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Research article

Lower and upper bounds for the p-(A-M)-norm of two operators in Hilbert spaces with applications

Najla Altwaijry[1,*] **and Silvestru Sever Dragomir**[2]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[2] Department of Mathematical and Geospatial Sciences, School of Science, RMIT University, GPO Box 2476V, Melbourne, Victoria 3001, Australia

* **Correspondence:** Email: najla@ksu.edu.sa.

Abstract: For $\nu \in [0,1]$, $p \geq 1$ and $A, B \in \mathcal{B}(H)$, we define the p-arithmetic-mean (A-M)-norm for the pair of operators (A, B) by

$$\|(A,B)\|_{p,\nu} := \sup_{\|x\|=1} \left((1-\nu)\|Ax\|^p + \nu\|Bx\|^p\right)^{1/p}.$$

In this paper, we obtain several lower and upper bounds for this norm. Some inequalities for the numerical radius of the off-diagonal operator matrix are given. In the case when $(A, B) = (T, T^*)$ and $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$, where $\operatorname{Re} T := \frac{T+T^*}{2}$ is the real part of T and $\operatorname{Im} T := \frac{T-T^*}{2\mathrm{i}}$ is the imaginary part of T, respectively, some inequalities for one operator are also provided.

Keywords: inner product spaces; operator norm; numerical radius; off-diagonal operator matrix; norms for pairs of operators; A-G inequality
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1. Introduction

The numerical radius $w(X)$ of an operator X on H is given by

$$w(X) = \sup\{|\langle Xx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$, one has

$$|\langle Xx, x\rangle| \leq w(X)\|x\|^2. \tag{1.2}$$

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$w(X) \leq \|X\| \leq 2w(X) \tag{1.3}$$

for any $X \in \mathcal{B}(H)$.

In 2003, Kittaneh [1] showed that for any operator $X \in B(H)$, we have the following refinement of the first inequality in (1.3):

$$w(X) \leq \frac{1}{2}\left(\|X\| + \left\|X^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, Kittaneh in [2] improved the inequality (1.3) as follows:

$$\frac{1}{4}\left\||X|^2 + |X^*|^2\right\| \leq w^2(X) \leq \frac{1}{2}\left\||X|^2 + |X^*|^2\right\| \tag{1.5}$$

for any operator $X \in \mathcal{B}(H)$.

In [3], the authors introduced a new norm, named the (α,β)-norm, on the space of bounded linear operators on a complex Hilbert space, which generalizes the numerical radius norm, the usual operator norm, and the recently introduced modified Davis-Wielandt radius (see [4]).

Let (α,β) with $\alpha,\beta \geq 0$ such that $(\alpha,\beta) \neq (0,0)$. Define the mapping $\|\cdot\|_{\alpha,\beta} : \mathcal{B}(H) \to [0,\infty)$ by

$$\|X\|_{\alpha,\beta} := \sup_{\|x\|=1}\left(\alpha\,|\langle Xx, x\rangle|^2 + \beta\,\|Xx\|^2\right)^{1/2}.$$

We observe that $\|\cdot\|_{\alpha,\beta}$ defines a norm on $\mathcal{B}(H)$. We also observe that for $(\alpha,\beta) = (1,0)$, we obtain $\|X\|_{\alpha,\beta} = w(X)$, and for $(\alpha,\beta) = (0,1)$, then $\|X\|_{\alpha,\beta} = \|X\|$. Furthermore, for $(\alpha,\beta) = (1,1)$, we obtain the modified Davis-Wielandt radius of X, namely

$$\|X\|_{\alpha,\beta} = dw^*(X) := \sup_{\|x\|=1}\left(|\langle Xx, x\rangle|^2 + \|Xx\|^2\right)^{1/2}; \text{ see also [4].}$$

As pointed out in [3, Theorem 2.1], the (α,β)-norm satisfies the following bounds in terms of the numerical radius and the operator norm:

$$(\alpha+\beta)^{1/2}\,w(X) \leq \|X\|_{\alpha,\beta} \leq (\alpha+4\beta)^{1/2}\,w(X)$$

and

$$\max\left\{\frac{1}{2}(\alpha+\beta)^{1/2}, \beta^{1/2}\right\}\|X\| \leq \|X\|_{\alpha,\beta} \leq (\alpha+\beta)^{1/2}\,\|X\|$$

for all $X \in \mathcal{B}(H)$.

Recall that the Crawford number of an operator $X \in \mathcal{B}(H)$ is defined by $c(X) := \inf_{\|x\|=1}|\langle Xx, x\rangle|$. In [3, Theorem 2.7], the authors proved the following results:

$$\|X\|_{\alpha,\beta} \geq \max\left\{\alpha w^2(X) + \beta c\left(|X|^2\right), \alpha c^2(X) + \beta\,\|X\|^2\right\}$$

and

$$\|X\|_{\alpha,\beta} \geq 2\sqrt{\alpha\beta}\max\left\{w(X)\,c^{1/2}\left(|X|^2\right), c(X)\,\|X\|\right\}$$

for all $X \in \mathcal{B}(H)$. For more results concerning various bounds for the (α,β)-norm of a product, see [3].

The case when $\beta = 1 - \alpha$, $\alpha \in [0, 1]$ was investigated in [5], where the authors considered the α-norm

$$\|X\|_\alpha := \sup_{\|x\|=1} \left(\alpha |\langle Xx, x\rangle|^2 + (1 - \alpha) \|Xx\|^2\right)^{1/2}$$

and obtained some upper bounds for the α-norm of $n \times n$ operator matrices, which generalize the existing numerical radius inequalities and the usual operator norm inequalities of $n \times n$ operator matrices. As an application of these results, they provided new upper bounds for the numerical radius and the usual operator norm of $n \times n$ operator matrices.

Motivated by the above concepts and results, in this paper we introduce a new norm for a pair of operators and provide some fundamental inequalities as follows.

For $\nu \in [0, 1]$, $p \geq 1$ and $A, B \in \mathcal{B}(H)$, we define the p-arithmetic-mean (A-M)-norm for the pair of operators (A, B) by

$$\|(A, B)\|_{p,\nu} := \sup_{\|x\|=1} \left((1 - \nu) \|Ax\|^p + \nu \|Bx\|^p\right)^{1/p} \tag{1.6}$$

and

$$\|(A, B)\|_p := \sup_{\|x\|=1} \left(\|Ax\|^p + \|Bx\|^p\right)^{1/p}.$$

We observe that

$$\|(A, B)\|_{p,1/2}^p = \frac{1}{2} \|(A, B)\|_p^p \text{ for all } p \geq 1.$$

We also consider

$$\begin{aligned}
\|(A, B)\|_{e,\nu} &:= \|(A, B)\|_{2,\nu} = \sup_{\|x\|=1} \left((1 - \nu) \|Ax\|^2 + \nu \|Bx\|^2\right)^{1/2} \\
&= \sup_{\|x\|=1} \left((1 - \nu) \left\langle |A|^2 x, x\right\rangle + \nu \left\langle |B|^2 x, x\right\rangle\right)^{1/2} \\
&= \sup_{\|x\|=1} \left\langle \left[(1 - \nu) |A|^2 + \nu |B|^2\right] x, x\right\rangle^{1/2} \\
&= \left\|(1 - \nu) |A|^2 + \nu |B|^2\right\|^{1/2} = \left\||A|^2 \nabla_\nu |B|^2\right\|^{1/2}
\end{aligned}$$

and

$$\|(A, B)\|_\nu := \|(A, B)\|_{1,\nu} := \sup_{\|x\|=1} \left((1 - \nu) \|Ax\| + \nu \|Bx\|\right).$$

Moreover, we put

$$\|(A, B)\|_e := \|(A, B)\|_2 = \sup_{\|x\|=1} \left(\|Ax\|^2 + \|Bx\|^2\right)^{1/2} = \left\||A|^2 + |B|^2\right\|^{1/2}$$

and

$$\|(A, B)\| := \|(A, B)\|_1 = \sup_{\|x\|=1} \left(\|Ax\| + \|Bx\|\right).$$

We also observe that

$$\max\left\{(1 - \nu)^{1/p} \|A\|, \nu^{1/p} \|B\|\right\} \leq \|(A, B)\|_{p,\nu} \leq \left((1 - \nu) \|A\|^p + \nu \|B\|^p\right)^{1/p} \tag{1.7}$$

for $\nu \in [0, 1]$, $p \geq 1$.

Let $\nu \in [0,1]$, $p \geq 1$. Observe that for all $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$, we have $\|(A, B)\|_{p,\nu} \geq 0$, and if $\|(A, B)\|_{p,\nu} = 0$, then $(A, B) = (0, 0)$. Also,

$$\begin{aligned}\|\alpha (A, B)\|_{p,\nu} &= \|(\alpha A, \alpha B)\|_{p,\nu} = \sup_{\|x\|=1} \left((1-\nu)\|\alpha A x\|^p + \nu \|\alpha B x\|^p\right)^{1/p} \\ &= |\alpha| \sup_{\|x\|=1} \left((1-\nu)\|A x\|^p + \nu \|B x\|^p\right)^{1/p} = |\alpha| \|(A, B)\|_{p,\nu}.\end{aligned}$$

By weighted Minkowski's inequality, it follows that

$$\begin{aligned}\|(A, B) + (C, D)\|_{p,\nu} &= \|(A + C, B + D)\|_{p,\nu} \\ &= \sup_{\|x\|=1} \left((1-\nu)\|Ax + Cx\|^p + \nu \|Bx + Dx\|^p\right)^{1/p} \\ &\leq \sup_{\|x\|=1} \left[\left((1-\nu)\|Ax\|^p + \nu \|Bx\|^p\right)^{1/p} + \left((1-\nu)\|Cx\|^p + \nu \|Dx\|^p\right)^{1/p}\right] \\ &\leq \sup_{\|x\|=1} \left[\left((1-\nu)\|Ax\|^p + \nu \|Bx\|^p\right)^{1/p}\right] + \sup_{\|x\|=1} \left((1-\nu)\|Cx\|^p + \nu \|Dx\|^p\right)^{1/p} \\ &= \|(A, B)\|_{p,\nu} + \|(C, D)\|_{p,\nu},\end{aligned}$$

which shows that $\|(\cdot,\cdot)\|_{p,\nu}$ is a norm on $\mathcal{B}(H) \times \mathcal{B}(H)$. From the above, we have that $\|(\cdot,\cdot)\|_{e,\nu}$ and $\|(\cdot,\cdot)\|_{\nu}$ are norms on $\mathcal{B}(H) \times \mathcal{B}(H)$ for all $\nu \in (0,1)$.

We observe that, for $(A, B) = (T, T^*)$, we can introduce, for $p \geq 1$ and $\nu \in [0,1]$, the following functional:

$$\sigma_{p,\nu}(T) := \|(T, T^*)\|_{p,\nu} := \sup_{\|x\|=1} \left((1-\nu)\|Tx\|^p + \nu \|T^* x\|^p\right)^{1/p}.$$

We observe that $\sigma_{p,\nu}(\cdot)$ are norms on $\mathcal{B}(H)$ for $p \geq 1$ and $\nu \in [0,1]$ and therefore provide two parameter generalizations of the operator norm that is obtained for either $\nu = 0$ or $\nu = 1$.

If T is normal, namely $TT^* = T^*T$, then $\|Tx\| = \|T^*x\|$ for all $x \in H$, which gives that $\sigma_{p,\nu}(T) = \|T\|$ for all $p \geq 1$ and $\nu \in [0,1]$.

We observe that for $p = 2$, we obtain the following connection with the weighted arithmetic mean of the nonnegative operators $|T|^2$ and $|T^*|^2$:

$$\sigma^2_{e,\nu}(T) = \left\||T|^2 \nabla_\nu |T^*|^2\right\| \text{ and } \sigma^2_e(T) = \left\||T|^2 + |T^*|^2\right\|.$$

For $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$, we can also introduce, for $p \geq 1$ and $\nu \in [0,1]$, the following functional:

$$\rho_{p,\nu}(T) := \|(\operatorname{Re} T, \operatorname{Im} T)\|_{p,\nu} := \sup_{\|x\|=1} \left((1-\nu)\|\operatorname{Re} Tx\|^p + \nu \|\operatorname{Im} Tx\|^p\right)^{1/p}.$$

Since for a real number a, we have that $\rho_{p,\nu}(aT) = |a| \rho_{p,\nu}(T)$, $T \in \mathcal{B}(H)$. Then, we can conclude that $\rho_{p,\nu}(T)$ are real norms on $\mathcal{B}(H)$ that give for $\nu = 0$ the real seminorms $\rho_0(T) := \|\operatorname{Re} T\|$ and for $\nu = 1$, $\rho_1(T) := \|\operatorname{Im} T\|$.

The famous Young inequality for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.8}$$

with equality if and only if $a = b$. The inequality (1.8) is also called a ν-weighted arithmetic-geometric mean inequality.

We recall that Specht's ratio is defined by [6]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1,\infty), \\ 1 & \text{if } h = 1. \end{cases} \tag{1.9}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality:

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \le (1-\nu)a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.10}$$

where $a, b > 0$, $\nu \in [0,1]$, and $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.10) is due to Tominaga [7], while the first one is due to Furuichi [8].

In this paper, we show among others that

$$\begin{aligned} \|(A,B)\|_{p,\nu}^2 \ge{}& \max\left\{\left\|(1-\nu)^2 A^2 + \nu^2 B^2\right\|, \nu(1-\nu)\|AB+BA\|\right\} \\ &+ \max\{(1-\nu)\|A\|, \nu\|B\|\} \times |\|(1-\nu)A + \nu B\| - \|(1-\nu)A - \nu B\|| \end{aligned}$$

for $A, B \in \mathcal{B}(H)$, $p \ge 1$ and $\nu \in [0,1]$.

Also, if $0 < m|B|^2 \le |A|^2 \le M|B|^2$ for some constants $0 < m < M$, then for $p \ge 1$ and $\nu \in [0,1]$, we have

$$\|(A,B)\|_{p,\nu}^p \le \alpha(m,M;p)\|A\|^{(1-\nu)p}\|B\|^{\nu p},$$

where

$$\alpha(m,M;p) := \begin{cases} S\left(m^{p/2}\right) & \text{if } M \le 1; \\ \max\left\{S\left(m^{p/2}\right), S\left(M^{p/2}\right)\right\} & \text{if } m \le 1 \le M; \\ S\left(M^{p/2}\right) & \text{if } 1 \le m. \end{cases} \tag{1.11}$$

Some inequalities for the numerical radius of the off-diagonal operator matrix are given. In the case when $(A,B) = (T,T^*)$ and $(A,B) = (\operatorname{Re} T, \operatorname{Im} T)$, where $\operatorname{Re} T := \frac{T+T^*}{2}$ is the real part of T and $\operatorname{Im} T := \frac{T-T^*}{2i}$ is the imaginary part of T, respectively, some inequalities for one operator are also provided.

2. Some general results

We recall the following vector inequality for positive operators $A \ge 0$, obtained by McCarthy in [9]:

$$\langle Ax, x\rangle^p \le \langle A^p x, x\rangle, \ p \ge 1$$

for $x \in H$, $\|x\| = 1$, and Buzano's inequality [10]

$$\frac{1}{2}\left[\|x\|\|y\| + |\langle x,y\rangle|\right] \ge |\langle x,e\rangle\langle e,y\rangle| \tag{2.1}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

The following result provides some basic lower and upper bounds for the power of the norm $\|(\cdot,\cdot)\|_{p,\nu}$.

Theorem 2.1. *Let $A, B \in \mathcal{B}(H)$. For $p \geq 2$ and $\nu \in [0,1]$, we have*

$$\|A\nabla_\nu B\|^p \leq \|(A,B)\|_{p,\nu}^p \leq \||A|^p \nabla_\nu |B|^p\|. \tag{2.2}$$

In particular,

$$\|A+B\|^p \leq 2^{p-1} \|(A,B)\|_p^p \leq 2^{p-1} \||A|^p + |B|^p\|. \tag{2.3}$$

If $1 \leq p < 2$, *then*

$$\|A\nabla_\nu B\| \leq \|(A,B)\|_{p,\nu} \leq \left\||A|^2 \nabla_\nu |B|^2\right\|^{1/2}. \tag{2.4}$$

In particular,

$$\|A+B\| \leq 2^{(p-1)/p} \|(A,B)\|_p \leq \sqrt{2} \left\||A|^2 + |B|^2\right\|^{1/2}. \tag{2.5}$$

Proof. For $p \geq 1$, we have by the convexity of the power function that

$$\begin{aligned} \|(1-\nu)Ax + \nu Bx\|^p &\leq ((1-\nu)\|Ax\| + \nu\|Bx\|)^p \\ &\leq (1-\nu)\|Ax\|^p + \nu\|Bx\|^p \\ &= (1-\nu)\left(\|Ax\|^2\right)^{p/2} + \nu\left(\|Bx\|^2\right)^{p/2} \\ &= (1-\nu)\left\langle |A|^2 x, x\right\rangle^{p/2} + \nu\left\langle |B|^2 x, x\right\rangle^{p/2}. \end{aligned} \tag{2.6}$$

If we use McCarthy's inequality, then for $p \geq 2$, we have

$$\begin{aligned} &(1-\nu)\left\langle |A|^2 x, x\right\rangle^{p/2} + \nu\left\langle |B|^2 x, x\right\rangle^{p/2} \\ &\leq (1-\nu)\langle |A|^p x, x\rangle + \nu\langle |B|^p x, x\rangle \\ &= \langle [(1-\nu)|A|^p + \nu|B|^p] x, x\rangle \leq \|(1-\nu)|A|^p + \nu|B|^p\| \end{aligned} \tag{2.7}$$

for $x \in H, \|x\| = 1$. Therefore, by (2.6) and (2.7), we obtain

$$\|(1-\nu)Ax + \nu Bx\|^p \leq (1-\nu)\|Ax\|^p + \nu\|Bx\|^p \leq \|(1-\nu)|A|^p + \nu|B|^p\|,$$

and by taking the supremum over $x \in H, \|x\| = 1$, we obtain the desired result (2.2).

If $1 \leq p < 2$, then $\frac{p}{2} < 1$, and since the function $g(t) = t^{p/2}$ is concave, then

$$\begin{aligned} (1-\nu)\left\langle |A|^2 x, x\right\rangle^{p/2} + \nu\left\langle |B|^2 x, x\right\rangle^{p/2} &\leq \left[(1-\nu)\left\langle |A|^2 x, x\right\rangle + \nu\left\langle |B|^2 x, x\right\rangle\right]^{p/2} \\ &= \left\langle \left[(1-\nu)|A|^2 + \nu|B|^2\right] x, x\right\rangle^{p/2} \\ &\leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{p/2} \end{aligned}$$

for $x \in H$.

Therefore,

$$\|(1-\nu)Ax + \nu Bx\|^p \leq (1-\nu)\|Ax\|^p + \nu\|Bx\|^p \leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{p/2},$$

and on taking the supremum over $x \in H, \|x\| = 1$, we obtain

$$\|A\nabla_\nu B\|^p \leq \|(A,B)\|_{p,\nu}^p \leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{p/2},$$

which is equivalent to (2.4).

Remark 2.1. *For $p = 2$, we get*

$$\|A\nabla_\nu B\|^2 \le \left\||A|^2 \nabla_\nu |B|^2\right\|, \ \nu \in [0,1].$$

In particular,

$$\|A + B\|^2 \le 2\left\||A|^2 + |B|^2\right\|.$$

For $p = 1$, we obtain

$$\|A\nabla_\nu B\| \le \|(A,B)\|_\nu \le \left\||A|^2 \nabla_\nu |B|^2\right\|^{1/2}, \ \nu \in [0,1].$$

In particular,

$$\|A + B\| \le \|(A,B)\| \le \sqrt{2}\left\||A|^2 + |B|^2\right\|^{1/2}.$$

The next result gives some lower bounds for $\|(A,B)\|_{p,\nu}^2$, where $A, B \in \mathcal{B}(H)$.

Theorem 2.2. *Let $A, B \in \mathcal{B}(H)$. Then for $p \ge 1$ and $\nu \in [0,1]$, we have*

$$\begin{aligned}\|(A,B)\|_{p,\nu}^2 \ge& \max\left\{\left\|(1-\nu)^2 A^2 + \nu^2 B^2\right\|, \nu(1-\nu)\|AB + BA\|\right\} \\ &+ \max\{(1-\nu)\|A\|, \nu\|B\|\} \times |\|(1-\nu)A + \nu B\| - \|(1-\nu)A - \nu B\||.\end{aligned} \tag{2.8}$$

In particular, for $\nu = 1/2$, we obtain that

$$\begin{aligned}\|(A,B)\|_p^2 \ge& \frac{1}{4^{\frac{p-1}{p}}}\max\left\{\left\|A^2 + B^2\right\|, \|AB + BA\|\right\} \\ &+ \frac{1}{4^{\frac{p-1}{p}}}\max\{\|A\|, \|B\|\}\,|\|A + B\| - \|A - B\||.\end{aligned} \tag{2.9}$$

Proof. From Theorem 2.1, we have for $p \ge 1$ and $\nu \in [0,1]$ that

$$\|(A,B)\|_{p,\nu}^2 \ge \|(1-\nu)A + \nu B\|^2$$

and

$$\|(A,B)\|_{p,\nu}^2 \ge \|(1-\nu)A - \nu B\|^2.$$

This implies that

$$\begin{aligned}\|(A,B)\|_{p,\nu}^2 \ge& \max\left\{\|(1-\nu)A + \nu B\|^2, \|(1-\nu)A - \nu B\|^2\right\} \\ =& \frac{1}{2}\left(\|(1-\nu)A + \nu B\|^2 + \|(1-\nu)A - \nu B\|^2\right) \\ &+ \frac{1}{2}\left|\|(1-\nu)A + \nu B\|^2 - \|(1-\nu)A - \nu B\|^2\right|\end{aligned} \tag{2.10}$$

for $p \ge 1$ and $\nu \in [0,1]$.

Observe that

$$\begin{aligned}&\|(1-\nu)A + \nu B\|^2 + \|(1-\nu)A - \nu B\|^2 \\ \ge& \left\|((1-\nu)A + \nu B)^2\right\| + \left\|((1-\nu)A - \nu B)^2\right\| \\ \ge& \max\left\{\left\|((1-\nu)A + \nu B)^2 + ((1-\nu)A - \nu B)^2\right\|, \ \left\|((1-\nu)A + \nu B)^2 - ((1-\nu)A - \nu B)^2\right\|\right\}.\end{aligned}$$

Since

$$\begin{aligned}&((1-\nu)A+\nu B)^2+((1-\nu)A-\nu B)^2\\&=(1-\nu)^2A^2+\nu(1-\nu)(AB+BA)+\nu^2B^2+(1-\nu)^2A^2-\nu(1-\nu)(AB+BA)+\nu^2B^2\\&=2\left[(1-\nu)^2A^2+\nu^2B^2\right]\end{aligned}$$

and

$$\begin{aligned}&((1-\nu)A+\nu B)^2-((1-\nu)A-\nu B)^2\\&=(1-\nu)^2A^2+\nu(1-\nu)(AB+BA)+\nu^2B^2-(1-\nu)^2A^2+\nu(1-\nu)(AB+BA)-\nu^2B^2\\&=2\nu(1-\nu)(AB+BA),\end{aligned}$$

then

$$\begin{aligned}&\|(1-\nu)A+\nu B\|^2+\|(1-\nu)A-\nu B\|^2\\&\geq 2\max\left\{\left\|(1-\nu)^2A^2+\nu^2B^2\right\|,\nu(1-\nu)\|AB+BA\|\right\}.\end{aligned}\tag{2.11}$$

Also,

$$\begin{aligned}&\left|\|(1-\nu)A+\nu B\|^2-\|(1-\nu)A-\nu B\|^2\right|\\&=(\|(1-\nu)A+\nu B\|+\|(1-\nu)A-\nu B\|)\times|\|(1-\nu)A+\nu B\|-\|(1-\nu)A-\nu B\||.\end{aligned}$$

Since

$$\begin{aligned}&\|(1-\nu)A+\nu B\|+\|(1-\nu)A-\nu B\|\geq\max\{\|(1-\nu)A+\nu B+(1-\nu)A-\nu B\|,\\&\|(1-\nu)A+\nu B-(1-\nu)A+\nu B\|\}=2\max\{(1-\nu)\|A\|,\nu\|B\|\},\end{aligned}$$

then

$$\begin{aligned}&\left|\|(1-\nu)A+\nu B\|^2-\|(1-\nu)A-\nu B\|^2\right|\\&\geq 2\max\{(1-\nu)\|A\|,\nu\|B\|\}\times|\|(1-\nu)A+\nu B\|-\|(1-\nu)A-\nu B\||.\end{aligned}\tag{2.12}$$

By making use of (2.10)–(2.12), we then obtain (2.8).

Now, from (2.8), we obtain for $\nu=1/2$ that

$$\|(A,B)\|_{p,1/2}^2\geq\frac{1}{4}\max\left\{\left\|A^2+B^2\right\|,\|AB+BA\|\right\}+\frac{1}{4}\max\{\|A\|,\|B\|\}\,|\|A+B\|-\|A-B\||,$$

and since $\|(A,B)\|_{p,1/2}^2=\frac{1}{2^{2/p}}\|(A,B)\|_p^2$, then we obtain the desired bound (2.9).

Remark 2.2. *If we make $p=1$ in Theorem 2.2, then we get*

$$\begin{aligned}\|(A,B)\|_\nu^2\geq&\max\left\{\left\|(1-\nu)^2A^2+\nu^2B^2\right\|,\nu(1-\nu)\|AB+BA\|\right\}\\&+\max\{(1-\nu)\|A\|,\nu\|B\|\}\times|\|(1-\nu)A+\nu B\|-\|(1-\nu)A-\nu B\||\end{aligned}$$

for $\nu \in [0,1]$ *and*

$$\|(A,B)\|^2 \geq \max\left\{\left\|A^2+B^2\right\|, \|AB+BA\|\right\} + \max\left\{\|A\|,\|B\|\right\}\left|\|A+B\|-\|A-B\|\right|.$$

For $p=2$ *in Theorem 2.2, we obtain*

$$\begin{aligned}\left\||A|^2\nabla_\nu |B|^2\right\| \geq &\max\left\{\left\|(1-\nu)^2A^2+\nu^2B^2\right\|, \nu(1-\nu)\|AB+BA\|\right\}\\ &+\max\left\{(1-\nu)\|A\|,\nu\|B\|\right\}\times\left|\|(1-\nu)A+\nu B\|-\|(1-\nu)A-\nu B\|\right|\end{aligned}$$

for $\nu \in [0,1]$ *and*

$$\left\||A|^2+|B|^2\right\| \geq \frac{1}{2}\max\left\{\left\|A^2+B^2\right\|, \|AB+BA\|\right\} + \frac{1}{2}\max\left\{\|A\|,\|B\|\right\}\left|\|A+B\|-\|A-B\|\right|. \quad (2.13)$$

Some interesting lower bounds for the norm of the sum $|C|^2+|D|^2$ are as follows:

Corollary 2.3. *For all* $C,D \in \mathcal{B}(H)$*, we have*

$$\left\||C|^2+|D|^2\right\| \geq \frac{1}{2}\max\left\{\left\|C^2+D^2\right\|, \left\|C^2-D^2\right\|\right\} + \frac{1}{2}\max\left\{\|C+D\|,\|C-D\|\right\}\left|\|C\|-\|D\|\right|. \quad (2.14)$$

Proof. We observe that

$$\begin{aligned}\|(C+D,C-D)\|_e^2 &= \sup_{\|x\|=1}\left(\|Cx+Dx\|^2+\|Cx-Dx\|^2\right) \quad (2.15)\\ &= 2\sup_{\|x\|=1}\left(\|Cx\|^2+\|Dx\|^2\right) = 2\|(C,D)\|_e^2\end{aligned}$$

for $C,D \in \mathcal{B}(H)$.

From (2.13), for $A=C+D$ and $B=C-D$, we obtain that

$$\begin{aligned}\|(C+D,C-D)\|_e^2 \geq &\frac{1}{2}\max\left\{\left\|(C+D)^2+(C-D)^2\right\|, \|(C+D)(C-D)+(C-D)(C+D)\|\right\}\\ &+\max\left\{\|C+D\|,\|C-D\|\right\}\left|\|C\|-\|D\|\right|. \quad (2.16)\end{aligned}$$

Since

$$(C+D)^2+(C-D)^2 = 2\left(C^2+D^2\right)$$

and

$$(C+D)(C-D)+(C-D)(C+D) = 2\left(C^2-D^2\right),$$

then by (2.15) and (2.16), we derive (2.14).

3. Upper bounds via Tominaga's inequality

There are many results connected with operator Young's inequality for positive operators P satisfying the condition

$$0 < kI \leq P \leq KI \quad (3.1)$$

in terms of the positive constants $k < K$; see for instance [7, 8] and the references therein. This condition (3.1) is obviously equivalent to

$$0 < k_1 I \leq P^2 \leq K_1 I, \tag{3.2}$$

where $k_1 = k^2$ and $K_1 = K^2$.

Motivated by (3.2), it is natural to consider the following more general condition for the operators $A, B \in \mathcal{B}(H)$ satisfying the double inequality:

$$0 < m|B|^2 \leq |A|^2 \leq M|B|^2 \tag{3.3}$$

for some constants $0 < m < M$.

If we make $B = I$ and $A = P \geq 0$ in (3.3), then we obtain (3.2).

By employing a result due to Tominaga, we can derive the following upper bounds for $\|(A, B)\|_{p,\nu}^p$.

Theorem 3.1. *Let $A, B \in \mathcal{B}(H)$ satisfy condition (3.3) for some constants $0 < m < M$. For $p \geq 1$ and $\nu \in [0, 1]$, we have*

$$\|(A, B)\|_{p,\nu}^p \leq \alpha(m, M; p)\|A\|^{(1-\nu)p}\|B\|^{\nu p}, \tag{3.4}$$

where $\alpha(m, M; p)$ is defined in (1.11).

For $\nu \in (0, 1)$, put $r := \min\{\nu, 1-\nu\}$. If $p \geq \frac{1}{r}$, then

$$\|(A, B)\|_{p,\nu}^p \leq \frac{1}{2}\alpha(m, M; p)\left\||A|^{2(1-\nu)p} + |B|^{2\nu p}\right\|. \tag{3.5}$$

Proof. Since

$$0 < m|B|^2 \leq |A|^2 \leq M|B|^2$$

for $0 < m < M$, then for $x \in H$ with $\|x\| = 1$, we obtain that

$$0 < m\left\langle |B|^2 x, x\right\rangle \leq \left\langle |A|^2 x, x\right\rangle \leq M\left\langle |B|^2 x, x\right\rangle,$$

which gives

$$0 < m^{p/2} \leq \left(\frac{\left\langle |A|^2 x, x\right\rangle}{\left\langle |B|^2 x, x\right\rangle}\right)^{p/2} \leq M^{p/2}$$

for $x \in H$ with $\|x\| = 1$.

By Tominaga's inequality (3.4), for $a = \left\langle |A|^2 x, x\right\rangle^{p/2}$ and $b = \left\langle |B|^2 x, x\right\rangle^{p/2}$, we have

$$\begin{aligned} &(1-\nu)\left\langle |A|^2 x, x\right\rangle^{p/2} + \nu\left\langle |B|^2 x, x\right\rangle^{p/2} \\ &\leq S\left(\left(\frac{\left\langle |A|^2 x, x\right\rangle}{\left\langle |B|^2 x, x\right\rangle}\right)^{p/2}\right)\left\langle |A|^2 x, x\right\rangle^{(1-\nu)p/2}\left\langle |B|^2 x, x\right\rangle^{\nu p/2} \end{aligned} \tag{3.6}$$

for $x \in H$, $\|x\| = 1$.

Since the function S is decreasing on $(0,1)$ and increasing on $(1,\infty)$, then we have the bounds

$$S\left(\left(\frac{\left\langle |A|^2 x,x\right\rangle}{\left\langle |B|^2 x,x\right\rangle}\right)^{p/2}\right)\le\begin{cases} S\left(m^{p/2}\right) & \text{if } M^{p/2}\le 1;\\ \max\left\{S\left(m^{p/2}\right),S\left(M^{p/2}\right)\right\} & \text{if } m^{p/2}\le 1\le M^{p/2};\\ S\left(M^{p/2}\right) & \text{if } 1\le m^{p/2}\end{cases}$$
$$=\alpha\left(m,M;p\right) \tag{3.7}$$

for $x\in H$, $\|x\|=1$.

By making use of the inequalities (2.6), (3.6), and (3.7), we obtain that

$$(1-\nu)\|Ax\|^p+\nu\|Bx\|^p\le\alpha\left(m,M;p\right)\left\langle |A|^2 x,x\right\rangle^{(1-\nu)p/2}\left\langle |B|^2 x,x\right\rangle^{\nu p/2} \tag{3.8}$$

for $x\in H$, $\|x\|=1$.

Observe that

$$\left\langle |A|^2 x,x\right\rangle^{(1-\nu)p/2}\left\langle |B|^2 x,x\right\rangle^{\nu p/2}=\|Ax\|^{(1-\nu)p}\|Bx\|^{\nu p}\le\|A\|^{(1-\nu)p}\|B\|^{\nu p}$$

for $x\in H$, $\|x\|=1$, and by (3.8), we get

$$(1-\nu)\|Ax\|^p+\nu\|Bx\|^p\le\alpha\left(m,M;p\right)\|A\|^{(1-\nu)p}\|B\|^{\nu p}$$

for $x\in H$, $\|x\|=1$. By taking the supremum over $x\in H$, $\|x\|=1$, we derive the desired result (3.4).

By the elementary inequality

$$\sqrt{ab}\le\frac{a+b}{2},\ a,b\ge 0,$$

we have

$$\left\langle |A|^2 x,x\right\rangle^{(1-\nu)p/2}\left\langle |B|^2 x,x\right\rangle^{\nu p/2}\le\frac{1}{2}\left[\left\langle |A|^2 x,x\right\rangle^{(1-\nu)p}+\left\langle |B|^2 x,x\right\rangle^{\nu p}\right]$$

for $x\in H$, $\|x\|=1$.

If we use McCarthy's inequality, since $(1-\nu)p,\ \nu p\ge 1$, then we have

$$\begin{aligned}\frac{1}{2}\left[\left\langle |A|^2 x,x\right\rangle^{(1-\nu)p}+\left\langle |B|^2 x,x\right\rangle^{\nu p}\right]&\le\frac{1}{2}\left[\left\langle |A|^{2(1-\nu)p}x,x\right\rangle+\left\langle |B|^{2\nu p}x,x\right\rangle\right]\\&=\frac{1}{2}\left\langle\left(|A|^{2(1-\nu)p}+|B|^{2\nu p}\right)x,x\right\rangle\\&\le\frac{1}{2}\left\||A|^{2(1-\nu)p}+|B|^{2\nu p}\right\|\end{aligned}$$

for $x\in H$, $\|x\|=1$.

By (3.8), we then have

$$(1-\nu)\|Ax\|^p+\nu\|Bx\|^p\le\frac{1}{2}\alpha\left(m,M;p\right)\left\||A|^{2(1-\nu)p}+|B|^{2\nu p}\right\|$$

for $x\in H$, $\|x\|=1$. By taking the supremum over $x\in H$, $\|x\|=1$, we obtain the desired result (3.5).

Remark 3.1. *If we make $p = 1$ in (3.4), then we get*

$$\|(A, B)\|_\nu \leq \alpha(m, M; 1) \|A\|^{1-\nu} \|B\|^{\nu}, \tag{3.9}$$

where

$$\alpha(m, M; 1) := \begin{cases} S\left(\sqrt{m}\right) & \text{if } M \leq 1; \\ \max\left\{S\left(\sqrt{m}\right), S\left(\sqrt{M}\right)\right\} & \text{if } m \leq 1 \leq M; \\ S\left(\sqrt{M}\right) & \text{if } 1 \leq m. \end{cases}$$

If we make $\nu = 1/2$ in (3.9), then we get

$$\|(A, B)\| \leq 2\alpha(m, M; 1) \|A\|^{1/2} \|B\|^{1/2}.$$

For $p = 2$, we get

$$\left\||A|^2 \nabla_\nu |B|^2\right\| \leq \alpha(m, M; 2) \|A\|^{2(1-\nu)} \|B\|^{2\nu}, \tag{3.10}$$

where

$$\alpha(m, M; 2) := \begin{cases} S(m) & \text{if } M \leq 1; \\ \max\{S(m), S(M)\} & \text{if } m \leq 1 \leq M; \\ S(M) & \text{if } 1 \leq m. \end{cases}$$

If we make $\nu = 1/2$ in (3.10), then we get

$$\left\||A|^2 + |B|^2\right\| \leq 2\alpha(m, M; 2) \|A\| \|B\|.$$

We can also state more upper bounds as follows.

Theorem 3.2. *With the assumptions of Theorem 3.1 and for $R = \max\{\nu, 1-\nu\}$, $\nu \in (0, 1)$ with $Rp \geq 2$, then*

$$\|(A, B)\|_{p,\nu}^p \leq \frac{1}{2}\alpha(m, M; p)\left[w^{Rp/2}\left(|B|^2 |A|^2\right) + \|A\|^{Rp} \|B\|^{Rp}\right]. \tag{3.11}$$

Also, we have

$$\|(A, B)\|_{p,\nu}^2 \leq \alpha^{2/p}(m, M; p) \left\||A|^2 \nabla_\nu |B|^2\right\|^R \left\||B|^2 \nabla_\nu |A|^2\right\|^R \tag{3.12}$$

and

$$\|(A, B)\|_{p,\nu}^2 \leq \alpha^{2/p}(m, M; p) \left\||A|^2 \nabla_\nu |B|^2\right\|^R \|A\|^{2\nu R} \|B\|^{2(1-\nu)R}. \tag{3.13}$$

Proof. Observe that, for $R = \max\{\nu, 1-\nu\}$,

$$\left\langle |A|^2 x, x\right\rangle^{(1-\nu)p/2} \left\langle |B|^2 x, x\right\rangle^{\nu p/2} \leq \left[\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle\right]^{Rp/2} \tag{3.14}$$

for $x \in H$, $\|x\| = 1$.

By Buzano's inequality (2.1), we have

$$\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle \leq \frac{1}{2}\left(\left|\left\langle |A|^2 x, |B|^2 x\right\rangle\right| + \left\||A|^2 x\right\| \left\||B|^2 x\right\|\right)$$

for $x \in H$, $\|x\| = 1$.

Therefore, since $Rp/2 \geq 1$, we deduce that

$$\begin{aligned}\left[\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle\right]^{Rp/2} &\leq \left(\frac{\left|\left\langle |A|^2 x, |B|^2 x\right\rangle\right| + \left\||A|^2 x\right\| \left\||B|^2 x\right\|}{2}\right)^{Rp/2} \\ &\leq \frac{\left|\left\langle |B|^2 |A|^2 x, x\right\rangle\right|^{Rp/2} + \left\||A|^2 x\right\|^{Rp/2} \left\||B|^2 x\right\|^{Rp/2}}{2} \\ &\leq \frac{1}{2}\left[w^{Rp/2}\left(|B|^2 |A|^2\right) + \|A\|^{Rp} \|B\|^{Rp}\right]\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By making use of (3.8) and (3.14), we deduce the desired result (3.11).

Observe that

$$\begin{aligned}\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle &= \left\langle |A|^2 x, x\right\rangle^{1-\nu} \left\langle |B|^2 x, x\right\rangle^{\nu} \left\langle |A|^2 x, x\right\rangle^{\nu} \left\langle |B|^2 x, x\right\rangle^{1-\nu} \\ &\leq \left((1-\nu)\left\langle |A|^2 x, x\right\rangle + \nu\left\langle |B|^2 x, x\right\rangle\right) \times \left(\nu\left\langle |A|^2 x, x\right\rangle + (1-\nu)\left\langle |B|^2 x, x\right\rangle\right) \\ &= \left\langle\left[(1-\nu)|A|^2 + \nu|B|^2\right] x, x\right\rangle \times \left\langle\left[\nu|A|^2 + (1-\nu)|B|^2\right] x, x\right\rangle \\ &\leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\| \left\|\nu|A|^2 + (1-\nu)|B|^2\right\|\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

Therefore,

$$\left[\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle\right]^{Rp/2} \leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{Rp/2} \left\|\nu|A|^2 + (1-\nu)|B|^2\right\|^{Rp/2}$$

for $x \in H$, $\|x\| = 1$.

By making use of (3.8) and (3.14), we deduce

$$\begin{aligned}&(1-\nu)\|Ax\|^p + \nu\|Bx\|^p \\ &\leq \alpha(m, M; p) \times \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{Rp/2} \left\|\nu|A|^2 + (1-\nu)|B|^2\right\|^{Rp/2},\end{aligned}$$

which gives the desired result (3.12).

From the above, we also have

$$\begin{aligned}&\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle \\ &= \left\langle |A|^2 x, x\right\rangle^{1-\nu} \left\langle |B|^2 x, x\right\rangle^{\nu} \left\langle |A|^2 x, x\right\rangle^{\nu} \left\langle |B|^2 x, x\right\rangle^{1-\nu} \\ &\leq \left((1-\nu)\left\langle |A|^2 x, x\right\rangle + \nu\left\langle |B|^2 x, x\right\rangle\right) \left\langle |A|^2 x, x\right\rangle^{\nu} \left\langle |B|^2 x, x\right\rangle^{1-\nu} \\ &\leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\| \|A\|^{2\nu} \|B\|^{2(1-\nu)},\end{aligned}$$

giving

$$\left[\left\langle |A|^2 x, x\right\rangle \left\langle |B|^2 x, x\right\rangle\right]^{Rp/2} \leq \left\|(1-\nu)|A|^2 + \nu|B|^2\right\|^{Rp/2} \|A\|^{\nu Rp} \|B\|^{(1-\nu)Rp}$$

for $x \in H$, $\|x\| = 1$, which proves (3.13).

Remark 3.2. *For $p = 4$, we obtain from (3.11) that*

$$\|(A,B)\|_{4,\nu}^{4} \le \frac{1}{2}\alpha(m,M;4)\left[w^{2R}\left(|B|^{2}|A|^{2}\right) + \|A\|^{4R}\|B\|^{4R}\right], \tag{3.15}$$

where

$$\alpha(m,M;4) := \begin{cases} S\left(m^{2}\right) & \text{if } M \le 1; \\ \max\left\{S\left(m^{2}\right), S\left(M^{2}\right)\right\} & \text{if } m \le 1 \le M; \\ S\left(M^{2}\right) & \text{if } 1 \le m. \end{cases}$$

For $\nu = 1/2$ in (3.15), we obtain

$$\|(A,B)\|_{4}^{4} \le \alpha(m,M;4)\left[w\left(|B|^{2}|A|^{2}\right) + \|A\|^{2}\|B\|^{2}\right].$$

If we make $p = 4$ in (3.12) and (3.13), then we get

$$\|(A,B)\|_{4,\nu}^{2} \le \sqrt{\alpha(m,M;4)}\left\||A|^{2}\nabla_{\nu}|B|^{2}\right\|^{R}\left\||B|^{2}\nabla_{\nu}|A|^{2}\right\|^{R} \tag{3.16}$$

and

$$\|(A,B)\|_{4,\nu}^{2} \le \sqrt{\alpha(m,M;4)}\left\||A|^{2}\nabla_{\nu}|B|^{2}\right\|^{R}\|A\|^{2\nu R}\|B\|^{2(1-\nu)R}. \tag{3.17}$$

From (3.17), we derive for $\nu = 1/2$ that

$$\|(A,B)\|_{4}^{2} \le 2\sqrt{\alpha(m,M;4)}\left\||A|^{2}\nabla_{\nu}|B|^{2}\right\|^{R}\left\||B|^{2}\nabla_{\nu}|A|^{2}\right\|^{R}$$

and

$$\|(A,B)\|_{4}^{2} \le \sqrt{\alpha(m,M;4)}\left\|\frac{|A|^{2}+|B|^{2}}{2}\right\|^{1/2}\|A\|^{1/2}\|B\|^{1/2}.$$

We observe that if $0 < m_1 \le |A|^2 \le M_1$ and $0 < m_2 \le |B|^2 \le M_2$, then for $x \in H$, $\|x\| = 1$, we have

$$0 < \frac{m_1}{M_2} \le \frac{\left\langle |A|^{2}x, x\right\rangle}{\left\langle |B|^{2}x, x\right\rangle} \le \frac{M_1}{m_2}.$$

By taking $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$, we get $0 < m|B|^2 \le |A|^2 \le M|B|^2$, and by setting

$$\alpha(m_1,M_1,m_2,M_2;p) := \begin{cases} S\left(\left(\frac{m_1}{M_2}\right)^{p/2}\right) & \text{if } \frac{M_1}{m_2} \le 1; \\ \max\left\{S\left(\left(\frac{m_1}{M_2}\right)^{p/2}\right), S\left(\left(\frac{M_1}{m_2}\right)^{p/2}\right)\right\} & \text{if } \frac{m_1}{M_2} \le 1 \le \frac{M_1}{m_2}; \\ S\left(\left(\frac{M_1}{m_2}\right)^{p/2}\right) & \text{if } 1 \le \frac{m_1}{M_2}, \end{cases}$$

we get from (3.4) that

$$\|(A,B)\|_{p,\nu}^{p} \le \alpha(m_1,M_1,m_2,M_2;p)\|A\|^{(1-\nu)p}\|B\|^{\nu p}$$

for $p \ge 1$ and $\nu \in [0,1]$.

For $\nu \in (0,1)$ and $p \geq \frac{1}{r}$, then by (3.5), we get

$$\|(A,B)\|_{p,\nu}^{p} \leq \frac{1}{2}\alpha\left(m_1, M_1, m_2, M_2; p\right)\left\||A|^{2(1-\nu)p} + |B|^{2\nu p}\right\|. \tag{3.18}$$

If $p \geq 1$ and $\nu \in [0,1]$ with $Rp \geq 2$, then by (3.11), we have

$$\|(A,B)\|_{p,\nu}^{p} \leq \frac{1}{2}\alpha\left(m_1, M_1, m_2, M_2; p\right) \times \left[w^{Rp/2}\left(|B|^2|A|^2\right) + \|A\|^{Rp}\|B\|^{Rp}\right], \tag{3.19}$$

while from (3.13),

$$\|(A,B)\|_{p,\nu}^{p} \leq \alpha^{2/p}\left(m_1, M_1, m_2, M_2; p\right) \times \left\||A|^2 \nabla_\nu |B|^2\right\|^{R}\|A\|^{2\nu R}\|B\|^{2(1-\nu)R}. \tag{3.20}$$

4. Inequalities for off-diagonal operator matrix

Consider now the off-diagonal part $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ of a 2×2 operator matrix $\begin{bmatrix} Z & X \\ Y & W \end{bmatrix}$ defined on $\mathcal{H}\oplus\mathcal{H}$. It is well known that, for $X, Y \in \mathcal{B}(\mathrm{H})$,

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right),$$

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right) \text{ for } \theta \in \mathbb{R},$$

$$w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y) \text{ and } w\left(\begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}\right) = \|Y\|.$$

In 2011, Hirzallah, Kittaneh, and Shebrawi [11] proved among others the following double inequality:

$$\frac{1}{2}\max\{w(X+Y), w(X-Y)\} \leq w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left[w(X+Y) + w(X-Y)\right]. \tag{4.1}$$

They also showed that

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \min\{w(X), w(Y)\} + \frac{1}{2}\min\{\|X+Y\|, \|X-Y\|\}. \tag{4.2}$$

Several other interesting inequalities of this type were also obtained.

In 2015, Kittaneh, Moslehian, and Yamazaki [12] showed that the following refinement for the triangle inequality holds:

$$\left\|\frac{X+Y}{2}\right\| \leq w\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\right) \leq \frac{\|X\| + \|Y\|}{2}$$

for all $X, Y \in \mathcal{B}(\mathcal{H})$. Several upper and lower bounds for the numerical radius of the operator matrix

$$\begin{bmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{bmatrix}$$

under various assumptions for the operators involved were also given.

We use the following equality employed in the proof of Theorem 2.3 of [12]:

$$w\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}A + e^{-i\theta}B\right\| \tag{4.3}$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

We can state now the following result that provides some upper bounds for the numerical radius of the off-diagonal operator matrix:

$$\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix},$$

where $\nu \in [0,1]$.

Proposition 4.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. For $p \geq 2$ and $\nu \in [0,1]$, we have*

$$w^p\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{2^p}\|(A, B^*)\|_{p,\nu}^p \leq \frac{1}{2^p}\left\||A|^p \nabla_\nu |B^*|^p\right\|. \tag{4.4}$$

If $1 \leq p < 2$, then

$$w\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\|(A, B^*)\|_{p,\nu} \leq \frac{1}{2}\left\||A|^2 \nabla_\nu |B^*|^2\right\|^{1/2}. \tag{4.5}$$

Proof. From (2.3), we get for $p \geq 2$ that

$$\begin{aligned}\left\|\frac{(1-\nu)e^{i\theta}A + \nu e^{-i\theta}B^*}{2}\right\|^p &= \frac{1}{2^p}\left\|(1-\nu)e^{i\theta}A + \nu e^{-i\theta}B^*\right\|^p \\ &\leq \frac{1}{2^p}\left\|\left(e^{i\theta}A, e^{-i\theta}B^*\right)\right\|_{p,\nu}^p \\ &= \frac{1}{2^p}\|(A, B^*)\|_{p,\nu}^p \leq \frac{1}{2^p}\left\||A|^p \nabla_\nu |B^*|^p\right\|\end{aligned}$$

for all $\theta \in \mathbb{R}$. By taking the supremum over $\theta \in \mathbb{R}$, we obtain (4.4).

From (2.4), we also have for $1 \leq p < 2$ that

$$\begin{aligned}\left\|\frac{(1-\nu)e^{i\theta}A + \nu e^{-i\theta}B^*}{2}\right\| &= \frac{1}{2}\left\|(1-\nu)e^{i\theta}A + \nu e^{-i\theta}B^*\right\| \\ &\leq \frac{1}{2}\|(A, B^*)\|_{p,\nu} \leq \frac{1}{2}\left\||A|^2 \nabla_\nu |B|^2\right\|^{1/2}\end{aligned}$$

for all $\theta \in \mathbb{R}$. By taking the supremum over $\theta \in \mathbb{R}$, we obtain (4.5).

We observe that for $p = 2$ in (4.4), we obtain

$$w^2\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \leq \frac{1}{4}\left\||A|^2 \nabla_\nu |B^*|^2\right\|, \tag{4.6}$$

while for $p = 1$ in (4.5), then we get

$$w^2\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \le \frac{1}{4}\|(A,B^*)\|_{1,\nu}^2. \tag{4.7}$$

Since

$$((1-\nu)\|Ax\| + \nu\|B^*x\|)^2 \le (1-\nu)\|Ax\|^2 + \nu\|B^*x\|^2 \tag{4.8}$$

for $\nu \in [0,1]$, $x \in H$, then

$$\|(A,B^*)\|_{1,\nu}^2 \le \|(A,B^*)\|_{e,\nu}^2,$$

and we can state that

$$w^2\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B & 0 \end{bmatrix}\right) \le \frac{1}{4}\|(A,B^*)\|_{1,\nu}^2 \le \frac{1}{4}\left\||A|^2 \nabla_\nu |B^*|^2\right\|$$

for $\nu \in [0,1]$, which shows that (4.7) is better than (4.6).

If we make $\nu = 1/2$ in (4.4), then we get for $p \ge 2$ that

$$w^p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \frac{1}{2}\|(A,B^*)\|_p^p \le \left\|\frac{|A|^p + |B^*|^p}{2}\right\|, \tag{4.9}$$

and for $p = 2$,

$$w^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|.$$

If we make $\nu = 1/2$ in (4.4), then we get for $1 \le p < 2$ that

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \frac{1}{2^{1/p}}\|(A,B^*)\|_p \le \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|^{1/2}, \tag{4.10}$$

and for $p = 1$,

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \frac{1}{2}\|(A,B^*)\| \le \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|^{1/2}.$$

From (4.9), we obtain for $B = A$ the following upper bounds for the numerical radius:

$$w^p(A) \le \frac{1}{2}\|(A,A^*)\|_p^p \le \left\|\frac{|A|^p + |A^*|^p}{2}\right\|$$

for $p \ge 2$, and from (4.10),

$$w(A) \le \frac{1}{2^{1/p}}\|(A,A^*)\|_p \le \left\|\frac{|A|^2 + |A^*|^2}{2}\right\|^{1/2}$$

for $1 \le p < 2$.

Since from Proposition 4.1, we have

$$w^p\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right) \le \frac{1}{2^p}\|(A,B)\|_{p,\nu}^p$$

for $p \ge 1$ and $\nu \in [0,1]$. Then the upper bounds from Theorems 3.1 and 3.2 for $\|(A,B)\|_{p,\nu}^p$ will give the following upper bounds for

$$w^p\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right).$$

Proposition 4.2. *Let $A, B \in \mathcal{B}(H)$ such that $0 < m|B|^2 \le |A|^2 \le M|B|^2$ for some constants $0 < m < M$. For $p \ge 1$ and $\nu \in [0,1]$, we have*

$$w^p\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right) \le \frac{1}{2^p}\alpha(m, M; p)\|A\|^{(1-\nu)p}\|B\|^{\nu p}, \tag{4.11}$$

where $\alpha(m, M; p)$ is defined by (1.11).

For $\nu \in (0,1)$, put $r := \min\{\nu, 1-\nu\}$. If $p \ge \frac{1}{r}$, then

$$w^p\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right) \le \frac{1}{2^p}\alpha(m, M; p)\left\|\frac{|A|^{2(1-\nu)p} + |B|^{2\nu p}}{2}\right\|. \tag{4.12}$$

For $\nu \in (0,1)$, put $R := \max\{\nu, 1-\nu\}$. If $Rp \ge 2$, then

$$w^p\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right) \le \frac{1}{2^p}\alpha(m, M; p)\left[\frac{w^{Rp/2}\left(|B|^2|A|^2\right) + \|A\|^{Rp}\|B\|^{Rp}}{2}\right]. \tag{4.13}$$

Also, we have

$$w^2\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right) \le \frac{1}{4}\alpha^{2/p}(m, M; p)\left\||A|^2\nabla_\nu|B|^2\right\|^R\left\||B|^2\nabla_\nu|A|^2\right\|^R \tag{4.14}$$

and

$$w^2\left(\begin{bmatrix} 0 & (1-\nu)A \\ \nu B^* & 0 \end{bmatrix}\right) \le \frac{1}{4}\alpha^{2/p}(m, M; p)\left\||A|^2\nabla_\nu|B|^2\right\|^R\|A\|^{2\nu R}\|B\|^{2(1-\nu)R}. \tag{4.15}$$

If we make $\nu = 1/2$ in (4.11), then we get for $p \ge 1$ that

$$w^p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \le \alpha(m, M; p)\|A\|^{p/2}\|B\|^{p/2}.$$

If $p \ge 2$, then from (4.12), we get

$$w^p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \le \alpha(m, M; p)\left\|\frac{|A|^p + |B|^p}{2}\right\|.$$

If $p \ge 4$, then by (4.13)–(4.15), we have

$$w^p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \le \alpha(m, M; p)\left[\frac{w^{p/4}\left(|B|^2|A|^2\right) + \|A\|^{p/2}\|B\|^{p/2}}{2}\right].$$

Also, we have

$$w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \le \alpha^{2/p}(m, M; p)\left\|\frac{|A|^2 + |B|^2}{2}\right\|$$

and

$$w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \le \alpha^{2/p}(m, M; p)\left\|\frac{|A|^2 + |B|^2}{2}\right\|^{1/2}\|A\|^{1/2}\|B\|^{1/2}.$$

5. Inequalities for one operator

In this section, we give some applications for one operator as follows.
From Theorem 2.1, we have for $(A, B) = (T, T^*)$ that

$$\|T\nabla_\nu T^*\| \le \sigma_{p,\nu}(T) \le \left\||T|^p \nabla_\nu |T^*|^p\right\|^{1/p},$$

where $p \ge 2$. If $1 \le p < 2$, then also

$$\|T\nabla_\nu T^*\| \le \sigma_{p,\nu}(T) \le \left\||T|^2 \nabla_\nu |T^*|^2\right\|^{1/2}.$$

From Theorem 2.2, we obtain for $p \ge 1$ and $\nu \in [0, 1]$ that

$$\begin{aligned}\sigma_{p,\nu}^2(T) \ge &\max\left\{\left\|(1-\nu)^2 T^2 + \nu^2 (T^*)^2\right\|, \nu(1-\nu)\left\||T|^2 + |T^*|^2\right\|\right\}\\ &+ \max\{(1-\nu), \nu\}\|T\| \times \left|\|(1-\nu)T + \nu T^*\| - \|(1-\nu)T - \nu T^*\|\right|.\end{aligned}$$

In particular, for $\nu = 1/2$, we get

$$\sigma_p^2(T) \ge \frac{1}{4^{\frac{p-1}{p}}}\max\left\{\left\|T^2 + (T^*)^2\right\|, \left\||T|^2 + |T^*|^2\right\|\right\} + \frac{1}{4^{\frac{p-1}{p}}}\|T\|\left|\|T + T^*\| - \|T - T^*\|\right|.$$

For $p = 2$, we derive

$$\left\||T|^2 + |T^*|^2\right\| \ge \frac{1}{2}\max\left\{\left\|T^2 + (T^*)^2\right\|, \left\||T|^2 + |T^*|^2\right\|\right\} + \frac{1}{2}\|T\|\left|\|T + T^*\| - \|T - T^*\|\right|.$$

From (2.14), we also get

$$\left\||T|^2 + |T^*|^2\right\| \ge \frac{1}{2}\max\left\{\left\|T^2 + (T^*)^2\right\|, \left\|T^2 - (T^*)^2\right\|\right\}.$$

Assume that $0 < n|T^*|^2 \le |T|^2 \le N|T^*|^2$ for some constants $0 < n < N$. If $p \ge \frac{1}{r}$, then by Theorem 3.1, we get

$$\sigma_{p,\nu}^p(T) \le \frac{1}{2}\alpha(m, M; p)\left\||T|^{2(1-\nu)p} + |T^*|^{2\nu p}\right\|,$$

where

$$\alpha(n, N; p) := \begin{cases} S\left(n^{p/2}\right) & \text{if } N \le 1; \\ \max\left\{S\left(n^{p/2}\right), S\left(M^{p/2}\right)\right\} & \text{if } n \le 1 \le N; \\ S\left(N^{p/2}\right) & \text{if } 1 \le n. \end{cases}$$

If $Rp \ge 2$, then by Theorem 3.2, we obtain

$$\sigma_{p,\nu}^p(T) \le \frac{1}{2}\alpha(n, N; p)\left[w^{Rp/2}\left(|T^*|^2|T|^2\right) + \|T\|^{2Rp}\right].$$

Also, we have

$$\sigma_{e,\nu}^2(T) \le \alpha^{2/p}(n, N; p)\left\||T|^2 \nabla_\nu |T^*|^2\right\|^R \left\||T^*|^2 \nabla_\nu |T|^2\right\|^R$$

and

$$\sigma_{e,\nu}^{2}(T) \le \alpha^{2/p}(n, N; p) \left\| |T|^{2} \nabla_{\nu} |T^{*}|^{2} \right\|^{R} \|T\|^{2}.$$

For an operator $T \in \mathcal{B}(H)$, we consider the real and imaginary parts of T defined by

$$\operatorname{Re} T := \frac{T + T^{*}}{2} \text{ and } \operatorname{Im} T = \frac{T - T^{*}}{2i}.$$

For $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$, we can consider, for $p \ge 1$ and $\nu \in [0, 1]$, the following functional:

$$\rho_{p,\nu}(T) := \|(\operatorname{Re} T, \operatorname{Im} T)\|_{p,\nu} := \sup_{\|x\|=1} \left((1-\nu)\|\operatorname{Re} T x\|^{p} + \nu \|\operatorname{Im} T x\|^{p}\right)^{1/p}.$$

Since for a real number a, we have that $\rho_{p,\nu}(aT) = |a| \rho_{p,\nu}(T)$, $T \in \mathcal{B}(H)$. Then, we can conclude that $\rho_{p,\nu}(T)$ is a real norm on $\mathcal{B}(H)$ for $\nu \in (0, 1)$.

For $p = 2$, we obtain that

$$\rho_{e,\nu}^{2}(T) = \left\| \operatorname{Re}^{2} T \nabla_{\nu} \operatorname{Im}^{2} T \right\|, \ \nu \in [0, 1]$$

and

$$\rho_{e}^{2}(T) = \left\| \operatorname{Re}^{2} T + \operatorname{Im}^{2} T \right\| = \frac{1}{2} \left\| |T|^{2} + |T^{*}|^{2} \right\| = \frac{1}{2} \sigma_{e}^{2}(T).$$

From Theorem 2.1, we get for $p \ge 2$ and $\nu \in [0, 1]$ that

$$\|\operatorname{Re} T \nabla_{\nu} \operatorname{Im} T\| \le \rho_{p,\nu}(T) \le \left\| |\operatorname{Re} T|^{p} \nabla_{\nu} |\operatorname{Im} T|^{p} \right\|^{1/p}.$$

If $1 \le p < 2$, then

$$\|\operatorname{Re} T \nabla_{\nu} \operatorname{Im} T\| \le \rho_{p,\nu}(T) \le \left\| \operatorname{Re}^{2} T \nabla_{\nu} \operatorname{Im}^{2} T \right\|^{1/2}.$$

From Theorem 2.2, for $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$, we obtain for $p \ge 1$ and $\nu \in [0, 1]$ that

$$\begin{aligned} \rho_{p,\nu}^{2}(T) \ge & \max\left\{ \left\| (1-\nu)^{2} \operatorname{Re}^{2} T + \nu^{2} \operatorname{Im}^{2} T \right\|, \nu(1-\nu) \|\operatorname{Re} T \operatorname{Im} T + \operatorname{Im} T \operatorname{Re} T\| \right\} \\ & + \max\left\{ (1-\nu)\|\operatorname{Re} T\|, \nu \|\operatorname{Im} T\| \right\} \times \left| \|(1-\nu)\operatorname{Re} T + \nu \operatorname{Im} T\| - \|(1-\nu)\operatorname{Re} T - \nu \operatorname{Im} T\| \right|. \end{aligned}$$

In particular, for $\nu = 1/2$, we get

$$\begin{aligned} \rho_{p}^{2}(T) \ge & \frac{1}{4^{\frac{p-1}{p}}} \max\left\{ \left\| \operatorname{Re}^{2} T + \operatorname{Im}^{2} T \right\|, \|\operatorname{Re} T \operatorname{Im} T + \operatorname{Im} T \operatorname{Re} T\| \right\} \\ & + \frac{1}{4^{\frac{p-1}{p}}} \max\left\{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \right\} \left| \|\operatorname{Re} T + \operatorname{Im} T\| - \|\operatorname{Re} T - \operatorname{Im} T\| \right|. \end{aligned}$$

From (2.14), we also get

$$\begin{aligned} \left\| |T|^{2} + |T^{*}|^{2} \right\| \ge & \max \left\| \operatorname{Re}^{2} T + \operatorname{Im}^{2} T \right\|, \left\| \operatorname{Re}^{2} T - \operatorname{Im}^{2} T \right\| \\ & + \max\left\{ \|\operatorname{Re} T + \operatorname{Im} T\|, \|\operatorname{Re} T - \operatorname{Im} T\| \right\} \left| \|\operatorname{Re} T\| - \|\operatorname{Im} T\| \right|. \end{aligned}$$

Assume that there exists constants $0 < k < K$ such that $0 < k \operatorname{Im}^{2} T \le \operatorname{Re}^{2} T \le K \operatorname{Im}^{2} T$.

From Theorem 3.1, we obtain for $p \ge 1$ and $\nu \in [0, 1]$ that

$$\rho_{p,\nu}^{p}(T) \le \alpha(k, K; p) \|\operatorname{Re} T\|^{(1-\nu)p} \|\operatorname{Im} T\|^{\nu p},$$

where

$$\alpha(k, K; p) := \begin{cases} S\left(k^{p/2}\right) & \text{if } K \le 1; \\ \max\left\{S\left(k^{p/2}\right), S\left(K^{p/2}\right)\right\} & \text{if } k \le 1 \le K; \\ S\left(K^{p/2}\right) & \text{if } 1 \le k. \end{cases}$$

If $p \ge \frac{1}{r}$, then by Theorem 3.2, we get

$$\rho_{p,\nu}^{p}(T) \le \frac{1}{2}\alpha(k, K; p)\left\| |\operatorname{Re} T|^{2(1-\nu)p} + |\operatorname{Im} T|^{2\nu p} \right\|.$$

If $Rp \ge 2$, then also

$$\rho_{p,\nu}^{p}(T) \le \frac{1}{2}\alpha(k, K; p)\left[w^{Rp/2}\left(\operatorname{Im}^2 T \operatorname{Re}^2 T\right) + \|\operatorname{Re} T\|^{Rp} \|\operatorname{Im} T\|^{Rp}\right].$$

Also, we have

$$\left\|\operatorname{Re}^2 T \nabla_\nu \operatorname{Im}^2 T\right\| \le \alpha^{2/p}(k, K; p)\left\|\operatorname{Re}^2 T \nabla_\nu \operatorname{Im}^2 T\right\|^R \left\|\operatorname{Im}^2 T \nabla_\nu \operatorname{Re}^2 T\right\|^R$$

and

$$\left\|\operatorname{Re}^2 T \nabla_\nu \operatorname{Im}^2 T\right\| \le \alpha^{2/p}(k, K; p)\left\|\operatorname{Re}^2 T \nabla_\nu \operatorname{Im}^2 T\right\|^R \|\operatorname{Re} T\|^{2\nu R} \|\operatorname{Im} T\|^{2(1-\nu)R}.$$

6. Conclusions

In this paper, we considered the p-arithmetic-mean (A-M)-norm for the pair of operators (A, B) in Hilbert spaces and provided several lower and upper bounds for it. Some natural applications for the off-diagonal operator matrix and in the case when $(A, B) = (T, T^*)$ and $(A, B) = (\operatorname{Re} T, \operatorname{Im} T)$ were also given.

Author contributions

Najla Altwaijry: Funding, Resources, Writing–review & editing, Project administration; Silvestru Sever Dragomir: Conceptualization, Visualization, Formal analysis, Validation, Investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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