AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 11031–11049.
DOI: 10.3934/math.2026453
Received: 05 February 2026
Revised: 02 April 2026
Accepted: 13 April 2026
Published: 21 April 2026

Research article

Solutions to two systems of constrained matrix equations with an application to image processing over quaternion algebra

Long-Sheng Liu*, **Xiao-Xiao Ma and Xiao-Quan Chen**

School of Mathematics and Statistics, Anqing Normal University, Anqing 246011, China

* **Correspondence:** Email: 062207@aqnu.edu.cn.

Abstract: This study primarily investigated the equivalence conditions for the existence of solutions to two quaternion matrix systems under constraints, as well as their general solutions. As an application, it focused on studying the reducibility of solutions to classical matrix equations and their applications in image processing, such as image encryption and decryption. Finally, an example is provided to validate the main results presented in this paper.

Keywords: image processing; matrix equation; rank; Moore-Penrose
Mathematics Subject Classification: 15A24, 15A23, 15A33

1. Introduction

The solvability conditions for the following systems are established in this study:

$$\begin{cases} E_1X_1 = C_1, E_1X_1B_3 + E_1X_2B_2 + A_2X_3B_2 = C_2, \\ A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_3, X_3B_4 = C_4, \end{cases} \tag{1.1}$$

and

$$\begin{cases} E_1X_1 = C_1,\ E_1X_2B_2 + A_2X_3B_2 = C_2, \\ A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_3. \end{cases} \tag{1.2}$$

As applications, we investigate the classical quaternion system:

$$AXB = E, CXD = F, \tag{1.3}$$

regarding the existence of a reducible solution. Furthermore, using the solution of System (1.1), we examine image encryption and decryption.

In this study, the real number field is denoted by $\mathbb{R}$. The set of all $m \times n$ quaternion matrices is represented by $\mathbb{H}^{m\times n}$, where

$$\mathbb{H} = \{v = v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, v_0, v_1, v_2, v_3 \in \mathbb{R}\}.$$

In fields like color video processing, control systems, and mechanics, the noncommutative division ring $\mathbb{H}$ is extensively utilized (see, e.g., [1–3]).

In this study, the symbols $r(A)$, 0, and I are used to denote the rank of matrix A, a zero matrix of appropriate size, and an identity matrix of suitable size, respectively. $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real and imaginary parts of A, respectively. The conjugate transpose of A is denoted by A^*. For any matrix $C \in \mathbb{H}^{l\times k}$, a unique matrix Y exists, satisfying

$$CYC = C, YCY = Y, (CY)^* = CY, (YC)^* = YC.$$

This matrix Y is termed the Moore-Penrose (M-P) inverse of C; it is denoted by $C^\dagger$. Furthermore, we introduce the following projectors:

$$L_C = I - C^\dagger C, R_C = I - CC^\dagger.$$

A matrix Z is called reducible when a permutation matrix K exists, satisfying

$$Z = K\begin{pmatrix} Z_1 & Z_2 \\ 0 & Z_3 \end{pmatrix}K^{-1}.$$

With square quaternion matrices Z_1 and Z_3 of appropriate dimensions. If Z_3 is of order k, then Z is termed k-reducible. For an arbitrary k such that $1 \le k < n$, define the set as

$$\mathbb{H}_k^{n\times n} = \left\{ Z = K\begin{pmatrix} Z_1 & Z_2 \\ 0 & Z_3 \end{pmatrix}K^{-1} \middle| 1 \le k < n, Z_1 \in \mathbb{H}^{(n-k)\times(n-k)}, Z_3 \in \mathbb{H}^{k\times k} \right\}.$$

Reducible matrices have applications in various areas, including directed graphs, compartmental analysis, and stochastic processes (see, e.g., [4, 5]). Recently, several authors have studied reducible solutions to matrix equations. For example, Xie and Wang [6] derived a reducible solution to the quaternion equation

$$AXB = E. \tag{1.4}$$

Nie et al. [5] considered the reducible solution of the system

$$AX = C, XB = D. \tag{1.5}$$

Wang [7] determined the k-reducible solution of the following system over $\mathbb{H}$:

$$A_1X = C_1, A_2XB_2 = C_2. \tag{1.6}$$

As far as we know, System (1.3) has not been studied with regard to its reducible solutions. Notably, the reducibility of the solution to System (1.3) is related to the general solution of System (1.2).

In mathematics, linear matrix equations represent a deeply investigated subject, owing to their critical applications in numerous areas such as singular system control theory [8], neural networks [9], feedback control [10], graph theory [11], and color image processing [12]. This topic has been widely studied (see, e.g., [13–15]). Xie and Wang [6] studied the quaternion equation

$$A_{11}X_1B_{11} + A_{11}X_2B_{22} + A_{22}X_3B_{22} = C_1, \tag{1.7}$$

obtaining solvability conditions using M-P inverses. Nie [16] derived solutions over $\mathbb{H}$ for System (1.6) and

$$A_{11}X_1 = C_1,\ B_{11}X_1C_{11} = C_2,\ E_{11}X_1F_{11} = C_3. \tag{1.8}$$

As an extension of (1.4), (1.6), and (1.8), we propose System (1.1).

Driven by the broad applications of linear matrix equations and reducible matrices, as well as the need for further theoretical development, we examined solvability conditions for the quaternion Systems (1.1) and (1.2) using the M-P inverses and rank equalities and then derived their general solutions when solvable. As an application of System (1.2), we also established conditions under which System (1.3) admits a reducible solution and presented its general solution. Moreover, we used System (1.1) to process images, including image encryption and decryption.

We structure the remainder of this paper as follows: Section 2 presents the key lemmas required for the subsequent analysis. Section 3 analyzes the solvability conditions for Systems (1.1) and (1.2) using the M-P inverses and rank equalities and derives their general solutions. In Section 4, to demonstrate an application of System (1.2), we analyze a reducible solution to System (1.3) and illustrate the main results with an example. By applying the solutions to System (1.1), Section 5 demonstrates their use in encrypting and decrypting color images. We conclude the paper in Section 6.

2. Preliminaries

The key theoretical foundations underpinning the arguments of this paper are presented in this section. First, we introduce the result of the matrix rank equality.

Lemma 2.1. *[17] Let $D_1 \in \mathbb{C}^{m\times n}$, $F_1 \in \mathbb{C}^{m\times k}$, $G_1 \in \mathbb{C}^{l\times n}$, $N_1 \in \mathbb{C}^{j\times k}$, and $M_1 \in \mathbb{H}^{l\times i}$ be known matrices. Then the following rank equality holds:*

$$r\begin{pmatrix} D_1 & F_1L_{N_1} \\ R_{M_1}G_1 & 0 \end{pmatrix} = r\begin{pmatrix} D_1 & F_1 & 0 \\ G_1 & 0 & M_1 \\ 0 & N_1 & 0 \end{pmatrix} - r(N_1) - r(M_1).$$

Remark 2.2. *Lemma 2.1 can be extended to the quaternion algebra $\mathbb{H}$.*

Next, we proceed to discuss the solvability of the following important matrix equations.

Lemma 2.3. *[18] Let A_1, A_2, B_1, and B_2 be given matrices with adequate shapes over $\mathbb{H}$. The equations $A_1X = A_2$ and $YB_1 = B_2$ are solvable if and only if*

$$R_{A_1}A_2 = 0,\ B_2L_{B_1} = 0. \tag{2.1}$$

When (2.1) *holds, the complete solutions for these equations are*

$$X = A_1^{\dagger}A_2 + L_{A_1}U_1,\ Y = B_2B_1^{\dagger} + U_2R_{B_1},$$

where U_1 and U_2 denote arbitrary quaternion matrices of suitable dimensions.

Lemma 2.4. *[15] Let $A_{11},\ B_{11},\ C_{33},\ D_{33},\ C_{44},\ D_{44},\ E_{11}$ be given. Put*

$$A_1 = R_{A_{11}}C_{33},\ B_1 = D_{33}L_{B_{11}},\ C_1 = R_{A_{11}}C_{44},\ D_1 = D_{44}L_{B_{11}},$$
$$E_1 = R_{A_{11}}E_{11}L_{B_{11}},\ M_1 = R_{A_1}C_1,\ N_1 = D_1L_{B_1},\ S_1 = C_1L_{M_1}.$$

The following expressions are equivalent:

(i) *The equation*

$$A_{11}X_1 + X_2B_{11} + C_{33}X_3D_{33} + C_{44}X_4D_{44} = E_{11}$$

is solvable.

(ii)

$$R_{M_1}R_{A_1}E_1 = 0,\ E_1L_{B_1}L_{N_1} = 0,\ R_{A_1}E_1L_{D_1} = 0,\ R_{C_1}E_1L_{B_1} = 0.$$

(iii)

$$r\begin{pmatrix} E_{11} & C_{44} & C_{33} & A_{11} \\ B_{11} & 0 & 0 & 0 \end{pmatrix} = r(B_{11}) + r\begin{pmatrix} C_{44} & C_{33} & A_{11} \end{pmatrix},$$

$$r\begin{pmatrix} E_{11} & A_{11} \\ D_{33} & 0 \\ D_{44} & 0 \\ B_{11} & 0 \end{pmatrix} = r\begin{pmatrix} D_{33} \\ D_{44} \\ B_{11} \end{pmatrix} + r(A_{11}),$$

$$r\begin{pmatrix} E_{11} & C_{33} & A_{11} \\ D_{44} & 0 & 0 \\ B_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{11} & C_{33} \end{pmatrix} + r\begin{pmatrix} D_{44} \\ B_{11} \end{pmatrix},$$

$$r\begin{pmatrix} E_{11} & C_{44} & A_{11} \\ D_{33} & 0 & 0 \\ B_{11} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_{11} & C_{44} \end{pmatrix} + r\begin{pmatrix} D_{33} \\ B_{11} \end{pmatrix}.$$

Under the condition of (i)*, the solution of the equation is*

$$X_1 = A_{11}^{\dagger}(E_{11} - C_{33}X_3D_{33} - C_{44}X_4D_{44}) - A_{11}^{\dagger}T_7B_{11} + L_{A_{11}}T_6,$$
$$X_2 = R_{A_{11}}(E_{11} - C_{33}X_3D_{33} - C_{44}X_4D_{44})B_{11}^{\dagger} + A_{11}A_{11}^{\dagger}T_7 + T_8R_{B_{11}},$$
$$X_3 = A_1^{\dagger}E_1B_1^{\dagger} - A_1^{\dagger}C_1M_1^{\dagger}E_1B_1^{\dagger} - A_1^{\dagger}S_1C_1^{\dagger}E_1N_1^{\dagger}D_1B_1^{\dagger} - A_1^{\dagger}S_1T_2R_{N_1}D_1B_1^{\dagger} + L_{A_1}T_4 + T_5R_{B_1},$$
$$X_4 = M_1^{\dagger}E_1D_1^{\dagger} + S_1^{\dagger}S_1C_1^{\dagger}E_1N_1^{\dagger} + L_{M_1}L_{S_1}T_1 + L_{M_1}T_2R_{N_1} + T_3R_{D_1},$$

where $T_1, \ldots, T_8$ denote arbitrary quaternion matrices of appropriate dimensions.

3. Solvability conditions and expression of general solutions for System (1.1)

The equivalence conditions for solvability and the explicit form of the general solution to System (1.1) are the focus of this section.

Let A_i, B_j, C_j $(i = \overline{1,2}, j = \overline{1,4})$, and E_1 be coefficient matrices of System (1.1). To facilitate the discussion, we introduce the following notations:

$$\begin{cases} M_1 = R_{E_1}A_2,\ N_2 = EL_{B_2},\ S_1 = A_2L_{M_1},\ E = R_{B_4}B_2, \\ F = C_2 - C_1B_3 - A_2C_4B_4^{\dagger}B_2,\ A_3 = A_1L_{E_1},\ A_4 = A_1 - E_1, \\ R_{A_3}A_4 = M_2,\ A_4L_{M_2} = S_2,\ C = C_3 - C_2 - A_1E_1^{\dagger}C_1B_1 + C_1B_3, \\ N_1 = B_2L_{B_1},\ A_5 = \begin{pmatrix} L_{M_2}L_{S_2} & -L_{E_1} \end{pmatrix},\ B = \begin{pmatrix} R_{B_2} \\ -R_{B_2} \end{pmatrix}. \end{cases} \tag{3.1}$$

$$\begin{cases} A_6 = L_{M_2},\ B_6 = R_{N_1},\ A_7 = E_1^{\dagger}S_1, B_5 = R_{N_2}EB_2^{\dagger}, \\ A_{11} = R_{A_5}A_6,\ B_{11} = B_6L_B,\ C_{11} = R_{A_5}A_7,\ D_{11} = B_5L_B, \\ E_{11} = R_{A_5}DL_B,\ M_{11} = R_{A_{11}}C_{11},\ N_{11} = D_{11}L_{B_{11}}, \\ S_{11} = C_{11}L_{M_{11}},\ D = E_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger}EB_2^{\dagger} \\ -M_2^{\dagger}CB_2^{\dagger} - S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}. \end{cases} \tag{3.2}$$

Next, we give the following main theorem:

Theorem 3.1. *The following three equivalent propositions can be stated as:*

(i) *System* (1.1) *is solvable.*

(ii)

$$R_{E_1}C_1 = 0,\ C_4L_{B_4} = 0, \tag{3.3}$$

$$R_{M_1}R_{E_1}F = 0,\ FL_{B_2}L_{N_2} = 0, R_{E_1}FL_E = 0, R_{A_2}FL_{B_2} = 0, \tag{3.4}$$

$$R_{M_2}R_{A_3}C = 0,\ CL_{B_1}L_{N_1} = 0,\ R_{A_3}CL_{B_2} = 0,\ R_{A_4}CL_{B_1} = 0, \tag{3.5}$$

$$\begin{gathered} R_{M_{11}}R_{A_{11}}E_{11} = 0,\ E_{11}L_{B_{11}}L_{N_{11}} = 0, \\ R_{A_{11}}E_{11}L_{D_{11}} = 0,\ R_{C_{11}}E_{11}L_{B_{11}} = 0. \end{gathered} \tag{3.6}$$

(iii)

$$r\begin{pmatrix} E_1 & C_1 \end{pmatrix} = r(E_1),\ r\begin{pmatrix} C_4 \\ B_4 \end{pmatrix} = r(B_4), \tag{3.7}$$

$$r\begin{pmatrix} G & A_2 & E_1 \end{pmatrix} = r\begin{pmatrix} A_2 & E_1 \end{pmatrix},\ r\begin{pmatrix} G \\ B_2 \end{pmatrix} = r(B_2), \tag{3.8}$$

$$r\begin{pmatrix} G & E_1 & A_2C_4 \\ B_2 & 0 & B_4 \end{pmatrix} = r(E_1) + r\begin{pmatrix} B_2 & B_4 \end{pmatrix}, \tag{3.9}$$

$$r\begin{pmatrix} G & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2), \tag{3.10}$$

$$r\begin{pmatrix} A & A_1 - E_1 & A_1 \\ C_1B_1 & 0 & E_1 \end{pmatrix} = r\begin{pmatrix} A_1 - E_1 & A_1 \\ 0 & E_1 \end{pmatrix}, \tag{3.11}$$

$$r\begin{pmatrix} A \\ B_1 \\ B_2 \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{3.12}$$

$$r\begin{pmatrix} A & A_1 \\ B_2 & 0 \\ C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} A_1 \\ E_1 \end{pmatrix} + r\begin{pmatrix} B_2 \end{pmatrix}, \tag{3.13}$$

$$r\begin{pmatrix} A & A_1 - E_1 \\ B_1 & 0 \end{pmatrix} = r\left(A_1 - E_1\right) + r\left(B_1\right), \tag{3.14}$$

$$r\begin{pmatrix} 0 & A_1 - E_1 & A & A_1 \\ A_2 & E_1 & G & 0 \\ 0 & 0 & C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 & A_1 \\ A_2 & E_1 & 0 \\ 0 & 0 & E_1 \end{pmatrix}, \tag{3.15}$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_4 \\ 0 & B_2 & B_2 & 0 \\ E_1 - A_1 & A & 0 & 0 \\ E_1 & 0 & G & A_2C_4 \\ 0 & B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & B_2 & B_4 \\ B_2 & B_2 & 0 \\ B_1 & 0 & 0 \end{pmatrix} + r\begin{pmatrix} A_1 - E_1 \\ E_1 \end{pmatrix}, \tag{3.16}$$

$$r\begin{pmatrix} 0 & B_2 & 0 & B_4 \\ E_1 & A & A_1 & 0 \\ -E_1 & G & 0 & A_2C_4 \\ E_1 & C_1B_1 & E_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 \\ E_1 & 0 \\ 0 & E_1 \end{pmatrix} + r\begin{pmatrix} B_2 & B_4 \end{pmatrix}, \tag{3.17}$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_2 \\ 0 & A_1 - E_1 & A & 0 \\ A_2 & E_1 & 0 & -G \\ 0 & 0 & B_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 \\ A_2 & E_1 \end{pmatrix} + r\begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}, \tag{3.18}$$

where $A = C_3 - C_2 + C_1B_3,\ G = C_2 - C_1B_3$. *If* (i) *holds, the general solution of System* (1.1) *is given by*

$$\begin{aligned} &X_1 = E_1^{\dagger}C_1 + L_{E_1}U_1,\ X_3 = C_4B_4^{\dagger} + U_2R_{B_4}, \\ &X_2 = E_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger}EB_2^{\dagger} - E_1^{\dagger}S_1T_1R_{N_2}EB_2^{\dagger} + L_{E_1}T_2 + T_3R_{B_2}, \\ &\text{or} \\ &X_2 = M_2^{\dagger}CB_2^{\dagger} + S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger} + L_{M_2}L_{S_2}W_4 + L_{M_2}W_1R_{N_1} + W_5R_{B_2}, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} T_1 &= M_{11}^{\dagger}E_{11}D_{11}^{\dagger} + S_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger} + L_{M_{11}}L_{S_{11}}W_{12} + L_{M_{11}}W_9R_{N_{11}} + W_{13}R_{D_{11}}, \\ T_2 &= (0,\ I_m)\left[A_5^{\dagger}(D - A_6W_1B_6 - A_7T_1B_5) - A_5^{\dagger}W_6B + L_{A_5}W_7\right], \\ T_3 &= \left[R_{A_5}(D - A_6W_1B_6 - A_7T_1B_5)B^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_B\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix}, \\ U_1 &= A_3^{\dagger}CB_1^{\dagger} - A_3^{\dagger}A_4M_2^{\dagger}CB_1^{\dagger} - A_3^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}B_2B_1^{\dagger} - A_3^{\dagger}S_2W_1R_{N_1}B_2B_1^{\dagger} + L_{A_3}W_2 + W_3R_{B_1}, \\ U_2 &= M_1^{\dagger}FE^{\dagger} + S_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger} + L_{M_1}L_{S_1}T_4 + L_{M_1}T_1R_{N_2} + T_5R_E, \\ W_1 &= A_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}C_{11}M_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger}D_{11}B_{11}^{\dagger} \\ &\quad - A_{11}^{\dagger}S_{11}W_9R_{N_{11}}D_{11}B_{11}^{\dagger} + L_{A_{11}}W_{10} + W_{11}R_{B_{11}}, \\ W_4 &= (I_m,\ 0)\left[A_5^{\dagger}(D - A_6W_1B_6 - A_7T_1B_5) - A_5^{\dagger}W_6B + L_{A_5}W_7\right], \\ W_5 &= \left[R_{A_5}(D - A_6W_1B_6 - A_7T_1B_5)B^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_B\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \end{aligned}$$

$W_2,\ W_3,\ W_j(j=\overline{6,13})$, T_4, and T_5 are any quaternion matrices of suitable shapes, where m and n denote the numbers of columns of A_1 and rows of B_2, respectively.

Proof. (i) ⇔ (ii): System (1.1) is solvable precisely when the equations

$$E_1X_1 = C_1,\ X_3B_4 = C_4, \tag{3.20}$$

$$E_1X_1B_3 + E_1X_2B_2 + A_2X_3B_2 = C_2, \tag{3.21}$$

$$A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_3 \tag{3.22}$$

have common solutions. Therefore, it is necessary to establish equivalent conditions for the existence of common solutions to Eqs (3.20)–(3.22). According to Lemma 2.3, Eq (3.20) is solvable if and only if condition (3.3) is satisfied. The general solutions to Eq (3.20) are given by:

$$X_1 = E_1^{\dagger}C_1 + L_{E_1}U_1,\ X_3 = C_4B_4^{\dagger} + U_2R_{B_4}. \tag{3.23}$$

Using the relation $E_1X_1 = C_1$ and substituting the expressions for X_1 and X_3 from (3.23) into (3.21) and (3.22) yields:

$$A_2U_2E + E_1X_2B_2 = F, \tag{3.24}$$

$$A_3U_1B_1 + A_4X_2B_2 = C. \tag{3.25}$$

Hence, we now focus on studying Eqs (3.24) and (3.25). By Lemma 2.4, the quaternion Eqs (3.24) and (3.25) are consistent if and only if conditions (3.4) and (3.5) hold, respectively. Under these conditions, the solutions to Eqs (3.24) and (3.25) are:

$$U_2 = M_1^{\dagger}FE^{\dagger} + S_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger} + L_{M_1}L_{S_1}T_4 + L_{M_1}T_1R_{N_2} + T_5R_E, \tag{3.26}$$

$$X_2 = E_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger}EB_2^{\dagger} - E_1^{\dagger}S_1T_1R_{N_2}EB_2^{\dagger} + L_{E_1}T_2 + T_3R_{B_2}, \tag{3.27}$$

and

$$U_1 = A_3^{\dagger}CB_1^{\dagger} - A_3^{\dagger}A_4M_2^{\dagger}CB_1^{\dagger} - A_3^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}B_2B_1^{\dagger} - A_3^{\dagger}S_2W_1R_{N_1}B_2B_1^{\dagger} + L_{A_3}W_2 + W_3R_{B_1},$$

$$X_2 = M_2^{\dagger}CB_2^{\dagger} + S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger} + L_{M_2}L_{S_2}W_4 + L_{M_2}W_1R_{N_1} + W_5R_{B_2}, \tag{3.28}$$

where $A_3,\ A_4,\ C,\ E,\ F,\ M_i,\ S_i$, and N_i $(i=\overline{1,2})$ are defined in (3.1), and W_j, T_j $(j=\overline{1,5})$ denote arbitrary quaternion matrices of suitable dimensions.

Finally, equating the two expressions for X_2 from (3.27) and (3.28), we obtain:

$$A_5\begin{pmatrix} W_4 \\ T_2 \end{pmatrix} + (W_5,\ T_3)B + A_6W_1B_6 + A_7T_1B_5 = D. \tag{3.29}$$

By Lemma 2.4, Eq (3.29) is solvable precisely when condition (3.6) holds. Its general solution is given by:

$$W_4 = (I_m,\ 0)\left[A_5^{\dagger}(D - A_6W_1B_6 - A_7T_1B_5) - A_5^{\dagger}W_6B + L_{A_5}W_7\right],$$

$$T_2 = (0,\ I_m)\left[A_5^{\dagger}(D - A_6W_1B_6 - A_7T_1B_5) - A_5^{\dagger}W_6B + L_{A_5}W_7\right],$$

$$W_5 = \Big[R_{A_5}(D - A_6W_1B_6 - A_7T_1B_5)B^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_B\Big]\begin{pmatrix} I_n \\ 0 \end{pmatrix},$$
$$T_3 = \Big[R_{A_5}(D - A_6W_1B_6 - A_7T_1B_5)B^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_B\Big]\begin{pmatrix} 0 \\ I_n \end{pmatrix},$$
$$W_1 = A_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}C_{11}M_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger}D_{11}B_{11}^{\dagger}$$
$$- A_{11}^{\dagger}S_{11}W_9R_{N_{11}}D_{11}B_{11}^{\dagger} + L_{A_{11}}W_{10} + W_{11}R_{B_{11}},$$
$$T_1 = M_{11}^{\dagger}E_{11}D_{11}^{\dagger} + S_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger} + L_{M_{11}}L_{S_{11}}W_{12} + L_{M_{11}}W_9R_{N_{11}} + W_{13}R_{D_{11}},$$

where $A_5, A_6, A_7, B, B_5, A_{11}, B_{11}, C_{11}, D_{11}, E_{11}, S_{11}$, and N_{11} are defined in (3.1) and (3.2).

To sum up, we obtain that Eqs (3.20)–(3.22) have common solutions if and only if conditions (3.3)–(3.6) hold, i.e., the necessary and sufficient conditions for the System (1.1) to have a solution are that conditions (3.3)–(3.6) all hold. When these conditions hold, System (1.1) yields the general solution (3.19).

(ii) $\Leftrightarrow$ (iii): We will prove the following:

$$R_{E_1}C_1 = 0 \Leftrightarrow (3.7),\ C_4L_{B_4} = 0 \Leftrightarrow (3.7),\ R_{M_1}R_{E_1}F = 0 \Leftrightarrow (3.8),$$
$$FL_{B_2}L_{N_2} = 0 \Leftrightarrow (3.8),\ R_{E_1}FL_E = 0 \Leftrightarrow (3.9),\ R_{A_2}FL_{B_2} = 0 \Leftrightarrow (3.10),$$
$$R_{M_2}R_{A_3}C = 0 \Leftrightarrow (3.11),\ CL_{B_1}L_{N_1} = 0 \Leftrightarrow (3.12),\ R_{A_3}CL_{B_2} = 0 \Leftrightarrow (3.13),$$
$$R_{A_4}CL_{B_1} = 0 \Leftrightarrow (3.14),\ R_{M_{11}}R_{A_{11}}E_{11} = 0 \Leftrightarrow (3.15),\ E_{11}L_{B_{11}}L_{N_{11}} = 0 \Leftrightarrow (3.16),$$
$$R_{A_{11}}E_{11}L_{D_{11}} = 0 \Leftrightarrow (3.17),\ R_{C_{11}}E_{11}L_{B_{11}} = 0 \Leftrightarrow (3.18).$$

It is easy to know that there is a particular solution $X_2^1,\ U_2^1$ of the matrix equation (3.24) and particular solution $X_2^0,\ U_1^0$ of the matrix equation (3.25) such that

$$E_1X_2^1B_2 + A_2U_2^1E = F,\ A_3U_1^0B_1 + A_4X_2^0B_2 = C,$$

where

$$X_2^1 = E_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}FB_2^{\dagger} - E_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger}EB_2^{\dagger},\ X_2^0 = M_2^{\dagger}CB_2^{\dagger} + S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger},$$
$$U_2^1 = M_1^{\dagger}FE^{\dagger} + S_1^{\dagger}S_1A_2^{\dagger}FN_2^{\dagger},\ U_1^0 = A_3^{\dagger}CB_1^{\dagger} - A_3^{\dagger}A_4M_2^{\dagger}CB_1^{\dagger} - A_3^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}B_2B_1^{\dagger}.$$

By employing Lemma 2.1 and applying elementary operations, we obtain

$$R_{E_1}C_1 = 0 \Leftrightarrow r(R_{E_1}C_1) = 0 \Leftrightarrow r\begin{pmatrix} C_1 & E_1 \end{pmatrix} = r(E_1) \Leftrightarrow (3.7),$$
$$C_4L_{B_4} = 0 \Leftrightarrow r(C_4L_{B_4}) = 0 \Leftrightarrow r\begin{pmatrix} C_4 \\ B_4 \end{pmatrix} = r(B_4) \Leftrightarrow (3.7),$$
$$R_{M_1}R_{E_1}F = 0 \Leftrightarrow r(R_{M_1}R_{E_1}F) = 0$$
$$\Leftrightarrow r\begin{pmatrix} R_{E_1}F & R_{E_1}A_2 \end{pmatrix} = r(R_{E_1}A_2) \Leftrightarrow r\begin{pmatrix} F & A_2 & E_1 \end{pmatrix} = r\begin{pmatrix} A_2 & E_1 \end{pmatrix} \Leftrightarrow (3.8),$$
$$FL_{B_2}L_{N_2} = 0 \Leftrightarrow r(FL_{B_2}L_{N_2}) = 0 \Leftrightarrow r\begin{pmatrix} F \\ B_2 \end{pmatrix} = r(B_2) \Leftrightarrow (3.8),$$
$$R_{E_1}FL_E = 0 \Leftrightarrow r(R_{E_1}FL_E) = 0 \Leftrightarrow r\begin{pmatrix} F & E_1 \\ E & 0 \end{pmatrix} = r(E_1) + r(E)$$

$$\Leftrightarrow r\begin{pmatrix} C_2 - C_1B_3 - A_2C_4B_4^{\dagger}B_2 & E_1 & 0 \\ B_2 & 0 & B_4 \end{pmatrix} = r(E_1) + r\begin{pmatrix} B_2 & B_4 \end{pmatrix} \Leftrightarrow (3.9),$$

$$R_{A_2}FL_{B_2} = 0 \Leftrightarrow r(R_{A_2}FL_{B_2}) = 0 \Leftrightarrow r\begin{pmatrix} F & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2) \Leftrightarrow (3.10),$$

$$R_{M_2}R_{A_3}C = 0 \Leftrightarrow r(R_{M_2}R_{A_3}C) = 0 \Leftrightarrow r\begin{pmatrix} C & A_4 & A_3 \end{pmatrix} = r\begin{pmatrix} A_4 & A_3 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} C_3 - C_2 + C_1B_3 & A_1 - E_1 & A_1 \\ C_1B_1 & 0 & E_1 \end{pmatrix} = \begin{pmatrix} A_1 - E_1 & A_1 \\ 0 & E_1 \end{pmatrix} \Leftrightarrow (3.11),$$

$$CL_{B_1}L_{N_1} = 0 \Leftrightarrow r(CL_{B_1}L_{N_1}) = 0 \Leftrightarrow r\begin{pmatrix} C_3 - C_2 + C_1B_3 \\ B_2 \\ B_1 \end{pmatrix} = \begin{pmatrix} B_2 \\ B_1 \end{pmatrix} \Leftrightarrow (3.12),$$

$$R_{A_3}CL_{B_2} = 0 \Leftrightarrow r(R_{A_3}CL_{B_2}) = 0 \Leftrightarrow r\begin{pmatrix} C_3 - C_2 + C_1B_3 & A_1 \\ B_2 & 0 \\ C_1B_1 & E_1 \end{pmatrix} = \begin{pmatrix} A_1 \\ E_1 \end{pmatrix} + r(B_2) \Leftrightarrow (3.13),$$

$$R_{A_4}CL_{B_1} = 0 \Leftrightarrow r\begin{pmatrix} C_3 - C_2 + C_1B_3 & A_1 - E_1 \\ B_1 & 0 \end{pmatrix} = \begin{pmatrix} A_1 - E_1 \end{pmatrix} + r(B_1) \Leftrightarrow (3.14),$$

$$R_{M_{11}}R_{A_{11}}E_{11} = 0 \Leftrightarrow r\begin{pmatrix} R_{A_{11}}E_{11} & R_{A_{11}}C_{11} \end{pmatrix} = r(R_{A_{11}}C_{11}) \Leftrightarrow r\begin{pmatrix} E_{11} & C_{11} & A_{11} \end{pmatrix} = r\begin{pmatrix} C_{11} & A_{11} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} R_{A_5}DL_B & R_{A_5}A_7 & R_{A_5}A_6 \end{pmatrix} = r\begin{pmatrix} R_{A_5}A_7 & R_{A_5}A_6 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} D & A_7 & A_6 & A_5 \\ B & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_7 & A_6 & A_5 \end{pmatrix} + r(B)$$

$$\Leftrightarrow r\begin{pmatrix} X_2^1 - X_2^0 & E_1^{\dagger}S_1 & I & -I & 0 \\ I & 0 & 0 & 0 & B_2 \\ 0 & 0 & M_2 & 0 & 0 \\ 0 & 0 & 0 & E_1 & 0 \end{pmatrix} = r\begin{pmatrix} E_1^{\dagger}S_1 & I & -I \\ 0 & M_2 & 0 \\ 0 & 0 & E_1 \end{pmatrix} + r\begin{pmatrix} I & B_2 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} X_2^1 - X_2^0 & 0 & I & -I & 0 \\ I & 0 & 0 & 0 & B_2 \\ 0 & 0 & M_2 & 0 & 0 \\ 0 & A_2 & 0 & E_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & I & -I \\ 0 & M_2 & 0 \\ A_2 & 0 & E_1 \end{pmatrix} + r\begin{pmatrix} I & B_2 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} 0 & 0 & I & -I & 0 & 0 \\ I & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & A_4 & 0 & C_2 - C_1B_3 - C_3 & A_1 \\ 0 & A_2 & 0 & E_1 & -C_2 + C_1B_3 & 0 \\ 0 & 0 & 0 & 0 & -C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} 0 & I & -I & 0 \\ 0 & A_4 & 0 & A_1 \\ A_2 & 0 & E_1 & 0 \\ 0 & 0 & 0 & E_1 \end{pmatrix} + r\begin{pmatrix} I & B_2 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} 0 & A_1 - E_1 & C_2 - C_1B_3 - C_3 & A_1 \\ A_2 & E_1 & -C_2 + C_1B_3 & 0 \\ 0 & 0 & -C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 & A_1 \\ A_2 & E_1 & 0 \\ 0 & 0 & E_1 \end{pmatrix} \Leftrightarrow (3.15).$$

Similarly, we have

$$E_{11}L_{B_{11}}L_{N_{11}} = 0 \Leftrightarrow (3.16),\ R_{A_{11}}E_{11}L_{D_{11}} = 0 \Leftrightarrow (3.17),\ R_{C_{11}}E_{11}L_{B_{11}} = 0 \Leftrightarrow (3.18).$$

In conclusion, System (1.1) is solvable precisely when (ii) or (iii) is true. □

Next, we consider a special case of System (1.1),

$$M_1 = R_{E_1}A_2,\ S_1 = A_2L_{M_1},\ A_3 = A_1L_{E_1},\ A_4 = A_1 - E_1,\ R_{A_3}A_4 = M_2,\ A_4L_{M_2} = S_2,$$
$$C = C_3 - C_2 - A_1E_1^{\dagger}C_1B_1,\ N_1 = B_2L_{B_1},\ A_5 = \begin{pmatrix} L_{M_2}L_{S_2} & -L_{E_1}\end{pmatrix},\ B = \begin{pmatrix} R_{B_2} \\ -R_{B_2}\end{pmatrix},$$
$$A_6 = L_{M_2},\ B_6 = R_{N_1},\ A_7 = E_1^{\dagger}S_1,\ B_5 = B_2B_2^{\dagger},\ A_{11} = R_{A_5}A_6,\ B_{11} = B_6L_B,\ C_{11} = R_{A_5}A_7,$$
$$D_{11} = B_5L_B,\ E_{11} = R_{A_5}DL_B,\ M_{11} = R_{A_{11}}C_{11},\ N_{11} = D_{11}L_{B_{11}},\ S_{11} = C_{11}L_{M_{11}},$$
$$D = E_1^{\dagger}C_2B_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}C_2B_2^{\dagger} - M_2^{\dagger}CB_2^{\dagger} - S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}.$$

Theorem 3.2. *We present three equivalent statements as follows:*
(i) *System* (1.2) *admits a solution.*
(ii)

$$R_{E_1}C_1 = 0,\ R_{M_1}R_{E_1}C_2 = 0,\ C_2L_{B_2} = 0,$$
$$R_{M_2}R_{A_3}C = 0,\ CL_{B_1}L_{N_1} = 0,\ R_{A_3}CL_{B_2} = 0,\ R_{A_4}CL_{B_1} = 0,$$
$$R_{M_{11}}R_{A_{11}}E_{11} = 0,\ E_{11}L_{B_{11}}L_{N_{11}} = 0,\ R_{A_{11}}E_{11}L_{D_{11}} = 0,\ R_{C_{11}}E_{11}L_{B_{11}} = 0.$$

(iii)

$$r\begin{pmatrix} E_1 & C_1\end{pmatrix} = r(E_1),\ r\begin{pmatrix} C_2 & A_2 & E_1\end{pmatrix} = r\begin{pmatrix} A_2 & E_1\end{pmatrix},\ r\begin{pmatrix} C_2 \\ B_2\end{pmatrix} = r(B_2),$$
$$r\begin{pmatrix} C_3 - C_2 & A_1 - E_1 & A_1 \\ C_1B_1 & 0 & E_1\end{pmatrix} = r\begin{pmatrix} A_1 - E_1 & A_1 \\ 0 & E_1\end{pmatrix},$$
$$r\begin{pmatrix} C_3 - C_2 \\ B_1 \\ B_2\end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2\end{pmatrix},$$
$$r\begin{pmatrix} C_3 - C_2 & A_1 \\ B_2 & 0 \\ C_1B_1 & E_1\end{pmatrix} = r\begin{pmatrix} B_2\end{pmatrix} + r\begin{pmatrix} A_1 \\ E_1\end{pmatrix},$$
$$r\begin{pmatrix} C_3 - C_2 & A_1 - E_1 \\ B_1 & 0\end{pmatrix} = r\begin{pmatrix} B_1\end{pmatrix} + r\begin{pmatrix} A_1 - E_1\end{pmatrix},$$
$$r\begin{pmatrix} 0 & A_1 - E_1 & C_3 - C_2 & A_1 \\ A_2 & E_1 & C_2 & 0 \\ 0 & 0 & C_1B_1 & E_1\end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 & A_1 \\ A_2 & E_1 & 0 \\ 0 & 0 & E_1\end{pmatrix},$$
$$r\begin{pmatrix} 0 & 0 & B_2 \\ 0 & B_2 & 0 \\ E_1 - A_1 & C_3 - C_2 & 0 \\ E_1 & 0 & C_2 \\ 0 & B_1 & 0\end{pmatrix} = r\begin{pmatrix} 0 & B_2 \\ B_2 & 0 \\ B_1 & 0\end{pmatrix} + r\begin{pmatrix} A_1 - E_1 \\ E_1\end{pmatrix},$$
$$r\begin{pmatrix} 0 & B_2 & 0 \\ E_1 & C_3 - C_2 & A_1 \\ -E_1 & C_2 & 0 \\ E_1 & C_1B_1 & E_1\end{pmatrix} = r\begin{pmatrix} 0 & A_1 \\ E_1 & 0 \\ 0 & E_1\end{pmatrix} + r\begin{pmatrix} B_2\end{pmatrix},$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_2 \\ 0 & A_1-E_1 & C_3-C_2 & 0 \\ A_2 & E_1 & 0 & -C_2 \\ 0 & 0 & B_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1-E_1 \\ A_2 & E_1 \end{pmatrix} + r\begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}.$$

When (i) *holds, the solution of System* (1.2) *is*

$$\begin{aligned} &X_1 = E_1^{\dagger}C_1 + L_{E_1}U_1, \\ &X_2 = E_1^{\dagger}C_2B_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}C_2B_2^{\dagger} - E_1^{\dagger}S_1T_1B_2B_2^{\dagger} + L_{E_1}T_2 + T_3R_{B_2}, \\ &or \\ &X_2 = M_2^{\dagger}CB_2^{\dagger} + S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger} + L_{M_2}L_{S_2}W_4 + L_{M_2}W_1R_{N_1} + W_5R_{B_2}, \\ &X_3 = M_1^{\dagger}C_2B_2^{\dagger} + L_{M_1}L_{S_1}T_4 + L_{M_1}T_1 + T_5R_{B_2}, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} &T_1 = M_{11}^{\dagger}E_{11}D_{11}^{\dagger} + S_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger} + L_{M_{11}}L_{S_{11}}W_{12} + L_{M_{11}}W_9R_{N_{11}} + W_{13}R_{D_{11}}, \\ &T_2 = (0,\ I_m)\left[A_5^{\dagger}(D - A_6W_1B_6 - A_7T_1B_5) - A_5^{\dagger}W_6B + L_{A_5}W_7\right], \\ &T_3 = \left[R_{A_5}(D - A_6W_1B_6 - A_7T_1B_5)B^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_B\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix}, \\ &U_1 = A_3^{\dagger}CB_1^{\dagger} - A_3^{\dagger}A_4M_2^{\dagger}CB_1^{\dagger} - A_3^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}B_2B_1^{\dagger} - A_3^{\dagger}S_2W_1R_{N_1}B_2B_1^{\dagger} + L_{A_3}W_2 + W_3R_{B_1}, \\ &W_1 = A_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}C_{11}M_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger}D_{11}B_{11}^{\dagger} \\ &- A_{11}^{\dagger}S_{11}W_9R_{N_{11}}D_{11}B_{11}^{\dagger} + L_{A_{11}}W_{10} + W_{11}R_{B_{11}}, \\ &W_4 = (I_m,\ 0)\left[A_5^{\dagger}(D - A_6W_1B_6 - A_7T_1B_5) - A_5^{\dagger}W_6B + L_{A_5}W_7\right], \\ &W_5 = \left[R_{A_5}(D - A_6W_1B_6 - A_7T_1B_5)B^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_B\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \end{aligned}$$

$W_2,\ W_3,\ W_j(j = \overline{6, 13})$*,* T_4*, and* T_5 *are any quaternion matrices of suitable shapes, with* m *and* n *being the column count of* A_1 *and the row count of* B_2*, respectively.*

Proof. Let B_3, B_4, and C_4 in Theorem 3.1 be vanished, and we can prove it. □

4. On the reducible solution to classical System (1.3)

In this section, we employ the conclusion of Theorem 3.2 to investigate the reducible solution and the expressions for System (1.3).

Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{m\times n}$, $D \in \mathbb{H}^{n\times(n-k+q)}$, $E = C_3 \in \mathbb{H}^{m\times q}$, $F = [C_1,\ C_2] \in \mathbb{H}^{m\times(n-k+q)}$, $C_1 \in \mathbb{H}^{m\times(n-k)}$, $C_2 \in \mathbb{H}^{m\times q}$ be given matrices, $K \in \mathbb{H}^{n\times n}$ be a permutation matrix, $1 \le k < n$. Put

$$\begin{cases} AK = [A_1,\ A_2],\ K^{-1}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ CK = [E_1,\ A_2], \\ K^{-1}D = \begin{pmatrix} I & 0 \\ 0 & B_2 \end{pmatrix},\ A_1 \in \mathbb{H}^{m\times(n-k)},\ A_2 \in \mathbb{H}^{m\times k}, \\ B_1 \in \mathbb{H}^{(n-k)\times m},\ E_1 \in \mathbb{H}^{m\times(n-k)},\ B_2 \in \mathbb{H}^{k\times m},\ I \in \mathbb{H}^{(n-k)\times(n-k)}, \end{cases} \tag{4.1}$$

$$R_{E_1}A_2 = M_1,\ A_2L_{M_1} = S_1,\ A_3 = A_1L_{E_1},\ A_4 = A_1 - E_1,\ C = C_3 - C_2 - A_1E_1^{\dagger}C_1B_1,$$
$$R_{A_3}A_4 = M_2,\ A_4L_{M_2} = S_2,\ N_1 = B_2L_{B_1},\ A_5 = \begin{pmatrix} L_{M_2}L_{S_2} & -L_{E_1} \end{pmatrix},\ B_3 = \begin{pmatrix} R_{B_2} \\ -R_{B_2} \end{pmatrix}, \tag{4.2}$$
$$A_6 = L_{M_2},\ B_4 = R_{N_1},\ A_7 = E_1^{\dagger}S_1,\ B_5 = B_2B_2^{\dagger},\ A_{11} = R_{A_5}A_6,\ B_{11} = B_4L_{B_3},$$
$$C_{11} = R_{A_5}A_7,\ D_{11} = B_5L_{B_3},\ E_{11} = R_{A_5}DL_{B_3},\ M_{11} = R_{A_{11}}C_{11},\ N_{11} = D_{11}L_{B_{11}},$$
$$S_{11} = C_{11}L_{M_{11}},\ D = E_1^{\dagger}C_2B_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}C_2B_2^{\dagger} - M_2^{\dagger}CB_2^{\dagger} - S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}.$$

Theorem 4.1. *Consider the following equivalent propositions:*

(i) *System* (1.3) *has a solution that is reducible.*

(ii)

$$R_{E_1}C_1 = 0,\ R_{M_1}R_{E_1}C_2 = 0,\ C_2L_{B_2} = 0,$$
$$R_{M_2}R_{A_3}C = 0,\ CL_{B_1}L_{N_1} = 0,\ R_{A_3}CL_{B_2} = 0,\ R_{A_4}CL_{B_1} = 0,$$
$$R_{M_{11}}R_{A_{11}}E_{11} = 0,\ E_{11}L_{B_{11}}L_{N_{11}} = 0,\ R_{A_{11}}E_{11}L_{D_{11}} = 0,\ R_{C_{11}}E_{11}L_{B_{11}} = 0.$$

(iii)

$$r\begin{pmatrix} E_1 & C_1 \end{pmatrix} = r(E_1),\ r\begin{pmatrix} C_2 & A_2 & E_1 \end{pmatrix} = r\begin{pmatrix} A_2 & E_1 \end{pmatrix},\ r\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} = r(B_2),$$

$$r\begin{pmatrix} C_3 - C_2 & A_1 - E_1 & A_1 \\ C_1B_1 & 0 & E_1 \end{pmatrix} = r\begin{pmatrix} A_1 - E_1 & A_1 \\ 0 & E_1 \end{pmatrix},$$

$$r\begin{pmatrix} C_3 - C_2 \\ B_1 \\ B_2 \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

$$r\begin{pmatrix} C_3 - C_2 & A_1 \\ B_2 & 0 \\ C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} B_2 \end{pmatrix} + r\begin{pmatrix} A_1 \\ E_1 \end{pmatrix},$$

$$r\begin{pmatrix} C_3 - C_2 & A_1 - E_1 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_1 \end{pmatrix} + r\begin{pmatrix} A_1 - E_1 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & A_1 - E_1 & C_3 - C_2 & A_1 \\ A_2 & E_1 & C_2 & 0 \\ 0 & 0 & C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 & A_1 \\ A_2 & E_1 & 0 \\ 0 & 0 & E_1 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & 0 & B_2 \\ 0 & B_2 & 0 \\ E_1 - A_1 & C_3 - C_2 & 0 \\ E_1 & 0 & C_2 \\ 0 & B_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & B_2 \\ B_2 & 0 \\ B_1 & 0 \end{pmatrix} + r\begin{pmatrix} A_1 - E_1 \\ E_1 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & B_2 & 0 \\ E_1 & C_3 - C_2 & A_1 \\ -E_1 & C_2 & 0 \\ E_1 & C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 \\ E_1 & 0 \\ 0 & E_1 \end{pmatrix} + r\begin{pmatrix} B_2 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_2 \\ 0 & A_1-E_1 & C_3-C_2 & 0 \\ A_2 & E_1 & 0 & -C_2 \\ 0 & 0 & B_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1-E_1 \\ A_2 & E_1 \end{pmatrix} + r\begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}.$$

When (i) *holds, the reducible solution of System* (1.3) *with respect to K is*

$$X = K\begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}K^{-1}, \tag{4.3}$$

$$\begin{aligned} X_1 &= E_1^{\dagger}C_1 + L_{E_1}U_1, \\ X_2 &= E_1^{\dagger}C_2B_2^{\dagger} - E_1^{\dagger}A_2M_1^{\dagger}C_2B_2^{\dagger} - E_1^{\dagger}S_1T_1B_2B_2^{\dagger} + L_{E_1}T_2 + T_3R_{B_2}, \\ &\textit{or} \\ X_2 &= M_2^{\dagger}CB_2^{\dagger} + S_2^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger} + L_{M_2}L_{S_2}W_4 + L_{M_2}W_1R_{N_1} + W_5R_{B_2}, \\ X_3 &= M_1^{\dagger}C_2B_2^{\dagger} + L_{M_1}L_{S_1}T_4 + L_{M_1}T_1 + T_5R_{B_2}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} T_1 &= M_{11}^{\dagger}E_{11}D_{11}^{\dagger} + S_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger} + L_{M_{11}}L_{S_{11}}W_{12} + L_{M_{11}}W_9R_{N_{11}} + W_{13}R_{D_{11}}, \\ T_2 &= (0,\ I_m)\left[A_5^{\dagger}(D - A_6W_1B_4 - A_7T_1B_5) - A_5^{\dagger}W_6B_3 + L_{A_5}W_7\right], \\ T_3 &= \left[R_{A_5}(D - A_6W_1B_4 - A_7T_1B_5)B_3^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_{B_3}\right]\begin{pmatrix} 0 \\ I_n \end{pmatrix}, \\ U_1 &= A_3^{\dagger}CB_1^{\dagger} - A_3^{\dagger}A_4M_2^{\dagger}CB_1^{\dagger} - A_3^{\dagger}S_2A_4^{\dagger}CN_1^{\dagger}B_2B_1^{\dagger} - A_3^{\dagger}S_2W_1R_{N_1}B_2B_1^{\dagger} + L_{A_3}W_2 + W_3R_{B_1}, \\ W_1 &= A_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}C_{11}M_{11}^{\dagger}E_{11}B_{11}^{\dagger} - A_{11}^{\dagger}S_{11}C_{11}^{\dagger}E_{11}N_{11}^{\dagger}D_{11}B_{11}^{\dagger} - A_{11}^{\dagger}S_{11}W_9R_{N_{11}}D_{11}B_{11}^{\dagger} \\ &\quad + L_{A_{11}}W_{10} + W_{11}R_{B_{11}}, \\ W_4 &= (I_m,\ 0)\left[A_5^{\dagger}(D - A_6W_1B_4 - A_7T_1B_5) - A_5^{\dagger}W_6B_3 + L_{A_5}W_7\right], \\ W_5 &= \left[R_{A_5}(D - A_6W_1B_4 - A_7T_1B_5)B_3^{\dagger} + A_5A_5^{\dagger}W_6 + W_8R_{B_3}\right]\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \end{aligned}$$

$W_2,\ W_3,\ W_j(j = \overline{6, 13})$*,* T_4*, and* T_5 *are any matrices of suitable shapes over* $\mathbb{H}$*, where m and n denote the numbers of columns and rows of* A_1 *and* B_2*, respectively.*

Proof. (i) ⇔ (ii): If the solution of quaternion System (1.3) is X, then X has the form (4.3). It follows from (4.1) that

$$\begin{aligned} [A_1,\ A_2]\begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} &= C_3, \\ [E_1,\ A_2]\begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B_2 \end{pmatrix} &= [C_1,\ C_2], \end{aligned}$$

i.e.,

$$\begin{cases} E_1X_1 = C_1,\ E_1X_2B_2 + A_2X_3B_2 = C_2, \\ A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_3. \end{cases}$$

Thus, (X_1, X_2, X_3) forms a solution of Eq (1.2). According to Theorem 3.2, we have that (i) and (ii) hold; moreover, X_1, X_2, and X_3 can be expressed as (4.4). It is easy to see that if we substitute (4.4) into (4.3), X can be expressed in the form of (4.3). If (ii) holds, then system (1.3), by Theorem 3.2, has a solution X_1, X_2, and X_3 expressed as (3.30). It follows from (4.1) that X, with the form (4.3), is a K-reducible solution to Eq (1.2).

(ii) ⇔ (iii): This is true according to Theorem 3.2. □

Finally, we provide an example to demonstrate our main findings.

Example 4.2. *Let*

$$A_1 = \begin{pmatrix} 1 & \mathbf{i} \end{pmatrix},\ A_2 = \begin{pmatrix} \mathbf{j} & \mathbf{k} \end{pmatrix},\ E_1 = \begin{pmatrix} \mathbf{i} & \mathbf{j} \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0 & \mathbf{i} \\ 1 & \mathbf{j} \end{pmatrix},\ B_2 = \begin{pmatrix} \mathbf{i} & \mathbf{k} \\ 1 & 0 \end{pmatrix},\ B_3 = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{i} & 0 \end{pmatrix},\ B_4 = \begin{pmatrix} 1 & \mathbf{k} \\ \mathbf{j} & 0 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} 2\mathbf{i} - \mathbf{k} & 3\mathbf{j} \end{pmatrix},\ C_2 = \begin{pmatrix} -2\mathbf{k} + \mathbf{j} - 3 & 2 - 3\mathbf{j} \end{pmatrix},$$
$$C_3 = \begin{pmatrix} \mathbf{k} - \mathbf{j} + 7\mathbf{i} - 1 & \mathbf{i} + 4\mathbf{k} + 1 \end{pmatrix},\ C_4 = \begin{pmatrix} \mathbf{i} - 1 & -\mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix}.$$

Computation directly yields

$$r\begin{pmatrix} E_1 & C_1 \end{pmatrix} = r(E_1) = 1,\ r\begin{pmatrix} C_4 \\ B_4 \end{pmatrix} = r(B_4) = 2,$$
$$r\begin{pmatrix} G & A_2 & E_1 \end{pmatrix} = r\begin{pmatrix} A_2 & E_1 \end{pmatrix} = 1,\ r\begin{pmatrix} G \\ B_2 \end{pmatrix} = r(B_2) = 2,$$
$$r\begin{pmatrix} G & E_1 & A_2C_4 \\ B_2 & 0 & B_4 \end{pmatrix} = r(E_1) + r\begin{pmatrix} B_2 & B_4 \end{pmatrix} = 3,$$
$$r\begin{pmatrix} G & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2) = 3,$$
$$r\begin{pmatrix} A & A_1 - E_1 & A_1 \\ C_1B_1 & 0 & E_1 \end{pmatrix} = r\begin{pmatrix} A_1 - E_1 & A_1 \\ 0 & E_1 \end{pmatrix} = 2,$$
$$r\begin{pmatrix} A \\ B_2 \\ B_1 \end{pmatrix} = r\begin{pmatrix} B_2 \\ B_1 \end{pmatrix} = 2,$$
$$r\begin{pmatrix} A & A_1 \\ B_2 & 0 \\ C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} A_1 \\ E_1 \end{pmatrix} + r\begin{pmatrix} B_2 \end{pmatrix} = 4,$$
$$r\begin{pmatrix} A & A_1 - E_1 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 - E_1 \end{pmatrix} + r\begin{pmatrix} B_1 \end{pmatrix} = 3,$$
$$r\begin{pmatrix} 0 & A_1 - E_1 & A & A_1 \\ A_2 & E_1 & G & 0 \\ 0 & 0 & C_1B_1 & E_1 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 & A_1 \\ A_2 & E_1 & 0 \\ 0 & 0 & E_1 \end{pmatrix} = 3,$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_4 \\ 0 & B_2 & B_2 & 0 \\ E_1 - A_1 & A & 0 & 0 \\ E_1 & 0 & G & A_2C_4 \\ 0 & B_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & B_2 & B_4 \\ B_2 & B_2 & 0 \\ B_1 & 0 & 0 \end{pmatrix} + r\begin{pmatrix} A_1 - E_1 \\ E_1 \end{pmatrix} = 8,$$

$$r\begin{pmatrix} 0 & B_2 & 0 & B_4 \\ E_1 & A & A_1 & 0 \\ -E_1 & G & 0 & A_2C_4 \\ E_1 & C_1B_1 & E_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 \\ E_1 & 0 \\ 0 & E_1 \end{pmatrix} + r\begin{pmatrix} B_2 & B_4 \end{pmatrix} = 5,$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_2 \\ 0 & A_1 - E_1 & A & 0 \\ A_2 & E_1 & 0 & -G \\ 0 & 0 & B_1 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A_1 - E_1 \\ A_2 & E_1 \end{pmatrix} + r\begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix} = 6,$$

where $A = C_3 - C_2 + C_1B_3$, $G = C_2 - C_1B_3$. *Given that all rank equalities from* (3.7) *to* (3.18) *are satisfied, it follows from Theorem 3.1 that System* (1.1) *admits a solution, which is*

$$X_1 = \begin{pmatrix} 2 & 0 \\ \mathbf{i} & 3 \end{pmatrix}, \; X_2 = \begin{pmatrix} 1 & \mathbf{i} \\ 0 & 2 \end{pmatrix}, \; X_3 = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ 0 & 1 \end{pmatrix}.$$

5. An application of Theorem 3.1

As is well known, color images can be represented by quaternion matrices. In this section, building on System (1.1), we present a color image encryption and decryption scheme, illustrating its operation with a specific example. The encryption process is illustrated in Figure 1.

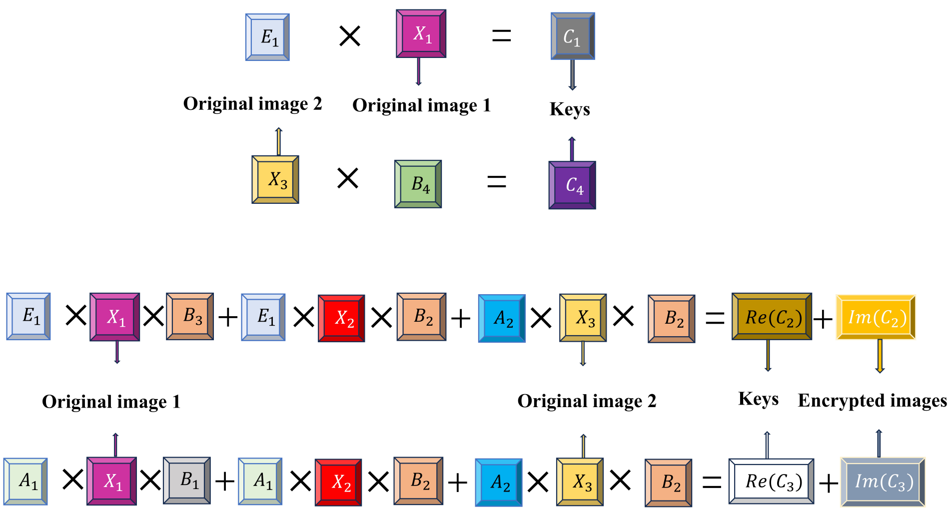

Figure 1. Encryption model.

The encryption process uses the two original images, as well as the encryption matrices $E_1, A_1, A_2, B_1, B_2, B_3$, and B_4. These matrices are processed using System (1.1) to generate the encrypted outputs $\mathrm{Im}(C_2)$ and $\mathrm{Im}(C_3)$. Although the encryption matrices are public, the keys are held solely by the intended recipient. Successful decryption requires both these public matrices and the private keys, as provided by Theorem 3.1. Algorithms 5.1 and 5.2 outline the encryption and decryption procedures, respectively. The results are illustrated in Figure 2.

Algorithm 5.1. The procedure for color image encryption.

1. **Input:** Two original color images along with the encryption matrices.
2. E_1, A_i ($i = \overline{1,2}$), and B_j ($j = \overline{1,4}$) are encryption matrices. We denote the two original color images by X_1 and X_3.
3. The encryption of X_1 and X_3 is performed via System (1.1).
4. The keys are represented by C_1, C_4, $\mathrm{Re}(C_2)$, and $\mathrm{Re}(C_3)$, while the two encrypted color images correspond to $\mathrm{Im}(C_2)$ and $\mathrm{Im}(C_3)$. Thus, we obtain the two color images that have been encrypted.
5. **Output:** The keys and the two color images that have been encrypted.

Algorithm 5.2. The procedure for color image decryption.

1. **Input:** Encryption matrices and keys, along with the two color images that have been encrypted.
2. The keys are denoted by C_1, C_4, $\mathrm{Re}(C_2)$, and $\mathrm{Re}(C_3)$; the encrypted color images are represented by $\mathrm{Im}(C_2)$ and $\mathrm{Im}(C_3)$. The encryption matrices are E_1, A_i ($i = \overline{1,2}$), and B_j ($j = \overline{1,4}$).
3. The Y_1 and Y_3 are obtained using Theorem 3.1.
4. The symbols Y_1 and Y_3 represent the two decrypted color images.
5. **Output:** Color images after decryption.

Figure 2. The encryption and decryption effects of Image 1 and Image 2.

The practical effectiveness of Figure 2 and the decryption scheme was rigorously evaluated using a set of image quality metrics, including Peak Signal-to-Noise Ratio (PSNR), Structural Similarity Index Measure (SSIM), and Feature Similarity Index Measure (FSIM). Table 1 presents the relevant data, which shows that the PSNR values for the two images are 274.4236 and 278.6477, respectively, and the SSIM and FSIM values are 1.0000 and 1, respectively. With regard to PSNR, the values for the two images significantly exceed 50. This demonstrates that the encryption and decryption scheme is highly practical and reliable.

Table 1. Data.

Original image	PSNR	SSIM	FSIM
Image 1	274.4236	1.0000	1
Image 2	278.6477	1.0000	1

6. Conclusions

Through the method of matrix rank equalities and M-P inverses, we investigated the compatibility conditions and derived the general solutions for the quaternion matrix equations (1.1) and (1.2) under solvability. As a theoretical application of the quaternion System (1.2), equivalence conditions for System (1.3) to admit a reducible solution are established. Ultimately, the developed color image encryption and decryption scheme confirms the practical utility of System (1.1).

Author contributions

The work, from conceptualization to manuscript finalization, saw the full involvement of Long-Sheng Liu, Xiao-Xiao Ma, and Xiao-Quan Chen in all stages. This included formal analysis, methodology design, software development, validation, and writing (drafting, reviewing, and editing). All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Key scientific research projects of universities in Anhui province (No. 2023AH050476).

Conflict of interest

The authors state that there are no competing interests.

References

1. Z. H. He, X. X. Wang, Y. F. Zhao, Eigenvalues of quaternion tensors with applications to color video processing, *J. Sci. Comput.*, **94** (2023), 1. https://doi.org/10.1007/s10915-022-02058-5
2. Z. G. Jia, S. T. Ling, M. X. Zhao, Color two-dimensional principal component analysis for face recognition based on quaternion model, In: D. S. Huang, V. Bevilacqua, P. Premaratne, P. Gupta, *Intelligent computing theories and application. ICIC 2017*, Lecture Notes in Computer Science, Springer, Cham, **10361** (2017), 177–189. https://doi.org/10.1007/978-3-319-63309-1_17
3. L. Q. Qi, Z. Y. Luo, Q. W. Wang, X. Z. Zhang, Quaternion matrix optimization: motivation and analysis, *J. Optim. Theory Appl.*, **193** (2022), 621–648. https://doi.org/10.1007/s10957-021-01906-y
4. J. Z. Lei, C. Y. Wang, On the reducibility of compartmental matrices, *Comput. Biol. Med.*, **38** (2008), 881–885. https://doi.org/10.1016/j.compbiomed.2008.05.004
5. X. R. Nie, Q. W. Wang, Y. Zhang, A system of matrix equations over the quaternion algebra with applications, *Algebra Colloq.*, **24** (2017), 233–253. https://doi.org/10.1142/S100538671700013X
6. M. Y. Xie, Q. W. Wang, Reducible solution to a quaternion tensor equation, *Front. Math. China*, **15** (2020), 1047–1070. https://doi.org/10.1007/s11464-020-0865-6
7. H. H. Wang, *A system of quaternion tensor equations with its applications*, MS. Thesis, University of Shanghai, 2019.
8. E. B. Castelan, V. G. da Silva, On the solution of a Sylvester equation appearing in descriptor systems control theory, *Syst. Control Lett.*, **54** (2005), 109–117. https://doi.org/10.1016/j.sysconle.2004.07.002
9. Y. N. Zhang, D. C. Jiang, J. Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients, *IEEE Trans. Neural Net.*, **13** (2002), 1053–1063. https://doi.org/10.1109/TNN.2002.1031938
10. V. L. Syrmos, F. L. Lewis, Output feedback eigenstructure assignment using two Sylvester equations, *IEEE Trans. Autom. Control*, **38** (2002), 495–499. https://doi.org/10.1109/9.210155
11. A. Dmytryshyn, B. K*å*gstr*ö*m, Coupled Sylvester-type matrix equations and block diagonalization, *SIAM J. Math. Anal.*, **36** (2015), 580–593. https://doi.org/10.1137/151005907
12. Z. H. He, W. L. Qin, J. Tian, X. X. Wang, Y. Zhang, A new Sylvester-type quaternion matrix equation model for color image data transmission, *Comput. Appl. Math.*, **43** (2024), 227. https://doi.org/10.1007/s40314-024-02732-4
13. L. S. Liu, The general solution to a System of linear coupled quaternion matrix equations with an application, *Adv. Appl. Clifford Algebras*, **33** (2023), 48. https://doi.org/10.1007/s00006-023-01283-x
14. M. S. Mehany, An analysis to some systems of matrix equations with ϕ-Hermitian solutions for some nonstandard involution ϕ over the real quaternion algebra, *Iran. J. Sci.*, **49** (2025), 795–809. https://doi.org/10.1007/s40995-024-01758-x
15. Q. W. Wang, Z. H. Gao, J. L. Gao, A comprehensive review on solving the system of equations $AX = C$ and $XB = D$, *Symmetry*, **17** (2025), 625. https://doi.org/10.3390/sym17040625

16. X. R. Nie, *Researches on general solutions to some generalized Sylvester matrix equations over the quaternion algebra with applications*, Ph.D. Thesis, University of Shanghai, 2017.

17. G. Matsaglia, G. P. H. Styan, Equalities and inequalities for ranks of matrices, *Linear Multilinear A.*, **2** (1974), 269–292. https://doi.org/10.1080/03081087408817070

18. S. Barnett, *Matrices: methods and applications*, Oxford: Oxford University Press, 1990. https://doi.org/10.1093/oso/9780198596653.001.0001

AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)