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Research article

Local and global properties of positive solutions for $\Delta u + f(u) = 0$ on Riemannian manifolds

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Abstract: We establish a local gradient estimate for positive solutions of the nonlinear elliptic equation $\Delta u + f(u) = 0$ on complete Riemannian manifolds with Ricci curvature bounded below. The proof relies on a new P-function and the Moser iteration technique. As its applications, we prove new Liouville theorems that extend several known results.

Keywords: gradient estimate; Liouville theorem; Moser iteration
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1. Introduction and main results

In this paper, we study the following nonlinear elliptic equation

$$\Delta u + f(u) = 0, \quad \text{in } M, \tag{1.1}$$

where M is an n-dimensional complete Riemannian manifold with $n > 2$ and Ricci curvature bounded below, and $f \in C^1(0, +\infty)$.

A particularly important special case arises when $f(u) = u^\sigma$, in which case Eq (1.1) reduces to the well-known Lane-Emden equation. In their seminal work, Gidas and Spruck [13, Theorem 1.1] showed that the Lane-Emden equation

$$\Delta u + u^\sigma = 0, \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

admits no positive solutions provided $n > 2$ and $1 < \sigma < \frac{n+2}{n-2}$. It also holds in complete Riemannian manifolds with nonnegative Ricci curvature. Recently, Lu [22] reproved the optimal Liouville theorem for Eq (1.2) on Riemannian manifolds with nonnegative Ricci curvature by establishing a priori estimates. The Lane-Emden type equation arises in fluid mechanics and conformal differential geometry and has been extensively studied. We refer the reader to [7–9] and the references therein for further details.

We also note that for the nonlinear heat equation

$$u_t - \Delta u = u^{\sigma}, \quad \text{in } \mathbb{R}^n, \quad \sigma > 1.$$

Quittner [28] addressed the conjecture regarding the nonexistence of positive classical solutions in the subcritical range $\sigma < \frac{n+2}{n-2}$.

The result of Gidas and Spruck has been extended to quasilinear elliptic equations of the form

$$\Delta_p u + f(u) = 0, \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

When $f(u) = u^{\sigma}$, Serrin and Zou [29] proved that the Lane-Emden-Fowler equation (1.3) admits no positive solutions under the condition

$$0 < \sigma < \frac{(n+1)p - n}{n - p} \text{ for } 1 < p < n.$$

More recently, He et al. [16] extended this Liouville-type theorem to the range $-\infty < \sigma < \frac{(n+1)p-n}{n-p}$ via the Moser iteration technique.

In 2007, McCoy [25] established that if

$$f'(u) \leq \frac{n+1}{n-1} \cdot \frac{f(u)}{u}, \quad \forall\, u > 0,$$

then any positive solution of Eq (1.1) must be constant. Later, Cuccu et al. [10] showed that if

$$f'(u) \leq (p-1)\frac{n+1}{n-1} \cdot \frac{f(u)}{u}, \quad \forall\, u > 0,$$

then any positive solution of (1.3) is constant. In 2015, Enache [12] further generalized this result: If

$$f'(u) \leq \beta(p-1)\frac{n+1}{n-1} \cdot \frac{f(u)}{u}, \quad \forall\, u > 0,$$

with $1 \leq \beta < \frac{n-1}{n-p}$ for $n > p$, then any positive solutions of (1.3) are constant. Equations (1.1) and (1.3) have been widely investigated; see, for example, [1–4,6,11,14,18–20,24,26,27] and references therein.

We now briefly recall the geometric context of Eq (1.1). When $f(u) = u^{\frac{n+2}{n-2}}$, this equation is related to the Yamabe problem, a classical question in differential geometry concerning the existence of a conformal metric with constant scalar curvature on a closed Riemannian manifold. For $n \geq 3$, the Yamabe problem asks whether there exists a conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ such that the scalar curvature of $\tilde{g}$ is constant. This leads to the partial differential equation

$$-\Delta_g u + \frac{n-2}{4(n-1)} R u = \widetilde{R} u^{\frac{n+2}{n-2}},$$

where R and $\widetilde{R}$ are the scalar curvatures of the metrics g and $\widetilde{g}$, respectively (see [15]).

We now recall some key results on Harnack inequalities and Liouville theorems on Riemannian manifolds. The celebrated Cheng-Yau gradient estimate [5] implies that any harmonic function bounded from above or below must be constant on a complete Riemannian manifold with nonnegative Ricci curvature. Using the method of Cheng and Yau, Li [18] first investigated Eq (1.2) and obtained several gradient estimates and Harnack inequalities on Riemannian manifolds with nonnegative Ricci curvature under the assumption $1 \le \sigma \le \frac{n}{n-2}$ for $n \ge 4$. In 2009, Kotschwar and Ni [17] extended Cheng–Yau's result to p-harmonic functions under the assumption of nonnegative sectional curvature. Later, Wang and Zhang [31] introduced the Nash-Moser iteration technique to improve gradient estimates and Harnack inequalities for p-harmonic functions on complete Riemannian manifolds with nonnegative Ricci curvature.

In 2023, Wang and Wei [32] applied the Moser iteration technique to establish that any positive solution of the equation

$$\Delta u + au^{\sigma} = 0 \quad (a > 0)$$

must be constant on Riemannian manifolds with nonnegative Ricci curvature, provided that

$$-\infty < \sigma < \frac{n+1}{n-1} + \frac{2}{\sqrt{n(n-1)}}.$$

Subsequently, He et al. [16] extended this Liouville-type result to the equation

$$\Delta_p u + au^{\sigma} = 0 \quad (a > 0),$$

under the condition

$$p - 1 < \sigma < \frac{(n+3)(p-1)}{n-1}. \tag{1.4}$$

In 2025, Lu [21] used the Bernstein method to show that any positive solution to Eq (1.1) is constant if there exists an exponent $\sigma \in \left(1, \frac{n+3}{n-1}\right)$ such that the function $u^{-\sigma} f(u)$ is nonincreasing on $(0, +\infty)$. More recently, He et al. [15] applied the Moser iteration technique to prove that, under condition (1.4), any positive weak solution of Eq (1.3) is constant on Riemannian manifolds with nonnegative Ricci curvature.

Definition 1.1. A function $f \in C^1(0, +\infty)$ is called subcritical with exponent σ if

$$\sigma f(u) - uf'(u) \ge 0, \quad \forall u > 0. \tag{1.5}$$

In the original definition by Serrin and Zou [29], f is required to be nonnegative and $\sigma \in \left(0, \frac{(n+1)p-n}{n-p}\right)$. In our setting, the nonnegativity of f is not assumed.

Indeed, whether Liouville theorems for Eq (1.1) subject to condition (1.5) on Riemannian manifolds with nonnegative Ricci curvature hold for exponents σ satisfying

$$1 < \sigma < \frac{n+2}{n-2} \quad \text{for} \quad n > 2 \tag{1.6}$$

remains an open problem.

Notably, under slightly stronger assumptions than (1.5), the Liouville theorems for Eq (1.1) on Riemannian manifolds with nonnegative Ricci curvature, which hold for exponents σ satisfying

$$-\infty < \sigma < \frac{n+2}{n-2} \quad \text{for} \quad n > 2, \tag{1.7}$$

have been established in Lu's recent work [23]. He also derived a gradient type estimate without any conditions on solutions, see Theorem 1.5 and Corollary 1.6 in that paper.

It is known from the results of He et al. [15] and Lu [21] that the Liouville-type theorem holds for Eq (1.1) for the subcritical exponent σ in the range

$$1 < \sigma < \frac{n+3}{n-1} \quad \text{for} \quad n > 2. \tag{1.8}$$

Since $\left(1, \frac{n+3}{n-1}\right)$ is a subset of $\left(1, \frac{n+2}{n-2}\right)$, the open problem (1.6) remains unsolved. One of the main goals of this paper is to improve upon the results of He et al. [15] and Lu [21]. Inspired by the work of Wang and Zhang [31] and that of Wang and Wei [32], we employ the Moser iteration technique to investigate gradient estimates and Liouville properties for Eq (1.1) on Riemannian manifolds.

Our method relies on the construction of a new auxiliary function $P = \frac{|\nabla u|^2}{2u^{2\beta}}$, where β provides a degree of freedom to optimize the constant $\lambda_{\alpha,\beta,\sigma}$ in Lemma 2.2, which ultimately extends the admissible range of σ in the final Theorem 1.6. By applying the Moser iteration technique to P, we obtain a local gradient estimate for positive solutions of Eq (1.1). The choice of this auxiliary function P is motivated by Lu's proof in [21], which corresponds to the case $\beta = 1$ in (2.4).

Let $B_R(x_0)$ denote the open geodesic ball of radius R centered at x_0; for simplicity, it is sometimes written as B_R. Let C represent a positive constant independent of R. The main result is the following gradient estimate for the solution of Eq (1.1).

Theorem 1.2. *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric}_g \geq -(n-1)\kappa g$ for some nonnegative constant κ. Assume that u is a C^2 solution to Eq (1.1) on $B_R(x_0)$ satisfying $u \geq c$ for some constant $c \geq 0$ and f is subcritical with exponent $\sigma \in (\sigma_1(\beta), \sigma_2(\beta))$, where*

$$\sigma_1(\beta) :\equiv \frac{\beta(n+1) - 2\sqrt{\beta(n-1-\beta(n-2))}}{n-1}, \tag{1.9}$$

and

$$\sigma_2(\beta) :\equiv \frac{\beta(n+1) + 2\sqrt{\beta(n-1-\beta(n-2))}}{n-1}. \tag{1.10}$$

Then, we have

$$\sup_{B_{R/2}(x_0)} \frac{|\nabla u|^2}{u^{2\beta}} \leq C\frac{(1+\sqrt{\kappa}R)^2}{R^2}, \tag{1.11}$$

where C depends on n, c, and β, with $1 \leq \beta < \frac{n-1}{n-2}$ when $c \neq 0$ and $\beta = 1$ when $c = 0$.

As an application of Theorem 1.2, by setting $c = 0$ (so that $\beta = 1$, $\sigma_1 = 1$, and $\sigma_2 = \frac{n+3}{n-1}$), we obtain the following Harnack inequality.

Corollary 1.3. *Let* (M, g) *be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by* $\mathrm{Ric}_g \geq -(n-1)\kappa g$ *for some nonnegative constant* κ*. Assume u is a* C^2 *positive solution to Eq* (1.1) *on* $B_R(x_0)$ *and f is subcritical with exponent* $\sigma \in \left(1, \frac{n+3}{n-1}\right)$*. Then, we have*

$$\sup_{B_{R/2}(x_0)} u \leq e^{C(1+\sqrt{\kappa R}+\sqrt{\kappa}R)} \cdot \inf_{B_{R/2}(x_0)} u, \tag{1.12}$$

where C depends only on n.

As a further application of Theorem 1.2, by taking $c = 0$ and letting $R \to \infty$, we derive the following Liouville theorem.

Corollary 1.4. *Let* (M, g) *be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Assume u is a* C^2 *positive solution to Eq* (1.1) *and f is subcritical with exponent* $\sigma \in \left(1, \frac{n+3}{n-1}\right)$*. Then, u is constant.*

Remark 1.5. The result in Corollary 1.4 was previously established by He et al. [15, Theorem 1.3 for $p = 2$] and Lu [21, Theorem 1.5].

Through a careful analysis of the condition $1 \leq \beta < \frac{n-1}{n-2}$ for $n > 2$, we establish the following Liouville theorem.

Theorem 1.6. *Let* (M, g) *be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Assume that u is a* C^2 *solution to Eq* (1.1) *satisfying* $u \geq c$ *for some constant* $c \geq 0$ *and f is subcritical with exponent* $\sigma \in \left(1, \frac{n+1+\sqrt{(n-1)(n+7)}}{2(n-2)}\right)$*. Then, u is constant.*

Remark 1.7. A direct computation shows that

$$\frac{n+3}{n-1} < \frac{n+1+\sqrt{(n-1)(n+7)}}{2(n-2)} < \frac{n+2}{n-2} \quad \text{for} \quad n > 2. \tag{1.13}$$

Therefore, Theorem 1.6 improves the admissible range of the subcritical exponent σ obtained by He et al. [15, Theorem 1.3 for $p = 2$] and Lu [21, Theorem 1.5]. This inequality (1.13) thus indicates that the conjecture stated in (1.6), that a Liouville theorem holds for the full subcritical range, is plausible.

Remark 1.8. When $f(u) = u^\sigma$, the subcritical condition (1.5) is automatically satisfied. Moreover, the assumption that $u \geq c$ for some constant $c \geq 0$ implies that Eq (1.2) admits no entire solutions with a positive lower bound $c > 0$ for any $\sigma \in \mathbb{R}$. In fact, even in the Euclidean case, this holds for $\sigma \geq \frac{n+2}{n-2}$, whereas the case $\sigma < \frac{n+2}{n-2}$ was established by Gidas–Spruck [13]. We have

$$\Delta u = -u^\sigma \leq -c^{\sigma-\beta} u^\beta,$$

where $n > 2$ and $\beta \in \left(1, \frac{n}{n-2}\right)$. By the classical Liouville theorem for elliptic inequalities (see Serrin and Zou [29]), we conclude that $u \equiv 0$.

The remainder of this paper is organized as follows. In Section 2, we present the necessary preliminary lemmas. The proofs of the main results are provided in Section 3.

2. Preliminary

Throughout this paper, let (M, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}_g \geq -(n-1)\kappa g$ for some constant $\kappa \geq 0$, and let ∇ denote the corresponding Levi-Civita connection. For any function $\psi \in C^1(M)$, we write $\nabla\psi \in \Gamma(T^*M)$ for the 1-form defined by $\nabla\psi(X) = \nabla_X\psi$ for all smooth vector fields X. The volume form is given locally by

$$\mathrm{dvol} = \sqrt{\det(g_{ij})}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

where $(x_1, ..., x_n)$ denotes a local coordinate system.

In our arguments on the gradient estimates of the solution to Eq (1.1), the following Saloff-Coste Sobolev inequalities play an important role.

Lemma 2.1. ([30]) *Let (M, g) be a complete manifold with $\mathrm{Ric}_g \geq -(n-1)\kappa g$. For $n > 2$, there exists a positive constant c_0 depends only on n, such that for all balls $B_R \subset M$ and volume V, we have*

$$\|h\|^2_{L^{\frac{2n}{n-2}}(B_R)} \leq e^{c_0(1+\sqrt{\kappa R})}V^{-\frac{2}{n}}R^2\left(\int_{B_R} |\nabla h|^2 + R^{-2}h^2\right), \tag{2.1}$$

for $h \in C_0^\infty(B_R)$.

Let u be a C^2 positive solution of Eq (1.1) on M. First, we set

$$w = \ln u. \tag{2.2}$$

Then, Eq (1.1) becomes

$$-\Delta w = |\nabla w|^2 + e^{-w}f(e^w). \tag{2.3}$$

We choose an auxiliary function

$$P = \frac{1}{2}|\nabla w|^2 e^{2(1-\beta)w}, \tag{2.4}$$

where $1 \leq \beta < (n-1)/(n-2)$ is to be determined later. The range of the parameter β is chosen to ensure that the equation $I_{\beta,\sigma} = 0$ has two distinct real roots, where $I_{\beta,\sigma}$ is defined in (2.7). Our main lemma is stated as follows.

Lemma 2.2. *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric}_g \geq -(n-1)\kappa g$ for some nonnegative constant κ. Assume u is a C^2 positive solution to Eq (1.1) on M and f is subcritical with exponent σ. For any $\alpha > 0$, we have*

$$\Delta P + \frac{\alpha}{2}\frac{|\nabla P|^2}{P} \geq \lambda_{\alpha,\beta,\sigma}P^2 e^{-2(1-\beta)w} - 2(n-1)\kappa P, \tag{2.5}$$

where

$$\lambda_{\alpha,\beta,\sigma} = \frac{\alpha I_{\beta,\sigma} + J_{\beta,\sigma}}{1+\alpha}, \tag{2.6}$$

and

$$I_{\beta,\sigma} = -(n-1)\sigma^2 + 2(n+1)\beta\sigma - (n+7)\beta^2 + 4\beta, \tag{2.7}$$

and

$$J_{\beta,\sigma} = -n\sigma^2 + 2(n+4)\beta\sigma + 4\beta - (n+16)\beta^2. \tag{2.8}$$

Proof. Let u be a C^2 positive solution of Eq (1.1) on M. Since $f \in C^1(0,+\infty)$, by the standard regularity theory, we know that $u \in C^3$. Chain rule gives

$$\nabla P = (\nabla^2 w \nabla w + (1-\beta)|\nabla w|^2 \nabla w) e^{2(1-\beta)w}, \tag{2.9}$$

and

$$\Delta P = \frac{1}{2}\Delta|\nabla w|^2 e^{2(1-\beta)w} + \langle \nabla|\nabla w|^2, \nabla e^{2(1-\beta)w}\rangle + \frac{1}{2}|\nabla w|^2 \Delta e^{2(1-\beta)w}. \tag{2.10}$$

We shall compute the above three terms, respectively. For the first term, by using the Bochner formula and Eq (2.3), we know

$$\begin{aligned}\Delta|\nabla w|^2 &= 2\|\nabla^2 w\|^2 + 2\langle \nabla w, \nabla \Delta w\rangle + 2\mathrm{Ric}(\nabla w, \nabla w) \\ &= 2\|\nabla^2 w\|^2 - 4\nabla^2 w(\nabla w, \nabla w) + 2|\nabla w|^2 e^{-w} f(e^w) \\ &- 2|\nabla w|^2 f'(e^w) + 2\mathrm{Ric}(\nabla w, \nabla w),\end{aligned} \tag{2.11}$$

where $\|\nabla^2 w\|$ denotes the Hilbert-Schmidt norm on matrices defined to be

$$\|\nabla^2 w\| = \left(\sum_{i,j=1}^{n} w_{ij}^2\right)^{\frac{1}{2}}.$$

For the second term,

$$\langle \nabla|\nabla w|^2, \nabla e^{2(1-\beta)w}\rangle = 4(1-\beta)e^{2(1-\beta)w}\nabla^2 w(\nabla w, \nabla w). \tag{2.12}$$

For the third term,

$$\begin{aligned}\Delta e^{2(1-\beta)w} &= 2(1-\beta)e^{2(1-\beta)w}\Big(2(1-\beta)|\nabla w|^2 + \Delta w\Big) \\ &= 2(1-\beta)e^{2(1-\beta)w}\Big((1-2\beta)|\nabla w|^2 - e^{-w} f(e^w)\Big).\end{aligned} \tag{2.13}$$

Plugging (2.11)–(2.13) into (2.10), we derive

$$\begin{aligned}\Delta P = e^{2(1-\beta)w}\Big(&\|\nabla^2 w\|^2 + (1-\beta)(1-2\beta)|\nabla w|^4 + 2(1-2\beta)\nabla^2 w(\nabla w, \nabla w) \\ &+ \beta|\nabla w|^2 e^{-w} f(e^w) - |\nabla w|^2 f'(e^w) + \mathrm{Ric}(\nabla w, \nabla w)\Big).\end{aligned} \tag{2.14}$$

We claim that

$$\|\nabla^2 w\|^2 \geq \frac{(\Delta w)^2}{n} + \frac{n}{n-1}\left(\frac{\Delta w}{n} - A_w\right)^2, \tag{2.15}$$

where $A_w = |\nabla w|^{-2}\nabla^2 w(\nabla w, \nabla w)$. In fact, we choose a local orthonormal frame $\{e_i\}$ at any given point such that $\nabla w = |\nabla w| e_1$. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|\nabla^2 w\|^2 &= w_{11}^2 + 2\sum_{j=2}^{n} w_{1j}^2 + \sum_{i,j=2}^{n} w_{ij}^2 \\ &\geq w_{11}^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} w_{ii}\right)^2. \end{aligned} \tag{2.16}$$

On the other hand, it is easily seen that

$$\begin{aligned} &\frac{(\Delta w)^2}{n} + \frac{n}{n-1}\left(\frac{\Delta w}{n} - A_w\right)^2 \\ &= \frac{1}{n}\left(w_{11} + \sum_{i=2}^{n} w_{ii}\right)^2 + \frac{n}{n-1}\left(-\frac{n-1}{n}w_{11} + \frac{1}{n}\sum_{i=2}^{n} w_{ii}\right)^2 \\ &= w_{11}^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} w_{ii}\right)^2. \end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17), we obtain the desired inequality (2.15).

Thus, from Eqs (2.3) and (2.15), we have

$$\begin{aligned} &\|\nabla^2 w\|^2 \\ &\geq \frac{(\Delta w)^2}{n} + \frac{n}{n-1}\left(\frac{\Delta w}{n} - A_w\right)^2 \\ &= \frac{(\Delta w)^2}{n-1} - \frac{2}{n-1}A_w\Delta w + \frac{n}{n-1}A_w^2 \\ &= \frac{1}{n-1}|\nabla w|^4 + \frac{2}{n-1}|\nabla w|^2 e^{-w} f(e^w) + \frac{1}{n-1}e^{-2w} f^2(e^w) \\ &+ \frac{2}{n-1}|\nabla w|^2 A_w + \frac{2}{n-1}e^{-w} f(e^w) A_w + \frac{n}{n-1}A_w^2. \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.14), the restriction on f gives

$$\begin{aligned} \Delta P \geq & e^{2(1-\beta)w}\Big(((1-\beta)(1-2\beta) + \frac{1}{n-1})|\nabla w|^4 + \frac{1}{n-1}e^{-2w} f^2(e^w) \\ & + (\beta + \frac{2}{n-1})|\nabla w|^2 e^{-w} f(e^w) - |\nabla w|^2 f'(e^w) + 2(1 - 2\beta + \frac{1}{n-1})|\nabla w|^2 A_w \\ & + \frac{2}{n-1}e^{-w} f(e^w) A_w + \frac{n}{n-1}A_w^2 + \mathrm{Ric}(\nabla w, \nabla w)\Big) \\ \geq & e^{2(1-\beta)w}\Big(((1-\beta)(1-2\beta) + \frac{1}{n-1})|\nabla w|^4 + \frac{1}{n-1}e^{-2w} f^2(e^w) \\ & + (\beta + \frac{2}{n-1} - \sigma)|\nabla w|^2 e^{-w} f(e^w) + 2(1 - 2\beta + \frac{1}{n-1})|\nabla w|^2 A_w \\ & + \frac{2}{n-1}e^{-w} f(e^w) A_w + \frac{n}{n-1}A_w^2 + \mathrm{Ric}(\nabla w, \nabla w)\Big). \end{aligned} \tag{2.19}$$

By using the equality $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$, we have

$$\begin{aligned}
&\frac{1}{n-1}\Big(e^{-w}f + A_w + \frac{1}{2}(n-1)(\beta+\frac{2}{n-1}-\sigma)|\nabla w|^2\Big)^2 \\
=&\frac{1}{n-1}e^{-2w}f^2(e^w) + \frac{1}{n-1}A_w^2 + \frac{1}{4}(n-1)(\beta+\frac{2}{n-1}-\sigma)^2|\nabla w|^4 \\
&+\frac{2}{n-1}e^{-w}f(e^w)A_w + (\beta+\frac{2}{n-1}-\sigma)|\nabla w|^2e^{-w}f(e^w) \\
&+(\beta+\frac{2}{n-1}-\sigma)|\nabla w|^2e^{-w}A_w.
\end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), it follows that

$$\begin{aligned}
\Delta P \geq & e^{2(1-\beta)w}\Big\{\Big((1-\beta)(1-2\beta)+\frac{1}{n-1}-\frac{1}{4}(n-1)(\beta+\frac{2}{n-1}-\sigma)^2\Big)|\nabla w|^4 \\
&+(2-5\beta+\sigma)|\nabla w|^2A_w + A_w^2 + \mathrm{Ric}(\nabla w,\nabla w)\Big\} \\
=&e^{2(1-\beta)w}\Big(l_{\beta,\sigma}|\nabla w|^4 + (2-5\beta+\sigma)|\nabla w|^2A_w + A_w^2 + \mathrm{Ric}(\nabla w,\nabla w)\Big),
\end{aligned} \tag{2.21}$$

where

$$l_{\beta,\sigma} := (1-\beta)(1-2\beta)+\frac{1}{n-1}-\frac{1}{4}(n-1)(\beta+\frac{2}{n-1}-\sigma)^2. \tag{2.22}$$

It follows from (2.4) and (2.9) that

$$\frac{|\nabla P|^2}{2P} = e^{2(1-\beta)w}\Big((1-\beta)^2|\nabla w|^4 + 2(1-\beta)A_w|\nabla w|^2 + A_w^2\Big). \tag{2.23}$$

Thus, combining (2.4), (2.21), and (2.23), for any $\alpha > 0$, we obtain

$$\begin{aligned}
&\Delta P + \frac{\alpha}{2}\frac{|\nabla P|^2}{P} \\
\geq\ & e^{2(1-\beta)w}\Big\{(l_{\beta,\sigma}+\alpha(1-\beta)^2)|\nabla w|^4 + (2-5\beta+\sigma+2\alpha(1-\beta))|\nabla w|^2A_w \\
&+(1+\alpha)A_w^2 + \mathrm{Ric}(\nabla w,\nabla w)\Big\} \\
=\ & e^{2(1-\beta)w}\Big\{\Big(l_{\beta,\sigma}+\alpha(1-\beta)^2-\frac{(2-5\beta+\sigma+2\alpha(1-\beta))^2}{4(1+\alpha)}\Big)|\nabla w|^4 \\
&+(1+\alpha)\Big(A_w+\frac{2-5\beta+\sigma+2\alpha(1-\beta)}{2(1+\alpha)}|\nabla w|^2\Big)^2 + \mathrm{Ric}(\nabla w,\nabla w)\Big\} \\
\geq\ & e^{2(1-\beta)w}\Big\{\Big(l_{\beta,\sigma}+\alpha(1-\beta)^2-\frac{(2-5\beta+\sigma+2\alpha(1-\beta))^2}{4(1+\alpha)}\Big)|\nabla w|^4 \\
&-(n-1)\kappa|\nabla w|^2\Big\} \\
=\ & \Big(4l_{\beta,\sigma}+4\alpha(1-\beta)^2-\frac{(2-5\beta+\sigma+2\alpha(1-\beta))^2}{1+\alpha}\Big)P^2e^{-2(1-\beta)w}
\end{aligned} \tag{2.24}$$

$$-2(n-1)\kappa P.$$

We define

$$\begin{aligned}\lambda_{\alpha,\beta,\sigma} &:= 4l_{\beta,\sigma} + 4\alpha(1-\beta)^2 - \frac{(2-5\beta+\sigma+2\alpha(1-\beta))^2}{1+\alpha} \\ &= \frac{4(l_{\beta,\sigma} + (1-\beta)^2 - (1-\beta)(2-5\beta+\sigma))\alpha + 4l_{\beta,\sigma} - (2-5\beta+\sigma)^2}{1+\alpha} \\ &= \frac{\alpha I_{\beta,\sigma} + J_{\beta,\sigma}}{1+\alpha},\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}I_{\beta,\sigma} : &= 4(l_{\beta,\sigma} + (1-\beta)^2 - (1-\beta)(2-5\beta+\sigma)) \\ &= -(n-1)\sigma^2 + 2(n+1)\beta\sigma - (n+7)\beta^2 + 4\beta,\end{aligned}$$

and

$$\begin{aligned}J_{\beta,\sigma} : &= 4l_{\beta,\sigma} - (2-5\beta+\sigma)^2 \\ &= -n\sigma^2 + 2(n+4)\beta\sigma + 4\beta - (n+16)\beta^2.\end{aligned}$$

Hence, Lemma 2.2 follows. □

3. Proof of main theorem

3.1. Proof of Theorem 1.2

This section presents the proof of Theorem 1.2, which is organized into three parts. First, we select a fixed constant $\alpha_0 > 0$ such that $\lambda_{\alpha_0,\beta,\sigma} > 0$ and derive a fundamental integral inequality for P. Second, we establish an L^γ estimate for P over a geodesic ball of radius $3R/4$, which provides the initial step for the Moser iteration. Finally, we complete the proof by using the Moser iteration method.

3.1.1. Integral inequality

One can obtain the two real roots of the equation $I_{\beta,\sigma} = 0$ with

$$\sigma_1(\beta) :\equiv \frac{\beta(n+1) - 2\sqrt{\beta(n-1-\beta(n-2))}}{n-1},$$

and

$$\sigma_2(\beta) :\equiv \frac{\beta(n+1) + 2\sqrt{\beta(n-1-\beta(n-2))}}{n-1}.$$

In order to guarantee $I_{\beta,\sigma} > 0$, we need that $\sigma \in (\sigma_1(\beta), \sigma_2(\beta))$. This implies

$$\lim_{\alpha\to\infty} \lambda_{\alpha,\beta,\sigma} = \lim_{\alpha\to\infty} \frac{\alpha I_{\beta,\sigma} + J_{\beta,\sigma}}{1+\alpha} > 0. \tag{3.1}$$

Thus, we can choose α_0 large enough such that

$$\lambda_{\alpha_0,\beta,\sigma} > 0. \tag{3.2}$$

To proceed further, we recall that for any fixed point $x_0 \in M$ and $R > 0$, there exists a cutoff function $\varphi = \varphi_R$, i.e., a function $\varphi \in C_0^\infty(B_R(x_0))$ such that

$$0 \le \varphi \le 1, \quad \varphi \equiv 1 \quad \text{in } B_{3R/4}, \tag{3.3}$$

and

$$|\nabla\varphi| \le \frac{C}{R}. \tag{3.4}$$

Now, we need to establish a key integral inequality of P.

Lemma 3.1. *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric}_g \ge -(n-1)\kappa g$ for some nonnegative constant κ. Assume that u is a C^2 solution to Eq (1.1) on $B_R(x_0)$ satisfying $u \ge c$ for some constant $c \ge 0$ and f is subcritical with exponent $\sigma \in (\sigma_1(\beta), \sigma_2(\beta))$. Then we have the following inequality*

$$\begin{aligned} & e^{-t_0}V^{\frac{2}{n}}R^{-2}\left\|P^{\frac{t+1}{2}}\varphi\right\|^2_{L^{\frac{2n}{n-2}}(B_R)} \\ \le\ & -4t\lambda_{\beta,\sigma}\int_{B_R}P^{t+2}\varphi^2 + 16\int_{B_R}P^{t+1}|\nabla\varphi|^2 + c_1t_0^2tR^{-2}\int_{B_R}P^{t+1}\varphi^2, \end{aligned} \tag{3.5}$$

where $t_0 = c_0(1+\sqrt{\kappa}R)$, and c_1 depends only on n, under the constraints $t > 2\alpha_0 + 1$ with α_0 satisfying the condition in (3.2), and $\lambda_{\beta,\sigma}$ is as defined in (3.6).

Proof. Let φ be a cutoff function as in (3.3) and (3.4). Since $u = e^w$ is a C^2 solution to Eq (1.1) with a lower bound $c \ge 0$, we define a positive constant L such that $e^{-2(1-\beta)w} \ge L$. When $c = 0$, we must set $\beta = 1$ (which yields $L = 1$). In fact, if $\beta > 1$ and $c = 0$, then u may tend to 0, so that $w \to -\infty$, and no positive lower bound L can exist, which would invalidate the estimate. By multiplying (2.5) by $P^t\varphi^\theta$ (for some positive constant θ to be chosen later and $t > 1$) and integrating by parts, we obtain

$$\begin{aligned} & \left(t - \frac{\alpha_0}{2}\right)\int_{B_R}P^{t-1}\varphi^\theta|\nabla P|^2 + \lambda_{\beta,\sigma}\int_{B_R}P^{t+2}\varphi^\theta \\ & \le -\theta\int_{B_R}P^t\varphi^{\theta-1}\langle\nabla P, \nabla\varphi\rangle + 2(n-1)\kappa\int_{B_R}P^{t+1}\varphi^\theta, \end{aligned} \tag{3.6}$$

where $\lambda_{\beta,\sigma} :\equiv L^{-1}\lambda_{\alpha_0,\beta,\sigma}$.

By using Young's inequality, we have

$$\begin{aligned} & -\theta\int_{B_R}P^t\varphi^{\theta-1}\langle\nabla P, \nabla\varphi\rangle \\ \le\ & \frac{t}{2}\int_{B_R}P^{t-1}\varphi^\theta|\nabla P|^2 + \frac{\theta^2}{2t}\int_{B_R}P^{t+1}\varphi^{\theta-2}|\nabla\varphi|^2. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), it follows that

$$\begin{aligned} &\frac{t-\alpha_0}{2}\int_{B_R} P^{t-1}\varphi^{\theta}|\nabla P|^2+\lambda_{\beta,\sigma}\int_{B_R} P^{t+2}\varphi^{\theta} \\ \leq\ &\frac{\theta^2}{2t}\int_{B_R} P^{t+1}\varphi^{\theta-2}|\nabla\varphi|^2+2(n-1)\kappa\int_{B_R} P^{t+1}\varphi^{\theta}. \end{aligned} \tag{3.8}$$

On the other hand, using the inequality $(a+b)^2 \leq 2a^2+2b^2$ yields

$$\begin{aligned} &\int_{B_R}\left|\nabla(P^{\frac{t+1}{2}}\varphi^{\frac{\theta}{2}})\right|^2 \\ \leq\ &\frac{(t+1)^2}{2}\int_{B_R} P^{t-1}\varphi^{\theta}|\nabla P|^2+\frac{\theta^2}{2}\int_{B_R} P^{t+1}\varphi^{\theta-2}|\nabla\varphi|^2. \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} &\frac{t-\alpha_0}{(t+1)^2}\int_{B_R}\left|\nabla(P^{\frac{t+1}{2}}\varphi^{\frac{\theta}{2}})\right|^2+\lambda_{\beta,\sigma}\int_{B_R} P^{t+2}\varphi^{\theta} \\ \leq\ &\theta^2\left(\frac{1}{2t}+\frac{t-\alpha_0}{2(t+1)^2}\right)\int_{B_R} P^{t+1}\varphi^{\theta-2}|\nabla\varphi|^2+2(n-1)\kappa\int_{B_R} P^{t+1}\varphi^{\theta}. \end{aligned} \tag{3.10}$$

We note that

$$\frac{1}{4t}\leq\frac{t-\alpha_0}{(t+1)^2}\leq\frac{1}{t},$$

for $t > 2\alpha_0+1$. Letting $\theta = 2$, it follows from (3.10) that

$$\begin{aligned} &\int_{B_R}\left|\nabla(P^{\frac{t+1}{2}}\varphi)\right|^2+4t\lambda_{\beta,\sigma}\int_{B_R} P^{t+2}\varphi^2 \\ \leq\ &16\int_{B_R} P^{t+1}|\nabla\varphi|^2+8(n-1)\kappa t\int_{B_R} P^{t+1}\varphi^2. \end{aligned} \tag{3.11}$$

Letting $h = P^{\frac{t+1}{2}}\varphi \in C_0^{\infty}(B_R)$, it follows from Saloff-Coste's Sobolev inequality (2.1) that

$$e^{-c_0(1+\sqrt{\kappa}R)}V^{\frac{2}{n}}R^{-2}\left\|P^{\frac{t+1}{2}}\varphi\right\|^2_{L^{\frac{2n}{n-2}}(B_R)}\leq\int_{B_R}\left|\nabla(P^{\frac{t+1}{2}}\varphi)\right|^2+R^{-2}\int_{B_R} P^{t+1}\varphi^2. \tag{3.12}$$

Substituting the above inequality into (3.11) yields

$$\begin{aligned} &e^{-c_0(1+\sqrt{\kappa}R)}V^{\frac{2}{n}}R^{-2}\left\|P^{\frac{t+1}{2}}\varphi\right\|^2_{L^{\frac{2n}{n-2}}(B_R)} \\ \leq\ &-4t\lambda_{\beta,\sigma}\int_{B_R} P^{t+2}\varphi^2+16\int_{B_R} P^{t+1}|\nabla\varphi|^2+(8(n-1)\kappa t+R^{-2})\int_{B_R} P^{t+1}\varphi^2. \end{aligned} \tag{3.13}$$

Set $t_0 = c_0(1+\sqrt{\kappa}R)$ and choose $c_1(n)$ such that

$$8(n-1)\kappa t+R^{-2}\leq c_1c_0^2(1+\sqrt{\kappa}R)^2tR^{-2}=c_1t_0^2tR^{-2}. \tag{3.14}$$

Combining (3.13) and (3.14), we can easily obtain (3.5). □

3.1.2. L^γ bound of P

Lemma 3.2. *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric}_g \geq -(n-1)\kappa g$ for some nonnegative constant κ. Assume that u is a C^2 solution to Eq* (1.1) *on $B_R(x_0)$ satisfying $u \geq c$ for some constant $c \geq 0$, and f is subcritical with exponent $\sigma \in (\sigma_1(\beta), \sigma_2(\beta))$. Let*

$$\gamma = \frac{n(1+t_0)}{n-2},$$

where t_0 is defined in Lemma 3.1 *(we note that this R-dependence is a key feature of their initial step for the Moser iteration). Then, we have*

$$\|P\|_{L^\gamma(B_{3R/4})} \leq c_4 V^{\frac{1}{\gamma}} \frac{t_0^2}{R^2}, \tag{3.15}$$

where c_4 depends on n, c and β.

Proof. Letting $t = t_0$ in (3.5), we have

$$\begin{aligned} & e^{-t_0} V^{\frac{2}{n}} R^{-2} \left\| P^{\frac{t_0+1}{2}} \varphi \right\|^2_{L^{\frac{2n}{n-2}}(B_R)} \\ \leq \;& -4t_0 \lambda_{\beta,\sigma} \int_{B_R} P^{t_0+2} \varphi^2 + 16 \int_{B_R} P^{t_0+1} |\nabla \varphi|^2 + c_1 t_0^3 R^{-2} \int_{B_R} P^{t_0+1} \varphi^2. \end{aligned} \tag{3.16}$$

For a pair of conjugate exponents $(p, q) = (\frac{t_0+2}{t_0+1}, t_0+2)$, by using Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq a^p + b^q$, we obtain

$$\begin{aligned} & c_1 t_0^3 R^{-2} \int_{B_R} P^{t_0+1} \varphi^2 \\ \leq \;& 2t_0 \lambda_{\beta,\sigma} \int_{B_R} P^{t_0+2} \varphi^2 + \frac{c_1 t_0^3}{R^2} \left(\frac{c_1 t_0^2}{2\lambda_{\beta,\sigma} R^2} \right)^{t_0+1} V, \end{aligned} \tag{3.17}$$

where V is the volume of B_R. Plugging (3.17) into (3.16), we derive

$$e^{-t_0} V^{\frac{2}{n}} R^{-2} \left\| P^{\frac{t_0+1}{2}} \varphi \right\|^2_{L^{\frac{2n}{n-2}}(B_R)} \leq -2t_0 \lambda_{\beta,\sigma} \int_{B_R} P^{t_0+2} \varphi^2 + 16 \int_{B_R} P^{t_0+1} |\nabla \varphi|^2 + \frac{c_1 t_0^3}{R^2} \left(\frac{c_1 t_0^2}{2\lambda_{\beta,\sigma} R^2} \right)^{t_0+1} V. \tag{3.18}$$

Let φ_1 be a cutoff function as in (3.3) and (3.4). Then, we define $\varphi = \varphi_1^{t_0+2}$ and choose $c_2(n)$ such that

$$16|\nabla \varphi|^2 \leq \frac{16C^2(t_0+2)^2}{R^2} \varphi^{\frac{2t_0+2}{t_0+2}} \leq \frac{c_2 t_0^2}{R^2} \varphi^{\frac{2t_0+2}{t_0+2}}. \tag{3.19}$$

For a pair of conjugate exponents $(p, q) = (\frac{t_0+2}{t_0+1}, t_0+2)$, by using Young's inequality again, we have

$$\begin{aligned} 16 \int_{B_R} P^{t_0+1} |\nabla \varphi|^2 \leq & \frac{c_2 t_0^2}{R^2} \int_{B_R} P^{t_0+1} \varphi^{\frac{2(t_0+1)}{t_0+2}} \\ \leq & 2t_0 \lambda_{\beta,\sigma} \int_{B_R} P^{t_0+2} \varphi^2 + \frac{c_2 t_0^2}{R^2} \left(\frac{c_2 t_0}{2\lambda_{\beta,\sigma} R^2} \right)^{t_0+1} V. \end{aligned} \tag{3.20}$$

Combining (3.18) and (3.20), we have

$$\begin{aligned} & e^{-t_0}V^{\frac{2}{n}}R^{-2}\left\|P^{\frac{t_0+1}{2}}\varphi\right\|^2_{L^{\frac{2n}{n-2}}(B_R)} \\ \le\ & \frac{c_2t_0^2}{R^2}\left(\frac{c_2t_0}{2\lambda_{\beta,\sigma}R^2}\right)^{t_0+1}V+\frac{c_1t_0^3}{R^2}\left(\frac{c_1t_0^2}{2\lambda_{\beta,\sigma}R^2}\right)^{t_0+1}V, \end{aligned}$$

i.e.,

$$\begin{aligned} & \left(\int_{B_R}P^{\frac{n(t_0+1)}{n-2}}\varphi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ \le\ & e^{t_0}V^{1-\frac{2}{n}}t_0^3\left(\frac{c_2}{t_0}\left(\frac{c_2}{2t_0\lambda_{\beta,\sigma}}\right)^{t_0+1}+c_1\left(\frac{c_1}{2\lambda_{\beta,\sigma}}\right)^{t_0+1}\right)\left(\frac{t_0^2}{R^2}\right)^{t_0+1} \\ \le\ & c_3^{t_0+1}e^{t_0}V^{1-\frac{2}{n}}t_0^3\left(\frac{t_0^2}{R^2}\right)^{t_0+1}, \end{aligned} \tag{3.21}$$

where the constant $c_3(n,c,\beta)$ satisfies

$$c_3^{t_0+1}\ge\frac{c_2}{t_0}\left(\frac{c_2}{2t_0\lambda_{\beta,\sigma}}\right)^{t_0+1}+c_1\left(\frac{c_1}{2\lambda_{\beta,\sigma}}\right)^{t_0+1}.$$

Taking the $\frac{1}{t_0+1}$ power of both sides of (3.21) yields

$$\left\|P\varphi^{\frac{2}{t_0+1}}\right\|_{L^\gamma(B_R)}\le c_3e^{\frac{t_0}{t_0+1}}V^{\frac{1}{\gamma}}t_0^{\frac{3}{t_0+1}}\frac{t_0^2}{R^2}\le c_4V^{\frac{1}{\gamma}}\frac{t_0^2}{R^2}, \tag{3.22}$$

where

$$c_4=c_3\sup_{t_0\ge1}e^{\frac{t_0}{t_0+1}}t_0^{\frac{3}{t_0+1}}.$$

Since $\varphi\equiv1$ in $B_{3R/4}$, we obtain

$$\|P\|_{L^\gamma(B_{3R/4})}\le c_4V^{\frac{1}{\gamma}}\frac{t_0^2}{R^2}. \tag{3.23}$$

Hence, the required bound follows. □

3.1.3. Nash-Moser iteration

Lemma 3.3. *Let (M,g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric}_g\ge-(n-1)\kappa g$ for some nonnegative constant κ. Assume that u is a C^2 solution to Eq (1.1) on $B_R(x_0)$ satisfying $u\ge c$ for some constant $c\ge0$ and f is subcritical with exponent $\sigma\in(\sigma_1(\beta),\sigma_2(\beta))$. Then, we have*

$$\|P\|_{L^\infty(B_{R/2})}\le c_7\frac{(1+\sqrt{\kappa}R)^2}{R^2}, \tag{3.24}$$

where c_7 depends on n, c, and β.

Proof. We discard the first term on the right-hand side of (3.5) to obtain

$$e^{-t_0}V^{\frac{2}{n}}R^{-2}\left\|P^{\frac{t+1}{2}}\varphi\right\|^2_{L^{\frac{2n}{n-2}}(B_R)} \leq 16\int_{B_R}P^{t+1}|\nabla\varphi|^2 + c_1t_0^2tR^{-2}\int_{B_R}P^{t+1}\varphi^2. \tag{3.25}$$

Let $\varphi_k \in C_0^\infty(B_{\rho_k})$ be a cutoff function as in (3.3) and (3.4) satisfying

$$\begin{cases} 0 \leq \varphi_k \leq 1, \\ |\nabla\varphi_k| \leq \frac{4^kC}{R}, \\ \varphi_k \equiv 1 \text{ in } B_{\rho_{k+1}}, \end{cases} \tag{3.26}$$

here $\rho_k = \frac{R}{2} + \frac{R}{4^k}$, $k = 1, 2, 3, \cdots$. Substituting φ_k into (3.25) instead of φ, we arrive at

$$\begin{aligned} e^{-t_0}V^{\frac{2}{n}}\left\|P^{\frac{t+1}{2}}\varphi_k\right\|^2_{L^{\frac{2n}{n-2}}(B_{\rho_k})} &\leq 16R^2\int_{B_{\rho_k}}P^{t+1}|\nabla\varphi_k|^2 + c_1t_0^2t\int_{B_{\rho_k}}P^{t+1}\varphi_k^2 \\ &\leq (C^216^{k+1} + c_1t_0^2t)\int_{B_{\rho_k}}P^{t+1}. \end{aligned} \tag{3.27}$$

By picking $\gamma_1 = \gamma$, $\gamma_{k+1} = \frac{n\gamma_k}{n-2}$, and $t = t_k$ such that

$$t_k + 1 = \gamma_k,$$

we can deduce from (3.27) that

$$\left(\int_{B_{\rho_k}}P^{\gamma_{k+1}}\varphi_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq e^{t_0}V^{-\frac{2}{n}}\left(C^216^{k+1} + c_1t_0^2(t_0+1)\left(\frac{n}{n-2}\right)^k\right)\int_{B_{\rho_k}}P^{\gamma_k}. \tag{3.28}$$

On the other hand, we can choose $c_5(n)$ which satisfies

$$c_5t_0^3 \geq \max\left\{c_1t_0^2(t_0+1), 16C^2\right\}. \tag{3.29}$$

Then, we have

$$\left(\int_{B_{\rho_k}}P^{\gamma_{k+1}}\varphi_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq 2c_5t_0^316^ke^{t_0}V^{-\frac{2}{n}}\int_{B_{\rho_k}}P^{\gamma_k}, \tag{3.30}$$

since $\frac{n}{n-2} < 16$. Taking $\frac{1}{\gamma_k}$ power of both sides of (3.30), we obtain

$$\begin{aligned} \|P\|_{L^{\gamma_{k+1}}(B_{\rho_{k+1}})} &\leq \left(2c_5t_0^3e^{t_0}V^{-\frac{2}{n}}\right)^{\frac{1}{\gamma_k}}16^{\frac{k}{\gamma_k}}\|P\|_{L^{\gamma_k}(B_{\rho_k})} \\ &\leq \left(2c_5t_0^3e^{t_0}\right)^{\sum_{k=1}^\infty\frac{1}{\gamma}_k}16^{\sum_{k=1}^\infty\frac{k}{\gamma_k}}\|P\|_{L^\gamma(B_{3R/4})}. \end{aligned} \tag{3.31}$$

Note

$$\sum_{k=1}^\infty\frac{1}{\gamma_k} = \frac{\frac{1}{\gamma_1}}{1-\frac{n-2}{n}} = \frac{n}{2\gamma} \tag{3.32}$$

and

$$\sum_{k=1}^{\infty}\frac{k}{\gamma_k}=\frac{n^2}{4\gamma}. \tag{3.33}$$

We now claim that the expression $\left(2c_5t_0^3e^{t_0}\right)^{\frac{n}{2\gamma}}16^{\frac{n^2}{4\gamma}}$ depends only on the dimension n. To verify this, first consider the case $\kappa=0$, which implies $t_0=c_0$. If $\kappa\neq 0$, then as $R\to\infty$, we have $t_0\to\infty$. Observe that in this limit,

$$\gamma=\frac{n(1+t_0)}{n-2}\sim\frac{nt_0}{n-2}.$$

It then follows that

$$\begin{aligned}\left(2c_5t_0^3e^{t_0}\right)^{\frac{n}{2\gamma}} &= \exp\left(\frac{n(\ln 2+\ln c_5+3\ln t_0+t_0)}{2\gamma}\right)\\ &\sim \exp\left(\frac{n-2}{2t_0}\cdot t_0\right)\\ &= e^{\frac{n-2}{2}}\end{aligned}$$

and

$$16^{\frac{n^2}{4\gamma}}\sim 16^{\frac{n(n-2)}{4t_0}}\to 16^0=1.$$

Then, we can derive

$$\|P\|_{L^\infty(B_{R/2})}\le c_6V^{-\frac{1}{\gamma}}\|P\|_{L^\gamma(B_{3R/4})}, \tag{3.34}$$

where $c_6(n)$ satisfies

$$c_6\ge\left(2c_5t_0^3e^{t_0}\right)^{\frac{n}{2\gamma}}16^{\frac{n^2}{4\gamma}}. \tag{3.35}$$

Combining (3.15) and (3.34), we obtain

$$\|P\|_{L^\infty(B_{R/2})}\le c_7\frac{(1+\sqrt{\kappa}R)^2}{R^2}, \tag{3.36}$$

where $c_7=c_0^2c_4c_6$ depends on n, c, and β.

Hence, Theorem 1.2 follows. □

3.2. Proof of Corollaries 1.3 and 1.4

Let $c=0$ so that $\beta=1$, $\sigma_1=1$, and $\sigma_2=\frac{n+3}{n-1}$. According to Lemma 2.2, we have

$$\Delta P+\frac{\alpha}{2}\frac{|\nabla P|^2}{P}\ge\lambda_{\alpha,\sigma}P^2-2(n-1)\kappa P. \tag{3.37}$$

Following the proof of Theorem 1.2, we obtain

$$\sup_{B_{R/2}(x_0)}\frac{|\nabla u|^2}{u^2}\le C\frac{(1+\sqrt{\kappa}R)^2}{R^2}. \tag{3.38}$$

A straightforward calculation then yields the desired Harnack inequality (1.12). According to the gradient estimate (3.38), by letting $R\to\infty$, we conclude that $\nabla u=0$, which yields $u\equiv C$ in M. Hence, Corollaries 1.3 and 1.4 follow.

3.3. Proof of Theorem 1.6

For $c > 0$ and $u \geq c$, it follows that $w = \ln u$ is bounded below. From the proof of Lemma 3.1, we see that this ensures L can be chosen as a positive constant, whereas $L = 1$ when $c = 0$. One can see $\sigma_1(\beta)$ is increasing on $[1, \frac{n-1}{n-2})$. Then, we have

$$\min \sigma_1(\beta) = \sigma_1(1) = 1. \tag{3.39}$$

One also can see $\sigma_2(\beta)$ is increasing on $[1, \beta_n]$ and decreasing on $[\beta_n, \frac{n-1}{n-2})$, where

$$\beta_n = \frac{(n-1)\sqrt{n+7} + (n+1)\sqrt{n-1}}{2(n-2)\sqrt{n+7}}.$$

Then, we also have

$$\max \sigma_2(\beta) = \sigma_2(\beta_n) = \frac{n+1+\sqrt{(n-1)(n+7)}}{2(n-2)}. \tag{3.40}$$

According to the gradient estimate (1.11), by letting $R \to \infty$, we conclude that $\nabla u = 0$, which yields $u \equiv C$ in M. Hence, Theorem 1.6 follows.

Author contributions

The authors declare that they have contributed equally to the conception, design, analysis, and writing of this manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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