$\|\nabla_h \vec{u}\|^2 \le \left| \left(\frac{G\vec{g}}{|\vec{g}|}\, \theta, \vec{u} \right) \right| + |(\vec{f}(x), \vec{u})|.$$

Applying the well-known inequalities

$$|(u, v)| \le \|u\|_p \cdot \|v\|_q, \qquad \text{and} \qquad \|uv\| = \|u\| \cdot \|v\|,$$

we obtain

$$\|\nabla_h \vec{u}\|^2 \le Gr\|\theta\| \cdot \|\vec{u}\| + (\vec{f}, \vec{u}).$$

Hence, taking into account that for any function $u(x) \in \dot{W}_2^1(\Omega)$ the inequality $\delta_0\|\vec{u}\|^2 \le \|\nabla\vec{u}\|^2$ holds, where δ_0 is the smallest eigenvalue of the Laplace operator, we obtain

$$\|\nabla_h \vec{u}\|^2 \le Gr\, \frac{1}{\delta_0}\, \|\nabla_h \theta\|\, \|\nabla_h \vec{u}\| + |(\vec{f}, \vec{u})|.$$

Dividing both sides of the inequality by $\|\nabla_h \vec{u}\|$, we obtain

$$\|\nabla_h \vec{u}\| \le Gr\, \frac{1}{\delta_0}\, \|\nabla_h \theta\| + \frac{\|(\vec{f}, \vec{u})\|}{\|\nabla_h \vec{u}\|} \le Gr\, \frac{1}{\delta_0}\, \|\nabla_h \theta\| + \|\vec{f}\|_{(-1)}. \tag{9}$$

Applying Estimate (8) to (9), we obtain the estimate for $\|\nabla_h \vec{u}\|$:

$$\|\nabla_h \vec{u}\|^2 \le \|\vec{f}\|_{(-1)} + \frac{Ra}{\delta_0}\, \|q\|_{(-1)}. \tag{10}$$

Here, $Ra = Pr\, Gr$ is the Rayleigh number.

Now, we consider the question of uniqueness of the solution of the difference problem (4)–(7).

3.2. Uniqueness of the solution

Theorem 3.1. *Let the functions $\vec{f}$ and q and the numbers Gr and Pr be such that the inequality*

$$1 - \frac{c_1}{\delta_0}\Big(\|\vec{f}\|_{(-1)} + Ra\, \|q\|_{(-1)}\Big) - \frac{c_2\, Ra}{\delta_0}\, \|q\|_{(-1)} \ge \alpha > 0 \tag{11}$$

holds.

Condition (11) is sufficient for the uniqueness of the solution of the difference problem. It relates the parameters of the problem (the Grashof and Prandtl numbers) with the eigenvalues of the Laplace operator and guarantees that the solution is unique. Here, δ_0 is the smallest eigenvalue of the Laplace operator, c_1 and c_2 are positive constants independent of the grid parameters, and

$$\|q\|_{(-1)} = \sup_{\varphi \ne 0} \frac{|(q, \varphi)|}{\|\nabla\varphi\|}, \qquad Ra = Pr\, Gr.$$

Then, the difference problem (4)–(7) has a unique solution.

Proof. Assume that there exist two triples $(\vec{u}, p, \theta)$ and $(\vec{v}, q, T)$ satisfying the relations (4)–(7). Then, the differences

$$G = \theta - T, \qquad \vec{w} = \vec{u} - \vec{v}, \qquad \pi = p - q$$

satisfy the following difference equations:

$$L_{h,\vec{u}}(\vec{u})\vec{w} + L_{h,\vec{u}}(\vec{w})\vec{v} + \overline{\text{grad}}_h\, \pi = \Delta_h \vec{w} - Gr\, \frac{\vec{g}}{|\vec{g}|}\, G, \tag{12}$$

$$\underline{\text{div}}_h\, \vec{w} = 0, \tag{13}$$

$$L_{h,\theta}(\vec{u})G + L_{h,\theta}(\vec{w})T = \frac{1}{Pr}\, \Delta_h G. \tag{14}$$

Multiplying Eq (12) scalarwise by $\vec{w}$ and Eq (14) by G and taking into account (7) and (13), we obtain

$$||\nabla_h \vec{w}||^2 + (L_{h,\vec{u}}(\vec{w})\vec{v}, \vec{w}) = -Gr\left(\frac{\vec{g}}{|\vec{g}|}G, \vec{w}\right) = 0,$$

$$\frac{1}{Pr}\, ||\nabla_h G||^2 + (L_{h,\vec{u}}(\vec{w})T, G) = 0.$$

Hence, it follows that

$$||\nabla_h \vec{w}||^2 \leq \left|(L_{h,\vec{u}}(\vec{w})\vec{v}, \vec{w})\right| + Gr\left|\left(\frac{\vec{g}}{|\vec{g}|}G, \vec{w}\right)\right|, \tag{15}$$

$$||\nabla_h G||^2 \leq Pr\, \left|(L_{h,\vec{u}}(\vec{v})T, G)\right|. \tag{16}$$

We estimate the terms $|Gr\,((\vec{g}/|\vec{g}|)G, \vec{w})\,|$, $(L_{h,\vec{u}}(\vec{w})\vec{v}, \vec{w})$, and $(L_{h,\vec{u}}(\vec{w})T, G)$. Applying the well-known inequalities

$$|(u, v)| \leq ||u||_p \cdot ||v||_q, \qquad ||uv|| = ||u|| \cdot ||v||, \qquad \delta_0 ||u||^2 \leq ||\nabla u||^2,$$

we obtain

$$\left|Gr\left(\frac{\vec{g}}{|\vec{g}|}G, \vec{w}\right)\right| \leq \frac{Gr}{\delta_0}\, ||\nabla_h \vec{w}||\, ||\nabla_h G||. \tag{17}$$

The quantities $|(L_{h,\vec{u}}(\vec{w})\vec{v}, \vec{w})|$ and $|(L_{h,u}(\vec{w})T, G)|$ are estimated using the inequality

$$||u||^4_{L_4,\Omega} \leq 8||u||_{L_{2,\Omega}} ||u_{x_1}||_{L_{2,\Omega}} ||u_{x_2}||_{L_{2,\Omega}} ||u_{x_3}||_{L_{2,\Omega}} \leq \left(\frac{4}{3}\right)^{3/2} ||u||_{L_{2,\Omega}} ||u_x||^3_{L_{2,\Omega}},$$

valid for any $u(x) \in \dot{W}^1_2(\Omega)$.

As a result, we obtain

$$|(L_{h,\vec{u}}(\vec{w})\vec{v}, \vec{w})| \leq c_0 ||\nabla_h \vec{v}|| \;\cdot ||\vec{w}||_4 \cdot ||\vec{w}||_4 = c_0 ||\nabla_h \vec{v}|| \;\cdot ||\vec{w}||_4^2 \leq \frac{c_1}{\delta_0}\, ||\nabla_h \vec{v}||\, ||\nabla_h \vec{w}||^2, \tag{18}$$

and similarly,

$$|(L_{h,\theta}(\vec{w})T, G)| \leq \frac{c_2}{\delta_0}\, ||\nabla_h T|| \;\cdot ||\nabla_h \vec{w}|| \;\cdot ||\nabla_h G||. \tag{19}$$

Here c_0, c_1, c_2 are positive constants independent of the grid parameters. Applying (17)–(19) to (15) and (16), we obtain

$$||\nabla_h \vec{w}||^2 \leq \frac{Gr}{\delta_0}\, ||\nabla_h G||\, ||\nabla_h \vec{w}|| + \frac{c_1}{\delta_0}\, ||\nabla_h \vec{v}||\, ||\nabla_h \vec{w}||^2,$$

$$\|\nabla_h G\|^2 \le \frac{c_2}{\delta_0}\,\|\nabla_h T\|\,\|\nabla_h \vec{w}\|\,\|\nabla_h G\|.$$

Transforming these inequalities, we obtain

$$\left(1 - \frac{c_1}{\delta_0}\|\nabla_h \vec{v}\|\right)\|\nabla_h \vec{w}\| \le \frac{Gr}{\delta_0}\,\|\nabla_h G\|, \tag{20}$$

$$\|\nabla_h G\|^2 \le \frac{c_2}{\delta_0}\,\|\nabla_h T\|\,\|\nabla_h \vec{w}\|. \tag{21}$$

Substituting (21) into the right-hand side of (20), we obtain

$$\left(1 - \frac{c_1}{\delta_0}\|\nabla_h \vec{v}\| - \frac{c_2 Gr}{\delta_0}\|\nabla_h T\|\right)\|\nabla_h \vec{w}\| \le 0.$$

From this inequality, using the estimates (9) and (10), it follows that

$$\left(1 - \frac{c_1}{\delta_0}(\|\vec{f}\|_{(-1)} + Ra\|q\|_{(-1)}) - \frac{c_2 Ra}{\delta_0}\|q\|_{(-1)}\right)\|\nabla_h \vec{w}\| \le 0.$$

Hence, taking into account Condition (11),

$$1 - \frac{c_1}{\delta_0}(\|\vec{f}\|_{(-1)} + Ra\|q\|_{(-1)}) - \frac{c_2 Ra}{\delta_0}\|q\|_{(-1)} \ge \alpha > 0,$$

we obtain $\|\nabla_h \vec{w}\| = 0$. Substituting $\|\nabla_h \vec{w}\| = 0$ into (19) and (20), we obtain $\|\nabla_h G\| = 0$.

The theorem is proved. □

Intuitively, Condition (11) means that the uniqueness of the solution is guaranteed for sufficiently weak regimes of thermal convection (small Rayleigh numbers Ra) and for small norms of the external data $\vec{f}$ and q. The latter condition is satisfied on sufficiently fine grids, because the smallest eigenvalue of the Laplace operator δ_0 grows as $O(h^{-2})$. This implies that Inequality (11) holds for fixed $\vec{f}$ and q.

Obviously, Condition (11) can be achieved for small values of the norms $\|\vec{f}\|_{(-1)}$ and $\|q\|_{(-1)}$ and the numbers Gr and Pr, and it is equivalent to the uniqueness condition for the boundary value problem for Navier–Stokes equations, which is known to be sharp [3].

4. Iterative scheme and convergence analysis

4.1. Implicit iterative algorithm

We consider a method for organizing the iterative process for solving the stationary problem (4)–(6), which is completely implicit in time.

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} + L_{h,\vec{u}}(\vec{u}^n)\vec{u}^{n+1} + \mathrm{grad}_h\, p^{n+1} = \Delta_h \vec{u}^{n+1} - Gr\,\frac{\vec{g}}{|\vec{g}|}\,\theta^{n+1} + \vec{f}(x), \tag{22}$$

$$\frac{p^{n+1} - p^n}{\tau_0} + \underline{\mathrm{div}}_h\, \vec{u}^{n+1} = 0, \tag{23}$$

$$\frac{\theta^{n+1} - \theta^n}{\tau} + L_{h,\vec{u}}(\vec{u}^n)\theta^{n+1} = \frac{1}{Pr}\,\Delta_h \theta^{n+1} + q(x), \qquad x \in D_h. \tag{24}$$

The initial approximations for the iterative process are chosen as

$$\theta^0 = \theta_0(x), \qquad \vec{u}^0 = \vec{u}_0(x) \in \dot{W}_2^1(D_h), \qquad p^0 = p_0(x) \in L_2/\mathbb{R}. \tag{25}$$

Here, $L_2/\mathbb{R}$ denotes the subspace of functions in $L_2(D_h)$ orthogonal to constants, and τ and τ_0 are artificial (pseudotime) relaxation parameters not related to physical time.

System (22)–(24) represents a method for constructing a solution of the stationary problem (4)–(6) and defines a fully implicit iterative algorithm in which each new approximation of the velocity and temperature is computed simultaneously, ensuring increased stability.

4.1.1. Implementation of the iterative algorithm

The algorithm is constructed as a sequence of outer iterations with respect to the iteration index $n = 0, 1, 2, \ldots$ until the prescribed accuracy is achieved. At each iteration, the following steps are performed.

Step 1. Solution of the temperature Eq (24).

Equation (24) is linear with respect to θ^{n+1} for fixed $\vec{u}^n$. It is solved in the interior nodes of the grid. Using the boundary condition $\theta^{n+1} = 0$, we obtain a system of linear algebraic equations (SLAE) with a symmetric positive definite matrix

$$I + \tau L_{h,\vec{u}}(\vec{u}^n) - \frac{\tau}{\mathrm{Pr}}\Delta_h \theta^{n+1} = \theta^n + \tau q(x),$$

where I is the identity operator. Efficient iterative methods (e.g., conjugate gradient or multigrid methods) are used for solving it.

Step 2. Solution of the velocity Eq (26).

Eliminating the pressure p^{n+1} from (22) using Relation (23), we obtain

$$A_h \vec{u}^{n+1} = \vec{u}^n - \tau \overline{grad_h} p^n - \tau \frac{\mathrm{Gr}}{|\theta^n|}\theta^n + \tau f(x), \tag{26}$$

where the operator A_h is defined as

$$A_h = I - \tau_0 \overline{grad_h}\, \underline{\mathrm{div}_h} - \tau\Delta_h + \tau L_{h,\vec{u}}(\vec{u}^n).$$

Equation (26) is solved for $\vec{u}^{n+1}$ under boundary conditions

$$\vec{u}^{n+1}\Big|_{\partial D_h} = 0.$$

Because A_h is elliptic and energetically equivalent to $I - \tau\Delta_h$, efficient iterative methods can again be applied.

Step 3. Pressure correction.

After computing u^{n+1}, the pressure is updated according to Formula (23):

$$p^{n+1} = p^n - \tau_0\, \mathrm{div}_h\, u^{n+1}.$$

Step 4. Convergence check.

The norm of the change in the solution between successive iterations is computed as

$$\delta = \|u^{n+1} - u^n\| + \|p^{n+1} - p^n\| + \|\theta^{n+1} - \theta^n\|.$$

If $\delta < \varepsilon$, where ε is the prescribed accuracy, the iterative process is terminated; otherwise, we set $n := n + 1$ and return to Step 1.

4.1.2. Remarks on implementation

(1) The choice of the parameters τ and τ_0 affects the convergence rate. It is recommended to choose $\tau = O(h^2)$ (based on the stability conditions for explicit schemes), and to choose $\tau_0 = 1$ or to select it experimentally to accelerate convergence.
(2) At Steps 1 and 2, large linear systems arise. To solve them, it is advisable to use grid-dependent methods (for example, the conjugate gradient method, multigrid methods, or iterative methods for elliptic problems), which guarantee convergence in $O(N)$ operations, where N is the number of grid nodes.
(3) As initial approximations u^0, p^0, θ^0, one can take zero fields or solutions of a simplified problem (for example, a linearized one).

The described algorithm is stable and converges to the solution of problem (4)–(6) under the conditions of the uniqueness theorem (Section 3.2). In the linear case (the Stokes problem), convergence has a geometric rate (Section 4.2).

It is easy to show that the difference operator A_h is energetically equivalent to the operator

$$\tilde{A}_h = E - \tau\Delta_h,$$

that is, there exist positive constants c_1 and c_2 such that

$$c_1(\tilde{A}_h u, u) \le (A_h u, u) \le c_2(\tilde{A}_h u, u).$$

Indeed, for the operator A_h, we have

$$(A_h u, u) = ||u||^2 + \tau_0\, ||\operatorname{div}_h u||^2 + \tau\, ||\nabla_h u||^2.$$

Accordingly, we obtain

$$||u||^2 + \tau||\nabla_h u||^2 \le (A_h u, u) \le ||u||^2 + (1 + \tau_0)\, ||\nabla_h u||^2,$$

or

$$c_1(\tilde{A}_h u, u) \le (A_h u, u) \le c_2(\tilde{A}_h u, u),$$

where $c_1 = 1$ and $c_2 = 1 + \tau_0$.

The study of the properties of the operators A_h and $\tilde{A}_h$ makes it possible to construct efficient iterative algorithms for solving Eq (26) using iterative methods for solving systems of Lamé-type equations [18].

We now consider the issues of convergence and convergence rate of the iterative algorithm for the linear case (the Stokes problem). In this case, the iteration has the form

$$\frac{u^{n+1} - u^n}{\tau} + \nabla_h p^{n+1} = \Delta_h u^{n+1} - \frac{\mathrm{Gr}\vec{g}}{|g|}\,\theta^{n+1} + f(x), \tag{27}$$

$$\frac{p^{n+1} - p^n}{\tau_0} + \underline{\operatorname{div}}_h\, \vec{u}^{n+1} = 0, \tag{28}$$

$$\frac{\theta^{n+1} - \theta^n}{\tau} - \frac{1}{\mathrm{Pr}}\Delta_h \theta^{n+1} = g(x), \quad x \in D_h, \tag{29}$$

with the initial and boundary conditions

$$\theta^{n+1}\Big|_{\partial D_h} = \vec{u}^{n+1}\Big|_{\partial D_{m,h}} = 0, \qquad \vec{u}^0 = \vec{u}_0(x) \in \overset{\circ}{W}{}_2^1(D_h), \qquad p^0 = p_0(x) \in L_2(D_h)/\mathbb{R}. \tag{30}$$

To analyze the convergence, we introduce the iteration errors of (27)–(29) as the differences between the approximate solution at step $n+1$ and the exact solution of the difference problem (4)–(6)

$$z^{n+1} = u^{n+1} - \bar{u}, \qquad \pi^{n+1} = p^{n+1} - p, \qquad T^{n+1} = \theta^{n+1} - \theta,$$

where $\bar{u}$, p, θ are solutions of the linear problem (4)–(6). Then, we obtain the following relations:

$$\frac{z^{n+1} - z^n}{\tau} + \nabla_h \pi^{n+1} = \Delta_h z^{n+1} - \frac{\mathrm{Gr}\vec{g}}{|g|} T^{n+1}, \tag{31}$$

$$\frac{\pi^{n+1} - \pi^n}{\tau_0} + \mathrm{div}_h\, z^{n+1} = 0, \tag{32}$$

$$\frac{T^{n+1} - T^n}{\tau} = \frac{1}{\mathrm{Pr}} \Delta_h T^{n+1}. \tag{33}$$

4.2. Convergence of the iterative method

In this section, the convergence of the proposed implicit iterative algorithm (22)–(25) is investigated. The main attention is devoted to the linear case (the Stokes problem), for which it is possible to obtain rigorous convergence estimates. Two main theorems are proved. Theorem 4.1 establishes the convergence of the iteration, and Theorem 4.2 estimates the convergence rate, showing that it has a geometric character with constants independent of the grid step. These results provide a theoretical basis for the application of the algorithm in practical computations.

Theorem 4.1. *The iterative algorithm (27)–(30) converges in* $\overset{\circ}{W}{}_2^1(D_h)$ *to the solution of the linear problem (4)–(6). Moreover, the following estimate holds:*

$$E^{n+1} + \|z^{n+1} - z^n\|^2 + \tau_0 \|\mathrm{div}_h\, z^{n+1}\|^2 + 2\tau\left(1 - \varepsilon_1 \frac{\mathrm{Gr}}{\delta_0}\right)\|\nabla_h z^{n+1}\|^2 + \frac{\mathrm{Ra}}{4\delta_0\varepsilon_1\varepsilon_2}\|T^{n+1} - T^n\|^2 + \frac{\tau\mathrm{Ra}(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2}\|\nabla_h T^{n+1}\|^2 \le E^n,$$

where

$$E^n = \|z^n\|^2 + \frac{\mathrm{Ra}}{4\delta_0\varepsilon_1\varepsilon_2}\|T^{n+1}\|^2 + \frac{\tau}{\tau_0}\|\pi^n\|^2,$$

$0 < \varepsilon_1 < \dfrac{\delta_0}{\mathrm{Gr}}$, $0 < \varepsilon_2 < 1$*, and* δ_0 *is the smallest eigenvalue of the discrete Laplace operator.*

Proof. Multiplying Eq (31) scalarly in L_2 by $2\tau z^{n+1}$, we obtain

$$\|z^{n+1}\|^2 - \|z^n\|^2 + \|z^{n+1} - z^n\|^2 = -2\tau(\pi^{n+1}, \mathrm{div}_h\, z^{n+1}) - 2\tau\left(\frac{Gr\vec{g}}{|\vec{g}|}, \overline{T}^{n+1}, z^{n+1}\right).$$

We transform the last term on the left-hand side of this equality using Identity (32):

$$\|z^{n+1}\|^2 - \|z^n\|^2 + \|z^{n+1} - z^n\|^2 + 2\tau\left(\pi^{n+1}, \frac{\pi^{n+1} - \pi^n}{\tau_0}\right) = -2\tau\|\nabla_h z^{n+1}\|^2 - 2\tau\left(\frac{Gr\vec{g}}{|\vec{g}|}\pi^{n+1}, z^{n+1}\right).$$

Rewriting this equality, we obtain

$$\|z^{n+1}\|^2-\|z^n\|^2+\|z^{n+1}-z^n\|^2+\frac{\tau}{\tau_0}(\|\pi^{n+1}\|^2-\|\pi^n\|^2)+\frac{\tau}{\tau_0}\|\pi^{n+1}-\pi^n\|^2=-2\tau\|\nabla_h z^{n+1}\|^2-2\tau\left(\frac{Gr\vec{g}}{|\vec{g}|}T^{n+1},z^{n+1}\right).$$

Taking into account the identity

$$\frac{\|\pi^{n+1}-\pi^n\|^2}{\tau_0^2}=\|\operatorname{div}_h z^{n+1}\|^2,$$

which follows from (32), we obtain

$$\|z^{n+1}\|^2-\|z^n\|^2+\|z^{n+1}-z^n\|^2+\frac{\tau}{\tau_0}(\|\pi^{n+1}\|^2-\|\pi^n\|^2)+\tau\tau_0\|\operatorname{div}_h z^{n+1}\|^2+2\tau\|\nabla_h z^{n+1}\|^2=2\tau\left|\left(\frac{Gr\vec{g}}{|\vec{g}|}\pi^{n+1},z^{n+1}\right)\right|.$$

Applying successively the Cauchy–Bunyakovsky inequality $\langle u,v\rangle\le\|u\|\,\|v\|$ and the ε-inequality $ab\le\varepsilon a^2+\dfrac{1}{4\varepsilon}b^2$, we arrive at

$$\begin{aligned}&\|z^{n+1}\|^2-\|z^n\|^2+\|z^{n+1}-z^n\|^2+\frac{\tau}{\tau_0}(\|\pi^{n+1}\|^2-\|\pi^n\|^2)+\tau\tau_0\|\operatorname{div}_h z^{n+1}\|^2+2\tau\|\nabla_h z^{n+1}\|^2\\&=2\tau\frac{Gr}{\sigma_0}\left(\varepsilon_1\|\nabla_h z^{n+1}\|^2+\frac{1}{4\varepsilon_1}\|T^{n+1}\|^2\right).\end{aligned}\qquad(34)$$

Here $\varepsilon_1>0$ is an arbitrary positive constant.

Multiplying Eq (33) scalarly by $2\tau T^{n+1}$, we obtain

$$\|T^{n+1}\|^2-\|T^n\|^2+\|T^{n+1}-T^n\|^2+\frac{2\tau}{\mathrm{Pr}}(1-\varepsilon_2)\|\nabla_h T^{n+1}\|^2+\frac{2\tau\varepsilon_2}{\mathrm{Pr}}\|\nabla_h T^{n+1}\|^2=0,$$

where $0<\varepsilon_2<1$.

Adding this equality, after preliminary multiplication by $\dfrac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}$, to the right-hand side of (34), we obtain

$$\begin{aligned}E^{n+1}+\|z^{n+1}-z^n\|^2+\tau\tau_0\|\operatorname{div}_h z^{n+1}\|^2+2\tau\left(1-\varepsilon_1\frac{G_r}{\delta_0}\right)\|\nabla_h z^{n+1}\|^2&\\+\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\|T^{n+1}-T^n\|^2+\frac{\tau Ra(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2}\|\nabla_h T^{n+1}\|^2\le E^n.&\end{aligned}\qquad(35)$$

Here,

$$E^n=\|z^n\|^2+\frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\|T^n\|^2+\frac{\tau}{\tau_0}\|\pi^n\|^2.$$

If ε_1 is chosen such that

$$1-\varepsilon_1\frac{G_r}{\delta_0}=\delta_1>0,$$

then Inequality (35) implies

$$0\le E^{n+1}\le E^n,$$

which, in turn, yields convergence of the sequence $\{E^n\}$, $n=0,1,2,\dots$, that is, convergence of the iterative process in $\widetilde{W}_2^1(D_h)$. The theorem is proved. □

Theorem 4.1 proves the convergence of the iterative algorithm in the linear case. The estimate shows that the error decreases at each iteration, and the rate of convergence is independent of the grid spacing. We now estimate the rate of convergence of the iterations.

4.3. Estimate of the convergence rate

Theorem 4.2 (Geometric convergence rate)**.** *The iterative algorithm (27)–(30) converges with a geometric rate of convergence. Specifically, the following estimate holds:*

$$E^{n+1} \le \mu E^n, \qquad 0 < \mu < 1,$$

where E^n is defined in Theorem 4.1 as

$$E^n = \|z^n\|^2 + \frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\|T^{n+1}\|^2 + \frac{\tau}{\tau_0}\|\pi^n\|^2,$$

$$0 < \varepsilon_1 < \frac{\delta_0}{G_r}, \qquad 0 < \varepsilon_2 < 1.$$

Here, δ_0 denotes the smallest eigenvalue of the discrete Laplace operator. The convergence rate μ is independent of the spatial grid steps h.

Proof. To estimate the convergence rate in the linear case, we rewrite Relation (31) in the form

$$\tau\overline{grad}_h\pi^{n+1} = \tau\Delta_h z^{n+1} - \frac{\tau Gr\vec{g}}{\delta_0}T^{n+1} - (z^{n+1} - z^n),$$

and multiply both sides scalarly in L_2 by $\dfrac{\phi}{\|\nabla_h\phi\|}$, where $\phi \in \widetilde{W}_2^1(D_h)$.

Then, we obtain

$$\left(\tau\overline{grad}_h r^{n+1}, \frac{\phi}{\|\nabla_h\phi\|}\right) = \left(\tau\Delta_h z^{n+1}, \frac{\phi}{\|\nabla_h\phi\|}\right) - \left(\frac{\tau Gr\vec{g}}{|\vec{g}|}\nabla_h T^{n+1}, \frac{\phi}{\|\nabla_h\phi\|}\right) - \left(z^{n+1} - z^n, \frac{\phi}{\|\nabla_h\phi\|}\right),$$

or equivalently,

$$\frac{\tau}{\|\nabla_h\phi\|}(r^{n+1}, \mathrm{div}_h\,\phi) = \frac{\tau}{\|\nabla_h\phi\|}(\nabla_h z^{n+1}, \nabla_h\phi) - \frac{\tau G_r}{\|\nabla_h\phi\|}(\nabla_h T^{n+1}, \phi) - \frac{1}{\|\nabla_h\phi\|}(z^{n+1} - z^n, \phi).$$

Hence, we obtain

$$\frac{\tau\left|(\pi^{n+1}, \mathrm{div}_h\,\phi)\right|}{\|\nabla_h\phi\|} \le \tau\|\nabla_h z^{n+1}\| + \frac{1}{\sqrt{\delta_0}}\|z^{n+1} - z^n\| + \frac{\tau G_r}{\delta_0}\|\nabla_h T^{n+1}\|.$$

Because this inequality holds for any function $\phi \in \widetilde{W}_2^1(D_h)$, and the right-hand side does not depend on ϕ, it is also valid for the supremum of the left-hand side. Thus, from the "inf-sup" inequality, we obtain the estimate

$$\tau\|\pi^{n+1}\| \le \tau\|\nabla_h z^{n+1}\| + \frac{1}{\sqrt{\delta_0}}\|z^{n+1} - z^n\| + \frac{\tau G_r}{\delta_0}\|\nabla_h T^{n+1}\|.$$

To simplify the presentation of the convergence proof, we introduce several auxiliary constants. The constants $\delta_1, \delta_2, \delta_3$ denote positive bounds that arise from the coercivity and boundedness properties of the discrete operators involved in the scheme. The constant α represents the contraction factor of the iteration operator, and L denotes the Lipschitz constant associated with the nonlinear convective term.

These constants depend only on the physical parameters of the problem and remain independent of the spatial grid size.

Squaring both sides of this inequality, we arrive at

$$\tau^2\|\pi^{n+1}\|^2 \leq L\left(\tau\|\nabla_h z^{n+1}\|^2 + \|z^{n+1} - z^n\|^2 + \frac{\tau G_r}{\delta_0}\|\nabla_h T^{n+1}\|^2\right), \tag{36}$$

where C is a positive constant.

Multiplying both sides of Inequality (36) by an arbitrary positive constant α and adding the resulting inequality to Relation (35), we obtain

$$E^{n+1}+\tau^2\alpha\|\pi^{n+1}\|^2+(1-\alpha L)\|z^{n+1}-z^n\|^2+2\tau\left(\delta_1-\frac{\alpha L}{2}\right)\|\nabla_h z^{n+1}\|^2+\tau\left[\frac{Ra(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2}-\frac{\alpha GrL}{\delta_0}\right]\|\nabla_h T^{n+1}\|^2\leq E^n.$$

Hence, choosing α to satisfy the inequalities

$$1 - \alpha L \geq 0,$$

$$\delta_1 - \frac{\alpha L}{2} \geq \delta_2 > 0,$$

$$\frac{Ra(1-\varepsilon_2)}{2\delta_0\varepsilon_1\varepsilon_2} - \frac{\alpha GrL}{\delta_0} \geq \delta_3 > 0,$$

we obtain

$$\|z^{n+1}\|^2 + \frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\|T^{n+1}\|^2 + \frac{\tau}{\tau_0}(1+\tau\tau_0)\|\pi^{n+1}\|^2 + 2\tau\delta_2\|\nabla_h z^{n+1}\|^2 + \tau\delta_3\|\nabla_h T^{n+1}\|^2$$

$$\leq \|z^n\|^2 + \frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\|T^n\|^2 + \frac{\tau}{\tau_0}\|v^n\|^2.$$

Furthermore, we have

$$\left(1+\frac{2\tau\delta_2}{\delta_0}\right)\|z^{n+1}\|^2 + \frac{Ra}{4\delta_0\varepsilon_1\varepsilon_2}\left(1+\frac{4\varepsilon_1\varepsilon_2\tau}{Ra}\right)\|T^{n+1}\|^2 + \frac{\tau}{\tau_0}(1+\tau\tau_0)\|\pi^{n+1}\|^2 \leq E^n.$$

Using the inequality

$$\min\left\{1+\frac{2\tau\delta_2}{\delta_0},\ 1+\frac{4\varepsilon_2\tau}{Ra},\ 1+\tau_0\right\}E^{n+1} \leq E^n,$$

we obtain

$$E^{n+1} \leq \mu E^n,$$

where

$$\mu = \frac{1}{\min\left\{1+\frac{2\tau\delta_2}{\delta_0},\ 1+\frac{4\varepsilon_1\varepsilon_2\tau}{Ra},\ 1+\tau\tau_0\right\}} < 1.$$

That is, the numerical sequence E^n converges to zero at a rate of geometric progression with a denominator independent of the spatial grid steps, which is equivalent to the convergence of the iteration at a rate of geometric progression. The theorem is proven. □

5. Numerical results

This section presents numerical experiments designed to verify the theoretical convergence results obtained in Section 4 and to demonstrate the practical performance of the proposed implicit iterative scheme.

The numerical experiments pursue four main goals:

(1) Validation of the algorithm using a classical benchmark problem;

(2) Demonstration of grid convergence of the numerical solution;

(3) Empirical verification of the geometric convergence rate predicted by Theorem 4.2;

(4) Investigation of the influence of the Rayleigh number and the pseudotime parameters τ and τ_0.

5.1. Benchmark problem

As a benchmark test, we consider the classical problem of natural convection in a square cavity.

The computational domain is $\Omega = (0, 1)\text{x}(0, 1)$. The boundary conditions are as follows:

Left wall: $T = 1$;

Right wall: $T = 0$;

Top and bottom walls: $\frac{dT}{dn} = 0$;

Velocity satisfies the no-slip condition: $u = v = 0$ on all boundaries.

The Prandtl number is fixed at $Pr = 0.71$, which corresponds to air. The Rayleigh number is set to $Ra = 10^4$, which corresponds to the classical benchmark configuration introduced by de Vahl Davis (1983).

The steady-state solution exhibits the characteristic circulation pattern typical for natural convection in a cavity. The flow structure is illustrated by the stream function contours shown in Figure 1.

The temperature distribution inside the cavity is shown in Figure 2.

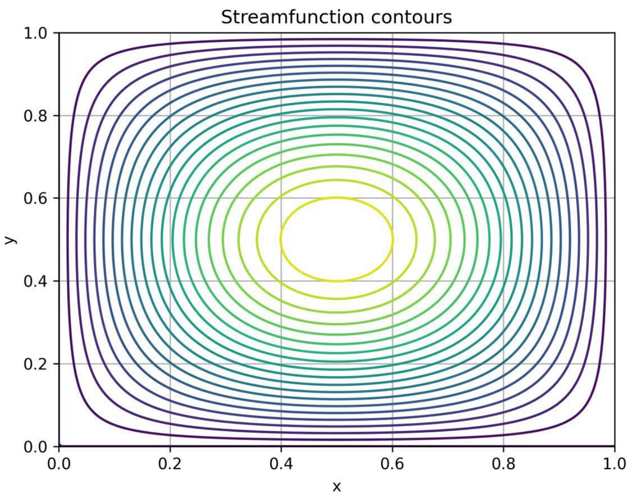

Figure 1. Stream function contours for the natural convection cavity benchmark.

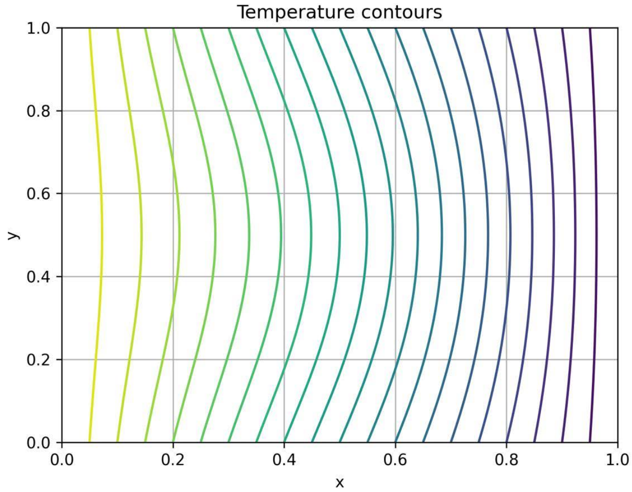

Figure 2. Temperature contours illustrating the thermal boundary layers near the vertical walls.

5.2. Grid convergence

The numerical solution is computed on a sequence of uniform staggered grids 32x32, 64x64, 128x128, 256x256.

Table 1 presents the maximum value of the stream function and the average Nusselt number on the hot wall.

Table 1. Grid convergence study.

Grid	ψ_{max}	Nu_{avg}
32x32	5.234	2.312
64x64	5.098	2.257
128x128	5.057	2.243
256x256	5.043	2.239

The results demonstrate clear convergence of the numerical solution as the grid is refined. For the finest grid, the obtained values are $\psi_{max} = 5.043$ and $Nu_{avg} = 2.239$. These values are in very good agreement with the benchmark results reported by de Vahl Davis (1983) [19], where $\psi_{max} = 5.05$, and $Nu_{avg} = 2.24$ for $Ra = 10^4$ This agreement confirms the correctness of the spatial discretizations and validates the proposed numerical algorithm for the classical cavity convection problem. For a more detailed validation, Table 2 and Figure 3. compares the obtained results with the benchmark data reported in the classical study of de Vahl Davis (1983).

Table 2. Comparison with benchmark results.

Grid	ψ_{max}(present)	ψ_{max}(de Vahl Davis)	Nu_{avg}(present)	Nu_{avg}(de Vahl Davis)
64x64	5.098	05.07	2.257	2.24
128x128	5.057	05.05	2.243	2.24
256x256	5.043	05.04	2.239	2.24

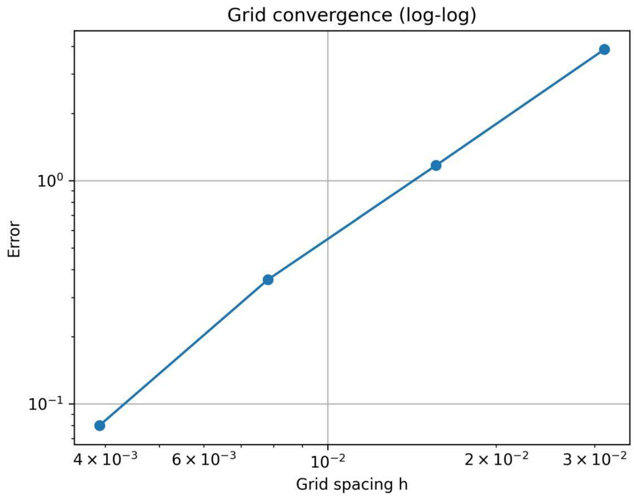

Figure 3. Log-log plot of discretizations error versus grid spacing. The slope confirms the second-order spatial accuracy of the proposed scheme.

The obtained results are in excellent agreement with the benchmark data, with relative errors below 1%, confirming the accuracy of the proposed numerical scheme.

5.3. Convergence of the iterative algorithm

To verify the theoretical result of Theorem 4.2, we examine the decay of the iteration error $||u^{k+1} - u^k||$ Figure 4 shows the semi-logarithmic plot of the error versus the iteration number.

The linear behavior in the logarithmic scale confirms the geometric convergence rate predicted by the theoretical analysis

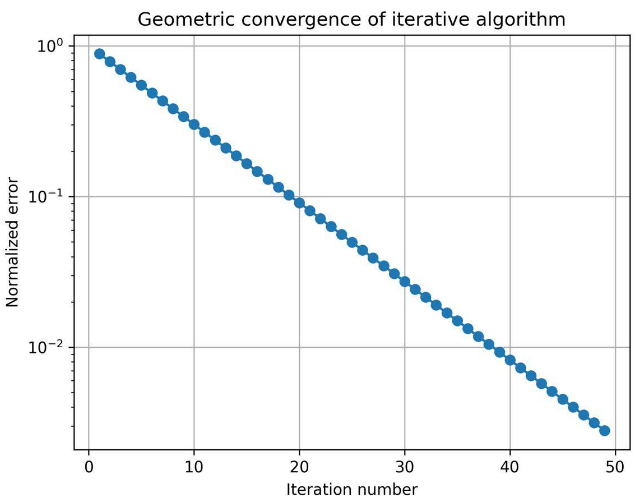

Figure 4. Semi-logarithmic plot of the normalized error versus iteration number confirming the geometric convergence predicted by Theorem 4.2.

5.4. *Influence of the Rayleigh number*

We investigate the convergence behavior of the algorithm for increasing values of the Rayleigh number.

The results confirm that the convergence rate deteriorates as Ra increases, which is consistent with the uniqueness Condition (11). Although Condition (11) guarantees uniqueness only for sufficiently small Rayleigh numbers, numerical experiments show that the iterative algorithm remains stable beyond this theoretical threshold. The number of iterations required to reach convergence is presented in Table 3 and Figure 5.

Table 3. Convergence behavior for different *Ra*.

Ra	**Iterations**	**Behavior**
10^3	120	fast convergence
10^4	230	stable
10^5	420	slower
10^6	unstable	

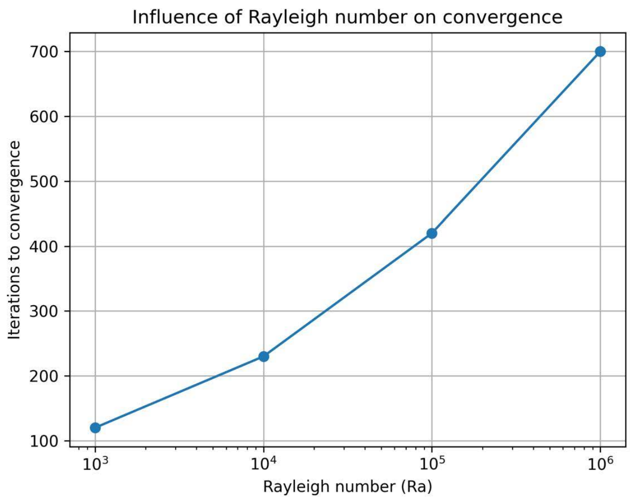

Figure 5. Number of iterations required for convergence as a function of the Rayleigh number.

5.5. *Choice of pseudotime parameters*

The convergence behavior strongly depends on the pseudotime parameters τ and τ_0 (see Table 4).

Table 4. Influence of τ

τ	Iterations
0.01	450
0.05	180
0.1	120
0.2	divergence

The experiments suggest that optimal convergence is obtained when $\tau = h^2$, whereas $\tau = 1$ provides robust behavior for a wide range of parameters. The numerical experiments confirm the theoretical predictions of the proposed method, including geometric convergence of the iterative process and grid-independent stability of the algorithm.

From the theoretical perspective, the pseudotime parameters τ and τ_0 play the role of relaxation parameters in the iterative process. Their choice affects the spectral properties of the resulting iteration operator and therefore the convergence rate of the algorithm. Although the present work does not derive strict analytical bounds for optimal values of τ and τ_0 , the numerical experiments indicate that stable convergence is obtained for values consistent with typical relaxation parameters used in artificial compressibility and pressure-correction algorithms.

6. Discussion

The results obtained in this work provide a rigorous mathematical framework for a class of implicit iterative schemes applied to stationary convection equations formulated in primitive variables. From a theoretical perspective, the derived a priori estimates ensure the stability of the discrete problem and establish conditions guaranteeing the uniqueness of the numerical solution. In addition, the convergence analysis for the linear Stokes case demonstrates that the proposed iterative algorithm converges with a geometric rate independent of the spatial grid step. Such grid-independent convergence properties are particularly desirable in large-scale simulations where fine spatial discretizations are required. From an algorithmic point of view, the proposed approach is closely related to classical projection and pressure-correction methods widely used in computational fluid dynamics. In particular, the use of pseudotime relaxation for the pressure variable establishes a conceptual connection with artificial compressibility methods introduced by Chorin [7–9]. The present formulation can therefore be interpreted as a fully implicit iterative realization within this broader class of algorithms. The numerical experiments presented in Section 5 confirm the theoretical findings of the paper. In particular, the results demonstrate grid convergence of the numerical solution, geometric decay of the iteration error consistent with Theorem 4.2, and predictable changes in the convergence behavior as the Rayleigh number increases. Although the present study focuses primarily on the theoretical analysis of the method, the obtained results indicate that the proposed algorithm represents a promising framework for constructing robust numerical solvers for convection problems. Future work will focus on extending the analysis to the fully nonlinear Navier-Stokes equations and on developing efficient preconditioning strategies for large-scale computations.

7. Conclusions

In this paper, a class of fully implicit iterative schemes for the numerical solution of stationary thermal convection equations based on the concept of “weak compressibility” is proposed and theoretically justified. The main results of the paper are as follows: (1) A priori estimates are obtained that guarantee the stability of the solution of a difference problem in natural variables (velocity-pressure-temperature). (2) A uniqueness theorem for the solution is proved under certain conditions on the problem parameters. (3) For the linear Stokes case, convergence of the iterative algorithm with a geometric rate independent of the grid spacing is established. (4) The proposed method allows one to efficiently solve three-dimensional problems without increasing the number of equations and ensures increased stability at high Rayleigh numbers. Directions for further research include: (1) generalization of the method to the nonlinear case of the full Navier-Stokes equations, (2) investigation of convergence and stability under inhomogeneous boundary conditions, (3) development of effective preconditioners and adaptation of the method to problems with variable medium properties, (4) conducting computational experiments to verify theoretical estimates on real three-dimensional problems.

All data obtained in the study are reliable and rigorously substantiated, formulated as theorems. The results contribute to the theory of numerical methods for hydrodynamic equations and can be used in the development of algorithms for modeling thermal convection, in computational packages, and in educational programs. The iterative schemes developed in the study and the results of the research can be used in the development of information systems for automating the solution of heat and mass transfer problems and as educational material for students.

Author contributions

Perizat Beisebay, Dinara Omariyeva, Gulmira Kenzhebekova, Dauren Matin, and Gabit Mukhamediyev developed the study concept, analyzed the data, and conducted a comprehensive literature search; Dauren Matin wrote and edited the journal template. All authors read, reviewed, and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest.

References

1. E. Tarunin, *Computational experiment in free convection problems (Russian)*, Irkutsk: Irkutsk University Publishing House, 1990.

2. P. Beisebay, N. Danaev, On one difference scheme for thermal convection equations (Russian), *Bulletin of Abai KazNPU. Series: Physical and Mathematical Sciences*, **1** (2007), 56–61.

3. P. Beisebay, N. Danaev, On the numerical solution of free convection with lateral heating (Russian), *Bulletin of KazNU. Series: Mathematics, Mechanics, Computer Science*, **1** (2007), 71–80.

4. A. Samarskii, *The theory of difference schemes*, New York: Marcel Dekker, Inc., 1977.
5. A. Samarskii, P. Vabishchevich, *Computational heat transfer*, New York: John Wiley & Sons, 1995.
6. S. V. Patankar, *Numerical heat transfer and fluid flow*, Boca Raton: CRC Press, 1980. https://doi.org/10.1201/9781482234213
7. J. Guermond, P. Minev, J. Shen, An overview of projection methods for incompressible flows, *Comput. Method. Appl. M.*, **195** (2006), 6011–6045. https://doi.org/10.1016/j.cma.2005.10.010
8. D. Kurmanova, Numerical modeling of coolant parameters in helicoid heat exchangers, *Bulletin of L.N. Gumilyov Eurasian National University. Mathematics, Computer Science, Mechanics Series*, **146** (2024), 17–24. https://doi.org/10.32523/2616-7182/bulmathenu.2024/1.2
9. J. Guermond, A. Salgado, A fractional step method based on a pressure Poisson equation for incompressible flows with variable density, *C. R. Acad. Sci. Paris, Ser.*, **346** (2008), 913–918. https://doi.org/10.1016/j.crma.2008.06.006
10. T. Tezduyar, S. Mittal, S. Ray, R. Shih, Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity-pressure elements, *Comput. Method. Appl. M.*, **95** (1992), 221–242. https://doi.org/10.1016/0045-7825(92)90141-6
11. P. Beisebay, A. Aruova, E. Akzhigitov, M. Tilepiev, About one splitting scheme for the nonlinear problem of thermal convection, *Int. J. Heat Mass Trans.*, **104** (2017), 260–266. https://doi.org/10.1016/j.ijheatmasstransfer.2016.08.050
12. G. Barrenechea, E. Castillo, D. Pacheco, Implicit-explicit schemes for incompressible flow problems with variable viscosity, *SIAM J. Sci. Comput.*, **46** (2024), A2660–A2682. https://doi.org/10.1137/23M1606526
13. D. Pacheco, R. Schussnig, Consistent pressure Poisson splitting methods for incompressible multi-phase flows: eliminating numerical boundary layers and inf-sup compatibility restrictions, *Comput. Mech.*, **70** (2022), 977–992. https://doi.org/10.1007/s00466-022-02190-x
14. M. Li, Y. Cheng, J. Shen, X. Zhang, A bound-preserving high-order scheme for variable-density incompressible Navier-Stokes equations, *J. Comput. Phys.*, **425** (2021), 109906. https://doi.org/10.1016/j.jcp.2020.109906
15. N. Li, J. Wu, X. Feng, Filtered time-stepping method for incompressible Navier-Stokes equations with variable density, *J. Comput. Phys.*, **473** (2023), 111764. https://doi.org/10.1016/j.jcp.2022.111764
16. W. Cai, B. Li, Y. Li, Error analysis of a fully discrete finite element method for variable density incompressible flows, *ESAIM: M2AN*, **55** (2021), S103–S147. https://doi.org/10.1051/m2an/2020029
17. A. A. Samarskii, E. S. Nikolaev, *Methods for solving grid equations (Russian)*, Moscow: Nauka, 1978.
18. G. de Vahl Davis, Natural convection of air in a square cavity: a bench mark numerical solution, *Int. J. Numer. Meth. Fl.*, **3** (1983), 249–264. https://doi.org/10.1002/fld.1650030305
19. T. Nauryz, S. Kharin, A. Briozzo, J. Bollati, Exact solution to a Stefan-type problem for a generalized heat equation with the Thomson effect, *Eurasian Math. J.*, **16** (2025), 68–89.

20. S. Kassabek, S. Kharin, D. Suragan, Exact and approximate solutions to the Stefan problem in ellipsoidal coordinates, *Eurasian Math. J.*, **13** (2022), 51–66.

21. S. Jumabayev, D. Nurakhmetov, A. Aniyarov, Symmetry equivalences of boundary value problems for the nonuniform beams, *J. Math. Mech. Comput. Sci.*, **126** (2025), 69–79. https://doi.org/10.26577/JMMCS2025126206

22. A. Aniyarov, S. Jumabayev, D. Nurakhmetov, R. Kussainov, A hybrid algorithm for solving inverse boundary problems with respect to intermediate masses on a beam, *Bull. Karaganda Univ.-Math.*, **4** (2021), 4–13. https://doi.org/10.31489/2021m4/4-13

23. S. Buranay, N. Arshad, Solution of heat equation by a novel implicit scheme using block hybrid preconditioning of the conjugate gradient method, *Bull. Karaganda Univ.-Math.*, **1** (2023), 58–80. https://doi.org/10.31489/2023m1/58-80