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Research article

Portfolio selection and risk control for an insurer with uncertain time horizon under inside information

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Abstract: This paper is devoted to studying the optimal investment and risk control strategy for an insurer with uncertain time under inside information. The jump process is incorporated into our research framework, and the correlation between the risky asset and the risk process is considered. Assuming the exit time is uncertain, we use forward calculus and Malliavin calculus to derive a characterization of the optimal investment and risk control under the criterion of maximizing the logarithmic utility of the terminal wealth in a pure jump market and a mixed market. Moreover, we apply filtration enlargement techniques to several interesting special cases and derive the corresponding explicit solutions. Finally, we conduct numerical simulations to analyze the impact of correlation coefficient and insider information on the investment strategy.

Keywords: optimal strategy; random time; jump-diffusion model; inside information; Malliavin calculus
Mathematics Subject Classification: 91G80, 93E20

1. Introduction

The optimal investment and risk control problem for insurers has been a crucial subject in financial theory and practice, and numerous works have focused on this topic. For example, Zhou and Zhang [1] investigated the optimal investment and risk control problem in an incomplete market using the martingale approach, where the risky asset price process was described by the geometric

Brownian motion model. They obtained the explicit expression under the mean-variance criterion. Bo and Wang [2] considered the same problem under stochastic parameters. They derived the optimal investment strategy by maximizing the expected power utility of terminal wealth. Furthermore, Shen and Yin [3] concentrated on the jump-diffusion model with stochastic parameters. The closed-form solutions for the optimal strategy were deduced under the logarithmic utility criterion of terminal wealth. Peng and Chen [4] presented sufficient and necessary conditions for the optimal investment and risk control strategy by using Malliavin calculus, which was different from the classical stochastic control method. In some special cases, the analytic solutions were provided. Deng and Yao [5] studied an optimal investment and risk control problem with a delay in the default market, where the controlled wealth process followed a stochastic delay differential equation. Peng and Yan [6] explored the optimal control problem for an insurer aiming to maximize the expected power utility of terminal wealth, and they provided the solutions in closed-form through the enlargement of filtration and Hamilton-Jacobi-Bellman (HJB) techniques. The literature mentioned above focused on controlling the number of policies to reduce the insurer's risk. However, there is another way to spread risk through reinsurance; for related research, see Schmidli [7], Bai and Guo [8], Peng and Chen [9], and the references therein.

All the works mentioned above assume that investors know exactly when to exit the market. In reality, the classic fixed-term investment model often fails. In practice, insurance institutions are tend to encounter unexpected business termination due to scenarios such as bankruptcy liquidation, regulatory mandatory intervention, and the large-scale collective termination of insurance policies. It is precisely due to the influence of internal and external factors that the timing of investment decisions is uncertain, which brings greater complexity and uncertainty to investment decisions. Recently, many scholars realized the critical influence of the uncertain time horizon on investment decisions, and adopted stochastic control, dynamic programming, and backward stochastic differential equation (BSDE) to solve related problems. Yu [10] extended the work of Lim and Zhou [11] to a continuous-time mean-variance portfolio selection problem with random parameters and a random time horizon, and they derived the closed-form solutions for the investment strategy and the efficient frontier. Subsequently, Lv, Wu, and Yu [12] generalized the problem in Yu [10] to incomplete markets, where the problem was formulated as a constrained stochastic linear quadratic optimal control objective function. Huang and Wu [13] considered the optimal investment strategy under inflation and an uncertain time horizon, and an explicit expression for the optimal strategy was provided under the maximum utility criterion.

Classical models generally assume that investors possess completely accurate market information. However, in practice, some insurers have access to inside information, that is, they possess additional information beyond what is publicly available in the market, which enables them to make more informed decisions under conditions of risk uncertainty. In general, inside information is represented by an enlarged filtration. In the past decades, many researchers have studied the optimization problem of insurers under inside information. Di Nunno et al. [14] explored an insurer with inside information in a Lévy market, and they derived explicit solutions under the logarithmic utility criterion. Peng et al. [15] assumed that the risk process was correlated with the risky asset process. They obtained the optimal investment and risk control strategies for an insurer who had inside information. Xiong and Zhang [16] investigated the optimal investment and proportional reinsurance strategy for an insurer with inside information. Closed-form solutions were obtained by solving the HJB equations. Additionally, for references of inside information, see also Cao and Peng [17], Danilova and Monoyios [18], and Kohatsu-Higa and Sulem [19].

Compared with the work in [20], our paper focuses on the situation of inside information rather than partial information and adopts the expected utility maximization of terminal wealth as the objective function instead of the mean-variance criterion. Meanwhile, we incorporate random time and correlation into the research framework, which makes up for the research limitations of the literature [21]. In view of the above considerations, we have incorporated both random time and inside information into our research framework. By using Malliavin calculus and variational methods, we derive some necessary and sufficient conditions for the optimal strategy of an insurer with insider information in both pure jump markets and mixed markets. Furthermore, by employing the enlargement of the filtration technique, for the special case of $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma(W_1(T_0), W_2(T_0), \eta_1(T_0), \eta_2(T_0)) : t \in [0, T]$, we provide an explicit solution related to the future signal $W_1(T_0), W_2(T_0)$. The results show that when $\bar{F(s)} \neq 0$, the conclusions of this paper can be reduced to Property 5.3 in [21]. In this paper, the jump process is introduced into the market model, and insider information and random time are incorporated into the research framework. By means of Malliavin calculus, the necessary and sufficient conditions for the optimal investment and risk control strategies of insurance companies are derived. In addition, some special cases are discussed, and explicit expressions for the optimal strategies are obtained. Finally, we perform numerical simulations to analyze the impacts of the correlation coefficient between the financial market and the insurance market, as well as the effect of insider information on the optimal strategy.

The rest of this paper is arranged as follows: Section 2 is devoted to providing the optimal strategy in the pure jump market; Section 3 investigates the optimal strategy in the mixed market; Section 4 considers the optimal strategies in some particular cases; Section 5 provides the numerical simulation; and Section 6 concludes this paper.

2. Optimal portfolio in a pure jump market

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\mathcal{F}_t$ represents the information available in the market up to time t. The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is complete and satisfies the usual hypotheses. T is a constant. We suppose that there are no transaction costs or taxes and can be continuously traded. In this section, we concentrate on the optimal portfolio problem under inside information in the pure jump market. The financial market consists of a risk-free asset and a risky asset, where the risk-free asset price process S_0 can be described by the following:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \\ S_0(0) = 1. \end{cases} \tag{2.1}$$

Here, $r(t)$ is a deterministic function of t, which denotes the risk-free interest rate. The risky asset price process $S(t)$ satisfies the following:

$$\begin{cases} dS(t) = S(t-)\Big[\mu(t)dt + \int_{\mathbb{R}_0} \gamma_1(t, z)\tilde{N}^1(d^- t, dz)\Big], \\ S(0) > 0, \end{cases} \tag{2.2}$$

where $\mu(t)$ represents the risk premium, and we assume that $\mu(t)$ and $\gamma_1(t, z)$ are caglad and uniformly bounded $\mathcal{F}_t$ adapted processes. $\tilde{N}^1(dt, dz) = N_1(dt, dz) - \nu_1(dz)dt$ is a compensated Poisson random measure, and $\nu_1(dz)$ is the corresponding Lévy measure. Suppose that $\gamma_1(t, z) > -1, dt \times \nu_1(dz)$ -a.e..

The risk process (per policy) for the insurer evolves according to the following:

$$\begin{cases} dR_t = p(t)dt + \int_{\mathbb{R}_0} \gamma_2(t,z)\tilde{N}^2(dt,dz), \\ R_0 = 0, \end{cases} \tag{2.3}$$

where $p(t)$ denotes the claim rate per insurance policy, and the parameter processes $p(t), \gamma_2(t,z)$ are caglad, uniformly bounded, and $\mathcal{F}_t$ adapted. $\tilde{N}^2(dt,dz) = N_2(dt,dz) - \nu_2(dz)dt$ is a compensated Poisson random measure independent of $\tilde{N}^1(dt,dz)$. We assume that $\gamma_2(t,z) > -1, dt \times \nu_2(dz)-$ a.s..

Denote by the proportion of wealth invested in the risky asset at time t by $\pi(t)$, while the rest is invested in the risk-free asset. Insurers manage risks by adjusting the number of policies sold, which is a key regulatory requirement of asset-liability management (ALM) in the insurance industry and also a main means for institutions to conduct risk control. Let $L(t)$ denote the number of policies sold by the insurer at time t. For technical convenience, this paper adopts the approach in [4] and transforms the policy quantity $L(t)$ into a ratio of liability at time t. Namely, $k(t) = \frac{L(t)}{X(t)}$. We allow that $L(t) < 0$, which implies that the insurer could purchase the insurance policies from other insurers. In practice, red some insurers who possess red inside information make their decisions based on the filtration enlargement. The enlargement information flow is denoted by $\mathbb{G} = \{\mathcal{G}_t \subset \mathcal{F}\}_{0\leqslant t\leqslant T}$, which is larger than $\mathbb{F} = \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$(i.e., $\mathcal{F}_t \subseteq \mathcal{G}_t \subset \mathcal{F}$, for each $t \in [0,T]$). The strategy processes $\pi(t)$ and $\kappa(t)$ adopted by insurers are adapted to the filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\leqslant t\leqslant T}$.

In the present paper, we use τ to represent the uncertainty of the time horizon, which is assumed to be a stochastic positive variable. Let $F(t) \triangleq P(\tau \leqslant t \mid \mathcal{G}_t)$ be the conditional distribution function of τ. Moreover, $F(\cdot)$ has a nonnegative and uniformly bounded density process $\delta(\cdot)$, such that $F(t) = \int_0^t \delta(s)ds$ for each $t \in [0,T]$. We assume that all parameter processes $\mu(t), p(t), \lambda(t), \sigma(t), q(t), \gamma_1(t,z), \gamma_2(t,z), \delta(t)$ involved in this paper are Malliavin differentiable with respect to W^i and N^i, for each $t \in [0,T], i = 1,2$, and satisfies $\mu(t), p(t), \lambda(t), \sigma(t), q(t), \delta(t) \in L^2(P), \gamma_1(t,z), \gamma_2(t,z) \in L^2(P \times \mu)$, where μ denotes the Lebesgue measure.

Let $u(t) = (\pi(t), \kappa(t))$ denote the strategy process for the insurer, with $\pi(t)$ as the wealth allocation proportion to risky assets and $\kappa(t)$ as the liability ratio at time t. The definition of an admissible strategy is defined below.

Definition 2.1. *$u(t) = (\pi(t), \kappa(t)) \in \mathcal{A}_{\mathcal{G}}$ is said to be an admissible strategy if it satisfies the following constraints:*

(1) $u(t)$ is caglad and adapted to the filtration $\mathbb{G}$.

(2) $\pi(s)\gamma_1(s,z), \log(1+\pi(s)\gamma_1(s,z))$ and $\frac{\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}, s \in [0,T], z \in \mathbb{R}_0$ are forward integrable with respect to $\tilde{N}^1(d^-t,dz)$. $\kappa(s)\gamma_2(s,z)$, $\log(1-\kappa(s)\gamma_2(s,z))$, and $\frac{\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}, s \in [0,T], z \in \mathbb{R}_0$ are forward integrable with respect to $\tilde{N}^2(d^-t,dz)$.

(3) $\pi(s)\gamma_1(s,z) > -1 + \varepsilon_\pi$ for a.s. (t,z) with respect to $dt \times \nu_1(dz)$ for some $\varepsilon_\pi \in (0,1)$ depending on π. $\kappa(s)\gamma_2(s,z) < 1 - \varepsilon_\kappa$ for a.s. (t,z) with respect to $dt \times \nu_2(dz)$ for some $\varepsilon_\kappa \in (0,1)$ depending on κ.

(4) $E\left[\int_0^T \int_{\mathbb{R}_0} (\pi(t)\gamma_1(t,z))^2 \nu_1(dz)dt\right] < \infty$, $E\left[\int_0^T \int_{\mathbb{R}_0} (\kappa(t)\gamma_2(t,z))^2 \nu_2(dz)dt\right] < \infty$.

(5) $\pi(t)$ is Malliavin differentiable and $D_{t^+,z}\pi(t) = \lim_{s\to t^+} D_{s,z}\pi(t)$ exists for a.s. (t,z), κ is Malliavin differentiable, and $D_{t^+,z}\kappa(t) = \lim_{s\to t^+} D_{s,z}\kappa(t)$ exist for a.s. (t,z).

(6) $\gamma_1(t,z)(\pi(t) + D_{t^+,z}\pi(t)) > -1 + \varepsilon_\pi$ for a.s. (t,z) for some $\varepsilon_\pi > 0$ depending on π, $\gamma_2(t,z)(\kappa(t) + D_{t^+,z}\kappa(t)) > -1 + \varepsilon_\kappa$ for a.s. (t,z) for some $\varepsilon_\kappa > 0$ depending on κ.

(7) $E\left[\int_0^T \int_{\mathbb{R}_0} \left|\gamma_1(t,z)D_{t^+,z}\pi(t)\right| \nu_1(dz)dt\right] < \infty$, $E\left[\int_0^T \int_{\mathbb{R}_0} \left|\gamma_2(t,z)D_{t^+,z}\kappa(t)\right| \nu_2(dz)dt\right] < \infty$.

(8) $\pi(t)$ and $\kappa(t)$ are progressively measurable with respect to the enlarged filtration $\mathbb{G}$, and for any $t \in [0, T]$,

$$E\left[\pi^2(t) \mid \mathcal{G}_t\right] < \infty, \quad E\left[\kappa^2(t) \mid \mathcal{G}_t\right] < \infty.$$

(9) The forward integrals $\int_0^T \pi(s)\gamma_1(s,z)d^-\tilde{N}^1(s,z)$ and $\int_0^T \kappa(s)\gamma_2(s,z)d^-\tilde{N}^2(s,z)$ are martingales with respect to $\mathbb{G}$, and satisfy

$$E\left[\sup_{0\le t\le T}\left|\int_0^t \pi(s)\gamma_1(s,z)d^-\tilde{N}^1(s,z)\right|^2\right] < \infty,$$

$$E\left[\sup_{0\le t\le T}\left|\int_0^t \kappa(s)\gamma_2(s,z)d^-\tilde{N}^2(s,z)\right|^2\right] < \infty, \textit{respectively.}$$

(10) For any $t \in [0, T]$,

$$E\left[\int_0^t \bar{F}(s)^2\pi(s)^2 ds\right] < \infty, \quad E\left[\int_0^t \bar{F}(s)^2\kappa(s)^2 ds\right] < \infty.$$

There is a collection of all admissible strategies by $\mathcal{A}_{\mathcal{G}}$.

Remark 2.1. *Condition (5) ensures that forward integrals are well-defined via Malliavin differentiability. Condition (9) guarantees their martingale property under the enlarged filtration, which supports semimartingale decomposition and the variational argument.*

The wealth process $X^{u(t)}$ the corresponds to the admissible strategy $u(t)$ is given by the following:

$$\begin{aligned} dX^u(t) =& \frac{\pi(t)X^u(t-)}{S(t)}dS(t) - \kappa(t)X^u(t-)dR(t) + \kappa(t)X^u(t-)\lambda(t)dt + \frac{X^u(t-) - \pi(t)X^u(t-)}{S_0(t)}dS_0(t) \\ =& X^u(t-)\Big\{[(\mu(t) - r(t))\pi(t) + r(t) + (\lambda(t) - p(t))\kappa(t)]dt \\ &+ \pi(t)\int_{\mathbb{R}_0}\gamma_1(t,z)\tilde{N}^1(d^-t, dz) - \kappa(t)\int_{\mathbb{R}_0}\gamma_2(t,z)\tilde{N}^2(d^-t, dz)\Big\}, \end{aligned}$$

$$\begin{aligned} X^u(t) =& x\exp\Big\{\int_0^t [(\mu(s) - r(s))\pi(s) + r(s) + (\lambda(s) - p(s))\kappa(s)]\, ds \\ &+ \int_0^t\int_{\mathbb{R}_0}\log(1 + \pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s, dz) + \int_0^t\int_{\mathbb{R}_0}\log(1 - \kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s, dz) \\ &+ \int_0^t\int_{\mathbb{R}_0}[\log(1 + \pi(s)\gamma_1(s,z)) - \pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds \\ &+ \int_0^t\int_{\mathbb{R}_0}[\log(1 - \kappa(s)\gamma_2(s,z)) + \kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\Big\}, \end{aligned} \tag{2.4}$$

where the initial wealth is $X^u(0) = x > 0$, and $\lambda(t)$ represents the premium per policy for the insurer at time t. Within the time interval dt, the premium that the insurer can collect is expressed as $\lambda(t)L(t)dt$. $\lambda(t)$ is a caglad process adapted to $(\mathcal{F}_t)_{0\le t\le T}$ that satisfies $\lambda(t) > p(t) > 0$. Due to the no-arbitrage principle in financial markets, it follows that $\mu(t) > r(t) > 0$, for each $t \in [0, T]$. Let

$$
\begin{aligned}
J^u(t) = &\int_0^t [(\mu(s) - r(s))\pi(s) + (\lambda(s) - p(s))\kappa(s)]\, ds \\
&+ \int_0^t \int_{\mathbb{R}_0} \log[1 + \pi(s)\gamma_1(s,z)]\, \tilde{N}^1(d^- s, dz) + \int_0^t \int_{\mathbb{R}_0} \log[1 - \kappa(s)\gamma_2(s,z)]\, \tilde{N}^2(d^- s, dz) \\
&+ \int_0^t \int_{\mathbb{R}_0} \Big[\log(1 + \pi(s)\gamma_1(s,z)) - \pi(s)\gamma_1(s,z) \Big] \nu_1(dz)ds \\
&+ \int_0^t \int_{\mathbb{R}_0} \Big[\log(1 - \kappa(s)\gamma_2(s,z)) + \kappa(s)\gamma_2(s,z) \Big] \nu_2(dz)ds.
\end{aligned} \tag{2.5}
$$

It follows from the Itô formula for the forward integrals that

$$
X^u(t) = x \exp\left\{ \int_0^t r(s)\, ds \right\} \exp\{J^u(t)\}. \tag{2.6}
$$

In practice, since insurers cannot know the exact time of exiting investments, the true terminal time for the insurer to make decisions is $\tau \wedge T$, which is an uncertain time horizon. We aim to find an admissible strategy that maximizes the expected logarithmic utility of terminal wealth, that is,

$$
\sup_{u \in \mathcal{A}_{\mathcal{G}}} E[\log X^u(T \wedge \tau)], \tag{2.7}
$$

with $X^u(t)$ being expressed by Eq (2.6). Furthermore, we have the following:

$$
\begin{aligned}
E[\log X^u(T \wedge \tau)] &= E[I_{\{\tau > T\}} \log X^u(T)] + E[I_{\{\tau \le T\}} \log X^u(\tau)] \\
&= E\left[\int_T^\infty \log X^u(T) dF(t) \right] + E\left[\int_0^T \log X^u(t) dF(t) \right] \\
&= E\left[\int_0^T \delta(t) \log X^u(t) dt + (1 - F(T)) \log X^u(T) \right].
\end{aligned} \tag{2.8}
$$

From Eqs (2.6) and (2.8), we can easily conclude that the optimal problem (2.7) is equivalent to the following:

$$
\sup_{u \in \mathcal{A}_{\mathcal{G}}} E\left[\int_0^T \delta(t) J^u(t) dt + \bar{F}(T) J^u(T) \right], \tag{2.9}
$$

where $\bar{F}(T) = 1 - F(T)$, and $J^u(t)$ satisfies Eq (2.5).

Theorem 2.1. *Let $u = (\pi(s), \kappa(s))$ be an admissible strategy. We have the following:*

$$
\begin{aligned}
&E\left[\int_0^T \delta(t) J^u(T) dt + \bar{F}(T) J^u(T) \right] \\
=&E\left\{ \int_0^T \bar{F}(s) \Big\{ [(\mu(s) - r(s))\pi(s) + (\lambda(s) - p(s))\kappa(s)]\, ds \right. \\
&+ \int_{\mathbb{R}_0} \log(1 + \pi(s)\gamma_1(s,z))\, \tilde{N}^1(ds, dz) + \int_{\mathbb{R}_0} \log(1 - \kappa(s)\gamma_2(s,z))\, \tilde{N}^2(ds, dz) \\
&+ \int_{\mathbb{R}_0} [\log(1 + \pi(s)\gamma_1(s,z)) - \pi(s)\gamma_1(s,z)]\, \nu_1(dz)ds \\
&\left. + \int_{\mathbb{R}_0} [\log(1 - \kappa(s)\gamma_2(s,z)) + \kappa(s)\gamma_2(s,z)]\, \nu_2(dz)ds \Big\} \right\}.
\end{aligned} \tag{2.10}
$$

Proof. Substituting Eq (2.5) into Eq (2.8) yields the following:

$$
\begin{aligned}
&E\left[\int_0^T \delta(t)\left(\int_0^t [(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right)dt\right.\\
&\quad\left.+\bar{F}(T)\left(\int_0^T [(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right)\right]\\
&+E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}\log(1+\pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s,dz)\,dt\right.\\
&\quad\left.+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}\log(1+\pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s,dz)\right]\\
&+E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z))-\pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds\,dt\right.\\
&\quad\left.+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z))-\pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds\right]\\
&+E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}\log(1-\kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s,dz)\,dt\right.\\
&\quad\left.+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}\log(1-\kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s,dz)\right]\\
&+E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}[\log(1-\kappa(s)\gamma_2(s,z))+\kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\,dt\right.\\
&\quad\left.+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}[\log(1-\kappa(s)\gamma_2(s,z))+\kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\right],
\end{aligned}
$$

Applying the Fubini theorem to each of the above five expected value terms, we have the following:

$$
\begin{aligned}
&E\left[\int_0^T \delta(t)\left(\int_0^t [(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right)dt\right.\\
&\quad\left.+\bar{F}(T)\left(\int_0^T [(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right)\right]\\
=&E\left[\int_0^T\int_s^T \delta(t)\,[(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,dt\,ds\right.\\
&\quad\left.+\int_0^T \bar{F}(T)\,[(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right]\\
=&E\left[\int_0^T \left(F(T)-F(s)+\bar{F}(T)\right)[(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right]\\
=&E\left[\int_0^T \bar{F}(s)\,[(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds\right],\text{respectively.}
\end{aligned}
$$

Similarly, we can derive the following results:

$$E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}\log(1+\pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s,dz)dt+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}\log(1+\pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s,dz)\right]$$
$$=E\left[\int_0^T \bar{F}(s)\int_{\mathbb{R}_0}\log(1+\pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s,dz)ds\right],$$

$$\begin{aligned}&E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z))-\pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds\,dt\right.\\&\left.\quad+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z))-\pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds\right]\\&=E\left[\int_0^T \bar{F}(s)\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z))-\pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds\right],\end{aligned}$$

$$E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}\log(1-\kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s,dz)\,dt+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}\log(1-\kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s,dz)\right]$$
$$=E\left[\int_0^T \bar{F}(s)\int_{\mathbb{R}_0}\log(1-\kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s,dz)\,ds\right],$$

$$\begin{aligned}&E\left[\int_0^T \delta(t)\int_0^t\int_{\mathbb{R}_0}[\log(1-\kappa(s)\gamma_2(s,z))+\kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\,dt\right.\\&\left.\quad+\bar{F}(T)\int_0^T\int_{\mathbb{R}_0}[\log(1-\kappa(s)\gamma_2(s,z))+\kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\right]\\&=E\left[\int_0^T \bar{F}(s)\int_{\mathbb{R}_0}[\log(1-\kappa(s)\gamma_2(s,z))+\kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\right].\end{aligned}$$

Summing up the above results, (2.10) can be obtained.

Note that Conditions (2), (3), (5), and (6) in Definition 2.1 ensure the forward integrability of $\log(1+\pi(t)\gamma_1(t,z))$ and $\log(1-\kappa(t)\gamma_2(t,z))$. We conclude that

$$E\left[\int_0^T\int_{\mathbb{R}_0}\log(1+\pi(s)\gamma_1(s,z))\tilde{N}^1\,(d^-s,dz)\right]=E\left[\int_0^T\int_{\mathbb{R}_0}D_{s^+,z}\log(1+\pi(s)\gamma_1(s,z))\nu_1(dz)ds\right], \tag{2.11}$$

$$E\left[\int_0^T\int_{\mathbb{R}_0}\log(1-\kappa(s)\gamma_2(s,z))\tilde{N}^2\,(d^-s,dz)\right]=E\left[\int_0^T\int_{\mathbb{R}_0}D_{s^+,z}\log(1-\kappa(s)\gamma_2(s,z))\nu_2(dz)ds\right]. \tag{2.12}$$

By combining Eqs (2.11) and (2.12) with (2.10), we have the following:

$$\begin{aligned}&\int_0^T \delta(t)J^u(t)dt+\bar{F}(T)J^u(T)\\&=\int_0^T \bar{F}(s)\Big\{[(\mu(s)-r(s))\pi(s)+(\lambda(s)-p(s))\kappa(s)]\,ds+\int_{\mathbb{R}_0}D_{s^+,z}\log(1+\pi(s)\gamma_1(s,z))\nu_1(dz)ds\\&\quad+\int_{\mathbb{R}_0}D_{s^+,z}\log(1-\kappa(s)\gamma_2(s,z))\nu_2(dz)ds+\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z))-\pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds\\&\quad+\int_{\mathbb{R}_0}[\log(1-\kappa(s)\gamma_2(s,z))+\kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\Big\}.\end{aligned} \tag{2.13}$$

Denote the following:

$$M^{(\pi)}(t)=\int_0^t \bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right\}ds+\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}\tilde{N}^1\left(d^-s,dz\right), \quad (2.14)$$

$$M^{(\kappa)}(t)=\int_0^t \bar{F}(s)\left\{\lambda(s)-p(s)-\int_{\mathbb{R}_0}\frac{\kappa(s)\gamma_2^2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\nu_2(dz)\right\}ds-\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\tilde{N}^2\left(d^-s,dz\right). \quad (2.15)$$

□

Theorem 2.2. *Suppose $u^* = (\pi^*,\kappa^*)$ is optimal for problem (2.7). Then, $M^{(\pi)}(t)$ and $M^{(\kappa)}(t)$ are martingales with respect to the filtration $\mathcal{G}_t$.*

Proof. Now, suppose $u^* = (\pi^*(s),\kappa^*(s))$ is optimal for the problem (2.7). Fix $t \in [0,T]$ and $h > 0$ such that $t + h \leq T$. Choose $(\beta_1,\beta_2) \in \mathcal{A}_{\mathcal{G}}$ of the form

$$\beta_1(s) = \chi_{[t,t+h]}(s)\beta_0, \quad \beta_2(s) = \chi_{[t,t+h]}(s)\beta_0', \quad 0 \leq s \leq T,$$

where β_0 and β_0' are bounded $\mathcal{G}_t$-measurable random variables such that $D_{t,z}\beta_1$ and $D_{t,z}\beta_2$ are bounded a.s.. Then, it is clear from Definition 2.1 that there exists a $\xi > 0$ such that $\pi^* + y\beta_1,\ \kappa^* + y\beta_2 \in \mathcal{A}_{\mathcal{G}}$ for all $y \in (-\xi,\xi)$. Then, the function

$$E\left[\int_0^T \delta(t)J^{(\pi^*+y\beta_1,\kappa^*+y\beta_2)}\,dt + \bar{F}(T)J^{(\pi^*+y\beta_1,\kappa^*+y\beta_2)}\right]$$

is maximal for $y = 0$. Hence, by Eq (2.12),

$$\begin{aligned}&\frac{d}{dy}E\left[\int_0^T \delta(t)J^{(\pi^*+y\beta_1,\kappa^*)}(t)\,dt + \bar{F}(T)J^{(\pi^*+y\beta_1,\kappa^*)}(T)\right]\\ =&E\left[\int_0^T\bar{F}(s)\left\{(\mu(s)-r(s))\beta_1(s)+\int_{\mathbb{R}_0}\left[\frac{\gamma_1(s,z)\beta_1(s)}{1+\pi(s)\gamma_1(s,z)}-\gamma_1(s,z)\beta_1(s)+D_{s^+,z}\left(\frac{\gamma_1(s,z)\beta_1(s)}{1+\pi(s)\gamma_1(s,z)}\right)\right]\nu_1(dz)\right\}ds\right]\\ =&0,\end{aligned} \quad (2.16)$$

$$\begin{aligned}&\frac{d}{dy}E\left[\int_0^T \delta(t)J^{(\pi^*,\kappa^*+y\beta_2)}(t)\,dt + \bar{F}(T)J^{(\pi^*,\kappa^*+y\beta_2)}(T)\right]\\ =&E\left[\int_0^T\bar{F}(s)\left\{(\lambda(s)-p(s))\beta_2(s)+\int_{\mathbb{R}_0}\left[\frac{-\gamma_2(s,z)\beta_2(s)}{1-\kappa(s)\gamma_2(s,z)}+\gamma_2(s,z)\beta_2(s)+D_{s^+z}\left(\frac{-\gamma_2(s,z)\beta_2(s)}{1-\kappa(s)\gamma_2(s,z)}\right)\right]\nu_2(dz)\right\}ds\right]\\ =&0.\end{aligned} \quad (2.17)$$

For convenience, we introduce the following notations:

$$C_1(s,z) = \frac{\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}, \qquad C_2(s,z) = \frac{\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}.$$

After some simple calculations, using Conditions (3) and (7) in Definition 2.1, Eq (2.16) can be transformed into the following:

$$0=E\left[\int_0^T\bar{F}(s)\beta_1(s)\{\mu(s)-r(s)\}\,ds-\int_0^T\int_{R_0}C_1(s,z)\gamma_1(s,z)\pi(s)\nu_1(dz)ds+\int_0^T\int_{R_0}\bar{F}(s)C_1(s,z)\beta_1(s)\tilde{N}^1(ds,dz)\right].$$

Substituting the expression for $\beta_1(s)$ into the preceding equation gives rise to the following:

$$0=E\left[\beta_0\left\{\int_t^{t+h}\bar{F}(s)\left(\mu(s)-r(s)-\int_{\mathbb{R}_0}C_1(s,z)\gamma(s,z)\pi(s)\nu_1(dz)\right)ds+\int_t^{t+h}\int_{\mathbb{R}_0}\bar{F}(s)C_1(s,z)\tilde{N}^1\left(d^-s,dz\right)\right\}\right].$$

Since β_0 is an arbitrarily bounded $\mathcal{G}_t$-measurable random variable, we have the following:

$$0=E\left[\int_t^{t+h}\bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}C_1(s,z)\gamma_1(s,z)\pi(s)\nu_1(dz)\right\}ds+\int_t^{t+h}\int_{\mathbb{R}_0}\bar{F}(s)C_1(s,z)\tilde{N}^1\left(d^-s,dz\right)\mid\mathcal{G}_t\right].$$

By similar arguments, Eq (3.20) can be rewritten as follows:

$$0=E\left[\int_t^{t+h}\bar{F}(s)\left\{\lambda(s)-p(s)-\int_{\mathbb{R}_0}C_2(s,z)\gamma_2(s,z)\kappa(s)\nu_2(dz)\right\}ds-\int_t^{t+h}\int_{\mathbb{R}_0}\bar{F}(s)C_2(s,z)\tilde{N}^2\left(d^-s,dz\right)\mid\mathcal{G}_t\right].$$

We can conclude that

$$E\left[\left(M^{(\pi^*)}(t+h)-M^{(\pi^*)}(t)\right)\mid\mathcal{G}_t\right]=0,\quad E\left[\left(M^{(\kappa^*)}(t+h)-M^{(\kappa^*)}(t)\right)\mid\mathcal{G}_t\right]=0,\quad u\in[0,T].$$

Therefore, the processes$M^{(\pi^*)}(t), M^{(\kappa^*)}(t)$ are both $\mathcal{G}_t$-martingales. □

Then, we have the following.

Theorem 2.3. *Suppose $\pi(t)=\pi^*(t)$, $\kappa(t)=\kappa^*(t)$ are the optimal solutions for problem (2.7). Furthermore, the process*

$$D_1:=\int_0^t\int_{\mathbb{R}_0}\frac{\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}\tilde{N}^1\left(d^-s,dz\right),D_2:=\int_0^t\int_{\mathbb{R}_0}\frac{\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\tilde{N}^2\left(d^-s,dz\right)$$

is a special $\mathcal{G}_t$-semimartingale with a decomposition given by (2.14) and (2.15).

However, note that this alone would not imply that D_1 and D_2 are $\mathcal{G}_t$-semimartingales, because $\nu^i_{\mathcal{G}}(dt,dz)$ need not integrate to a process of finite variation. We may write the following:

$$\begin{aligned}M^{(\pi)}(t)=\int_0^t\bar{F}(s)\Big\{&\int_{\mathbb{R}_0}\frac{\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}(N^1-\nu^1_{\mathcal{G}})(ds,dz)\\&+\int_{\mathbb{R}_0}\frac{\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}\left(\nu^1_{\mathcal{G}}-\nu_1\right)(ds,dz)+\left[\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right]ds\Big\},\end{aligned}\tag{2.18}$$

$$\begin{aligned}M^{(\kappa)}(t)=\int_0^t\bar{F}(s)\Big\{&\int_{\mathbb{R}_0}\frac{-\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}(N^2-\nu^2_{\mathcal{G}})(ds,dz)\\&+\int_{\mathbb{R}_0}\frac{\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\left(\nu_2-\nu^2_{\mathcal{G}}\right)(ds,dz)+\left[\lambda(s)-p(s)-\int_{\mathbb{R}_0}\frac{\kappa(s)\gamma_2^2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\nu_2(dz)\right]ds\Big\}.\end{aligned}\tag{2.19}$$

Hence, by the uniqueness of the semimartingale decomposition of the $\mathcal{G}_t$-semimartingale $M^{(\pi)}(t)$ and $M^{(\kappa)}(t)$, we conclude that the finite variation part above must be 0. Therefore, we get the following result.

Theorem 2.4. *Suppose $\pi(t), \kappa(t) \in \mathcal{A}_{\mathcal{G}}$ are optimal for problem (2.7). Then, $\pi(t), \kappa(t)$ solves the following equations:*

$$\int_0^t \bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right\}ds=\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}\left(\nu_1-\nu_{\mathcal{G}}^1\right)(ds,dz), \tag{2.20}$$

$$\int_0^t \bar{F}(s)\left\{\lambda(s)-p(s)-\int_{\mathbb{R}_0}\frac{\kappa(s)\gamma_2^2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\nu_2(dz)\right\}ds=\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)}\left(\nu_{\mathcal{G}}^2-\nu_2\right)(ds,dz). \tag{2.21}$$

This implies that when $\mathcal{F}_t \subset \mathcal{G}_t$, the optimal strategy depends on $\bar{F}(s)$, which characterizes the randomness of the exit time τ.

Proof. From Theorem 2.3,

$$\int_0^t \int_{\mathbb{R}_0} \frac{\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}\tilde{N}^1\left(d^- s, dz\right)$$

is a semimartingale. By Eq (2.14) and the semimartingale decomposition theorem, we obtain the following:

$$\int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}\tilde{N}^1(ds,dz) = M^{(\pi)}(t) + \int_0^t \bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right\}ds.$$

Here, $\tilde{N}^1(d^- s, dz) = N^1(d^- s, dz) - \nu_1(dz)$. It follows from Theorem 2.2 that $M^{(\pi)}(t)$ is a martingale. Furthermore, we perform the following identity transformation for the compound Poisson random measure:

$$\tilde{N}^1(d^- s, dz) = (N_1(d^- s, dz) - \nu_{\mathcal{G}}^1(dz)) + (\nu_{\mathcal{G}}^1(dz) - \nu_1(dz)).$$

Substituting the above change of Poisson measure into the semimartingale decomposition yields the following:

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}(N_1(d^- s,dz)-\nu_{\mathcal{G}}^1(dz)) + \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}(\nu_{\mathcal{G}}^1(dz)-\nu_1(dz))\\ =&M^{(\pi)}(t) + \int_0^t \bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right\}ds.\end{aligned}$$

By transposition, we have the following:

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}(N_1(d^- s,dz)-\nu_{\mathcal{G}}^1(dz)) - M^{(\pi)}(t)\\ =&\int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}(\nu_1(dz)-\nu_{\mathcal{G}}^1(dz)) + \int_0^t \bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right\}ds.\end{aligned}$$

Both terms on the left-hand side are martingales, so the finite-variation part on the right-hand side must be zero, i.e.,

$$\int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi(s)\gamma_1(s,z)}(\nu_1(dz)-\nu_{\mathcal{G}}^1(dz)) + \int_0^t \bar{F}(s)\left\{\mu(s)-r(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)}\nu_1(dz)\right\}ds = 0.$$

Therefore, we obtain Eq (2.20). Equation (2.21) can be similarly derived. □

Corollary 2.1. *Suppose $\mathcal{F}_t = \mathcal{G}_t, t \in [0,T]$. Then, there are two situations of optimal strategy $u^* = (\pi^*, \kappa^*)$:*
(1) If $\bar{F}(s)$=0, it means $\int_0^t \delta(s)ds$=1.
This could happen if and only if t=∞, which is mentioned in Chen et al. [20], who also considered the uncertainty of the time horizon in optimal investment and risk control.
(2) If $\bar{F}(s) \neq 0$, it means

$$\mu(s) - r(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma_1^2(s,z)}{1+\pi(s)\gamma_1(s,z)} \nu_1(dz) = 0, \tag{2.22}$$

$$\lambda(s) - p(s) - \int_{\mathbb{R}_0} \frac{\kappa(s)\gamma_2^2(s,z)}{1-\kappa(s)\gamma_2(s,z)} \nu_2(dz) = 0. \tag{2.23}$$

Remark 2.2. *Corollary 2.1 corresponds to the scenario without in information: In this scenario, the insurer can only obtain public market information, which is characterized by the original filtration $\{\mathcal{F}_t\}$, and cannot observe any additional information beyond the scope of public market information. Under this circumstance, $\nu = \nu_g$ holds. Consequently, the left-hand sides of the optimal strategy expressions (2.20) and (2.21) under inside information identically vanish. Since $\bar{F}(s) \neq 0$, this implies that the integrands on the left-hand sides must be identically zero. Hence, Eqs (2.22) and (2.23) are valid. The results are consistent with the classical insurance portfolio optimization. Theorem 2.4 corresponds to the scenario with inside information: in this scenario, the decision-maker possesses an information advantage richer than public market information. This advantageous information is characterized by the enlarged filtration $\{\mathcal{G}_t\}$, (i.e., for all $t \in [0,T]$, $\mathcal{F}_t \subset \mathcal{G}_t$ holds (where $\{\mathcal{G}_t\}$ is the enlarged filtration of $\{\mathcal{F}_t\}$), reflecting the supplementary effect of inside information on the information set).*

To clearly reveal the influence mechanism of inside information on the optimal investment strategy, we adopt the semimartingale decomposition technique under the enlarged filtration $\{\mathcal{G}_t\}$. In the specific derivation process, we use the method of adding and subtracting ν_g as a supplementary term, which not only strictly satisfies the theoretical conditions of semimartingale decomposition, but also subtly highlights the incremental impact of inside information on the optimal strategy, thus making the action path and effect of inside information more intuitively distinguishable through ν_g.

3. Optimal portfolio in a mixed market

In this section, we adopt a more general jump-diffusion models in the financial and insurance markets to investigate the optimal investment strategy. We assume that $\sigma(s)\pi(s)$ and $q(s)\kappa(s)$, $s \in [0,T]$ are caglad and forward integrable with respect to W^1, and $q(s)\kappa(s)$, $s \in [0,T]$ is caglad and forward integrable with respect to W^2. The risk-free asset price process S_0 is described by the following:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \\ S_0(0) = 1. \end{cases} \tag{3.1}$$

The dynamic evolution process of the risky asset is as follows:

$$\begin{cases} dS(t) = S(t-)\Big[\mu(t)dt + \sigma(t)d^-W_t^1 + \int_{\mathbb{R}_0} \gamma_1(t,z)\tilde{N}^1(d^-t,dz)\Big], \\ S(0) > 0. \end{cases} \tag{3.2}$$

The risk process per policy for the insurer is governed by the following:

$$\begin{cases} dR_t = p(t)dt + q(t)d\overline{W}_t + \int_{\mathbb{R}_0} \gamma_2(t,z)\tilde{N}^2(dt,dz), \\ R_0 = 0. \end{cases} \tag{3.3}$$

where $q(t), \sigma(t)$ are assumed to be bounded parameter processes that are caglad and adapted to $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$. $\overline{W}_t, t \in [0,T]$ is a Brownian motion given by $\overline{W}_t = \rho W^1(t) + \sqrt{1-\rho^2}W^2(t)$ with $\rho \in (-1,1)$ to describe the correlation between the insurer's liabilities and capital gains in the financial market. $W^1(t), W^2(t), \tilde{N}^1(dt,dz)$ and $\tilde{N}^2(dt,dz)$ are mutually independent.

For technical considerations, we further impose the following conditions on Definition 2.1:
(1)

$$\begin{aligned} E\Big[\int_0^T \Big(&|r(s)| + |\mu(s)| + \sigma^2(s) + \left|D^1_{s+}\sigma(s)\right| + |p(s)| + \sum_{i=1}^2 q_i^2(s) + \sum_{i=1}^2 \left|D^i_{s+}q_i(s)\right| \\ &+ \sum_{i=1}^2 \int_{\mathbb{R}_0} \Big(\left|\gamma_i(s,z)\right| + \left|\gamma_i(s,z)\right|^2 + \left|D^i_{s+,z}\gamma_i(s,z)\right| + \left|D^i_{s+,z}\gamma_i(s,z)\right|^2\Big)\nu_i(dz)\Big)ds\Big] < \infty. \end{aligned}$$

(2) $\sigma(t)$ and $q(t)$ are forward integrable with respect to W^1, and $q(t)$ is forward integrable with respect to W^2.

The wealth $X^{u(t)}(t)$ of the insurer in the mixed market evolves according to the following:

$$\begin{aligned} dX^u(t) = X^u(t-)\Big\{&[(\mu(t) - r(t))\pi(t) + r(t) + (\lambda(t) - p(t))\kappa(t)]dt + (\sigma(t)\pi(t) - \rho q(t)\kappa(t))d^-W^1_t \\ -& \sqrt{1-\rho^2}q(t)\kappa(t)d^-W^2_t + \pi(t)\int_{\mathbb{R}_0} \gamma_1(t,z)\tilde{N}^1(d^-t,dz) - \kappa(t)\int_{\mathbb{R}_0} \gamma_2(t,z)\tilde{N}^2(d^-t,dz)\Big\}, \end{aligned} \tag{3.4}$$

with $X^u(0) = x > 0$. By the Itô formula for forward integrals, we have the following:

$$\begin{aligned} X^u(t) = x\exp\Big\{&\int_0^t \Big[(\mu(s) - r(s))\pi(s) + r(s) + (\lambda(s) - p(s))\kappa(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) \\ &+\rho\sigma(s)q(s)\pi(s)\kappa(s) - \frac{1}{2}q^2(s)\kappa^2(s)\Big]ds + \int_0^t (\sigma(s)\pi(s) - \rho q(s)\kappa(s))d^-W^1_s \\ &- \int_0^t \sqrt{1-\rho^2}q(s)\kappa(s)d^-W^2_s + \int_0^t\int_{\mathbb{R}_0} \log(1 + \pi(s)\gamma_1(s,z))\,\tilde{N}^1(d^-s,dz) \\ &+\int_0^t\int_{\mathbb{R}_0} \log(1-\kappa(s)\gamma_2(s,z))\,\tilde{N}^2(d^-s,dz) + \int_0^t\int_{\mathbb{R}_0}[\log(1+\pi(s)\gamma_1(s,z)) - \pi(s)\gamma_1(s,z)]\,\nu_1(dz)ds \\ &+ \int_0^t\int_{\mathbb{R}_0} [\log(1-\kappa(s)\gamma_2(s,z)) + \kappa(s)\gamma_2(s,z)]\,\nu_2(dz)ds\Big\}. \end{aligned} \tag{3.5}$$

Let

$$
\begin{aligned}
J^u(T) = &\int_0^T \Big[(\mu(s) - r(s))\pi(s) + (\lambda(s) - p(s))\kappa(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) + \rho\sigma(s)q(s)\pi(s)\kappa(s) \\
&- \frac{1}{2}q^2(s)\kappa^2(s)\Big] ds + \int_0^T (\sigma(s)\pi(s) - \rho q(s)\kappa(s)) d^- W_s^1 - \int_0^T \sqrt{1-\rho^2} q(s)\kappa(s) d^- W_s^2 \\
&+ \int_0^T \int_{\mathbb{R}_0} \log(1 + \pi(s)\gamma_1(s,z)) \tilde{N}^1(d^-s, dz) + \int_0^T \int_{\mathbb{R}_0} \log(1 - \kappa(s)\gamma_2(s,z)) \tilde{N}^2(d^-s, dz) \\
&+ \int_0^T \int_{\mathbb{R}_0} [\log(1 + \pi(s)\gamma_1(s,z)) - \pi(s)\gamma_1(s,z)] \nu_1(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} [\log(1 - \kappa(s)\gamma_2(s,z)) + \kappa(s)\gamma_2(s,z)] \nu_2(dz)ds.
\end{aligned} \tag{3.6}
$$

Using a similar argument as in Theorem 2.1, we obtain the folloowing:

$$
\begin{aligned}
&\int_0^T \delta(t)J^u(t)dt + \bar{F}(T)J^u(T) \\
=&\Big[\int_0^T \bar{F}(s)\Big\{\Big[(\mu(s) - r(s))\pi(s) + (\lambda(s) - p(s))\kappa(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) + \rho\sigma(s)q(s)\pi(s)\kappa(s) \\
&- \frac{1}{2}q^2(s)\kappa^2(s)\Big]ds + (\sigma(s)\pi(s) - \rho q(s)\kappa(s))d^- W_s^1 - \sqrt{1-\rho^2} q(s)\kappa(s) d^- W_s^2 \\
&+ \int_{\mathbb{R}_0} \log(1 + \pi(s)\gamma_1(s,z)) \tilde{N}^1(d^-s, dz) + \int_{\mathbb{R}_0} \log(1 - \kappa(s)\gamma_2(s,z)) \tilde{N}^2(d^-s, dz) \\
&+ \int_{\mathbb{R}_0} [\log(1 + \pi(s)\gamma_1(s,z)) - \pi(s)\gamma_1(s,z)] \nu_1(dz)ds \\
&+ \int_{\mathbb{R}_0} [\log(1 - \kappa(s)\gamma_2(s,z)) + \kappa(s)\gamma_2(s,z)] \nu_2(dz)ds\Big\}\Big].
\end{aligned} \tag{3.7}
$$

Denote the following:

$$
\begin{aligned}
M_1^u(t) = &\int_0^t \bar{F}(s)\left[\mu(s) - r(s) - \sigma^2(s)\pi(s) + \rho\sigma(s)q(s)\kappa(s)\right] ds + \int_0^t \bar{F}(s)\sigma(s)dW_s^1 \\
&+ \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1 + \pi(s)\gamma_1(s,z)} \tilde{N}^1(d^-s, dz) - \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1^2(s,z)\pi(s)}{1 + \pi(s)\gamma_1(s,z)} \nu_1(dz)ds,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
M_2^u(t) = &\int_0^t \bar{F}(s)\left[\lambda(s) - p(s) + \rho\sigma(s)q(s)\pi(s) - q(s)\kappa(s)\right] ds + \int_0^t \bar{F}(s)\rho(s)q(s)dW_s^1 \\
&- \int_0^t \sqrt{1-\rho^2} q(s) d^- W_s^2 + \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa(s)\gamma_2(s,z)} \tilde{N}^2(d^-s, dz) - \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2^2(s,z)\kappa(s)}{1-\kappa(s)\gamma_2(s,z)} \nu_2(dz)ds.
\end{aligned} \tag{3.9}
$$

Remark 3.1. *For any t, let $\mathcal{G}_t$ be the σ-algebra generated by the random variables $W^i(s)$ and $\tilde{N}^i(ds, dz), z \in \mathbb{R}_0, s \le t$. A stochastic process $\gamma_i = \gamma_i(t,z), t \ge 0, z \in \mathbb{R}_0$ is called $\mathbb{G}$-adapted if for all $t \ge 0$ and for all $z \in \mathbb{R}_0$, the random variable $\gamma_i(t,z) = \gamma_i(t,z,\omega), \omega \in \Omega$, is $\mathcal{G}_t$-measurable for any*

$\mathbb{G}$*-adapted process* γ *such that*

$$E\left[\int_0^T \int_{\mathbb{R}_0} \gamma_i{}^2(t,z)v^i(dz)dt\right] < \infty \quad \text{for some } T > 0. \tag{3.10}$$

Then, we have the following Itô isometry:

$$E\left[\left(\int_0^T \int_{\mathbb{R}_0} \gamma_i(t,z)\tilde{N}^i(dt,dz)\right)^2\right] = E\left[\int_0^T \int_{\mathbb{R}_0} \gamma_i^2(t,z)v^i(dz)dt\right]. \tag{3.11}$$

By Condition (3) in Definition 2.1, we have $1+(\pi(s)+\xi\beta_1(s))\gamma_1(s,z) \geqslant \varepsilon_\pi+\xi\left(\varepsilon_{\beta_1}-1\right), dt\times v_1(dz)-$ a.e. and $1-(\kappa(s)+\xi\beta_1(s))\gamma_2(s,z) \geqslant \varepsilon_\kappa+\xi\left(\varepsilon_{\beta_2}-1\right), dt\times v_2(dz)-$ a.e.. Therefore, $\zeta > 0$ is small enough such that for all $\xi \in (-\zeta,\zeta)$, we have $1+(\pi(s)+\xi\beta_1(s))\gamma_1(s,z) \geqslant \varepsilon_\pi-\zeta_1, dt\times v_1(dz)$-a.e. for some $\zeta_1 \in (0,\varepsilon_\pi)$ and $1-(\kappa(s)+\xi\beta_2(s))\gamma_2(s,z) \geqslant \varepsilon_\kappa-\zeta_2, dt\times v_2(dz)$ a.e., for some $\zeta_2 \in (0,\varepsilon_k)$; we obtain the following:

$$E\left[\int_0^T\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+(\pi(s)+\xi\beta_1(s))\gamma_1(s,z)}\tilde{N}^1(d^-s,dz)\right]^2 \leqslant \frac{1}{(\varepsilon_\pi-\zeta_1)^2}E\left[\int_0^T\int_{\mathbb{R}_0} \bar{F}^2(s)\gamma_1^2(s,z)v_1(dz)ds\right]<\infty,$$

$$E\left[\int_0^T\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2(s,z)}{1-(\kappa(s)+\xi\beta_2(s))\gamma_2(s,z)}\tilde{N}^2(d^-s,dz)\right]^2 \leqslant \frac{1}{(\varepsilon_\kappa-\zeta_2)^2}E\left[\int_0^T\int_{\mathbb{R}_0} \bar{F}^2(s)\gamma_2^2(s,z)v_2(dz)ds\right]<\infty,$$

where$(\pi(s),\kappa(s))$ and $\beta_1(s)$, $\beta_2(s)$ all belong to $\mathcal{A}_{\mathcal{G}}$. Additionally, the coefficients $\mu(t), r(t), \sigma(t), \gamma_1(t), \gamma_2(t)$ are bounded, and we conclude $E\left(M_1^{(\pi+\xi\beta_1,\kappa)}\right)^2 < \infty$ and $E\left(M_2^{(\pi,\kappa+\xi\beta_2)}\right)^2 < \infty$ in $\xi \in (-\zeta,\zeta)$. Consequently, $\left(M_1^{(\pi+\xi\beta_1,\kappa)}\right)_{\xi\in(-\zeta,\zeta)}$ and $\left(M_2^{(\pi,\kappa+\xi\beta_2)}\right)_{\xi\in(-\zeta,\zeta)}$ are uniformly integrable.

Theorem 3.1. *The admissible strategy* $u^*(t) = (\pi^*(t),\kappa^*(t))$ *is optimal for problem (2.9) when processes* $\left\{M_1^{(\pi^*,\kappa^*)}(t)\right\}$ *and* $\left\{M_2^{(\pi^*,\kappa^*)}(t)\right\}$ *are* $(\mathcal{G}_t, P)$*-martingales.*

Proof. The proof is analogous to Theorem 2.2 and is omitted here. □

The integer valued random measure $N^1(dt,dz)$ and $N^2(dt,dz)$ have unique predictable compensators $v_{\mathcal{G}}^1(dt,dz)$ and $v_{\mathcal{G}}^2(dt,dz)$, respectively. Thus, $M_1^{u^*}(t)$ can be rewritten as follows:

$$\begin{aligned} M_1^{u^*}(t) &= \int_0^t \bar{F}(s)\sigma(s)dW_s^1 + \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi^*(s)\gamma_1(s,z)}\left(N^1-v_{\mathcal{G}}^1\right)(ds,dz) \\ &+\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi^*(s)\gamma_1(s,z)}\left(v_{\mathcal{G}}^1-v_{\mathcal{F}}^1\right)(ds,dz)+\int_0^t\bar{F}(s)\left[\mu(s)-r(s)-\sigma^2(s)\pi^*(s)+\rho\sigma(s)q(s)\kappa^*(s)\right]ds \\ &-\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\gamma_1^2(s,z)\pi^*(s)}{1+\pi^*(s)\gamma_1(s,z)}v_1(dz)ds, \end{aligned} \tag{3.12}$$

where $v_{\mathcal{F}}^1(ds,dz) = v_1(dz)ds$. Furthermore, we see that the orthogonal decomposition of $\left\{\boldsymbol{M}_1^{u^*}(t)\right\}$ into a continuous part $\left\{{}^c\boldsymbol{M}_1^{u^*}(t)\right\}$ and a discontinuous part $\left\{{}^dM_1^{u^*}(t)\right\}$ is given by the following:

$${}^cM_1^{u^*}(t) = \int_0^t \bar{F}(s)\sigma(s)dW_s^1 + \int_0^t \bar{F}(s)\sigma(s)\alpha_1(s)ds, \tag{3.13}$$

$$ {}^{d}M_1^{u^*}(t) = \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi^*(s)\gamma_1(s,z)} \left(N^1 - \nu_{\mathcal{G}}^1\right)(ds,dz), \tag{3.14} $$

where $\{\alpha_1(s)\}$ is an $\mathcal{G}_t$-adapted process. By the uniqueness of the semimartingale decomposition of the $\mathcal{G}_t$-semimartingale $\left\{M_1^{u^*}(t)\right\}$, we conclude that the finite variation part of $\left\{M_1^{u^*}(t)\right\}$ must be 0. Consequently, we obtain the following result:

$$ \begin{aligned} 0 &= \int_0^t \bar{F}(s)\sigma(s)\alpha_1(s)ds = \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1(s,z)}{1+\pi^*(s)\gamma_1(s,z)} \left(\nu_{\mathcal{G}}^1 - \nu_{\mathcal{F}}^1\right)(ds,dz) \\ &+\int_0^t \bar{F}(s)\left[\mu(s)-r(s)-\sigma^2(s)\pi^*(s)+\rho\sigma(s)q(s)\kappa^*(s)\right]ds - \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1^2(s,z)\pi^*(s)}{1+\pi^*(s)\gamma_1(s,z)}\nu_1(dz)ds. \end{aligned} \tag{3.15} $$

Similarly, we can write the following:

$$ \begin{aligned} M_2^{u^*}(t) &= -\int_0^t \bar{F}(s)\rho q(s)dW_s^1 - \int_0^t \bar{F}(s)\sqrt{1-\rho^2}q(s)dW_s^2 - \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa^*(s)\gamma_2(s,z)}\left(N^2-\nu_{\mathcal{G}}^2\right)(ds,dz) \\ &+\int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa^*(s)\gamma_2(s,z)}\left(\nu_{\mathcal{F}}^2-\nu_{\mathcal{G}}^2\right)(ds,dz) + \int_0^t \bar{F}(s)\left[\lambda(s)-p(s)+\rho\sigma(s)q(s)\pi^*(s)-q^2(s)\kappa^*(s)\right]ds \\ &- \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\kappa^*(s)\gamma_2^2(s,z)}{1-\kappa^*(s)\gamma_2(s,z)}\nu_2(dz)ds. \end{aligned} \tag{3.16} $$

$M_2^{u^*}(t)$ can be decomposed into a continuous part $\left\{{}^{c}\boldsymbol{M}_2^{u^*}(t)\right\}$ and a discontinuous part$\left\{{}^{d}M_2^{u^*}(t)\right\}$, is given by

$$ \begin{aligned} {}^{c}M_2^{u^*}(t) = &-\int_0^t \bar{F}(s)\rho q(s)dW_s^1 - \int_0^t \bar{F}(s)\rho q(s)\alpha_1(s)ds \\ &- \int_0^t \bar{F}(s)\sqrt{1-\rho^2}q(s)dW_s^2 - \int_0^t \bar{F}(s)\sqrt{1-\rho^2}q(s)\alpha_2(s)ds, \end{aligned} \tag{3.17} $$

$$ {}^{d}M_2^{u^*}(t) = -\int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa^*(s)\gamma_2(s,z)} \left(N^2 - \nu_{\mathcal{G}}^2\right)(ds,dz), \tag{3.18} $$

where $\{\alpha_2(s)\}$ is also a $\mathcal{G}_t$-adapted process. By the uniqueness of the semimartingale decomposition of the $\mathcal{G}_t$-semimartingale $\left\{M_2^{u^*}(t)\right\}$, we deduce that the finite variation part of $\left\{M_2^{u^*}(t)\right\}$ must be 0, that is,

$$ \begin{aligned} 0 &= \int_0^t \bar{F}(s)\rho q(s)\alpha_1(s)ds + \int_0^t \bar{F}(s)\sqrt{1-\rho^2}q(s)\alpha_2(s)ds + \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2(s,z)}{1-\kappa^*(s)\gamma_2(s,z)}\left(\nu_{\mathcal{F}}^2-\nu_{\mathcal{G}}^2\right)(ds,dz) \\ &+ \int_0^t \bar{F}(s)\left[\lambda(s)-p(s)+\rho\sigma(s)q(s)\pi^*(s)-q^2(s)\kappa^*(s)\right]ds - \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2^2(s,z)\kappa^*(s)}{1-\kappa^*(s)\gamma_2(s,z)}\nu_2(dz)ds, \end{aligned} \tag{3.19} $$

where $\nu_{\mathcal{F}}^2(ds,dz) = \nu_2(dz)ds$ with $s \in [0,T]$. Suppose that $u^* \in \mathcal{A}_{\mathcal{G}}$ is an optimal strategy for problem (2.9). Then, $u^* = (\pi^*, \kappa^*)$ solves Eqs (3.15) and (3.19), where $\{\alpha_1(s)\}$ and $\{\alpha_2(s)\}$ are $\mathcal{G}_t$-adapted processes, $\nu_{\mathcal{G}}^1$ and $\nu_{\mathcal{G}}^2$ are the $\mathcal{G}_t$-compensators of N^1 and N^2, respectively, and $\nu_{\mathcal{F}}^i(ds,dz) = \nu_i(dz)ds$, $i = 1,2$.

Corollary 3.1. *Suppose $\mathcal{G}_t = \mathcal{F}_t$, that is, the insurer has no inside information. Then, the necessary conditions for $u^* = (\pi^*, \kappa^*)$ to be optimal are as follows:*

$$ \int_0^t \bar{F}(s)\left[\mu(s)-r(s)-\sigma^2(s)\pi^*(s)+\rho\sigma(s)q(s)\kappa^*(s)\right]ds - \int_0^t\int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1^2(s,z)\pi^*(s)}{1+\pi^*(s)\gamma_1(s,z)}\nu_1(dz)ds = 0, \tag{3.20} $$

$$\int_0^t \bar{F}(s)\left[\lambda(s)-p(s)+\rho\sigma(s)q(s)\pi^*(s)-q^2(s)\kappa^*(s)\right]ds-\int_0^t\int_{\mathbb{R}_0}\frac{\bar{F}(s)\kappa^*(s)\gamma_2^2(s,z)}{1-\kappa^*(s)\gamma_2(s,z)}\nu_2(dz)ds=0. \tag{3.21}$$

Proof. Since $\mathcal{G}_t=\mathcal{F}_t$, $\nu_{\mathcal{G}}=\nu_{\mathcal{F}}$. Then, Eqs (3.15) and (3.19) become (3.20) and (3.21). □

Different from the results in Section 2, if the insurer has no inside information in a mixed market, then, the random exit time will affect the optimal strategy. However, we still can't provide explicit results. In the next section, we will present the explicit expression of the optimal investment and risk control strategies in a particular case.

Proposition 3.1. *Assume that there exists an optimal strategy $u^*=(\pi^*,\kappa^*)$ for our problem (2.9). Then, the processes $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_1(s,z)\tilde{N}^1(ds,dz)\right\}_{0\leqslant t\leqslant T}$, $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_2(s,z)\tilde{N}^2(ds,dz)\right\}_{0\leqslant t\leqslant T}$, $\left\{W^1(t)\right\}_{0\leqslant t\leqslant T}$, and $\left\{W^2(t)\right\}_{0\leqslant t\leqslant T}$ are all $\mathcal{G}_t$-semimartingales.*

Proof. Suppose $u^*=(\pi^*,\kappa^*)$ is an optimal strategy for problem (2.9). From item (4) in Definition 2.1, we have that $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_1(s,z)\left(\nu^1_{\mathcal{G}}-\nu^1_{\mathcal{F}}\right)(ds,dz)\right\}$ and $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_2(s,z)\left(\nu^2_{\mathcal{G}}-\nu^2_{\mathcal{F}}\right)(ds,dz)\right\}$ are of a finite variation.

Since $\int_0^t\int_{\mathbb{R}_0}\gamma_1(s,z)\tilde{N}^1(ds,dz)$ equals $\int_0^t\int_{\mathbb{R}_0}\gamma_1(s,z)\left(N^1-\nu^1_{\mathcal{G}}\right)(ds,dz)+\int_0^t\int_{\mathbb{R}_0}\left(\nu^1_{\mathcal{G}}-\nu^1_{\mathcal{F}}\right)(ds,dz)$ and $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_1(s,z)\left(N^1-\nu^1_{\mathcal{G}}\right)(ds,dz)\right\}$ is a $\mathcal{G}_t-$ martingale, we conclude that $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_1(s,z)\tilde{N}^1(ds,dz)\right\}$ is a $\mathcal{G}_t$-semimartingale. Similarly, we have that $\left\{\int_0^t\int_{\mathbb{R}_0}\gamma_2(s,z)\tilde{N}^2(ds,dz)\right\}$ is also a $\mathcal{G}_t$-semimartingale.

Furthermore, we have that $M_1^{u^*}(t)$ and $M_2^{u^*}(t)$ as $\mathcal{G}_t$-martingales. By (3.13) and (3.17), we have the following:

$$W_t^1=\int_0^t\frac{1}{\bar{F}(s)\sigma(s)}d^cM_1^{u^*}(t)-\int_0^t\alpha_1(s)ds,$$

$$W_t^2=-\int_0^t\frac{1}{\sqrt{1-l^2}\bar{F}(s)q(s)}d^cM_2^{u^*}(s)-\frac{\rho}{\sqrt{1-\rho^2}}W_t^1-\int_0^t\frac{\rho\alpha_1(s)}{\sqrt{1-l^2}}ds-\int_0^t\alpha_2(s)ds,$$

with $t\in[0,T]$. Therefore, $\left\{W_t^1\right\}_{0\leqslant t\leqslant T}$ and $\left\{W_t^2\right\}_{0\leqslant t\leqslant T}$ are $\mathcal{G}_t$-semimartingale. □

4. Optimal strategies in some particular cases

In this section, we assume that the insurer has access to future information regarding the financial and actuarial markets. This information flow is driven by the stochastic processes $W_1(T_0), W_2(T_0)$, $\eta_1(T_0), \eta_2(T_0)$ at a future time $T_0\geqslant T$, which implies that the inside information can be described as follows:

$$\mathbb{G}\triangleq\{\mathcal{G}_t\triangleq\mathcal{F}_t\vee\sigma\left(W_1(T_0),W_2(T_0),\eta_1(T_0),\eta_2(T_0)\right):t\in[0,T]\}. \tag{4.1}$$

The insurer who owns inside information makes decisions at time t based on the filtration $\mathcal{G}_t$, where $\eta_i(t)=\int_0^t\int_{\mathbb{R}_0}z\tilde{N}^i(dt,dz), i=1,2$. Referring to Property 5.1 from Peng and Wang [21], it follows that Proposition 4.1 holds.

Proposition 4.1. *Let $\mathcal{G}_t$ denote an inside filtration of the form given in Eq (4.1). Then, we have that*

$$W_i(t)-\int_0^t\frac{W_i(T_0)-W_i(s)}{T_0-s}ds, \tag{4.2}$$

and

$$\eta_i(t) - \int_0^t \frac{\eta_i(T_0) - \eta_i(s)}{T_0 - s} ds = \eta_i(t) - \int_0^t \int_s^{T_0} \int_{\mathbb{R}_0} \frac{z}{T_0 - s} \tilde{N}^i(dr, dz) ds, \tag{4.3}$$

i=1,2, are ($\mathcal{G}_t$,P)-martingales.

Proposition 4.1 means that in the present situation of the enlargement of filtration, the processes $\{\alpha_1(s)\}$ and $\{\alpha_2(s)\}$ in (3.13) and (3.17) are of the form $\left\{-\frac{W_1(T_0)-W_1(s)}{T_0-s}\right\}$ and $\left\{-\frac{W_2(T_0)-W_2(s)}{T_0-s}\right\}$, respectively.

Proposition 4.2. *The compensating measure $\nu^i_{\mathcal{G}}$ of the jump measure N^i is given by the following:*

$$\nu^i_{\mathcal{G}}(ds, dz) = \nu_i(dz)ds + \frac{1}{T_0 - s} \int_s^{T_0} \tilde{N}^i(dr, dz)ds \tag{4.4}$$

$$= \frac{1}{T_0 - s} \int_s^{T_0} N^i(dr, dz)ds, \quad i = 1, 2. \tag{4.5}$$

Proof. It is sufficient to show that if $\nu^i_{\mathcal{G}}$ is the right hand side of (4.4), then

$$\int_0^t \int_{\mathbb{R}_0} f(z) \left(N^i - \nu^i_{\mathcal{G}}\right)(ds, dz)$$

is a $\mathcal{G}_t$-martingale for all $f \in G$. Here, G is a set of bounded deterministic functions on $\mathbb{R}$, zero around zero, which determines a measure on $\mathbb{R}$ whose mass at the origin is zero. The same argument holds if we take G to be the set of invertible functions $f(z)$ that are integrable w.r.t. $\nu^i_{\mathcal{G}}$. Let $f(z)$ be such a function, and consider the Lévy processes $W_i(t) + \bar{\eta}(t)$ for $i = 1, 2$, whose σ-algebra is denoted by$\bar{\mathcal{F}}_t$, and where $\bar{\eta}(t) := \int_0^t \int_{\mathbb{R}} f(z)\tilde{N}(ds, dz)$.

Since $f(z)$ is invertible, we have $\bar{\mathcal{F}}_t = \mathcal{F}_t$ and $\bar{\mathcal{G}}_t = \mathcal{G}_t$, where $\bar{\mathcal{G}}_t = \bar{\mathcal{F}}_t \vee \sigma(W_1(T_0), W_2(T_0), \eta_1(T_0), \eta_2(T_0))$. From Proposition 4.1, we can obtain the following:

$$\bar{M}(t) := \int_0^t \int_{\mathbb{R}} f(z)\tilde{N}^i(ds, dz) - \int_0^t \int_s^{T_0} \int_{\mathbb{R}_0} \frac{f(z)}{T_0 - s} \tilde{N}^i(dr, dz)ds$$

is a $\mathcal{G}_t$-martingale.

Under the measure in (4.4), the optimal portfolio condition specified by (3.19) and (3.15) transforms into the following:

$$\begin{aligned}&\int_0^t \bar{F}(s)[\mu(s) - r(s) - \sigma^2(s)\pi^*(s) + \rho\sigma(s)q(s)\kappa^*(s)]ds + \int_0^t \bar{F}(s)\sigma(s)\frac{W_1(T_0) - W_1(s)}{T_0 - s}ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \int_s^{T_0} \frac{\bar{F}(s)\gamma_1(s,z)}{(1 + \pi^*(s)\gamma_1(s,z))(T_0 - s)} \tilde{N}^1(dr, dz)ds - \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_1^2(s,z)\pi^*(s)}{1 + \pi^*(s)\gamma_1(s,z)} \nu_1(dz)ds = 0,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&\int_0^t \bar{F}(s)\left[\lambda(s) - p(s) + \rho\sigma(s)q(s)\pi^*(s) - q^2(s)\kappa^*(s)\right]ds - \int_0^t \bar{F}(s)\rho q(s)\frac{W_1(T_0) - W_1(s)}{T_0 - s}ds \\ &- \int_0^t \bar{F}(s)\sqrt{1-\rho^2}q(s)\frac{W_2(T_0) - W_2(s)}{T_0 - s}ds - \int_0^t \int_{\mathbb{R}_0} \int_s^{T_0} \frac{\bar{F}(s)\gamma_2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))(T_0 - s)} \tilde{N}^2(dr, dz)ds \\ &- \int_0^t \int_{\mathbb{R}_0} \frac{\bar{F}(s)\gamma_2^2(s,z)\kappa^*(s)}{1 - \kappa^*(s)\gamma_2(s,z)} \nu_2(dz)ds = 0, \text{respectively.}\end{aligned} \tag{4.7}$$

Substituting Eq (4.4) into Eqs (4.6) and (4.7) leads to the following results:

$$\begin{aligned}&\int_0^t \bar{F}(s)\left[\mu(s)-r(s)-\sigma^2(s)\pi^*(s)+\rho\sigma(s)q(s)\kappa^*(s)+\sigma(s)\frac{W_1(T_0)-W_1(s)}{T_0-s}\right]ds\\&+\int_0^t\int_{\mathbb{R}_0}\int_s^{T_0}\frac{\bar{F}(s)\gamma_1(s,z)}{(1+\pi^*(s)\gamma_1(s,z))(T_0-s)}N^1(dr,dz)ds-\int_0^t\int_{\mathbb{R}_0}\bar{F}(s)\gamma_1(s,z)\nu_1(dz)ds=0,\end{aligned}\tag{4.8}$$

$$\begin{aligned}&\int_0^t \bar{F}(s)\left[\lambda(s)-p(s)+\rho\sigma(s)q(s)\pi^*(s)-q^2(s)\kappa^*(s)-\int_0^t\rho q(s)\frac{W_1(T_0)-W_1(s)}{T_0-s}\right.\\&\left.-\int_0^t\sqrt{1-\rho^2}q(s)\frac{W_2(T_0)-W_2(s)}{T_0-s}\right]ds-\int_0^t\int_{\mathbb{R}_0}\int_s^{T_0}\frac{\bar{F}(s)\gamma_2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))(T_0-s)}N^2(dr,dz)ds\\&+\int_0^t\int_{\mathbb{R}_0}\bar{F}(s)\gamma_2(s,z)\nu_2(dz)ds=0.\end{aligned}\tag{4.9}$$

The theorem below establishes that, under certain additional assumptions, Conditions (4.8) and (4.9) are also sufficient for the optimality of the strategy (π^*, κ^*). □

Proposition 4.3. *Let $\int_{\mathbb{R}_0}|\gamma_i(s,z)|\,\nu_i(dz)<\infty, i=1,2$ be the strategy (π^*,κ^*) that is optimal for the insurer if and only if $(\pi^*,\kappa^*)\in\mathcal{A}_{\mathcal{G}}$, and for a.s. $(\omega,s),(\pi^*,\kappa^*)$, it satisfies the following equations:*

$$\begin{aligned}&\bar{F}(s)\left[\mu(s)-r(s)-\sigma^2(s)\pi^*(s)+\rho\sigma(s)q(s)\kappa^*(s)+\sigma(s)\frac{W_1(T_0)-W_1(s)}{T_0-s}\right.\\&\left.+\int_{\mathbb{R}_0}\int_s^{T_0}\frac{\gamma_1(s,z)}{(1+\pi^*(s)\gamma_1(s,z))(T_0-s)}N^1(dr,dz)-\int_{\mathbb{R}_0}\gamma_1(s,z)\nu_1(dz)\right]=0,\end{aligned}\tag{4.10}$$

$$\begin{aligned}&\bar{F}(s)\left[\lambda(s)-p(s)+\rho\sigma(s)q(s)\pi^*(s)-q^2(s)\kappa^*(s)-\rho q(s)\frac{W_1(T_0)-W_1(s)}{T_0-s}\right.\\&-\sqrt{1-\rho^2}q(s)\frac{W_2(T_0)-W_2(s)}{T_0-s}-\int_{\mathbb{R}_0}\int_s^{T_0}\frac{\gamma_2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))(T_0-s)}N^2(dr,dz)\\&\left.+\int_{\mathbb{R}_0}\gamma_2(s,z)\nu_2(dz)\right]=0.\end{aligned}\tag{4.11}$$

Proof. From Propositions 4.1 and 4.2, the optimization problem is equivalent to the following:

$$
\begin{aligned}
&\sup_{u\in\mathcal{A}_{\mathcal{G}}} E\left[\int_0^T \delta(t)J^u(t)dt + \bar{F}(T)J^u(T)\right] \\
&= E\left\{\int_0^T \bar{F}(s)\left\{\left[(\mu(s)-r(s))\pi(s) + (\lambda(s)-p(s))\kappa(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) + \rho\sigma(s)q(s)\pi(s)\kappa(s)\right.\right.\right. \\
&\left.-\frac{1}{2}q^2(s)\kappa^2(s)\right]ds + (\sigma(s)\pi(s)-\rho q(s)\kappa(s))\left(dW_s^1 + \alpha_1(s)ds\right) \\
&- \sqrt{1-\rho^2}q(s)\kappa(s)\left(dW_s^2 + \alpha_2(s)ds\right) - (\sigma(s)\pi(s)-\rho q(s)\kappa(s))\alpha_1(s)ds \\
&+ \sqrt{1-\rho^2}q(s)\kappa(s)\alpha_2(s)ds + \int_{\mathbb{R}_0} \log(1+\pi(s)\gamma_1(s,z))\left(N^1 - \nu_{\mathcal{G}}^1\right)(ds,dz) \\
&+ \int_{\mathbb{R}_0} \log(1-\kappa(s)\gamma_2(s,z))\left(N^2 - \nu_{\mathcal{G}}^2\right)(ds,dz) + \int_{\mathbb{R}_0} \log(1+\pi(s)\gamma_1(s,z))\,\nu_{\mathcal{G}}^1(ds,dz) \\
&\left.\left.+ \int_{\mathbb{R}_0} \log(1-\kappa(s)\gamma_2(s,z))\,\nu_{\mathcal{G}}^2(ds,dz) + \int_{\mathbb{R}_0} \pi(s)\gamma_1(s,z)\nu_1(dz)ds + \int_{\mathbb{R}_0} \kappa(s)\gamma_2(s,z)\nu_2(dz)ds\right\}\right\} \qquad (4.12) \\
&= E\left\{\int_0^T \bar{F}(s)\left\{\left[(\mu(s)-r(s))\pi(s) + (\lambda(s)-p(s))\kappa(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) + \rho\sigma(s)q(s)\pi(s)\kappa(s)\right.\right.\right. \\
&\left.-\frac{1}{2}q^2(s)\kappa^2(s)\right]ds + (\sigma(s)\pi(s)-\rho q(s)\kappa(s))\frac{W_1(T_0)-W_1(s)}{T_0-s}ds \\
&- \sqrt{1-\rho^2}q(s)\kappa(s)\frac{W_2(T_0)-W_2(s)}{T_0-s}ds + \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\log(1+\pi(s)\gamma_1(s,z))}{T_0-s}N^1(dr,dz)ds \\
&+ \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\log(1-\kappa(s)\gamma_2(s,z))}{T_0-s}N^2(dr,dz)ds \\
&\left.\left.- \int_{\mathbb{R}_0} \pi(s)\gamma_1(s,z)\nu_1(dz)ds + \int_{\mathbb{R}_0} \kappa(s)\gamma_2(s,z)\nu_2(dz)ds\right\}\right\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
H(\pi,\kappa) =& \bar{F}(s)\Big[(\mu(s)-r(s))\pi + (\lambda(s)-p(s))\kappa - \frac{1}{2}\sigma^2(s)\pi^2 + \rho\sigma(s)q(s)\pi\kappa \\
&- \frac{1}{2}q^2(s)\kappa^2 + (\sigma(s)\pi - \rho q(s)\kappa)\frac{W_1(T_0)-W_1(s)}{T_0-s} - \sqrt{1-\rho^2}q(s)\kappa\frac{W_2(T_0)-W_2(s)}{T_0-s} \\
&+ \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\log(1+\pi\gamma_1(s,z))}{T_0-s}N^1(dr,dz) + \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\log(1-\kappa\gamma_2(s,z))}{T_0-s}N^2(dr,dz) \\
&- \int_{\mathbb{R}_0} \pi\gamma_1(s,z)\nu_1(dz) + \int_{\mathbb{R}_0} \kappa\gamma_2(s,z)\nu_2(dz)\Big].
\end{aligned}
$$

Based on the first-order optimality conditions, it follows that:

$$
\begin{aligned}
\left.\frac{\partial H}{\partial \pi}\right|_{u=u^*} =& \bar{F}(s)\Big[\mu(s) - r(s) - \sigma^2(s)\pi^*(s) + \rho\sigma(s)q(s)\kappa^*(s) + \sigma(s)\frac{W_1(T_0)-W_1(s)}{T_0-s} \\
&+ \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_1(s,z)}{(1+\pi^*(s)\gamma_1(s,z))(T_0-s)}N^1(dr,dz) - \int_{\mathbb{R}_0}\gamma_1(s,z)\nu_1(dz)\Big] = 0,
\end{aligned}
$$

$$\frac{\partial H}{\partial \kappa}\bigg|_{u=u^*} = \bar{F}(s)\Big[\lambda(s) - p(s) + \rho\sigma(s)q(s)\pi^*(s) - q^2(s)\kappa^*(s) - \rho q(s)\frac{W_1(T_0) - W_1(s)}{T_0 - s}$$
$$- \sqrt{1-\rho^2}q(s)\frac{W_2(T_0) - W_2(s)}{T_0 - s} - \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))(T_0 - s)}N^2(dr,dz)$$
$$+ \int_{\mathbb{R}_0} \gamma_2(s,z)\nu_2(dz)\Big] = 0.$$

The above equations satisfy (4.10) and (4.11). Moreover, we have the following:

$$\frac{\partial^2 H}{\partial \pi^2} = -\bar{F}(s)\Big[\sigma^2(s) + \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_1^2(s,z)}{(1+\pi(s)\gamma_1(s,z))^2(T_0 - s)}N^1(dr,dz)\Big] < 0,$$
$$\frac{\partial^2 H}{\partial \kappa^2} = -\bar{F}(s)\Big[q^2(s) + \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_2^2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))^2(T_0 - s)}N^2(dr,dz)\Big] < 0,$$

and

$$\frac{\partial^2 H}{\partial \pi^2}\cdot\frac{\partial^2 H}{\partial \kappa^2} - \left(\frac{\partial^2 H}{\partial \pi \partial \kappa}\right)^2 = \bar{F}^2(s)\Big[\left(1-\rho^2\right)\sigma^2(s)q^2(s) + \sigma^2(s)\int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_2^2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))^2(T_0 - s)}N^2(dr,dz)$$
$$+ q^2(s)\int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_1^2(s,z)}{(1+\pi(s)\gamma_1(s,z))^2(T_0 - s)}N^1(dr,dz)$$
$$+ \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_1^2(s,z)}{(1+\pi^*(s)\gamma_1(s,z))^2(T_0 - s)}N^1(dr,dz)\cdot\int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_2^2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))^2(T_0 - s)}N^2(dr,dz)\Big]$$
$$> 0.$$

Consequently, the optimal strategy $u^* = (\pi^*, \kappa^*)$ is as specified in (4.10) and (4.11). □

Remark 4.1. *From (4.10) and (4.11), it follows that if $\bar{F}(s) \neq 0$, then, the expression simplifies to*

$$\mu(s) - r(s) - \sigma^2(s)\pi^*(s) + \rho\sigma(s)q(s)\kappa^*(s) + \sigma(s)\frac{W_1(T_0) - W_1(s)}{T_0 - s}ds$$
$$+ \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_1(s,z)}{(1+\pi^*(s)\gamma_1(s,z))(T_0 - s)}N^1(dr,dz)ds - \int_{\mathbb{R}_0} \bar{F}(s)\gamma_1(s,z)\nu_1(dz)ds = 0, \tag{4.13}$$

and

$$\lambda(s) - p(s) + \rho\sigma(s)q(s)\pi^*(s) - q^2(s)\kappa^*(s) - \rho q(s)\frac{W_1(T_0) - W_1(s)}{T_0 - s}$$
$$- \sqrt{1-\rho^2}q(s)\frac{W_2(T_0) - W_2(s)}{T_0 - s} - \int_{\mathbb{R}_0}\int_s^{T_0} \frac{\gamma_2(s,z)}{(1-\kappa^*(s)\gamma_2(s,z))(T_0 - s)}N^2(dr,dz) \tag{4.14}$$
$$+ \int_{\mathbb{R}_0} \gamma_2(s,z)\nu_2(dz) = 0.$$

Our results extend Proposition 5.3 in Peng and Wang [21], and thus their conclusion is a special case of ours.

Proposition 4.4. *Suppose that* $v_1(dz) = v_2(dz) = 0$, $\gamma_1(s,z) = \gamma_2(s,z) = 0$. *Then, the optimal investment and risk control strategy* (π^*, κ^*) *for the insurer is expressed by the following:*

$$\pi^*(s) = \frac{\mu(s)-r(s)}{(1-\rho^2)\,\sigma^2(s)} + \frac{\rho(\lambda(s)-p(s))}{(1-\rho^2)\,q(s)\sigma(s)} + \frac{W_1(T_0)-W_1(s)}{(T_0-s)\,\sigma(s)} - \frac{\rho}{\sqrt{1-\rho^2}\sigma(s)}\frac{W_2(T_0)-W_2(s)}{T_0-s}, \tag{4.15}$$

and

$$\kappa^*(s) = \frac{\rho(\mu(s)-r(s))}{(1-\rho^2)\,q(s)\sigma(s)} + \frac{\lambda(s)-p(s)}{(1-\rho^2)\,q^2(s)} - \frac{W_2(T_0)-W_2(s)}{(T_0-s)\sqrt{1-\rho^2}q(s)}, \tag{4.16}$$

which coincides with Proposition 5.4 in Peng and Wang [21].

5. Numerical analysis

In this section, we conduct stochastic simulations based on the explicit solution of the optimal investment and risk control strategy derived from Proposition 4.4, to further analyze the impacts of the correlation coefficient ρ and inside information on the optimal strategies $\pi^*(s)$ and $k^*(s)$. For simplicity and without a loss of generality, we make the following assumptions: the investment horizon $T = 1$, and the moment of inside information awareness $T_0 = 1.2$ (with $T_0 > T$), the risk-free interest rate $r = 0.03$, the expected return of risky assets $\mu = 0.08$, and the volatility of risky assets $\sigma = 0.2$, the insurance premium rate $\lambda = 0.05$, the payout rate $p = 0.03$, and the insurance risk volatility $q = 0.15$. In Figures 3 and 4, we fix $\rho = 0.1$. The number of paths simulated by Monte Carlo is $N = 10,000$.

As can be seen from Figures 1 and 2, with the increase of the correlation between the financial market and the insurance market, the insurers with inside information will simultaneously increase their holdings of risky assets and risky liabilities. This implies that when the positive correlation between the financial market and the insurance market increases, the insurers tend to pursue higher returns by increasing the proportion of risky investments and raising the liability ratio level. From this point, the investment type of insurers belongs to “aggressive”. Meanwhile, Figures 1 and 2 show that under the same correlation degree between the two markets, the investment strategies of insurers are relatively stable overall, but the volatility increases in the middle and late stages. This further shows that as the inside information gradually becomes clear, insurers with inside information still tend to be aggressive in investment decisions.

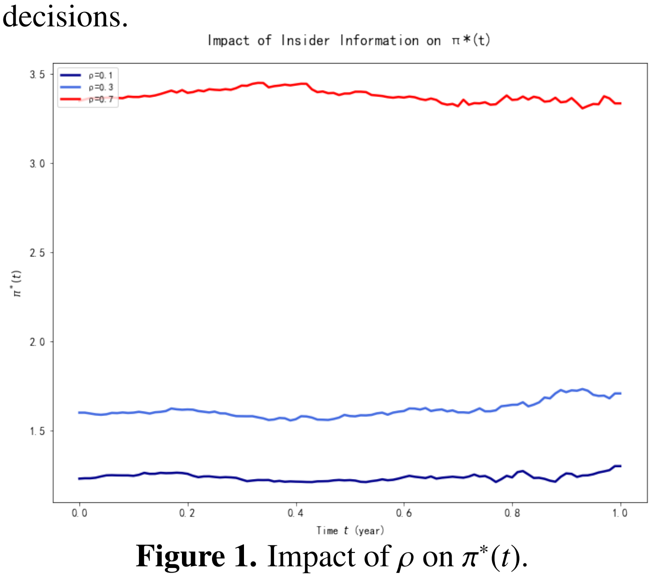

Figure 1. Impact of ρ on $\pi^*(t)$.

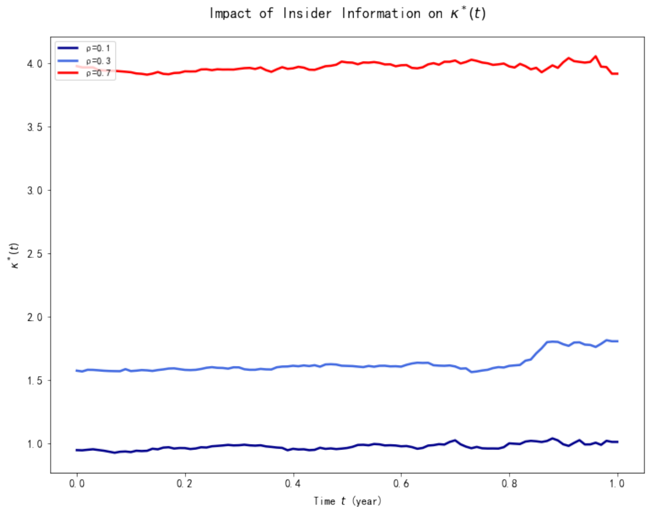

Figure 2. Impact of ρ on $\kappa^*(t)$.

It can be seen from Figures 3 and 4, compared with insurers without inside information, insurers with inside information can dynamically adjust their investment strategies to obtain excess returns. Meanwhile, Figures 3 and 4 show that although some insurers have inside information, they remain cautious in the initial stage of investment (e.g., during the period of 0–0.4), which is closely related to the small fluctuation of investment strategies in this stage. As the investment enters the middle and late stages, the inside information held by insurers gradually becomes clear. It can be seen from the significant increase in strategy volatility during this period that they are more inclined to pursue higher returns by adjusting significantly risky assets and the liability ratio. During this period, both the risky assets and the liability ratio held by insurers with inside information are higher than those without inside information. Meanwhile, it can be observed from Figure 3 that the risky assets held by insurers with inside information show an overall upward trend. This indicates that their investment strategies are more aggressive, and also reflects that insurers obtain higher excess returns by using inside information.

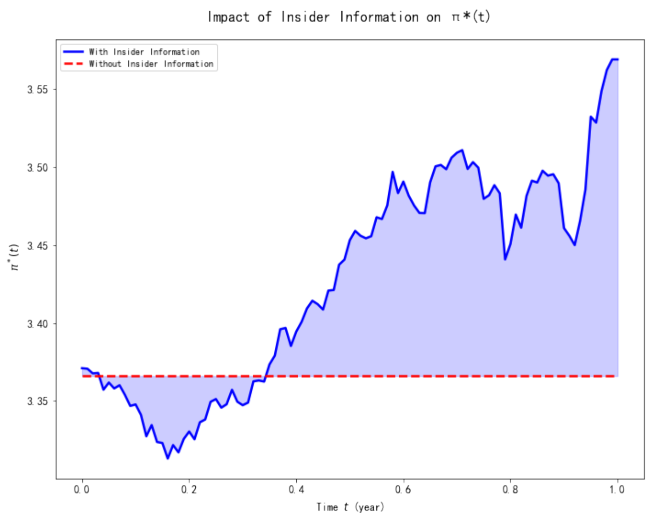

Figure 3. Impact of inside information on π^*.

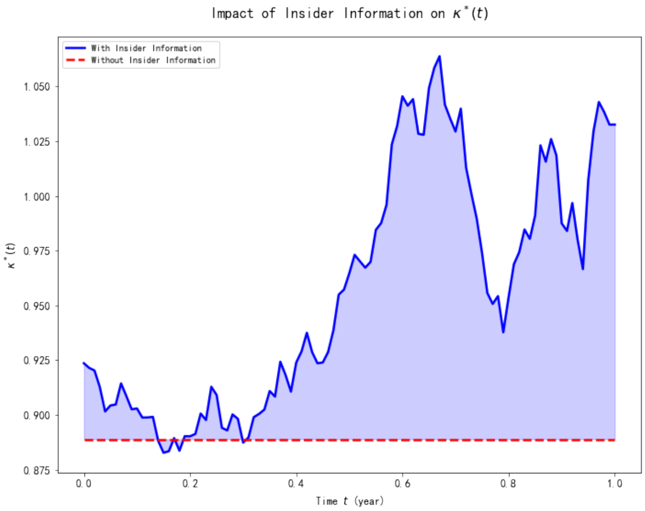

Figure 4. Impact of inside information on $\kappa^*(t)$.

6. Conclusions

Compared with existing works, this paper constructs a more analytical framework that incorporates important factors in real financial markets, such as random time, inside information, and jump processes. Under the criterion of maximizing the expected logarithmic utility of the terminal wealth, we employed Malliavin calculus and the variational method to derive the characterization of the optimal investment and risk control strategies in both pure jump markets and mixed markets. These results revealed that random time is a key factor for the optimal strategies. Furthermore, we conducted an analysis through several concrete examples. The insurer with inside information was described by $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma\left(W_1\left(T_0\right), W_2\left(T_0\right), \eta_1\left(T_0\right), \eta_2\left(T_0\right)\right) : t \in [0, T]$. We obtained an alternative characterization of the optimal strategy, which was independent of the diffusion termW_i, where $i = 1, 2$. The conclusions of this paper not only generalize the results in [21], but also demonstrate that Property 5.3 proposed in [21] is a special case of our conclusions if the tail distribution of the random time satisfies $\bar{F}(s) \neq 0$; simultaneously, the random time horizon does not impose any impact on the optimal strategy under the condition that$\bar{F}(s) \neq 0$. As can be seen from Figures 1 and 2, when the financial market and the insurance market are strongly correlated, insurers with inside information adopt more aggressive investment strategies. Meanwhile, Figure 3 showed that such insurance companies also pursue aggressive investments in the middle and late stages. This indicates that insurers seek higher excess returns by utilizing inside information. Our study was restricted to the logarithmic utility function and a set of restrictive assumptions based on Malliavin calculus techniques. A natural extension of our framework is to generalize the analysis to alternative utility forms, such as power utility (constant relative risk aversion, CRRA) or exponential utility (constant absolute risk aversion, CARA). However, it remains challenging to solve the optimization problems associated with such utility functions, which require more sophisticated mathematical tools. Furthermore, we can represent inside information in a more concrete form, such as $\mathcal{G}_t = \mathcal{F}_t \cup \sigma(Y)$, where $Y = \gamma W(T) + (1-\gamma)\varepsilon$. Under this framework, we can provide richer economic interpretations for inside information, and the derived optimal strategy also admits clear and reasonable economic intuition. These will be an important direction for our future research.

Author contributions

Hongwei Liu: Writing-original draft, Writing-review & editing, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis; Xinzhi Wang: Writing-original draft, Writing-review & editing, Methodology, Investigation, Formal analysis; Caibo Xiao: Writing-original draft; Writing-review & editing, Formal analysis.

Use of Generative-AI tools declaration

We declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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