Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 10936–10962.
DOI: 10.3934/math.2026449
Received: 22 January 2026
Revised: 24 March 2026
Accepted: 26 March 2026
Published: 20 April 2026

Research article

Exploring the exact soliton solutions and modulation instability analysis of the (3+1)-dimensional oceanic wave model

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Abstract: The oceanic wave equation plays an important role in modeling wave propagation phenomena related to weather forecasting and coastal dynamics. In this work, the Hirota trilinear scheme is employed to investigate the nonlinear (3+1)-dimensional fifth-order dynamical ocean equation. By applying this analytical technique, breather-wave and two-soliton solutions of the considered model are successfully derived and verified symbolically using Maple computational software. To illustrate the physical characteristics and propagation behavior of the obtained solutions, several structures are presented through two-dimensional, three-dimensional, and contour plots. The obtained wave structures represent new exact solutions that have not been reported in previous studies of this model. Furthermore, modulation instability analysis is performed to examine the stability of the steady-state solutions of the governing equation. The results provide deeper insight into nonlinear ocean wave dynamics and may contribute to applications in ocean engineering, coastal wave analysis, and related nonlinear physical systems. The effectiveness and simplicity of the Hirota trilinear scheme demonstrated in this study also suggest its applicability to other higher-dimensional nonlinear evolution equations arising in applied mathematics and fluid dynamics.

Keywords: nonlinear fifth-order model; trilinear form; breather-wave solutions; two-soliton solutions; modulation instability
Mathematics Subject Classification: 35Q51, 35Q92, 35R11

1. Introduction

The increasing demand for marine structures, ships, and wave energy devices to operate under high sea states and other energetic conditions underscores the importance of ocean wave theory. It is therefore essential to model and simulate nonlinear ocean wave fields in large-scale wave basins. Ocean waves significantly influence processes that govern fluxes across the air–sea interface and mixing in the upper ocean.

Various types of wave structures have been reported, including traveling waves, lump waves, kink waves, lump–kink waves, one-soliton waves, two-soliton waves, rogue waves, and many others. Numerous studies have significantly contributed to the fields of oceanography and related areas.

For instance, solitary wave solutions of a higher-dimensional wave model with gas bubbles were obtained using the modified Kudryashov method in [1]. In [2], novel wave solutions of the coupled Drinfel'd–Sokolov–Wilson model were derived using a new auxiliary equation scheme. Moreover, periodic, exponential, and soliton-type solutions of two coupled nonlinear Schrödinger equations were obtained using the generalized Kudryashov scheme, the modified Kudryashov technique, and the exponential rational function method in [3]. In addition, kink-type multisoliton solutions of the (1+1)-dimensional Mikhailov–Novikov–Wang equation were derived using the simplified Hirota method in [4].

There are also numerous analytical methods available in the literature, including the extended (G'/G)-expansion method [5, 6], the particle swarm optimization method [7], the extended Fan subequation technique [8], the simplest equation method [9], the new auxiliary equation scheme [10], the unified solver technique [11], the generalized exponential rational function technique [12], the $(G'/(G' + G + A))$-expansion technique [13], the modified simplest equation method [14], the Bäcklund transformation method [15], the Bernoulli (G'/G)-expansion technique [16], the generalized exponential rational function (GERF) method [17], Lie symmetry analysis [18], the extended hyperbolic function method [19], the Hirota bilinear technique [20], the Sardar subequation method [21], the separation of variables method [22], the $(1/G, G'/G)$-expansion method [23], the exp_a function method [24], the (G'/G^2)-expansion method [25], the Riccati subequation neural networks method [26], the Kudryashov method [27], and more.

There is a method known as the Hirota trilinear scheme, which is used to obtain different types of wave solutions. For instance, lump-, breather-wave-, breather–kink-, and lump–kink-type solutions of the (2+1)-dimensional Bogoyavlensky–Konopelchenko model were obtained using the Hirota trilinear scheme in [28]. The N-soliton solutions of the Bogoyavlensky–Schiff model were derived using the Hirota trilinear method in [29]. One-soliton and two-soliton solutions of the new (2+1)-dimensional Korteweg-de Vries (KdV) equation were obtained in [30]. Furthermore, breather-wave, rogue-wave, and solitary-wave solutions of the coupled nonlinear Schrödinger equation were obtained in [31]. Lump, breather-wave, and breather–kink wave solutions of the p^{-}-gKP equation were derived using Hirota trilinear analysis in [32]. Lump-type and breather–lump–kink interaction solutions of the (3+1)-dimensional generalized Bogoyavlensky-Konopelchenko (gBK) model were obtained using Hirota trilinear analysis in [33]. In addition, lump, breather-wave, and interaction-wave solutions were reported in [34, 35].

The model considered in this study is an integrable (3+1)-dimensional fifth-order nonlinear oceanic wave equation. Lump and kink-soliton solutions of the considered equation were obtained using the

Hirota trilinear and bilinear schemes in [36]. Multisoliton solutions were derived using the simplified Hirota technique [37, 38].

The main purpose of this work is to obtain breather-wave and two-soliton solutions of an integrable (3+1)-dimensional fifth-order nonlinear oceanic wave equation using the Hirota trilinear scheme. Furthermore, dynamical analysis is performed to investigate the behavior of the obtained solutions.

The motivation of this paper is that, to the best of our knowledge, the Hirota trilinear scheme is applied for the first time to the nonlinear (3+1)-dimensional fifth-order oceanic wave equation. This model is important in the fields of ocean engineering and other related areas, particularly in weather forecasting and coastal studies. The Hirota trilinear scheme provides various types of analytical solutions for nonlinear evolution equations. In this work, we determine breather-wave and two-soliton solutions of the considered model. Breather waves are pulsating localized structures that are often used to describe extreme wave phenomena in nonlinear dispersive media. The two-soliton solution exhibits an interaction structure similar to the superposition of two solitary waves. The Hirota trilinear scheme possesses several advantages. For example, it employs a convenient ansatz to transform the nonlinear equation into a bilinear or trilinear form, and the resulting equations can be solved perturbatively to obtain multisoliton solutions. Moreover, the Hirota method is not limited to soliton solutions; it can also be used to derive other nonlinear wave structures, including complexitons, positons, negatons, and lump solutions. The simplified Hirota method is particularly effective for constructing soliton solutions, which are highly regular wave structures with stable propagation properties. Additionally, we perform a modulation instability analysis to obtain the stable steady-state solutions for the concerned model.

This paper consists of several sections: Section 2 presents the description of the methodology, Section 3 introduces the model and its trilinear form, Section 4 discusses the breather-wave and two-soliton solutions, Section 5 provides graphical illustrations, Section 6 analyzes modulation instability, Section 7 presents the results and discussion, and Section 8 concludes the paper.

2. The Hirota trilinear scheme

Consider $\zeta = \zeta(x_1, x_2, ..., x_n)$ as a C^∞ function given as in [39],

$$\Upsilon_{n_1x_1,...,n_jx_j}(\zeta) \equiv \Upsilon_{n_1,...,n_j}(\zeta_{s_1x_1,...,s_jx_j}) = e^{-\zeta}\partial_{x_1}^{n_1}...\partial_{x_j}^{n_j}e^{\zeta}, \tag{2.1}$$

with binary Bell polynomials (BBPs) shown as

$$\zeta_{s_1x_1,...,s_jx_j} = \partial_{x_1}^{s_1}...\partial_{x_j}^{s_j}\zeta, \;\; \zeta_{0x_i} \equiv \zeta, \;\; s_1 = 0, ..., n_1; ...; s_j = 0, ..., n_j,$$

and we have

$$\begin{aligned} &\Upsilon_1(\zeta) = \zeta_x, \;\; \Upsilon_2(\zeta) = \zeta_{2x} + \zeta_x^2, \;\; \Upsilon_3(\zeta) = \zeta_{3x} + 3\zeta_x\zeta_{2x} + \zeta_x^3, ..., \;\; \zeta = \zeta(x,t), \\ &\Upsilon_{x,t}(\zeta) = \zeta_{x,t} + \zeta_x\zeta_t, \;\; \Upsilon_{2x,t}(\zeta) = \zeta_{2x,t} + \zeta_{2x}\zeta_t + 2\zeta_{x,t}\zeta_x + \zeta_x^2\zeta_t, ... \end{aligned} \tag{2.2}$$

The multidimensional BBPs can be shown as

$$\Sigma_{n_1x_1,...,n_jx_j}(G,H) = \Upsilon_{n_1,...,n_j}(\zeta)\Big|_{\zeta_{s_1x_1,...,s_jx_j}=\begin{cases} A_{s_1x_1,...,s_jx_j}, & s_1+s_2+...+s_j, \text{ is odd;} \\ B_{s_1x_1,...,s_jx_j}, & s_1+s_2+...+s_j, \text{ is even.}\end{cases}} \tag{2.3}$$

The rule depends on the total order of differentiation

$$S = s_1 + s_2 + \cdots + s_j.$$

Then, the parity rule can be written as

$$\zeta_{s_1 x_1,\ldots,s_j x_j} = \begin{cases} A_{s_1 x_1,\ldots,s_j x_j} = G_{s_1 x_1,\ldots,s_j x_j}, & \text{if } S \text{ is odd;} \\ B_{s_1 x_1,\ldots,s_j x_j} = H_{s_1 x_1,\ldots,s_j x_j}, & \text{if } S \text{ is even} \end{cases} \tag{2.4}$$

according to the conditions

$$\Sigma_x(G) = G_x, \;\; \Sigma_{2x}(G,H) = H_{2x} + G_x^2, \;\; \Sigma_{x,t}(G,H) = H_{x,t} + G_x G_t, \ldots \tag{2.5}$$

Here, G and H are the wave functions.

Proposition 2.1. *Suppose*

$$G = \ln(\Theta/\Delta), \quad H = \ln(\Theta\Delta). \tag{2.6}$$

Here, G describes the difference structure between the two tau-functions (Θ and Δ), and H describes the combined amplitude structure of the two tau-functions (Θ and Δ).

Then, the connection between the BBPs and Hirota D-operator may be written in the form

$$\Sigma_{n_1 x_1,\ldots,n_j x_j}(G,H)\Big|_{G=\ln(\Theta/\Delta),\;\; H=\ln(\Theta\Delta)} = (\Theta\Delta)^{-1} D_{x_1}^{n_1} \ldots D_{x_j}^{n_j} \Theta\Delta \tag{2.7}$$

with Hirota formula

$$\prod_{i=1}^{j} D_{x_i}^{n_i} g.\,\eta = \prod_{i=1}^{j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i'} \right)^{n_i} \Theta(x_1,\ldots,x_j)\Delta(x_1',\ldots,x_j')\Bigg|_{x_1=x_1',\ldots,x_j=x_j'}. \tag{2.8}$$

The Hirota trilinear D-operator is given as in [33]:

$$\prod_{i=1}^{3} D_{q^- \cdot \chi_i}^{n_i} g.\eta.\lambda. = \prod_{i=1}^{3} \left(\Lambda_{pi} \frac{\partial}{\partial \chi_i} + \Lambda_{pi}^{'} \frac{\partial}{\partial \chi_i'} + \Lambda_{pi''} \frac{\partial}{\partial \chi_i''} \right)^{n_i} g(x,y,t)\eta(x',y',t')\lambda(x'',y'',t'')\Bigg|_{(x,y,t)=(x',y',t')=(x'',y'',t'')}, \tag{2.9}$$

where the vectors χ, χ', χ'' *are arbitrary nonnegative integers. Also,* $q_i^- = (q_i, q_i^{'}, q_i^{''})$, $(1 \le i \le 3)$ *are natural numbers, the powers of* $\Lambda_m^n = (-1)^{Q_m^n}$, $(m \ge 1)$, *where* $n =\equiv Q_m^n modm$, $0 \le Q_m^n \le m$, $n \ge 0$. *The powers* Q_m^n *have the following signs:*

$$\begin{aligned} & m = 2k(k \in N) : 1, -1, 1, -1, \ldots, \\ & m = 1 : 1, 1, 1, 1, \ldots, \\ & m = 3 : 1, -1, 1, -1, \ldots, \\ & m = 5 : 1, -1, 1, -1, \ldots, \\ & \ldots\ldots. \end{aligned}$$

Proposition 2.2. *Take*

$$\Xi(G) = \sum_i \delta_i \mathfrak{P}_{s_1x_1,...,s_jx_j} = 0, G = \ln(\Theta/\Delta), \quad H = \ln(\Theta\Delta) \tag{2.10}$$

so that we have

$$\begin{cases} \sum_i \delta_{1i} \Upsilon_{n_1x_1,...,n_jx_j}(G,H) = 0, \\ \sum_i \delta_{1i} \Upsilon_{s_1x_1,...,s_jx_j}(G,H) = 0 \end{cases} \tag{2.11}$$

along conditions

$$\Re(\aleph', \aleph) = \Re(\aleph') - \Re(\aleph) = \Re(H+G) - \Re(H-G) = 0. \tag{2.12}$$

The generalized Bell polynomials $\Upsilon_{n_1x_1,...,n_jx_j}(\xi)$ can be expressed

$$\begin{aligned} (\Theta\Delta)^{-1} D_{x_1}^{n_1} ... D_{x_j}^{n_j} \Theta\Delta &= \Sigma_{n_1x_1,...,n_jx_j}(G,H)\Big|_{G=\ln(\Theta/\Delta), \quad H=\ln(\Theta\Delta)} \\ &= \Sigma_{n_1x_1,...,n_jx_j}(G, G+\gamma)\Big|_{G=\ln(\Theta/\Delta), \quad \gamma=\ln(\Theta\Delta)} \\ &= \sum_{k_1}^{n_1} ... \sum_{k_j}^{n_j} \prod_{i=1}^{j} \begin{pmatrix} n_i \\ k_i \end{pmatrix} \mathfrak{P}_{k_1x_1,...,k_jx_j}(\gamma) \Upsilon_{(n_1-k_1)x_1,...,(n_j-k_j)x_j}(A). \end{aligned} \tag{2.13}$$

The resulting Cole–Hopf transformation is given as

$$\Upsilon_{k_1x_1,...,k_jx_j}(A = \ln(f)) = \frac{f_{n_1x_1,...,n_jx_j}}{f}, \tag{2.14}$$

$$(\Theta\Delta)^{-1} D_{x_1}^{n_1} ... D_{x_j}^{n_j} \Theta\Delta\Big|_{\Delta=\exp(\gamma/2), \quad \Theta/\Delta=f} = f^{-1} \sum_{k_1}^{n_1} ... \sum_{k_j}^{n_j} \prod_{d=1}^{j} \begin{pmatrix} n_d \\ k_d \end{pmatrix} \mathfrak{P}_{k_1x_1,...,k_dx_d}(\gamma) f_{(n_1-k_1)x_1,...,(n_d-k_d)x_d} \tag{2.15}$$

with

$$\Upsilon_t(G) = \frac{f_t}{f}, \quad \Upsilon_{2x}(G,\beta) = \gamma_{2x} + \frac{f_{2x}}{f}, \quad \Upsilon_{2x,y}(G,H) = \frac{\gamma_{2x} f_y}{f} + \frac{2\gamma_{x,y} f_x}{f} + \frac{f_{2x,y}}{f}. \tag{2.16}$$

The Hirota trilinear scheme uses a rational form for the solution so that the convergence of the solution can be obtained easily. However, this aspect will not be discussed here. We will apply an appropriate dependent variable transformation, introduce a τ function ansatz, employ the Bell polynomial framework and trilinear D-operator to obtain the corresponding trilinear form and then perform coefficient matching to derive the required parameter constraints.

3. Model description and its trilinear form

Consider the nonlinear fifth-order integrable (3+1)-dimensional nonlinear oceanic wave equation proposed by A. M. Wazwaz, given as [36]

$$g_{ttt} - g_{txxxx} - 12 g_{xt} g_{xx} - 8 g_x g_{xxt} - 4 g_t g_{xxx} + (\lambda g_x + \Omega g_y + \mu g_z)_{xx} = 0, \tag{3.1}$$

which is an integrable equation applicable to problems such as the diurnal cycle in sea surface temperature. In the oceanic context, the dependent variable $g(x, y, z, t)$ represents the wave profile (e.g., free surface elevation or wave amplitude) of the propagating nonlinear ocean wave field. The parameters λ, Ω, and μ are nonzero constant coefficients that represent the linear dispersion or propagation effects of the wave in the spatial directions x, y, and z. λ characterizes the strength of the nonlinear interaction; Ω is associated with dispersion effects governing wave propagation; and μ accounts for higher-order dispersive or dissipative contributions, depending on the specific physical setting.

Consider the following logarithmic transformation:

$$g(x,y,z,t) = (\log(U(x,y,z,t)))_x, \tag{3.2}$$

where $U(x,y,z,t)$ is the unknown wave function to be found for constructing the solution of Eq (3.1).

We adopt the convention that $log(.)$ denotes the natural logarithm. Moreover, we assume that $U(x,y,z,t) > 0$ and is sufficiently smooth to ensure that $log(U)$ and its spatial derivatives, including $g(x,y,z,t) = (log(U(x,y,z,t)))_x$, are well-defined.

Theorem 3.1. *The Hirota trilinear form of Eq* (3.1) *is obtained by using Eq* (3.2)*.*

Proof. Substituting Eq (3.2) into Eq (3.1), we obtain the following trilinear form:

$$\begin{aligned} &U^2\left(U_{\text{ttt}} + \lambda U_{\text{xxx}} - U_{\text{xxxxt}} + \Omega U_{\text{xxy}} + \mu U_{\text{xxz}}\right) + U_t\left(U\left(U_{\text{xxxx}} - 3U_{\text{tt}}\right) + 2\left(U_{\text{xx}}^2 - 2U_x U_{\text{xxx}}\right)\right) \\ &+ 2U_t^3 + 2U_x\left(U_x\left(\lambda U_x - 2U_{\text{xxt}} + \Omega U_y + \mu U_z\right) + 2U_{\text{xt}}U_{\text{xx}}\right) \\ &- U\left(U_x\left(3\lambda U_{\text{xx}} - 4U_{\text{xxxt}} + 2\Omega U_{\text{xy}} + 2\mu U_{\text{xz}}\right) + U_{\text{xx}}\left(2U_{\text{xxt}} + \Omega U_y + \mu U_z\right)\right) = 0. \end{aligned} \tag{3.3}$$

The functions g, η, and λ appearing in the operator definition are all identified with the same τ function $U(x,y,z,t)$ used throughout the analysis. That is, the trilinear expressions are evaluated by setting $g = \eta = \lambda = U$ so that the operator acts repeatedly on a single function rather than on distinct functions. This convention is consistent with the subsequent derivations, where the bilinear/trilinear forms are expressed entirely in terms of U and its derivatives. □

4. Mathematical analysis

4.1. Breather-wave solutions

Theorem 4.1. *Breather-wave solutions of Eq* (3.1) *are obtained using Eq* (3.3)*.*

Proof. We consider the following relation to obtain the breather-wave solutions [34]:

$$U(x,y,z,t) = \kappa_2 e^{p_1(a_1y+b_1z+c_1t+x)} + \kappa_1 \cos\left(p\left(a_2y + b_2z + c_2t + x\right)\right) + e^{-p_1(a_1y+b_1z+c_1t+x)}. \tag{4.1}$$

We substitute Eq (4.1) into Eq (3.3) and collect coefficients of each power of $e^{p_1(a_1y+b_1z+c_1t+x)}$, $\cos\left(p\left(a_2y + b_2z + c_2t + x\right)\right)$, $e^{-p_1(a_1y+b_1z+c_1t+x)}$, and others, taking each equal to 0. Evaluating them, we find that a_1, a_2, p, and p_1 are the free parameters, and the values of derived constraints are given as follows.

Set 1:

$$\left\{p = \pm p_1, b_1 = \frac{-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda}{\mu}, \kappa_1 = 0\right\}. \tag{4.2}$$

By using Eq (4.2) in Eq (4.1) and applying Eq (3.2), we obtain

$$\begin{aligned}g_1(x,y,z,t) =&(\kappa_2 p_1 \exp(p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))\\&- p_1\exp(-p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x)))\\&/(\kappa_2\exp(p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))\\&+ \exp(-p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))), \qquad \mu \neq 0. \qquad (4.3)\end{aligned}$$

Set 2:

$$\begin{aligned}\{p &= \pm p_1, a_2 = \frac{-2b_2\mu + 4c_1p_1^2 + c_1^3 - 3c_2^2c_1 - 2\lambda}{2\Omega},\\ b_1 &= \frac{-2a_1\Omega - 4c_2p_1^2 + c_2^3 - 3c_1^2c_2 - 2\lambda}{2\mu},\\ \kappa_1 &= \frac{2\sqrt{\kappa_2\left(-c_1^2 + 2c_2c_1 - c_2^2 - 4p_1^2\right)\left(-c_1^2 + 2c_2c_1 - c_2^2 + 4p_1^2\right)}}{c_1^2 - 2c_2c_1 + c_2^2 + 4p_1^2}\}, \qquad (4.4)\end{aligned}$$

$$\begin{aligned}g_2(x,y,z,t) =&(\kappa_2p_1\exp(p_1(\frac{z(-2a_1\Omega - 4c_2p_1^2 + c_2^3 - 3c_1^2c_2 - 2\lambda)}{2\mu} + a_1y + c_1t + x))\\&+ p_1(-\exp(-p_1(\frac{z(-2a_1\Omega - 4c_2p_1^2 + c_2^3 - 3c_1^2c_2 - 2\lambda)}{2\mu} + a_1y + c_1t + x)))\\&- \kappa_1p_1\sin(p_1(\frac{y(-2b_2\mu + 4c_1p_1^2 + c_1^3 - 3c_2^2c_1 - 2\lambda)}{2\Omega} + b_2z + c_2t + x)))\\&/(\kappa_2\exp(p_1(\frac{z(-2a_1\Omega - 4c_2p_1^2 + c_2^3 - 3c_1^2c_2 - 2\lambda)}{2\mu}\\&+ a_1y + c_1t + x)) + \exp(-p_1(\frac{z(-2a_1\Omega - 4c_2p_1^2 + c_2^3 - 3c_1^2c_2 - 2\lambda)}{2\mu} + a_1y + c_1t + x))\\&+ \kappa_1\cos(p_1(\frac{y(-2b_2\mu + 4c_1p_1^2 + c_1^3 - 3c_2^2c_1 - 2\lambda)}{2\Omega} + b_2z + c_2t + x))),\\&\mu \neq 0, \Omega \neq 0, \qquad (4.5)\end{aligned}$$

where κ_1 is given in Eq (4.4).
Set 3:

$$\begin{aligned}\{p &= \pm p_1, a_2 = \frac{-2b_2\mu + 4c_1p_1^2 + c_1^3 - 3c_2^2c_1 - 2\lambda}{2\Omega},\\ b_1 &= \frac{-2a_1\Omega - 4c_2p_1^2 + c_2^3 - 3c_1^2c_2 - 2\lambda}{2\mu},\end{aligned}$$

$$\kappa_1 = -\frac{2\sqrt{\kappa_2\left(-c_1^2+2c_2c_1-c_2^2-4p_1^2\right)\left(-c_1^2+2c_2c_1-c_2^2+4p_1^2\right)}}{c_1^2-2c_2c_1+c_2^2+4p_1^2}\}, \tag{4.6}$$

$$\begin{aligned}
g_3(x,y,z,t) =&(\kappa_2 p_1 \exp(p_1(\frac{z(-2a_1\Omega-4c_2p_1^2+c_2^3-3c_1^2c_2-2\lambda)}{2\mu}+a_1y+c_1t+x))\\
&+p_1(-\exp(-p_1(\frac{z(-2a_1\Omega-4c_2p_1^2+c_2^3-3c_1^2c_2-2\lambda)}{2\mu}+a_1y+c_1t+x)))\\
&-\kappa_1 p_1 \sin(p_1(\frac{y(-2b_2\mu+4c_1p_1^2+c_1^3-3c_2^2c_1-2\lambda)}{2\Omega}+b_2z+c_2t+x)))\\
&/(\kappa_2 \exp(p_1(\frac{z(-2a_1\Omega-4c_2p_1^2+c_2^3-3c_1^2c_2-2\lambda)}{2\mu}\\
&+a_1y+c_1t+x))+\exp(-p_1(\frac{z(-2a_1\Omega-4c_2p_1^2+c_2^3-3c_1^2c_2-2\lambda)}{2\mu}+a_1y+c_1t+x))\\
&+\kappa_1 \cos(p_1(\frac{y(-2b_2\mu+4c_1p_1^2+c_1^3-3c_2^2c_1-2\lambda)}{2\Omega}+b_2z+c_2t+x))),\\
&\mu\neq 0, \Omega\neq 0,
\end{aligned} \tag{4.7}$$

where κ_1 is given in Eq (4.6).
Set 4:

$$\left\{b_1 = \frac{-a_1\Omega+4c_1p_1^2-c_1^3-\lambda}{\mu}, \kappa_1 = 0\right\}, \tag{4.8}$$

$$\begin{aligned}
g_4(x,y,z,t) =&(\kappa_2 p_1 \exp(p_1(\frac{z(-a_1\Omega+4c_1p_1^2-c_1^3-\lambda)}{\mu}+a_1y+c_1t+x))\\
&-p_1\exp(-p_1(\frac{z(-a_1\Omega+4c_1p_1^2-c_1^3-\lambda)}{\mu}+a_1y+c_1t+x)))\\
&/(\kappa_2 \exp(p_1(\frac{z(-a_1\Omega+4c_1p_1^2-c_1^3-\lambda)}{\mu}+a_1y+c_1t+x))\\
&+\exp(-p_1(\frac{z(-a_1\Omega+4c_1p_1^2-c_1^3-\lambda)}{\mu}+a_1y+c_1t+x))), \quad \mu\neq 0.
\end{aligned} \tag{4.9}$$

Set 5:

$$\begin{aligned}
&\{p = \pm\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}, a_2 = \frac{-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda}{\Omega},\\
&b_1 = \frac{-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda}{\mu}, \kappa_1 = \frac{2\left(\sqrt{3}\sqrt{c_1\kappa_2(2c_1+c_2)}\right)}{3c_1}, p_1 = -\frac{c_1}{2}-c_2\},
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&g_5(x,y,z,t)\\
=&((-\frac{c_1}{2}-c_2)\kappa_2 \exp((-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))
\end{aligned}$$

$$+(\frac{c_1}{2}+c_2)\exp((\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$-\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}\kappa_1\sin(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}+b_2z+c_2t+x)))$$
$$/(\kappa_2\exp((-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$+\exp(-(-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$+\kappa_1\cos(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}+b_2z+c_2t+x))),$$
$$\mu\neq 0,\Omega\neq 0, \tag{4.11}$$

where κ_1 is given in Eq (4.10).
Set 6:

$$\{p=\pm\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1},a_2=\frac{-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda}{\Omega},$$
$$b_1=\frac{-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda}{\mu},\kappa_1=-\frac{2\left(\sqrt{3}\sqrt{c_1\kappa_2\,(2c_1+c_2)}\right)}{3c_1},p_1=-\frac{c_1}{2}-c_2\}, \tag{4.12}$$

$$g_6(x,y,z,t)$$
$$=((-\frac{c_1}{2}-c_2)\kappa_2\exp((-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$+(\frac{c_1}{2}+c_2)\exp((\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$-\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}\kappa_1\sin(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}+b_2z+c_2t+x)))$$
$$/(\kappa_2\exp((-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$+\exp(-(-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))$$
$$+\kappa_1\cos(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}+b_2z+c_2t+x))),$$
$$\mu\neq 0,\Omega\neq 0, \tag{4.13}$$

where κ_1 is given in Eq (4.12).
Set 7:

$$\{p=\pm\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1},a_2=\frac{-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda}{\Omega},$$
$$b_1=\frac{-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda}{\mu},\kappa_1=\frac{2\left(\sqrt{3}\sqrt{c_1\kappa_2\,(2c_1+c_2)}\right)}{3c_1},p_1=\frac{c_1}{2}+c_2\}, \tag{4.14}$$

$$
\begin{aligned}
&g_7(x,y,z,t)\\
=&((\frac{c_1}{2}+c_2)\kappa_2\exp((\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y\\
&+c_1t+x))+(-\frac{c_1}{2}-c_2)\exp((-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&-(\sqrt{-3c_1^2-6c_2c_1}\sqrt{c_1\kappa_2(2c_1+c_2)}\sin(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}\\
&+b_2z+c_2t+x)))/(\sqrt{3}c_1))/(\kappa_2\exp((\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&+\exp(-(\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&+((2(\sqrt{3}\sqrt{c_1\kappa_2(2c_1+c_2)}))\cos(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}\\
&+b_2z+c_2t+x)))/(3c_1)),\ \mu\neq 0,\Omega\neq 0.
\end{aligned}\tag{4.15}
$$

Set 8:

$$
\begin{aligned}
&\{p=\pm\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1},a_2=\frac{-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda}{\Omega},\\
&b_1=\frac{-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda}{\mu},\kappa_1=-\frac{2(\sqrt{3}\sqrt{c_1(2c_1+c_2)\kappa_2})}{3c_1},p_1=\frac{c_1}{2}+c_2\},
\end{aligned}\tag{4.16}
$$

$$
\begin{aligned}
&g_8(x,y,z,t)\\
=&((\frac{c_1}{2}+c_2)\kappa_2\exp((\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&+(-\frac{c_1}{2}-c_2)\exp((-\frac{c_1}{2}-c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&+\frac{\sqrt{-3c_1^2-6c_2c_1}\sqrt{c_1(2c_1+c_2)\kappa_2}\sin(\frac{1}{2}\sqrt{-3c_1^2-6c_2c_1}(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}+b_2z+c_2t+x))}{\sqrt{3}c_1})\\
&/(\kappa_2\exp((\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&+\exp(-(\frac{c_1}{2}+c_2)(\frac{z(-a_1\Omega+4c_2c_1^2+4c_2^2c_1-\lambda)}{\mu}+a_1y+c_1t+x))\\
&-((2(\sqrt{3}\sqrt{c_1(2c_1+c_2)\kappa_2}))\cos(\frac{1}{2}(-\sqrt{-3c_1^2-6c_2c_1})(\frac{y(-b_2\mu+c_1^3+3c_2c_1^2+3c_2^2c_1+c_2^3-\lambda)}{\Omega}\\
&+b_2z+c_2t+x)))/(3c_1)),\ \ \mu\neq 0,\Omega\neq 0,c_1\neq 0.
\end{aligned}\tag{4.17}
$$

Set 9:

$$
\{p=\pm\frac{\sqrt{-3c_1(c_1+2c_2)}p_1}{c_1+2c_2},b_1=\frac{-a_1\Omega+4c_1p_1^2-c_1^3-\lambda}{\mu},
$$

$$a_2 = \frac{c_1(-2b_2\mu + 16c_2p_1^2 - 6c_2^3 - 2\lambda) - 4c_2(b_2\mu - c_2p_1^2 + \lambda) + c_1^2(4p_1^2 - 3c_2^2) + c_1^4 + 2c_2c_1^3}{2(c_1 + 2c_2)\Omega},$$

$$\kappa_1 = \frac{2(\sqrt{3}\sqrt{c_1\kappa_2(c_1^2 + 4c_2c_1 + 4c_2^2 + 4p_1^2)(4c_1p_1^2 + 8c_2p_1^2 + 3c_1^3 + 12c_2c_1^2 + 12c_2^2c_1)})}{3c_1(c_1^2 + 4c_2c_1 + 4c_2^2 + 4p_1^2)}\}, \tag{4.18}$$

$$\begin{aligned}g_9(x,y,z,t) =&(-\frac{\sqrt{3}\sqrt{-c_1(c_1+2c_2)}\kappa_1 p_1 \sin(\frac{\sqrt{3}\sqrt{-c_1(c_1+2c_2)}p_1(a_2y+b_2z+c_2t+x)}{c_1+2c_2})}{c_1+2c_2}\\
&+ \kappa_2 p_1 \exp(p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))\\
&+ p_1(-\exp(-p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu}\\
&+ a_1y + c_1t + x))))/(\kappa_1 \cos(\frac{(\sqrt{-3c_1(c_1+2c_2)}p_1)(a_2y + b_2z + c_2t + x)}{c_1+2c_2})\\
&+ \kappa_2 \exp(p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))\\
&+ \exp(-p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))),\quad \mu \neq 0, c_1 + 2c_2 \neq 0,\end{aligned} \tag{4.19}$$

where a_2 and κ_1 are given in Eq (4.18).
Set 10:

$$\{p = \pm\frac{\sqrt{-3c_1(c_1+2c_2)}p_1}{c_1+2c_2}, b_1 = \frac{-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda}{\mu},$$

$$a_2 = \frac{c_1(-2b_2\mu + 16c_2p_1^2 - 6c_2^3 - 2\lambda) - 4c_2(b_2\mu - c_2p_1^2 + \lambda) + c_1^2(4p_1^2 - 3c_2^2) + c_1^4 + 2c_2c_1^3}{2(c_1 + 2c_2)\Omega},$$

$$\kappa_1 = -\frac{2(\sqrt{3}\sqrt{c_1\kappa_2(c_1^2 + 4c_2c_1 + 4c_2^2 + 4p_1^2)(4c_1p_1^2 + 8c_2p_1^2 + 3c_1^3 + 12c_2c_1^2 + 12c_2^2c_1)})}{3c_1(c_1^2 + 4c_2c_1 + 4c_2^2 + 4p_1^2)}\}, \tag{4.20}$$

$$\begin{aligned}g_{10}(x,y,z,t) =&(-\frac{\sqrt{3}\sqrt{-c_1(c_1+2c_2)}\kappa_1 p_1 \sin(\frac{\sqrt{3}\sqrt{-c_1(c_1+2c_2)}p_1(a_2y+b_2z+c_2t+x)}{c_1+2c_2})}{c_1+2c_2}\\
&+ \kappa_2 p_1 \exp(p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))\\
&+ p_1(-\exp(-p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu}\\
&+ a_1y + c_1t + x))))/(\kappa_1 \cos(\frac{(\sqrt{-3c_1(c_1+2c_2)}p_1)(a_2y + b_2z + c_2t + x)}{c_1+2c_2})\\
&+ \kappa_2 \exp(p_1(\frac{z(-a_1\Omega + 4c_1p_1^2 - c_1^3 - \lambda)}{\mu} + a_1y + c_1t + x))\end{aligned}$$

$$+\exp(-p_1(\frac{z(-a_1\Omega+4c_1p_1^2-c_1^3-\lambda)}{\mu}+a_1y+c_1t+x))),\quad \mu\neq 0, c_1+2c_2\neq 0, \quad (4.21)$$

where a_2 and κ_1 are given in Eq (4.20).
Set 11:

$$\{p=\pm\frac{\sqrt{-3c_2(2c_1+c_2)}p_1}{3c_2}, a_2=\frac{-3b_2\mu+(8c_1+4c_2)p_1^2-3c_2^3-3\lambda}{3\Omega},$$
$$b_1=\frac{c_2\left(-6a_1\Omega+16c_1p_1^2-6\lambda\right)+c_2^2\left(4p_1^2-9c_1^2\right)+4c_1^2p_1^2+3c_2^4}{6(c_2\mu)},$$
$$\kappa_1=\frac{2\left(\sqrt{3}\sqrt{c_2\kappa_2\left(4c_2p_1^2+8c_1p_1^2+9c_2^3+18c_1c_2^2\right)\left(3c_2^2+6c_1c_2+4p_1^2\right)}\right)}{4c_2p_1^2+8c_1p_1^2+9c_2^3+18c_1c_2^2}\}, \quad (4.22)$$

$$g_{11}(x,y,z,t)$$
$$=(\kappa_2p_1\exp(p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6c_2\mu}+a_1y+c_1t+x))$$
$$+p_1(-\exp(-p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6c_2\mu}$$
$$+a_1y+c_1t+x)))-\frac{\sqrt{-c_2(2c_1+c_2)}\kappa_1p_1\sin(\frac{\sqrt{-c_2(2c_1+c_2)}p_1(\frac{y(-3b_2\mu+(8c_1+4c_2)p_1^2-3c_2^3-3\lambda)}{3\Omega}+b_2z+c_2t+x)}{\sqrt{3}c_2})}{\sqrt{3}c_2})$$
$$/(\kappa_2\exp(p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6(c_2\mu)}+a_1y+c_1t+x))$$
$$+\exp(-p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6(c_2\mu)}+a_1y+c_1t+x))$$
$$+\kappa_1\cos(\frac{(\sqrt{-3c_2(2c_1+c_2)}p_1)(\frac{y(-3b_2\mu+(8c_1+4c_2)p_1^2-3c_2^3-3\lambda)}{3\Omega}+b_2z+c_2t+x)}{3c_2})),$$
$$\mu\neq 0, c_2\neq 0, \quad (4.23)$$

where κ_1 is given in Eq (4.22).
Set 12:

$$\{p=\pm\frac{\sqrt{-3c_2(2c_1+c_2)}p_1}{3c_2}, a_2=\frac{-3b_2\mu+(8c_1+4c_2)p_1^2-3c_2^3-3\lambda}{3\Omega},$$
$$b_1=\frac{c_2\left(-6a_1\Omega+16c_1p_1^2-6\lambda\right)+c_2^2\left(4p_1^2-9c_1^2\right)+4c_1^2p_1^2+3c_2^4}{6(c_2\mu)},$$
$$\kappa_1=-\frac{2\left(\sqrt{3}\sqrt{c_2\kappa_2\left(4c_2p_1^2+8c_1p_1^2+9c_2^3+18c_1c_2^2\right)\left(3c_2^2+6c_1c_2+4p_1^2\right)}\right)}{4c_2p_1^2+8c_1p_1^2+9c_2^3+18c_1c_2^2}\}, \quad (4.24)$$

$$g_{12}(x,y,z,t)$$

$$
\begin{aligned}
=&(\kappa_2 p_1 \exp(p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6c_2\mu}+a_1y+c_1t+x))\\
&+p_1(-\exp(-p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6c_2\mu}\\
&+a_1y+c_1t+x)))-\frac{\sqrt{-c_2(2c_1+c_2)}\kappa_1 p_1\sin(\frac{\sqrt{-c_2(2c_1+c_2)}p_1(\frac{y(-3b_2\mu+(8c_1+4c_2)p_1^2-3c_2^3-3\lambda)}{3\Omega}+b_2z+c_2t+x)}{\sqrt{3}c_2})}{\sqrt{3}c_2})\\
&/(\kappa_2\exp(p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6(c_2\mu)}+a_1y+c_1t+x))\\
&+\exp(-p_1(\frac{z(c_2(-6a_1\Omega+16c_1p_1^2-6\lambda)+c_2^2(4p_1^2-9c_1^2)+4c_1^2p_1^2+3c_2^4)}{6(c_2\mu)}+a_1y+c_1t+x))\\
&+\kappa_1\cos(\frac{(\sqrt{-3c_2(2c_1+c_2)}p_1)(\frac{y(-3b_2\mu+(8c_1+4c_2)p_1^2-3c_2^3-3\lambda)}{3\Omega}+b_2z+c_2t+x)}{3c_2})),\\
&\mu\neq 0, c_2\neq 0,
\end{aligned}
\tag{4.25}
$$

where κ_1 is given in Eq (4.24).
Set 13:

$$
\begin{aligned}
\{a_2=&\frac{2p^2p_1^2(-b_2\mu+(c_1-\frac{c_2}{2})p^2+\frac{c_2^3}{2}-3c_1c_2^2+\frac{3}{2}c_1^2c_2-\lambda)}{(p^2+p_1^2)^2\Omega}\\
&+\frac{p_1^4(-b_2\mu+(4c_1+c_2)p^2+2c_1^3-3c_2c_1^2-\lambda)-p^4(b_2\mu+c_2p^2+c_2^3+\lambda)+(2c_1+c_2)p_1^6}{(p^2+p_1^2)^2\Omega},\\
b_1=&\frac{p^2p_1^2(-2a_1\Omega+(c_1-2c_2)p_1^2+c_1^3-6c_2c_1^2+3c_2^2c_1-2\lambda)}{\mu(p^2+p_1^2)^2}\\
&+\frac{p^4(-a_1\Omega+(-c_1-4c_2)p_1^2+2c_2^3-3c_1c_2^2-\lambda)+p_1^4(-a_1\Omega+c_1p_1^2-c_1^3-\lambda)+(-c_1-2c_2)p^6}{\mu(p^2+p_1^2)^2},
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
\kappa_1=&(2(p_1\sqrt{\kappa_2(-(c_1^2p_1^2+c_2^2p_1^2-2c_1c_2p_1^2+p^4+2p_1^2p^2+p_1^4))(-c_1^2p^2-c_2^2p^2+2c_1c_2p^2+p^4+2p_1^2p^2+p_1^4)}))\\
&/(p(c_1^2p_1^2+c_2^2p_1^2-2c_1c_2p_1^2+p^4+2p_1^2p^2+p_1^4))\},
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
g_{13}(x,y,z,t)=&(\kappa_2p_1e^{p_1(a_1y+b_1z+c_1t+x)}-\kappa_1p\sin(p(a_2y+b_2z+c_2t+x))\\
&+p_1(-e^{p_1(-(a_1y+b_1z+c_1t+x))}))/(\kappa_2e^{p_1(a_1y+b_1z+c_1t+x)}+\kappa_1\cos(p(a_2y+b_2z+c_2t+x))\\
&+e^{-p_1(a_1y+b_1z+c_1t+x)}),\quad \mu\neq 0,\Omega\neq 0,(p^2+p_1^2)\neq 0,
\end{aligned}
\tag{4.28}
$$

where a_2, b_1, and κ_1 are given in Eq (4.26).
Set 14:

$$
\{a_2=\frac{2p^2p_1^2(-b_2\mu+(c_1-\frac{c_2}{2})p^2+\frac{c_2^3}{2}-3c_1c_2^2+\frac{3}{2}c_1^2c_2-\lambda)}{(p^2+p_1^2)^2\Omega}
$$

$$+\frac{p_1^4(-b_2\mu+(4c_1+c_2)p^2+2c_1^3-3c_2c_1^2-\lambda)-p^4(b_2\mu+c_2p^2+c_2^3+\lambda)+(2c_1+c_2)p_1^6}{(p^2+p_1^2)^2\Omega},$$

$$b_1=\frac{p^2p_1^2(-2a_1\Omega+(c_1-2c_2)p_1^2+c_1^3-6c_2c_1^2+3c_2^2c_1-2\lambda)}{\mu(p^2+p_1^2)^2}$$
$$+\frac{p^4(-a_1\Omega+(-c_1-4c_2)p_1^2+2c_2^3-3c_1c_2^2-\lambda)+p_1^4(-a_1\Omega+c_1p_1^2-c_1^3-\lambda)+(-c_1-2c_2)p^6}{\mu(p^2+p_1^2)^2}, \tag{4.29}$$

$$\kappa_1=-(2(p_1\sqrt{\kappa_2(-(c_1^2p_1^2+c_2^2p_1^2-2c_1c_2p_1^2+p^4+2p_1^2p^2+p_1^4))(-c_1^2p^2-c_2^2p^2+2c_1c_2p^2+p^4+2p_1^2p^2+p_1^4)}))$$
$$/(p(c_1^2p_1^2+c_2^2p_1^2-2c_1c_2p_1^2+p^4+2p_1^2p^2+p_1^4))\}, \tag{4.30}$$

$$g_{14}(x,y,z,t)=(\kappa_2p_1e^{p_1(a_1y+b_1z+c_1t+x)}-\kappa_1p\sin(p(a_2y+b_2z+c_2t+x))+p_1(-e^{p_1(-(a_1y+b_1z+c_1t+x))}))$$
$$/(\kappa_2e^{p_1(a_1y+b_1z+c_1t+x)}+\kappa_1\cos(p(a_2y+b_2z+c_2t+x))+e^{-p_1(a_1y+b_1z+c_1t+x)}),$$
$$\mu\neq0,\Omega\neq0,(p^2+p_1^2)\neq0, \tag{4.31}$$

where a_2, b_1, and κ_1 are given in Eq (4.29). □

4.2. Two-soliton solutions

Theorem 4.2. *The two-soliton solutions of Eq* (3.1) *are produced by utilizing Eq* (3.3)*.*

Proof. To determine the two-soliton solutions, we assume the relation given as in [40],

$$U(x,y,z,t)=A_{12}\exp(\kappa_1(a_1x+b_1y+c_1z+d_1t)+\kappa_2(a_2x+b_2y+c_2z+d_2t)+\delta_1+\delta_2)$$
$$+e^{\kappa_1(a_1x+b_1y+c_1z+d_1t)+\delta_1}+e^{\kappa_2(a_2x+b_2y+c_2z+d_2t)+\delta_2}+1. \tag{4.32}$$

By applying Eq (4.32) to Eq (3.3) and collecting coefficients of every order of

$$\exp\left(\kappa_1(a_1x+b_1y+c_1z+d_1t)+\kappa_2(a_2x+b_2y+c_2z+d_2t)+\delta_1+\delta_2\right),$$
$$e^{\kappa_1(a_1x+b_1y+c_1z+d_1t)+\delta_1},$$
$$e^{\kappa_2(a_2x+b_2y+c_2z+d_2t)+\delta_2},$$

and setting them equal to 0, We achieve the following solution sets by solving the system with the help of the Maple tool.
Set 1:

$$\{b_1=-\frac{a_1^2c_1\mu-a_1^4d_1\kappa_1^2+a_1^3\lambda+d_1^3}{a_1^2\Omega},b_2=-\frac{a_2^2c_2\mu-a_2^4d_2\kappa_2^2+a_2^3\lambda+d_2^3}{a_2^2\Omega},$$
$$A_{12}=\frac{a_2^2\left(a_1^4\kappa_1^2+d_1^2\right)-2a_1a_2d_1d_2+a_1^2d_2^2+a_1^2a_2^4\kappa_2^2-2a_1^3a_2^3\kappa_1\kappa_2}{a_2^2\left(a_1^4\kappa_1^2+d_1^2\right)-2a_1a_2d_1d_2+a_1^2d_2^2+a_1^2a_2^4\kappa_2^2+2a_1^3a_2^3\kappa_1\kappa_2}\}, \tag{4.33}$$

$$
\begin{aligned}
g_{15}(x,y,z,t) =&(A_{12}(a_1\kappa_1 + a_2\kappa_2)\exp(\kappa_1(-\frac{y(a_1^2c_1\mu - a_1^4d_1\kappa_1^2 + a_1^3\lambda + d_1^3)}{a_1^2\Omega} + a_1x + c_1z + d_1t)\\
&+ \kappa_2(-\frac{y(a_2^2c_2\mu - a_2^4d_2\kappa_2^2 + a_2^3\lambda + d_2^3)}{a_2^2\Omega} + a_2x + c_2z + d_2t) + \delta_1 + \delta_2)\\
&+ a_1\kappa_1\exp(\kappa_1(-\frac{y(a_1^2c_1\mu - a_1^4d_1\kappa_1^2 + a_1^3\lambda + d_1^3)}{a_1^2\Omega} + a_1x + c_1z + d_1t) + \delta_1)\\
&+ a_2\kappa_2\exp(\kappa_2(-\frac{y(a_2^2c_2\mu - a_2^4d_2\kappa_2^2 + a_2^3\lambda + d_2^3)}{a_2^2\Omega} + a_2x + c_2z + d_2t)\\
&+ \delta_2))/(A_{12}\exp(\kappa_1(\frac{-y(a_1^2c_1\mu - a_1^4d_1\kappa_1^2 + a_1^3\lambda + d_1^3)}{a_1^2\Omega} + a_1x + c_1z + d_1t)\\
&+ \kappa_2(\frac{-y(a_2^2c_2\mu - a_2^4d_2\kappa_2^2 + a_2^3\lambda + d_2^3)}{a_2^2\Omega} + a_2x + c_2z + d_2t) + \delta_1 + \delta_2)\\
&+ \exp(\kappa_1(\frac{-y(a_1^2c_1\mu - a_1^4d_1\kappa_1^2 + a_1^3\lambda + d_1^3)}{a_1^2\Omega} + a_1x + c_1z + d_1t) + \delta_1)\\
&+ \exp(\kappa_2(\frac{-y(a_2^2c_2\mu - a_2^4d_2\kappa_2^2 + a_2^3\lambda + d_2^3)}{a_2^2\Omega} + a_2x + c_2z + d_2t) + \delta_2) + 1),\\
&a_1 \neq 0, a_2 \neq 0, \Omega \neq 0,
\end{aligned}
\tag{4.34}
$$

where A_{12} is given in Eq (4.33).

Set 2:

$$
\begin{aligned}
&\{a_1 = \frac{a_2d_1}{2d_2}, b_1 = -\frac{4a_2^2c_1d_2^2\mu - a_2^4d_1^3\kappa_1^2 + 2a_2^3d_1d_2\lambda + 16d_1d_2^4}{4a_2^2d_2^2\Omega},\\
&b_2 = -\frac{a_2^2c_2\mu - a_2^4d_2\kappa_2^2 + a_2^3\lambda + d_2^3}{a_2^2\Omega}, A_{12} = \frac{a_2^4d_1^2\kappa_1^2 + 4a_2^4d_2^2\kappa_2^2 - 4a_2^4d_1d_2\kappa_1\kappa_2 + 4d_2^4}{a_2^4d_1^2\kappa_1^2 + 4a_2^4d_2^2\kappa_2^2 + 4a_2^4d_1d_2\kappa_1\kappa_2 + 4d_2^4}\},
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
g_{16}(x,y,z,t) =&(A_{12}(\frac{a_2d_1\kappa_1}{2d_2} + a_2\kappa_2)\exp(\kappa_2(a_2x + b_2y + c_2z + d_2t)\\
&+ \kappa_1(-\frac{y(4a_2^2c_1d_2^2\mu - a_2^4d_1^3\kappa_1^2 + 2a_2^3d_1d_2\lambda + 16d_1d_2^4)}{4a_2^2d_2^2\Omega} + \frac{a_2d_1x}{2d_2} + c_1z + d_1t) + \delta_1 + \delta_2)\\
&+ \frac{a_2d_1\kappa_1\exp(\kappa_1(-\frac{y(4a_2^2c_1d_2^2\mu + a_2^4(-d_1^3)\kappa_1^2 + 2a_2^3d_1d_2\lambda + 16d_1d_2^4)}{4a_2^2d_2^2\Omega} + \frac{a_2d_1x}{2d_2} + c_1z + d_1t) + \delta_1)}{2d_2}\\
&+ a_2\kappa_2\exp(\kappa_2(-\frac{y(a_2^2c_2\mu - a_2^4d_2\kappa_2^2 + a_2^3\lambda + d_2^3)}{a_2^2\Omega} + a_2x + c_2z + d_2t) + \delta_2))\\
&/(A_{12}\exp(\kappa_2(a_2x + b_2y + c_2z + d_2t)\\
&+ \kappa_1(\frac{-y(4a_2^2c_1d_2^2\mu - a_2^4d_1^3\kappa_1^2 + 2a_2^3d_1d_2\lambda + 16d_1d_2^4)}{4a_2^2d_2^2\Omega}\\
&+ \frac{a_2d_1}{2d_2}x + c_1z + d_1t) + \delta_1 + \delta_2) + \exp(\kappa_1(\frac{-y(4a_2^2c_1d_2^2\mu - a_2^4d_1^3\kappa_1^2 + 2a_2^3d_1d_2\lambda + 16d_1d_2^4)}{4a_2^2d_2^2\Omega}
\end{aligned}
$$

$$+\frac{a_2d_1}{2d_2}x+c_1z+d_1t)+\delta_1)+\exp(\kappa_2(\frac{-y(a_2^2c_2\mu-a_2^4d_2\kappa_2^2+a_2^3\lambda+d_2^3)}{a_2^2\Omega}+a_2x+c_2z+d_2t)$$
$$+\delta_2)+1),\quad a_2\neq 0, d_2\neq 0, \Omega\neq 0, \tag{4.36}$$

where A_{12} is given in Eq (4.35).

Substituting solution Eq (4.36) into Eq (3.1) yields an identically vanishing residual under the constraint Eq (4.35), optionally reported that this was verified symbolically in Maple. □

5. Modulation instability (MI)

Consider the steady-state result for the (3+1)-dimensional dynamical oceanic wave model as presented in [41–44].

The modulation instability study is carried out by perturbing a continuous wave (CW) steady-state solution of the governing equation, which takes the form

$$g(x,y,z,t)=G(x,y,z,t)\,e^{\iota at}. \tag{5.1}$$

This state represents a uniform background of constant intensity over which small disturbances are introduced.

To make the steady background explicit and ensure consistency, we have revised the formulation by introducing the transformation

$$g(x,y,z,t)=\left(\epsilon G(x,y,z,t)+\sqrt{a}\right)e^{\iota at}, \tag{5.2}$$

where a is a real constant associated with the carrier phase (or intrinsic temporal frequency) of the background wave, and $G(x,y,z,t)$ denotes a small complex-valued perturbation (envelope) superimposed on the steady-state $\sqrt{a}$.

In this formulation, ϵ is a small bookkeeping parameter. This representation clearly separates the constant background $\sqrt{a}$ from the perturbation G. Subsequently, we substitute this form into the governing equation and retain terms up to first order in ϵ, neglecting higher-order contributions. This modification removes the earlier ambiguity and provides a consistent perturbation framework throughout the analysis.

Substituting Eq (5.2) into Eq (3.1), after linearizing, we obtain

$$-3a^2G_t+3a\iota G_{tt}-a\iota G_{xxxx}+\lambda G_{xxx}+\mu G_{xxz}+\Omega G_{xxy}+G_{ttt}-G_{xxxxt}-\iota a^3G=0. \tag{5.3}$$

Assume the solution of Eq (5.3) is given by

$$G(x,y,z,t)=A_1e^{\iota(-qt+\beta_1x+\beta_2y+\beta_3z)}+A_2e^{-\iota(-qt+\beta_1x+\beta_2y+\beta_3z)}, \tag{5.4}$$

where β_1,β_2, and β_3 are the perturbation wave-numbers, q is the corresponding temporal frequency, and A_1, and A_2 are small amplitudes. Substituting Eq (5.4) into Eq (5.3), collecting the coefficients of $e^{\iota(-qt+\beta_1x+\beta_2y+\beta_3z)}$ and $e^{-\iota(-qt+\beta_1x+\beta_2y+\beta_3z)}$, and setting the determinant of the resulting coefficient matrix to zero, we obtain

$$a^6+2a^4\beta_1^4-3a^4q^2+a^2\beta_1^8+3a^2q^4+6a^2\beta_1^3\lambda q+6a^2\beta_3\beta_1^2\mu q+6a^2\beta_2\beta_1^2q\Omega-\beta_1^6\lambda^2-2\beta_3\beta_1^5\lambda\mu$$

$$- 2\beta_2\beta_1^5\lambda\Omega - \beta_3^2\beta_1^4\mu^2 - 2\beta_2\beta_3\beta_1^4\mu\Omega - \beta_2^2\beta_1^4\Omega^2 - q^6 - 2\beta_1^4q^4 + 2\beta_1^3\lambda q^3 + 2\beta_3\beta_1^2\mu q^3 + 2\beta_2\beta_1^2q^3\Omega - \beta_1^8q^2 + 2\beta_1^7\lambda q + 2\beta_3\beta_1^6\mu q + 2\beta_2\beta_1^6q\Omega = 0. \tag{5.5}$$

Finding the dispersion result of Eq (5.5) yields

$$\omega = \frac{3a^2q - \beta_3\beta_1^2\mu - \beta_2\beta_1^2\Omega + q^3 + \beta_1^4q}{\beta_1^3} \pm \frac{\sqrt{a^6 + 2a^4\beta_1^4 + 6a^4q^2 + a^2\beta_1^8 + 9a^2q^4 + 6a^2\beta_1^4q^2}}{\beta_1^3}. \tag{5.6}$$

The solution will be unstable if

$$a^6 + 2a^4\beta_1^4 + 6a^4q^2 + a^2\beta_1^8 + 9a^2q^4 + 6a^2\beta_1^4q^2 < 0. \tag{5.7}$$

The modulation instability gain spectrum $G(p)$ is obtained as

$$G(q) = 2\,\mathrm{Im}(\omega) = \pm\sqrt{a^6 + 2a^4\beta_1^4 + 6a^4q^2 + a^2\beta_1^8 + 9a^2q^4 + 6a^2\beta_1^4q^2}. \tag{5.8}$$

6. Results and discussion

Here, we compare the solutions obtained in this work with those reported previously for the governing equation. In [36], lump and kink-soliton solutions of the governing equation were obtained using the Hirota trilinear and bilinear schemes. In [37, 38], multisoliton solutions were derived using the simplified Hirota technique. In [45], solitary wave solutions were obtained by using the modified extended tanh-function method. In our research, we obtained breather-wave and two-soliton solutions of the considered model. For a specified range of parameter values, various forms of exact soliton solutions including anti-kink soliton, double peaked soliton, solitary wave, kink soliton, breather-wave, and singular soliton solutions are illustrated in Figures 1–15. Singularities are frequently present in the mathematical models used in physics and can appear in many contexts. To better understand the behavior of the solutions, we also provide 2D, 3D, and contour plots of the obtained results. These solutions are expected to contribute to further studies of this equation and may have applications in ocean engineering and other related fields.

Figures 2 and 10 demonstrate the anti-kink soliton $g_1(x, y, z, t)$ and $g_{10}(x, y, z, t)$ for $a_1 = 1,\ a_2 = 2,\ c_1 = 1,\ \kappa_1 = 0,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ p_1 = 1,\ t = 1,\ \Omega = 3,\ z = 2$ and $a_1 = 1,\ b_2 = 2,\ c_1 = -1,\ c_2 = 3,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = 1,\ t = 1,\ z = 2$, respectively. while Figure 3 visualizes the double–peaked soliton structure $g_2(x, y, z, t)$ for $a_1 = 2,\ c_1 = 2,\ c_2 = -1,\ \kappa_2 = 1,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = -0.5,\ t = 1,\ z = 2$, whereas Figure 4 displays the solitary wave $g_3(x, y, z, t)$ for $a_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = -1,\ \kappa_2 = 1,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = -1,\ t = 1,\ z = 2$. Figures 5 and 6 represent the kink soliton $g_4(x, y, z, t)$ and $g_5(x, y, z, t)$ for $a_1 = 4,\ a_2 = 2,\ c_1 = 3,\ \kappa_1 = 0,\ \kappa_2 = 2,\ \lambda = -1,\ \mu = 2,\ \Omega = 1,\ p_1 = 1,\ t = 1,\ z = 2$ and $a_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = 0.5,\ \kappa_2 = 3,\ \lambda = 2,\ \mu = 1,\ \Omega = 0.5,\ p_1 = -\frac{c_1}{2} - c_2,\ t = 1,\ z = 2$, respectively. Figure 7 depicts the breather-wave structure $g_6(x, y, z, t)$ for $a_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = 0.5,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 1,\ \Omega = 0.5,\ p_1 = 0.5\ t = 1,\ z = 2$, and Figure 8 demonstrates the solitary wave $g_8(x, y, z, t)$ for $a_1 = 1,\ b_2 = 2,\ c_1 = -1,\ c_2 = 1,\ \kappa_2 = 2,\ \lambda = 1,\ \mu = 2,\ \Omega = 0.5,\ p_1 = \frac{c_1}{2} + c_2,\ t = 1,\ z = 2$. Figures 9 and 14 visualize the interaction between solitons $g_9(x, y, z, t)$ and $g_{15}(x, y, z, t)$ for $a_1 = 1,\ b_2 = 2,\ c_1 = -1,\ c_2 = 3,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = 1,\ t = 1,\ z = 2$ and

$a_1 = 3,\ a_2 = 2,\ c_1 = 2,\ c_2 = 4,\ \kappa_1 = 1,\ \kappa_2 = 2,\ \lambda = 4,\ \mu = 1,\ \Omega = 3,\ t = 1,\ z = 2$, respectively. Figure 11 demonstrates the kink soliton $g_{11}(x, y, z, t)$ for $a_1 = 1,\ b_2 = 2,\ c_1 = 3,\ c_2 = 1,\ \kappa_2 = 3,\ \lambda = 4,\ \mu = 2,\ \Omega = 1,\ p_1 = 0.5,\ t = 1,\ z = 2$; Figures 12 and 15 represent antikink waves $g_{12}(x, y, z, t)$ and $g_{16}(x, y, z, t)$ for $a_1 = 4,\ b_2 = 3,\ c_1 = 2,\ c_2 = -3,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = 0.5,\ t = 1,\ z = 2$ and $a_1 = 2,\ c_1 = 2,\ c_2 = 4,\ d_1 = 2,\ d_2 = 3,\ \kappa_1 = 1,\ \kappa_2 = 2,\ \lambda = 4,\ \mu = 1,\ \Omega = 3,\ t = 1,\ z = 2$, respectively. Figure 13 displays singular soliton $g_{14}(x, y, z, t)$ for $a_1 = 2,\ b_2 = 1,\ c_1 = 1,\ c_2 = 2,\ \kappa_2 = -1,\ \lambda = 1,\ \mu = 2,\ \Omega = 3,\ p_1 = 4,\ t = 1,\ z = 2$.

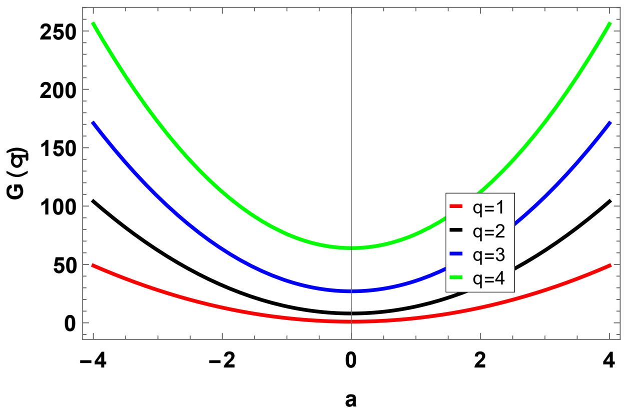
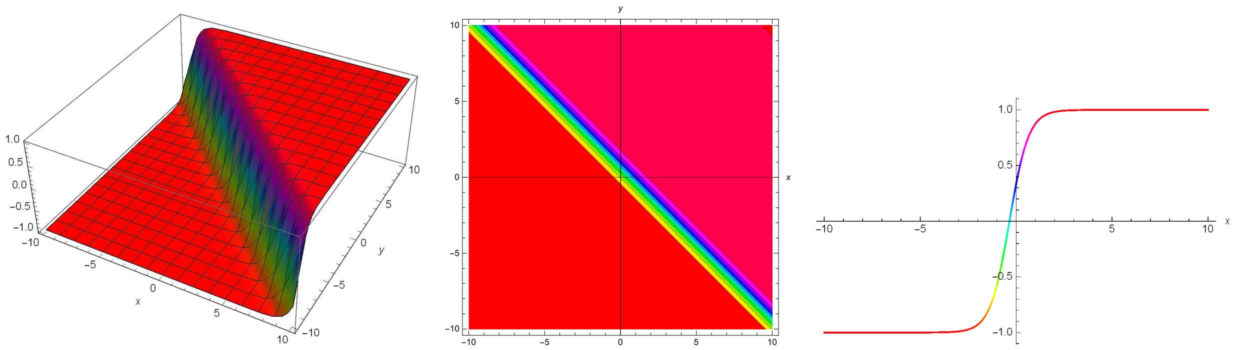
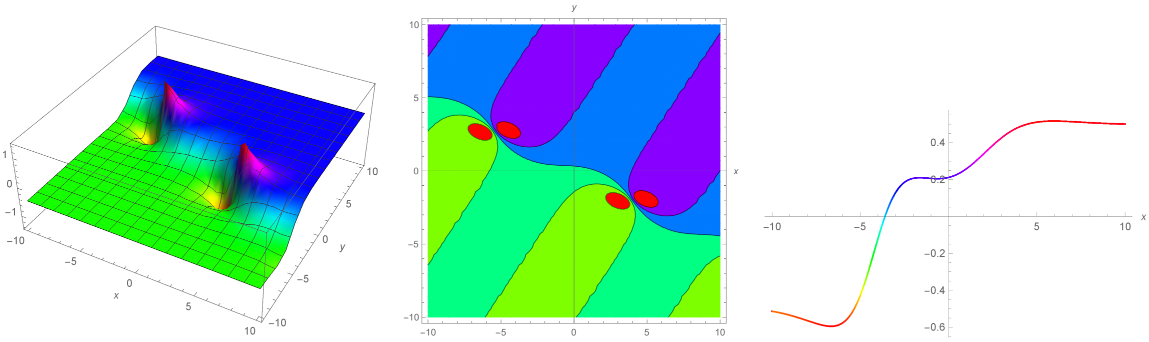


Figure 1. Gain spectrum of MI for different values of $q(q = 1, 2, 3, 4)$, $\beta_1 = 0.02$, and a $\in (-4, 4)$. The red line is for $q = 1$, the black line for $q = 2$, the blue line for $q = 3$, and the green line for $q = 4$. The graph is symmetric about the y-axis. This shows that the concerned model is stable.



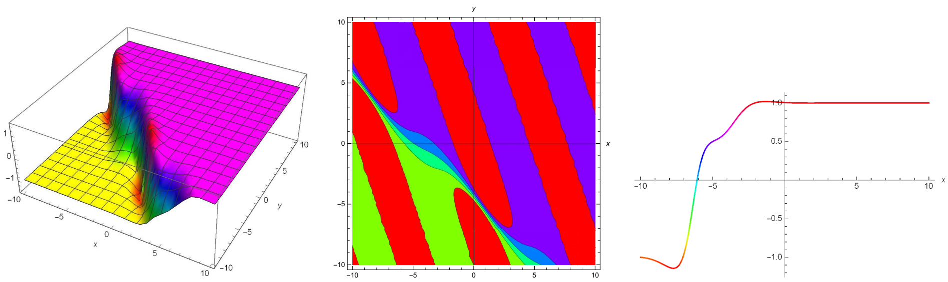
(a) (b) (c)

Figure 2. Graphical interpretations for solution $g_1(x, y, z, t)$ using $a_1 = 1,\ a_2 = 2,\ c_1 = 1,\ \kappa_1 = 0,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ p_1 = 1,\ t = 1,\ \Omega = 3,$ and $z = 2$. The figure illustrates a sharp transition between two asymptotic states, demonstrating the localized steep gradient and stable propagation profile.



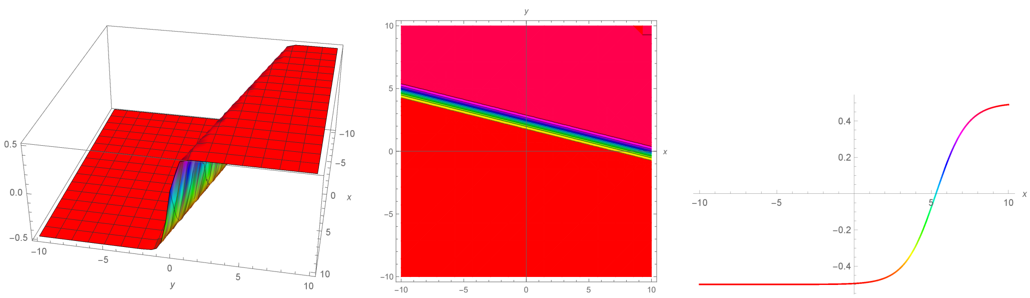
(a) (b) (c)

Figure 3. Graphical interpretations for solution $g_2(x, y, z, t)$ using $a_1 = 2,\ c_1 = 2,\ c_2 = -1,\ \kappa_2 = 1,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = -0.5,\ t = 1,$ and $z = 2$. The graph is showing a localized wave with an oscillatory envelope.



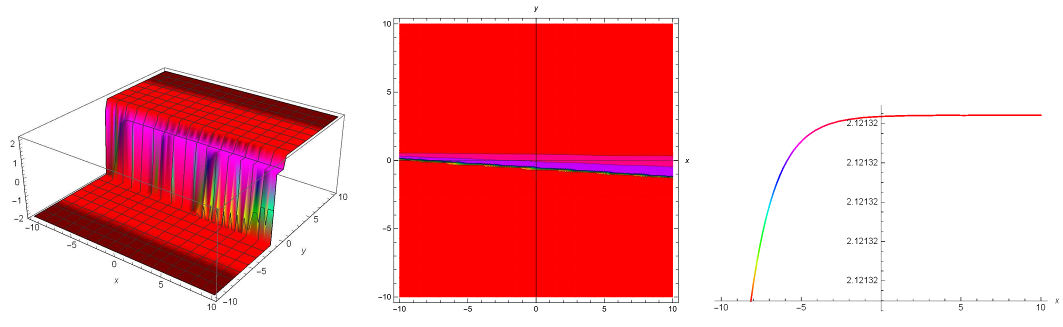
(a) (b) (c)

Figure 4. Graphical interpretations for solution $g_3(x, y, z, t)$ using $a_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = -1,\ \kappa_2 = 1,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = -1,\ t = 1,$ and $z = 2$. The figure demonstrates a sustained oscillatory profile, highlighting the regular spatial variation and persistence of the wave pattern.



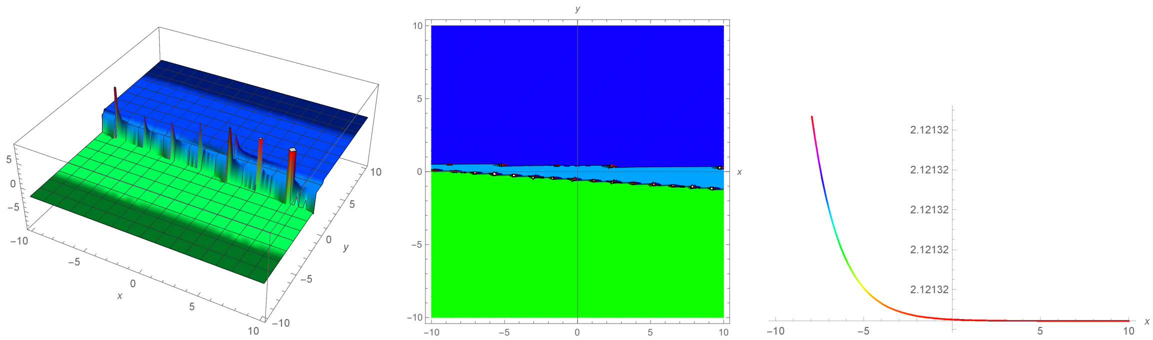
(a) (b) (c)

Figure 5. Graphical interpretations for solution $g_4(x, y, z, t)$ using $a_1 = 4,\ a_2 = 2,\ c_1 = 3,\ \kappa_1 = 0,\ \kappa_2 = 2,\ \lambda = -1,\ \mu = 2,\ \Omega = 1,\ p_1 = 1,\ t = 1,\ z = 2$. The figure illustrates a sharp transition between two asymptotic states, demonstrating the localized steep gradient and stable propagation profile.



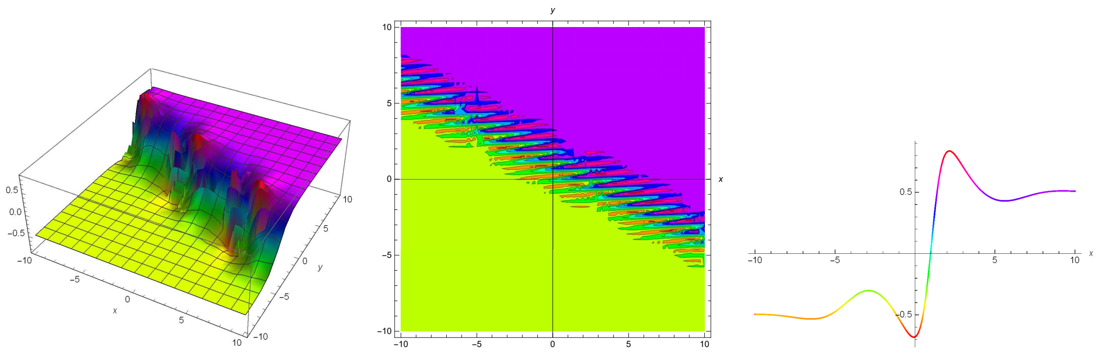
(a) (b) (c)

Figure 6. Graphical interpretations for solution $g_5(x, y, z, t)$ using $a_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = 0.5,\ \kappa_2 = 3,\ \lambda = 2,\ \mu = 1,\ \Omega = 0.5,\ p_1 = -\frac{c_1}{2} - c_2,\ t = 1,\ z = 2$. The graph is showing a localized wave with an oscillatory envelope.



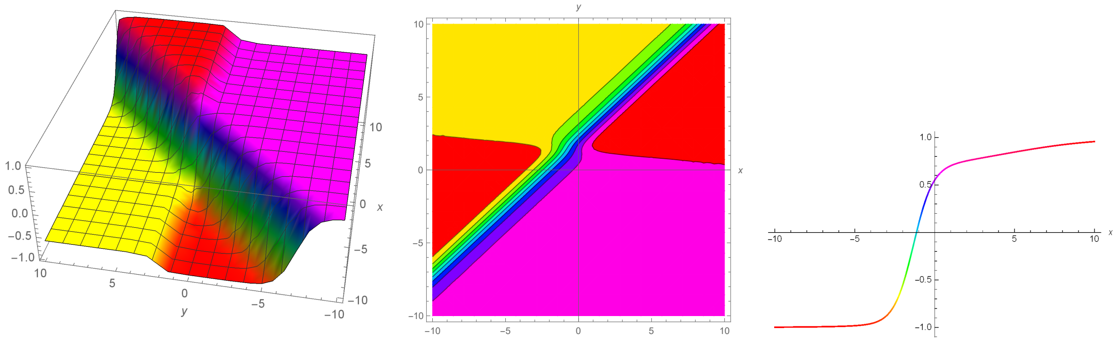
(a) (b) (c)

Figure 7. Graphical interpretations for solution $g_6(x, y, z, t)$ using $a_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = 0.5,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 1,\ \Omega = 0.5,\ p_1 = 0.5\ t = 1,\ z = 2$.



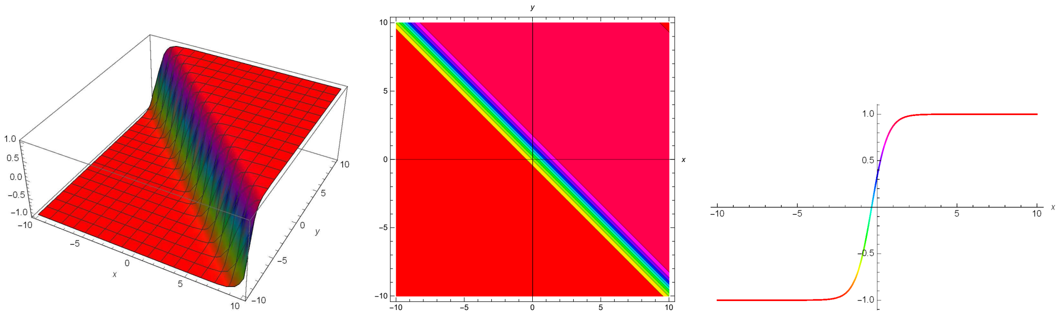
(a) (b) (c)

Figure 8. Graphical interpretations for solution $g_8(x, y, z, t)$ using $a_1 = 1,\ b_2 = 2,\ c_1 = -1,\ c_2 = 1,\ \kappa_2 = 2,\ \lambda = 1,\ \mu = 2,\ \Omega = 0.5,\ p_1 = \frac{c_1}{2} + c_2,\ t = 1,\ z = 2$.



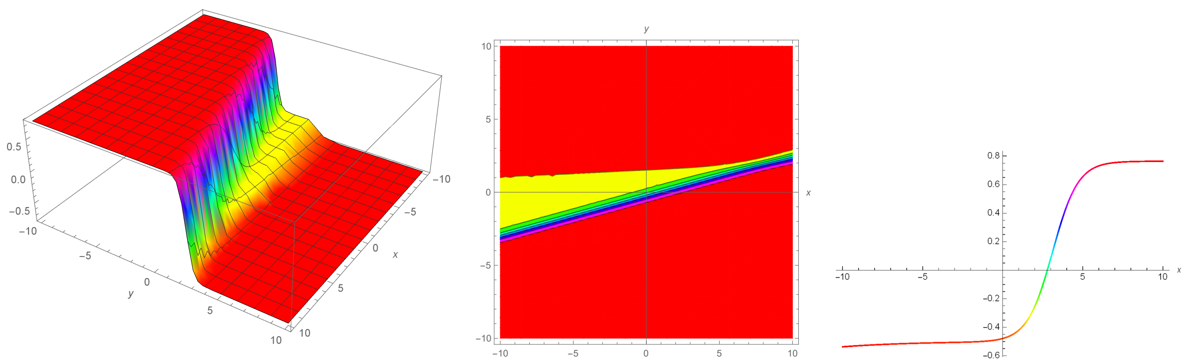
(a) (b) (c)

Figure 9. Graphical interpretations for solution $g_9(x, y, z, t)$ using $a_1 = 1,\ b_2 = 2,\ c_1 = -1,\ c_2 = 3,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = 1,\ t = 1,\ z = 2.$



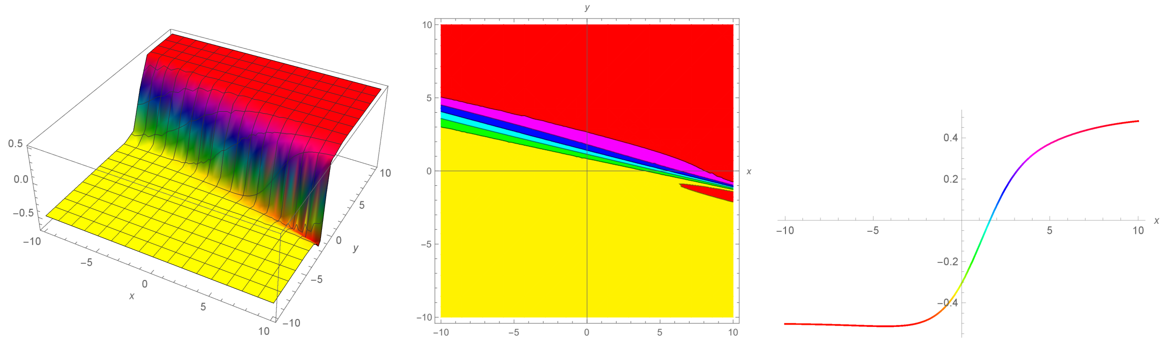
(a) (b) (c)

Figure 10. Graphical interpretations for solution $g_{10}(x, y, z, t)$ using $a_1 = 1,\ b_2 = 2,\ c_1 = -1,\ c_2 = 3,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = 1,\ t = 1,\ z = 2.$



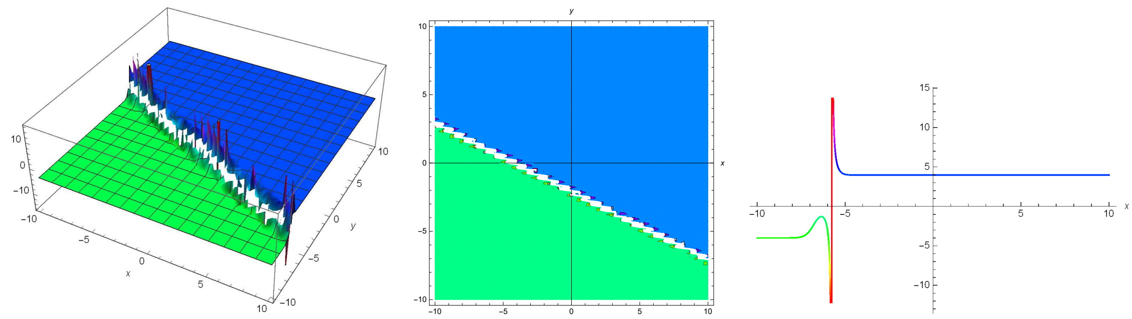
(a) (b) (c)

Figure 11. Graphical interpretations for solution $g_{11}(x, y, z, t)$ using $a_1 = 1,\ b_2 = 2,\ c_1 = 3,\ c_2 = 1,\ \kappa_2 = 3,\ \lambda = 4,\ \mu = 2,\ \Omega = 1,\ p_1 = 0.5,\ t = 1,\ z = 2.$



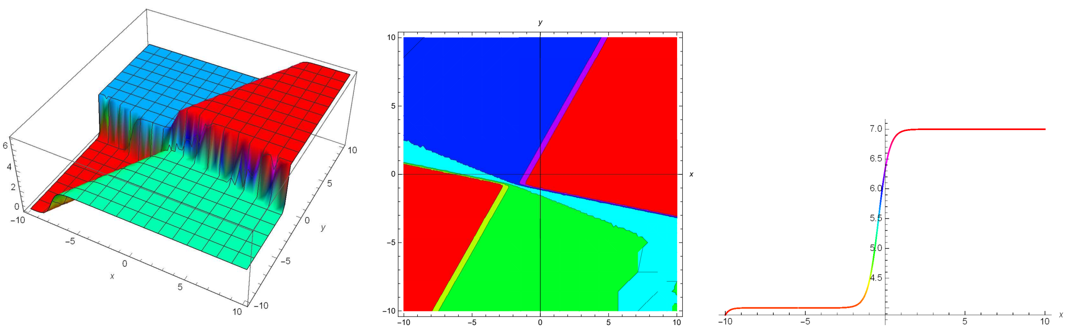
(a) (b) (c)

Figure 12. Graphical interpretations for solution $g_{12}(x, y, z, t)$ using $a_1 = 4,\ b_2 = 3,\ c_1 = 2,\ c_2 = -3,\ \kappa_2 = 2,\ \lambda = 2,\ \mu = 2,\ \Omega = 3,\ p_1 = 0.5,\ t = 1,\ z = 2.$



(a) (b) (c)

Figure 13. Graphical interpretations for solution $g_{14}(x, y, z, t)$ using $a_1 = 2,\ b_2 = 1,\ c_1 = 1,\ c_2 = 2,\ \kappa_2 = -1,\ \lambda = 1,\ \mu = 2,\ \Omega = 3,\ p_1 = 4,\ t = 1,\ z = 2.$



(a) (b) (c)

Figure 14. Graphical interpretations for solution $g_{15}(x, y, z, t)$ using $a_1 = 3,\ a_2 = 2,\ c_1 = 2,\ c_2 = 4,\ \kappa_1 = 1,\ \kappa_2 = 2,\ \lambda = 4,\ \mu = 1,\ \Omega = 3,\ t = 1,\ z = 2.$

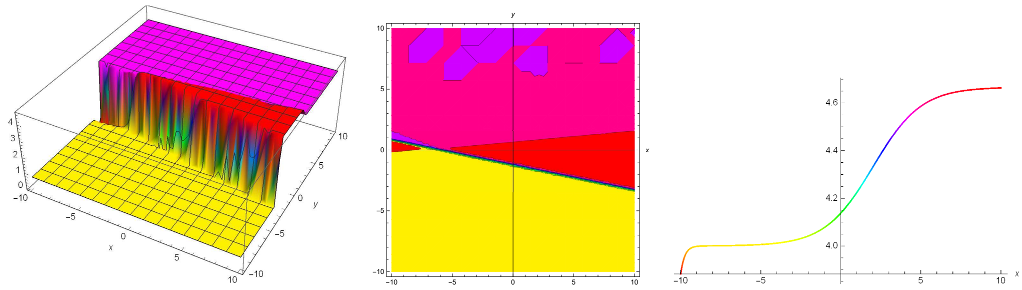


Figure 15. Graphical interpretations for solution $g_{16}(x, y, z, t)$ using $a_1 = 2,\ c_1 = 2,\ c_2 = 4,\ d_1 = 2,\ d_2 = 3,\ \kappa_1 = 1,\ \kappa_2 = 2,\ \lambda = 4,\ \mu = 1,\ \Omega = 3,\ t = 1,\ z = 2.$

7. Conclusions

We have developed new breather-wave and two-soliton solutions of the integrable (3+1)-dimensional fifth-order nonlinear oceanic wave equation by utilizing the Hirota trilinear scheme. A few of these solutions are visualized using two-dimensional, three-dimensional, and contour plots. The considered model has useful applications in weather forecasting and for coastal communities. The dynamics of breather-wave and solitons on various spatial backgrounds have been studied using the higher-dimensional nonlinear wave structure, which describes the development of oceanic waves with higher-order temporal dispersion. We added the conditions for parameters to avoid degeneracy of the specific parametric family. Moreover, the modulation instability (MI) of the considered model has been analyzed to examine the stability and accuracy of the obtained solutions. The results provide deeper insight into nonlinear ocean wave dynamics and may contribute to applications in ocean engineering, coastal wave analysis, and related nonlinear physical systems. Overall, the Hirota trilinear scheme is a reliable, straightforward, and effective tool for handling many dominant nonlinear dynamical models of the modern era. This study may be useful for further analysis and development of similar nonlinear wave models.

Author contributions

Haitham Qawaqneh: Conceptualization, supervision, project administration; Kalim U. Tariq: Software, validation, visualization, writing-review & editing; Abdulrahman Alomair: Methodology, resources, funding acquisition; Mohammed Ahmed Alomair: Formal analysis, investigation, writing–original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No.KFU261176].

Conflict of interest

The authors declare that they have no conflict of interest.

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