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Research article

General fractional differential equations with fixed memory length

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Abstract: This paper investigated the properties and general Laplace transform of general fractional operators with fixed memory length. Subsequently, a class of general fractional differential equations with fixed memory length was systematically studied. First, the existence of solutions to this type of equations was established via integral methods. Then, the uniqueness of the solution was rigorously proven by applying the Banach fixed-point theorem.

Keywords: general Laplace transform; general fractional differential equation; fixed memory length; existence; uniqueness
Mathematics Subject Classification: 26A33, 34A08, 34A12

1. Introduction

Fractional calculus began with the 1695 correspondence between Leibniz and L'Hôpital, making it a field with a history of over 300 years [1]. It was not until the late 20th century that fractional calculus began to develop rapidly with the progress of computational technology [2]. However, the core problem, that fractional-order systems cannot preserve periodicity, remains unresolved and has become a research frontier. Against this backdrop, developing novel fractional operators that inherently preserve periodicity has become crucial. The general fractional operators with fixed memory length proposed in this paper, which not only inherit the "periodicity preserving" characteristic of fixed memory length (FML) operators but also integrate the unified framework of general fractional operators, provide a new tool for the integration of fractional systems and periodic dynamics.

Specifically, the core of these operators resides in rectifying the "non-periodicity defect" of classical fractional operators via FML, while enhancing theoretical universality by leveraging the general fractional operator framework proposed by Fernandez et al. [3]. Their FML characteristic is demonstrated by introducing a fixed memory length to ensure the operators satisfies the assertion that a periodic function retains its periodicity after fractional operations: For any $T > 0$, $\alpha > 0$, if

$f(s+T) = f(s)$, then $D^{\alpha} f(s+T) = D^{\alpha} f(s)$. Their integration with the general framework is reflected by selecting special functions (e.g., $g(s) = s$ corresponding to the Riemann-Liouville operator [4] and $g(s) = lns$ corresponding to the Hadamard operator [5]), where the operators can degenerate into classical forms. This design combines the advantages of "periodicity preservation" and "definitional unity", thereby overcoming the limitation of singularity in existing FML operators [6–9].

To address the problem of periodicity preservation in fractional systems, preliminary explorations have been conducted in existing research: Abdelouahab et al. [6] proposed the Grünwald–Letnikov fractional derivative with FML, which achieves periodicity preservation by fine-tuning the classical derivative (introducing FML and modifying the lower limit). Wei et al. [7] extended it to Riemann-Liouville and Caputo fractional derivatives with FML and studied their basic properties. To refine this conclusion, Ledesma et al. [8] conducted a more in-depth study, considering a suitable function space based on [7], and investigating properties of fractional operators with FML. Furthermore, Ledesma et al. [9] studied the Laplace transform of these fractional operators with FML, considered the Riemann–Liouville FDEs with FML, and analyzed the existence and uniqueness of their solutions. However, existing work still has two limitations: (i) It only focuses on a single type of FML operator without forming a unified framework. (ii) It does not clarify the fusion path of FML and the general framework, resulting in limited theoretical universality. The general fractional operators with FML proposed in this paper are precisely a response to the above limitations.

The multiple definitions of fractional operators (e.g., Riemann-Liouville, Caputo, Hadamard) once led to theoretical redundancy [4]. To address this, Fernandez et al. [3] proposed general fractional operators, which provide a unified characterization via a function $g(s)$ and offer an approach for definition integration. Building on this framework, the general fractional operators in this paper incorporate FML. Not only do they retain the flexibility of degenerating into classical operators via special functions, but also solve the core issue of periodicity preservation through FML, thereby promoting the deep integration of fractional calculus and periodic dynamics. In addition, some properties of the general Caputo fractional derivative have been studied by Almeida [10]. The boundedness of general fractional integral operators within the function space X_C^p was systematically investigated by Fan [11]. Furthermore, a detailed discussion on the relationship between general fractional integrals and derivatives was conducted by Fu [12], which established critical theoretical foundations advancing the development of fractional calculus.

Building on the aforementioned literatures, this paper investigates the properties of general fractional operators with FML, considers a class of general FDEs with FML, and analyzes the existence and uniqueness of their solutions.

Control theory, as the core bridge connecting mathematics and engineering, has its central task of achieving precise guidance of system behavior through input regulation. In the modeling and analysis of complex dynamic systems, the existence and uniqueness of solutions have always been the cornerstone for constructing reliable control strategies. In recent years, fixed-point theory has become a key tool for breaking through traditional analytical frameworks due to its strong adaptability to nonlinear, non-smooth, and memory-dependent systems.

For example, Kattan et al. [13], innovatively combined the Bohnenblust-Karlin fixed-point theorem, constructed existence conditions for mild solutions in Hilbert space, and opened up a new path for the controllability analysis of non-convex energy functionals. Hammad et al. [14] proposed a joint framework between the cosine operator family and fixed-point theory. By embedding the time-delay term into phase space decomposition and combining the Schauder fixed-point theorem, it successfully characterized the global dynamic behavior of system solutions under non-local initial value conditions. Kattan et al. [15], for second-order non-autonomous systems with impulse jumps, pioneered the generalized cone compression fixed-point principle. By introducing piecewise Lyapunov functions and a measure of non-compactness indicators, it reconstructed stability conditions within impulse intervals, significantly enhancing the robustness of disturbance-resistant design for electromechanical systems.

The core demands of the above control theories for the existence and uniqueness of solutions and fixed-point methods exactly form a theoretical resonance with the generalized fractional differential equations (FDEs) with fixed memory length studied in this paper—the existence and uniqueness of solutions of the FDEs analyzed is precisely the prerequisite for constructing reliable control strategies for such complex systems. Meanwhile, fixed-point theory (such as the Banach fixed-point theorem) also provides a key mathematical tool for the proof of uniqueness of solutions of FDEs in this paper.

The remaining parts of this work are as follows. In Section 2, we introduce relevant function spaces and definitions, and present key properties of general fractional operators with FML. In Section 3, we focus on the general Laplace transform of the general fractional integral and derivatives with FML. In Section 4, we consider the following general FDEs with FML:

$$\begin{cases} {}_{s-L}D_s^{\alpha,g} x(s) = f(s, x(s)), \quad a \le s \le b, \\ x(s)|_{[a-L,a]} = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (0, 1)$, $a > L > 0$, $b > a + L$, $g(s) \in C^1[a - L, b]$, with $g(a - L) \ge 0$ and $g'(s) = g'(s - L) > 0$, and $f : [a, b] \times R \to R$ is a continuous function. We then establish the existence and uniqueness of solutions to Eq (1.1).

2. Preliminaries

Let $b > a > 0$. Then:

$$C[a, b] := \{f : f \text{ is continuous on } [a, b]\},$$

$$C^1[a, b] := \{f : f \text{ is differentiable and } f' \text{ is continuous on } [a, b]\},$$

$$AC[a, b] := \{f : f \text{ is absolute continuous on } [a, b]\},$$

$$AC^n_\delta[a, b] := \left\{ f : f \in C[a, b] \textit{ and } \delta^{n-1}[f(s)] \in AC[a, b], \delta = \frac{1}{g'(s)}\frac{d}{ds}, g'(s) > 0 \right\},$$

$$L^p[a, b] := \{f : f \text{ is } p\text{th Lebesgue integrable on } [a, b]\},$$

$$X^p_C[a, b] := \{f : f \text{ is complex-valued Lebesgue measuable on } [a, b]\}.$$

Definition 2.1. *[11] Let $c \in R$, and $1 \le p \le \infty$. The space $X^p_C[a, b]$ is defined to consist of those complex-valued Lebesgue measurable functions on [a, b] for which $\|f\|_{X^p_C} < \infty$, defined by*

$$\|f\|_{X^p_C[a,b]} = \left(\int_a^b |s^c f(s)|^p \frac{ds}{s} \right)^{\frac{1}{p}} \qquad (1 \le p < \infty, c \in R) \tag{2.1}$$

and

$$\|f\|_{X^\infty_C[a,b]} = \operatorname*{ess\,sup}_{s\in[a,b]} [s^c |f(s)|] \qquad (p = \infty, c \in R). \tag{2.2}$$

In particular, when $c = \frac{1}{p}(1 \le p \le \infty)$, the space $X_C^p[a,b]$ coincides with the space $L^p[a,b]$ with

$$\|f\|_{L^p[a,b]} = \left(\int_a^b |f(s)|^p ds\right)^{\frac{1}{p}} \qquad (1 \le p < \infty)$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{s\in[a,b]} |f(s)| \qquad (p = \infty).$$

Definition 2.2. *Suppose $L > 0$ and $a - L > 0$. $f(s) \in X_C^p[a-L,b]$ and $g(s) \in C^1[a-L,b]$ with $g(a-L) \ge 0$ and $g'(s) > 0$. The left-sided general fractional integral with FML L of f of order $\alpha(\alpha > 0)$ is defined by*

$${}_{s-L}I_s^{\alpha,g} f(s) = \frac{1}{\Gamma(\alpha)} \int_{s-L}^s (g(s) - g(\tau))^{\alpha-1} f(\tau) g'(\tau) d\tau, \quad a \le s \le b.$$

Note that the integral defined here is equivalent to

$${}_{s-L}I_s^{\alpha,g} f(s) = {}_{a-L}I_s^{\alpha,g} f(s) - {}_{a-L}R_s^{\alpha,g} f(s), \quad a \le s \le b,$$

with

$${}_{a-L}R_s^{\alpha,g} f(s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{a-L}^{s-L} (g(s) - g(\tau))^{\alpha-1} f(\tau) g'(\tau) d\tau, & a < s \le b, \\ 0, & s = a. \end{cases}$$

The purpose of this equivalent transformation is to decompose the fractional integral with FML into a classical fractional integral and a correction term, which facilitates the subsequent analysis of properties.

Definition 2.3. *Let $\alpha > 0$, $n = [\alpha] + 1$, $L > 0$, and $a - L > 0$. If $f(s) \in AC_\delta^n[a-L,b]$ and $g(s) \in C^1[a-L,b]$ with $g(a-L) \ge 0$ and $g'(s) > 0$, the left-sided general Riemann-Liouville fractional derivative with FML L of f of order α is given by*

$$\begin{aligned} {}_{s-L}D_s^{\alpha,g} f(s) &= \left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n {}_{s-L}I_s^{n-\alpha,g} f(s) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n \int_{s-L}^s (g(s) - g(\tau))^{n-\alpha-1} f(\tau) g'(\tau) d\tau \\ &= \delta^n {}_{s-L}I_s^{n-\alpha,g} f(s), \quad a \le s \le b. \end{aligned}$$

For $\alpha = n$, then ${}_{s-L}D_s^{\alpha,g} f(s) = \delta^n f(s)$.

Definition 2.4. *Let $\alpha > 0$, $n = [\alpha] + 1$, $L > 0$, and $a - L > 0$. If $f(s) \in AC_\delta^n[a-L,b]$ and $g(s) \in C^1[a-L,b]$ with $g(a-L) \ge 0$ and $g'(s) > 0$, the left-sided general Caputo fractional derivative with FML L of f of order α is given by*

$$\begin{aligned} {}_{s-L}^{\ C}D_s^{\alpha,g} f(s) &= {}_{s-L}I_s^{n-\alpha,g} \left(\frac{1}{g'(s)}\frac{d}{ds}\right)^n f(s) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_{s-L}^s (g(s) - g(\tau))^{n-\alpha-1} \left(\frac{1}{g'(\tau)}\frac{d}{d\tau}\right)^n f(\tau) g'(\tau) d\tau \\ &= {}_{s-L}I_s^{n-\alpha,g} \delta^n f(s), \quad a \le s \le b. \end{aligned}$$

For $\alpha = n$, then ${}_{s-L}^{\ C}D_s^{\alpha,g} f(s) = \delta^n f(s)$.

Lemma 2.5. *[16] Let $f(s) \in L^1[a,b]$. Then $\lim_{\alpha\to 0^+} {}_aI_s^{\alpha} f(s) = f(s)$ a.e. in $[a,b]$.*

Definition 2.6. *[12] Suppose $f(s) \in X_C^p[a,b]$, $g(s) \in C^1[a,b]$ with $g(a) \geq 0$ and $g'(s) > 0$. The general fractional integral of $f(s)$ is defined by*

$$ {}_aI_s^{\alpha,g} f(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (g(s) - g(\tau))^{\alpha-1} f(\tau) g'(\tau)\, d\tau, \quad a \leq s \leq b. $$

Recently, these general fractional operators have been linked to classical ones through conjugation relations, where the general operator is expressed as the conjugate of the classical operator [17], as follows:

$$ {}_aI_s^{\alpha,g} f(s) = \delta \circ {}_aI_s^{\alpha} \circ \delta^{-1}, $$

where

$$ \delta = \frac{1}{g'(s)} \frac{d}{ds}. $$

Through this conjugation relationship, some conclusions from the classical context can be directly derived in a generalized sense, thus simplifying the proof process [17]. Based on the conjugation relationship, we can see that the properties of the general fractional integral ${}_aI_s^{\alpha,g} f(s)$ can be found directly from the corresponding properties of the classical fractional integral ${}_aI_s^{\alpha} f(s)$. Therefore, by Lemma 2.5, the formal characteristics of the general fractional integral are established as follows:

Lemma 2.7. *Let $f(s) \in L^1[a,b]$, $g(s) \in C^1[a,b]$, and $g'(s) > 0$. Then $\lim_{\alpha\to 0^+} {}_aI_s^{\alpha,g} f(s) = f(s)$ a.e. in $[a,b]$.*

Lemma 2.8. *[3, 12] Let $\alpha \geq \beta > 0$, $g(s) \in C^1[a,b]$ with $g(a) \geq 0$ and $g'(s) > 0$. Suppose $f \in X_C^p[a,b]$, and then the following composition relationship holds:*

$$ {}_aD_s^{\beta,g} {}_aI_s^{\alpha,g} f(s) = {}_aI_s^{\alpha-\beta,g} f(s). $$

Particularly, when $\alpha = \beta$, we have

$$ {}_aD_s^{\alpha,g} {}_aI_s^{\alpha,g} f(s) = f(s). $$

Lemma 2.9. *[3, 12] Let $\alpha > 0$, $n = [\alpha] + 1$, and $g(s) \in C^1[a,b]$ with $g(a) \geq 0$ and $g'(s) > 0$. If $f(s) \in AC_\delta^n[a,b]$, then*

$$ {}_aI_s^{\alpha,g} {}_aD_s^{\alpha,g} f(s) = f(s) - \sum_{k=1}^{n} \frac{{}_aI_s^{k-\alpha,g} f(a) (g(s) - g(a))^{\alpha-k}}{\Gamma(\alpha - k + 1)}. $$

Lemma 2.10. *Let $L > 0$, $a - L > 0$, $\alpha \in (0,1)$, and $g(s) \in C^1[a-L,b]$ with $g(a-L) \geq 0$ and $g'(s) = g'(s-L) > 0$. If $f \in C[a-L,b]$ is such that ${}_{s-L}D_s^{\alpha,g} f$ exists, then*

$$ {}_{s-L}D_s^{\alpha,g} f(s) = {}_aD_s^{\alpha,g} f(s) - \frac{1}{\Gamma(-\alpha)} \int_a^{s-L} (g(s) - g(\tau))^{-\alpha-1} f(\tau) g'(\tau) d\tau - \frac{(g(s) - g(s-L))^{-\alpha} f(s-L)}{\Gamma(1-\alpha)}. $$

Proof. By Definition 2.3, we have

$$
\begin{aligned}
{}_{s-L}D_s^{\alpha,g} f(s) =& \frac{1}{\Gamma(1-\alpha)} \frac{1}{g'(s)} \frac{d}{ds} \int_{s-L}^{s} (g(s)-g(\tau))^{-\alpha} f(\tau) g'(\tau) d\tau \\
=& {}_aD_s^{\alpha,g} f(s) - \frac{1}{\Gamma(1-\alpha)} \frac{1}{g'(s)} \frac{d}{ds} \int_{a}^{s-L} (g(s)-g(\tau))^{-\alpha} f(\tau) g'(\tau) d\tau \\
=& {}_aD_s^{\alpha,g} f(s) - \frac{1}{\Gamma(-\alpha)} \int_{a}^{s-L} (g(s)-g(\tau))^{-\alpha-1} f(\tau) g'(\tau) d\tau \\
& - \frac{(g(s)-g(s-L))^{-\alpha} f(s-L)}{\Gamma(1-\alpha)}.
\end{aligned}
$$

□

Theorem 2.11. *Let $c > 0$, $\alpha > 0$, and $f \in X_C^p[a-L, b]$. Then ${}_{s-L}I_s^{\alpha,g} f \in X_C^p[a, b]$. We set $\nu = \frac{g(s)}{g(\tau)}$ where $a - L \le \tau \le s \le b$, $1 \le p \le \infty$. If there is a function $F(\nu) \in C[1, \frac{g(b)}{g(a-L)}]$ that satisfies the conditions $g^{-1}(\nu g(\tau)) \le F(\nu)\tau$ and $\frac{d(g^{-1}(\nu g(\tau)))}{d\tau} \le F(\nu)$, then*

$$
\|{}_{s-L}I_s^{\alpha,g} f\|_{X_C^p[a,b]} \le K \|f\|_{X_C^p[a-L,b]},
$$

where

$$
K = \int_1^{\frac{g(b)}{g(a-L)}} \frac{g(b)^\alpha (\nu-1)^\alpha}{\Gamma(\alpha)\nu^{\alpha+1}} F(\nu)^c d\nu. \tag{2.3}
$$

Proof. For $1 \le p < \infty$, we define the function H as

$$
H(s,\tau) = \frac{1}{\Gamma(\alpha)} (g(s)-g(\tau))^{\alpha-1} f(\tau) g'(\tau), \qquad \forall (s,\tau) \in [a,b] \times [s-L, s].
$$

By making the change of variable $\nu = \frac{g(s)}{g(\tau)}$, then

$$
\begin{aligned}
\left(\int_a^b \left| s^c \int_{s-L}^s H(s,\tau) d\tau \right|^p \frac{ds}{s} \right)^{\frac{1}{p}} &= \left(\int_a^b \left| \int_{s-L}^s \frac{s^{c-\frac{1}{p}}}{\Gamma(\alpha)} (g(s)-g(\tau))^{\alpha-1} f(\tau) g'(\tau) d\tau \right|^p ds \right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left| \int_1^{\frac{g(s)}{g(s-L)}} \frac{s^{c-\frac{1}{p}}}{\Gamma(\alpha)} \frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}} g(s)^\alpha f\left(g^{-1}\left(\frac{g(s)}{\nu} \right) \right) d\nu \right|^p ds \right)^{\frac{1}{p}}.
\end{aligned}
$$

By the general Minkowski inequality, we have

$$
\begin{aligned}
& \left(\int_a^b \left| s^c \int_{s-L}^s H(s,\tau) d\tau \right|^p \frac{ds}{s} \right)^{\frac{1}{p}} \\
\le & \int_1^{\frac{g(b)}{g(a-L)}} \frac{1}{\Gamma(\alpha)} \frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}} d\nu \left(\int_{g^{-1}(\nu(g(a-L))}^b \left| s^{c-\frac{1}{p}} (g(s))^\alpha f\left(g^{-1}\left(\frac{g(s)}{\nu} \right) \right) \right|^p ds \right)^{\frac{1}{p}} \\
\le & \int_1^{\frac{g(b)}{g(a-L)}} \frac{(g(b))^\alpha}{\Gamma(\alpha)} \frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}} d\nu \left(\int_{a-L}^{g^{-1}(\frac{g(b)}{\nu})} \left| (g^{-1}(\nu g(\tau)))^{c-\frac{1}{p}} f(\tau) \right|^p (g^{-1}(\nu g(\tau)))' d\tau \right)^{\frac{1}{p}}.
\end{aligned}
$$

It follows from the fact that $g^{-1}(\nu g(\tau)) \leq F(\nu)\tau$, $\frac{d(g^{-1}(\nu g(\tau)))}{d\tau} \leq F(\nu)$, $\nu \geq 1$, that

$$\left(\int_{a-L}^{g^{-1}(\frac{g(b)}{\nu})} \left|(g^{-1}(\nu g(\tau)))^{c-\frac{1}{p}} f(\tau)\right|^p (g^{-1}(\nu g(\tau)))' d\tau\right)^{\frac{1}{p}} \leq F(\nu)^c \left(\int_{a-L}^{b} \left|\tau^{c-\frac{1}{p}} f(\tau)\right|^p d\tau\right)^{\frac{1}{p}}.$$

According to $F(\nu) \in C[1, \frac{g(b)}{g(a-L)}]$, we have

$$K = \int_1^{\frac{g(b)}{g(a-L)}} \frac{g(b)^\alpha (\nu-1)^\alpha}{\Gamma(\alpha)\nu^{\alpha+1}} F(\nu)^c d\nu < \infty,$$

$$\left(\int_a^b \left|s^c \int_{s-L}^s H(s,\tau)d\tau\right|^p \frac{ds}{s}\right)^{\frac{1}{p}} \leq \int_1^{\frac{g(b)}{g(a-L)}} \frac{(g(b))^\alpha}{\Gamma(\alpha)} \frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}} F(\nu)^c d\nu \cdot \|f\|_{X_C^p[a-L,b]} < \infty.$$

Hence

$$g(s) := \int_{s-L}^s H(s,\tau)d\tau =_{s-L} I_s^{\alpha,g} f(s) \in X_C^p[a,b],$$

that is, ${}_{s-L}I_s^{\alpha,g} f \in X_C^p[a,b]$.

For $p = \infty$ and $\forall s \in [a,b]$, we have

$$s^c|_{s-L}I_s^{\alpha,g} f| = s^c \left|\int_{s-L}^s \frac{1}{\Gamma(\alpha)}(g(s)-g(\tau))^{\alpha-1} f(\tau)g'(\tau)d\tau\right|.$$

Making the change of variable $\nu = \frac{g(s)}{g(\tau)}$, then

$$\begin{aligned}
s^c|_{s-L}I_s^{\alpha,g} f| &\leq \int_1^{\frac{g(s)}{g(s-L)}} \left|\frac{s^c}{\Gamma(\alpha)}\left(\frac{g(s)}{\nu}\right)^{\alpha-1} (\nu-1)^{\alpha-1}\frac{g(s)}{\nu^2} f\left(g^{-1}\left(\frac{g(s)}{\nu}\right)\right)\right| d\nu \\
&\leq \int_1^{\frac{g(b)}{g(a-L)}} \left|\frac{s^c g(s)^\alpha}{\Gamma(\alpha)}\left(\frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}}\right) f\left(g^{-1}\left(\frac{g(s)}{\nu}\right)\right)\right| d\nu \\
&\leq g(b)^\alpha \int_1^{\frac{g(b)}{g(a-L)}} \left|\frac{1}{\Gamma(\alpha)}\frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}} F(\nu)^c \tau^c f(\tau)\right| d\nu \\
&\leq \int_1^{\frac{g(b)}{g(a-L)}} \frac{g(b)^\alpha}{\Gamma(\alpha)}\frac{(\nu-1)^{\alpha-1}}{\nu^{\alpha+1}} F(\nu)^c d\nu \cdot \|f\|_{X_c^\infty[a-L,b]} < \infty.
\end{aligned}$$

Hence, ${}_{s-L}I_s^{\alpha,g} f \in X_C^p[a,b]$. □

Example 2.12. *If $g(s) = \frac{s^\beta}{\beta}$ ($\beta > 0$), there exists a function $F(\nu) = \nu^{\frac{1}{\beta}}$ for which both $g^{-1}(\nu g(\tau)) \leq F(\nu)\tau$ and $\frac{d(g^{-1}(\nu g(\tau)))}{d\tau} \leq F(\nu)$ hold, and K is defined by Eq (2.3) where $F(\nu) = \nu^{\frac{1}{\beta}}$.*

Proof. If $g(s) = \frac{s^\beta}{\beta}$ ($\beta > 0$), then $g^{-1}(\nu g(\tau)) = g^{-1}(\frac{s^\beta}{\beta}) = s \leq \nu^{\frac{1}{\beta}} \cdot \tau$ and $\frac{d(g^{-1}(\nu g(\tau)))}{d\tau} \leq \nu^{\frac{1}{\beta}}$. Therefore, we can find a function $F(\nu) = \nu^{\frac{1}{\beta}}$. □

Theorem 2.13. *Let $\alpha > 0$ and $f \in L^1[a-L,b]$. Then $\lim_{\alpha \to 0^+} {}_{s-L}I_s^{\alpha,g} f(s) = f(s)$ almost everywhere in $[a,b]$.*

Proof. Since

$$ {}_{s-L}I_s^{\alpha,g}f(s) = {}_{a-L}I_s^{\alpha,g}f(s) - {}_{a-L}R_s^{\alpha,g}f(s), \quad a \le s \le b, $$

then for $s \in [a,b]$ and $\alpha \in (0,1)$, by using the mean value theorem, we have

$$ \begin{aligned} \left|{}_{a-L}R_s^{\alpha,g}f(s)\right| \le & \int_{a-L}^{s-L} \left|\frac{1}{\Gamma(\alpha)}(g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)\right| d\tau \\ \le & \frac{1}{\Gamma(\alpha)} \sup_{s\in[a,b]} \left[(g(s)-g(s-L))^{\alpha-1}\right] \cdot \sup_{\tau\in[a-L,b-L]} g'(\tau) \cdot \int_{a-L}^{b} |f(\tau)|\, d\tau \\ \le & \frac{1}{\Gamma(\alpha)} L^{\alpha-1} \cdot \sup_{\sigma\in[a-L,b]} \left[(g'(\sigma))^{\alpha-1}\right] \cdot \sup_{\tau\in[a-L,b-L]} g'(\tau) \cdot \|f\|_{L^1[a-L,b]}. \end{aligned} $$

Due to $\lim_{\alpha\to 0^+} \Gamma(\alpha) = +\infty$, we have

$$ 0 \le \lim_{\alpha\to 0^+} \left|{}_{a-L}R_s^{\alpha,g}f(s)\right| \le \lim_{\alpha\to 0^+} \frac{1}{\Gamma(\alpha)} L^{\alpha-1} \cdot \sup_{\sigma\in[a-L,b]} \left[(g'(\sigma))^{\alpha-1}\right] \cdot \sup_{\tau\in[a-L,b-L]} g'(\tau) \cdot \|f\|_{L^1[a-L,b]} = 0. $$

Hence

$$ \lim_{\alpha\to 0^+} \left|{}_{a-L}R_s^{\alpha,g}f(s)\right| = 0, \quad \forall s \in [a,b]. $$

According to Lemma 2.7, we have

$$ \lim_{\alpha\to 0^+} {}_{a-L}I_s^{\alpha,g}f(s) = f(s), \ \ a.e.\ in\ [a,b]. $$

Hence,

$$ \lim_{\alpha\to 0^+} {}_{s-L}I_s^{\alpha,g}f(s) = \lim_{\alpha\to 0^+} {}_{a-L}I_s^{\alpha,g}f(s) - \lim_{\alpha\to 0^+} {}_{a-L}R_s^{\alpha,g}f(s) = f(s), \quad a.e.\ in\ [a,b]. $$

□

Theorem 2.14. *Let $\alpha \ge 1$ and $f \in L^1[a-L,b]$. Then*

$$ \begin{aligned} {}_{s-L}I_s^{\alpha,g}f(s) = & {}_{a-L}I_a^{\alpha,g}f(a) \\ & + \int_a^s \left[g'(\tau)\,{}_{\tau-L}I_\tau^{\alpha-1,g}f(\tau) - \frac{(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)g'(\tau-L)}{\Gamma(\alpha)}\right] d\tau, \ \ s \in [a,b]. \end{aligned} \tag{2.4} $$

Hence,

$$ {}_{s-L}I_s^{\alpha,g}f(s) \in AC[a,b]. $$

Proof. If $\alpha = 1$, $\forall s \in [a,b]$, we have

$$ \begin{aligned} {}_{s-L}I_s^{1,g}f(s) = & \int_{a-L}^{a} f(\tau)g'(\tau)d\tau + \int_a^s f(\tau)g'(\tau)d\tau - \int_a^s f(\tau-L)g'(\tau-L)d\tau \\ = & {}_{a-L}I_a^{1,g}f(a) + \int_a^s [f(\tau)g'(\tau) - f(\tau-L)g'(\tau-L)]d\tau. \end{aligned} $$

By Theorem 2.13, we obtain that Eq (2.4) holds for $\alpha = 1$. When $\alpha > 1$, Theorem 2.11 implies ${}_{s-L}I_s^{\alpha,g} f \in L^1[a,b]$ and ${}_{s-L}I_s^{\alpha-1,g} f \in L^1[a,b]$. For $\tau \in [a,b]$, we get

$$\begin{aligned}
{}_{\tau-L}I_\tau^{\alpha-1,g} f(\tau) =& \frac{1}{\Gamma(\alpha-1)} \int_{a-L}^{\tau} (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) d\sigma \\
&- \frac{1}{\Gamma(\alpha-1)} \int_{a-L}^{\tau-L} (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) d\sigma \\
=& \frac{1}{\Gamma(\alpha-1)} \int_{a-L}^{a} (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) d\sigma \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_{a}^{\tau} (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) d\sigma \\
&- \frac{1}{\Gamma(\alpha-1)} \int_{a}^{\tau} (g(\tau)-g(\sigma-L))^{\alpha-2} f(\sigma-L) g'(\sigma-L) d\sigma.
\end{aligned}$$

Next, for $s \in (a,b]$, we obtain

$$\begin{aligned}
\int_a^s g'(\tau)_{\tau-L}I_\tau^{\alpha-1,g} f(\tau) d\tau =& \frac{1}{\Gamma(\alpha-1)} \int_a^s \int_{a-L}^{a} (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) g'(\tau) d\sigma d\tau \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_a^s \int_a^{\tau} (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) g'(\tau) d\sigma d\tau \\
&- \frac{1}{\Gamma(\alpha-1)} \int_a^s \int_a^{\tau} (g(\tau)-g(\sigma-L))^{\alpha-2} f(\sigma-L) g'(\sigma-L) g'(\tau) d\sigma d\tau \\
=& I_1 + I_2 - I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 =& \frac{1}{\Gamma(\alpha-1)} \int_{a-L}^{a} \int_a^s (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) g'(\tau) d\tau d\sigma \\
=& \frac{1}{\Gamma(\alpha)} \int_{a-L}^{a} (g(s)-g(\sigma))^{\alpha-1} f(\sigma) g'(\sigma) d\sigma - {}_{a-L}I_a^{\alpha,g} f(a), \\
I_2 =& \frac{1}{\Gamma(\alpha-1)} \int_a^s \int_\sigma^s (g(\tau)-g(\sigma))^{\alpha-2} f(\sigma) g'(\sigma) g'(\tau) d\tau d\sigma \\
=& \frac{1}{\Gamma(\alpha)} \int_a^s (g(s)-g(\sigma))^{\alpha-1} f(\sigma) g'(\sigma) d\sigma,
\end{aligned}$$

and

$$\begin{aligned}
I_3 =& \frac{1}{\Gamma(\alpha-1)} \int_a^s \int_\sigma^s (g(\tau)-g(\sigma-L))^{\alpha-2} f(\sigma-L) g'(\sigma-L) g'(\tau) d\tau d\sigma \\
=& \frac{1}{\Gamma(\alpha)} \int_{a-L}^{s-L} [(g(s)-g(\sigma))^{\alpha-1} f(\sigma) g'(\sigma)] d\sigma \\
&- \frac{1}{\Gamma(\alpha)} \int_a^s [(g(\sigma)-g(\sigma-L))^{\alpha-1} f(\sigma-L) g'(\sigma-L)] d\sigma.
\end{aligned}$$

So

$$\int_a^s g'(\tau)_{\tau-L}I_\tau^{\alpha-1,g}f(\tau)d\tau = I_1 + I_2 - I_3 = \frac{1}{\Gamma(\alpha)}\int_{s-L}^s (g(s)-g(\sigma))^{\alpha-1}f(\sigma)g'(\sigma)d\sigma$$
$$-_{a-L}I_a^{\alpha,g}f(a) + \frac{1}{\Gamma(\alpha)}\int_a^s [(g(\sigma)-g(\sigma-L))^{\alpha-1}f(\sigma-L)g'(\sigma-L)]d\sigma.$$

Through systematic rearrangement, for $s \in (a,b]$, we have

$$_{s-L}I_s^{\alpha,g}f(s) = {}_{a-L}I_a^{\alpha,g}f(a) + \int_a^s \left[g'(\tau)_{\tau-L}I_\tau^{\alpha-1,g}f(\tau) - \frac{(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)g'(\tau-L)}{\Gamma(\alpha)}\right]d\tau.$$

Besides, for $s = a$, this equality obviously holds. Consequently, this equality holds for $s \in [a,b]$. Furthermore, since

$$\tau \in [a,b] \to g'(\tau)_{\tau-L}I_\tau^{\alpha-1,g}f(\tau) - \frac{(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)g'(\tau-L)}{\Gamma(\alpha)} \in L^1[a,b],$$

then by Eq (2.4), we obtain $_{s-L}I_s^{\alpha,g}f \in AC[a,b]$. □

Theorem 2.15. *If $f \in AC[a-L,b]$, for $\forall\alpha > 0$, we have*

$$_{s-L}I_s^{\alpha,g}f(s) = \frac{f(s-L)}{\Gamma(\alpha+1)}(g(s)-g(s-L))^\alpha + {}_{s-L}I_s^{\alpha+1,g}\delta f(s), \qquad \forall s \in [a,b].$$

Proof.

$$\begin{aligned} _{s-L}I_s^{\alpha,g}f(s) &= \frac{1}{\Gamma(\alpha)}\int_{s-L}^s (g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)d\tau \\ &= \frac{1}{\Gamma(\alpha+1)}f(s-L)(g(s)-g(s-L))^\alpha + \frac{1}{\Gamma(\alpha+1)}\int_{s-L}^s (g(s)-g(\tau))^\alpha \frac{f'(\tau)}{g'(\tau)}g'(\tau)d\tau \\ &= \frac{f(s-L)}{\Gamma(\alpha+1)}(g(s)-g(s-L))^\alpha + {}_{s-L}I_s^{\alpha+1,g}\delta f(s). \end{aligned}$$

□

Theorem 2.16. *Let $\alpha > 0$ and $m \in Z^+$. If $f \in AC_\delta^m[a-L,b]$, $\forall r = 1,2,3,...,m$, we have*

$$_{s-L}I_s^{\alpha,g}f(s) = \sum_{k=0}^{r-1}\frac{\delta^k f(s-L)(g(s)-g(s-L))^{\alpha+k}}{\Gamma(\alpha+k+1)} + {}_{s-L}I_s^{\alpha+r,g}\delta^r f(s), \quad \forall s \in [a,b].$$

Proof. By Theorem 2.15, we obtain

$$_{s-L}I_s^{\alpha,g}f(s) = \frac{1}{\Gamma(\alpha+1)}(g(s)-g(s-L))^\alpha f(s-L) + {}_{s-L}I_s^{\alpha+1,g}\delta f(s).$$
$$_{s-L}I_s^{\alpha+1,g}\delta f(s) = \frac{1}{\Gamma(\alpha+2)}\delta f(s-L)(g(s)-g(s-L))^{\alpha+1} + {}_{s-L}I_s^{\alpha+2,g}\delta^2 f(s).$$
$$_{s-L}I_s^{\alpha+2,g}\delta^2 f(s) = \frac{1}{\Gamma(\alpha+3)}\delta^2 f(s-L)(g(s)-g(s-L))^{\alpha+2} + {}_{s-L}I_s^{\alpha+3,g}\delta^3 f(s).$$

Repeat the process above and we obtain

$$ {}_{s-L}I_s^{\alpha,g}f(s) = \sum_{k=0}^{r-1} \frac{\delta^k f(s-L)(g(s)-g(s-L))^{\alpha+k}}{\Gamma(\alpha+k+1)} + {}_{s-L}I_s^{\alpha+r,g}\delta^r f(s), \quad \forall s \in [a,b]. $$

□

Example 2.17. *If $\alpha > 0$, $f(s) = g^m(s)$, and $m \in Z^+$, then by Theorem 2.16,*

$$ {}_{s-L}I_s^{\alpha,g}f(s) = {}_{s-L}I_s^{\alpha,g}g^m(s) = \sum_{k=0}^{m} \frac{m!(g(s)-g(s-L))^{\alpha+k}g^{m-k}(s-L)}{(m-k)!\Gamma(\alpha+k+1)}. $$

In particular, if C is a constant, then

$$ {}_{s-L}I_s^{\alpha,g}C = \frac{1}{\Gamma(\alpha)}\int_{s-L}^{s}(g(s)-g(\tau))^{\alpha-1}Cg'(\tau)d\tau = \frac{C(g(s)-g(s-L))^{\alpha}}{\Gamma(\alpha+1)}. $$

Theorem 2.18. *If $\alpha > 0$ and $f \in AC[a-L,b]$, then*

$$ \begin{aligned} {}_{s-L}I_s^{\alpha,g}f(s) =& {}_{a-L}I_a^{\alpha,g}f(a) \\ &+ \int_a^s \left[g'(\tau)\,{}_{\tau-L}I_\tau^{\alpha,g}\delta f(\tau) + \frac{(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)(g'(\tau)-g'(\tau-L))}{\Gamma(\alpha)} \right] d\tau, \end{aligned} $$

which implies ${}_{s-L}I_s^{\alpha,g}f(s) \in AC[a,b]$.

Proof. By Theorem 2.15,

$$ {}_{s-L}I_s^{\alpha,g}f(s) = \frac{(g(s)-g(s-L))^{\alpha}f(s-L)}{\Gamma(\alpha+1)} + {}_{s-L}I_s^{\alpha+1,g}\delta f(s). \tag{2.5} $$

Since $\alpha + 1 > 1$ and $\delta f(s) \in L^1[a-L,b]$, by Theorem 2.14, for $\forall s \in [a,b]$, we have

$$ {}_{s-L}I_s^{\alpha+1,g}\delta f(s) = {}_{a-L}I_a^{\alpha+1,g}\delta f(a) + \int_a^s \left[g'(\tau)\,{}_{\tau-L}I_\tau^{\alpha,g}\delta f(\tau) - \frac{(g(\tau)-g(\tau-L))^{\alpha+1}f'(\tau-L)}{\Gamma(\alpha+1)} \right] d\tau. $$

Moreover, we obtain

$$ \begin{aligned} f(s-L)(g(s)-g(s-L))^{\alpha} =& (g(a)-g(a-L))^{\alpha}f(a-L) + \int_a^s (g(\tau)-g(\tau-L))^{\alpha}f'(\tau-L)d\tau \\ &+ \int_a^s \alpha(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)(g'(\tau)-g'(\tau-L))d\tau. \end{aligned} $$

Hence,

$$ \begin{aligned} {}_{s-L}I_s^{\alpha,g}f(s) =& {}_{a-L}I_a^{\alpha,g}f(a) \\ &+ \int_a^s \left[g'(\tau)\,{}_{\tau-L}I_\tau^{\alpha,g}\delta f(\tau) + \frac{(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)(g'(\tau)-g'(\tau-L))}{\Gamma(\alpha)} \right] d\tau. \end{aligned} $$

Consider the function

$$ \psi(\tau) = g'(\tau)\,{}_{\tau-L}I_\tau^{\alpha,g}\delta f(\tau) + \frac{(g(\tau)-g(\tau-L))^{\alpha-1}f(\tau-L)(g'(\tau)-g'(\tau-L))}{\Gamma(\alpha)}. $$

According to $\delta f(\tau) \in L^1[a-L,b]$ and Theorem 2.11, we derive $\psi(\tau) \in L^1[a,b]$, hence ${}_{s-L}I_s^{\alpha,g}f(s) \in AC[a,b]$. □

Theorem 2.19. *Let* $\alpha > 0$ *and* $\forall n \in N_+$, $g(s) \in C^1[a-L,b]$ *with* $g(a-L) \geq 0$ *and* $g'(s) > 0$. *If* $f \in AC_\delta^n[a-L,b]$, *then*

$$
\begin{aligned}
{}_{s-L}I_s^{\alpha,g} f(s) = &\sum_{k=0}^{n-1} \frac{{}_{a-L}I_a^{\alpha,g}\delta^k f(a)}{k!}(g(s)-g(a))^k +_a I_s^{n,g}\,{}_{s-L}I_s^{\alpha,g}\delta^n f(s) \\
&+ \sum_{k=1}^{n} {}_aI_s^{k,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}[\delta^{n-1}]f(s-L)(g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right].
\end{aligned}
$$

Proof. By Theorem 2.18, for each $k = 0, 1, 2, ..., n-1$, we get

$$
\begin{aligned}
{}_{s-L}I_s^{\alpha,g}\delta^k f(s) =&{}_{a-L}I_a^{\alpha,g}\delta^k f(a) + \int_a^s g'(\tau)\,{}_{\tau-L}I_\tau^{\alpha,g}\delta^{k+1} f(\tau)d\tau \\
&+ \int_a^s \frac{(g(\tau)-g(\tau-L))^{\alpha-1}[\delta^k f(\tau-L)](g'(\tau)-g'(\tau-L))}{\Gamma(\alpha)}d\tau.
\end{aligned}
$$

By Definition 2.6, we have ${}_aI_s^{1,g}f(s) = \int_a^s f(\tau)g'(\tau)d\tau$. Then, for $s \in [a,b]$, we obtain

$$
\begin{aligned}
{}_{s-L}I_s^{\alpha,g}\delta^k f(s) =&{}_{a-L}I_a^{\alpha,g}\delta^k f(a) +_a I_s^{1,g}\,{}_{s-L}I_s^{\alpha,g}\delta^{k+1} f(s) \\
&+_a I_s^{1,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}[\delta^k f(s-L)](g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right].
\end{aligned}
$$

Using ${}_aI_s^{k,g}1 = \frac{(g(s)-g(a))^k}{k!}$, $\forall s \in [a,b]$, we have

$$
\begin{aligned}
{}_aI_s^{k,g}\,{}_{s-L}I_s^{\alpha,g}\delta^k f(s) =&\frac{(g(s)-g(a))^k}{k!}\,{}_{a-L}I_a^{\alpha,g}\delta^k f(a) +_a I_s^{k+1,g}\,{}_{s-L}I_s^{\alpha,g}\delta^{k+1} f(s) \\
&+_a I_s^{k+1,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}[\delta^k f(s-L)](g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{}_{s-L}I_s^{\alpha,g} f(s) =&{}_{s-L}\ I_s^{\alpha,g}\delta^0 f(s) ={}_{a-L}I_a^{\alpha,g} f(a) +_a I_s^{1,g}\,{}_{s-L}I_s^{\alpha,g}\delta f(s) \\
&+_a I_s^{1,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}f(s-L)(g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right] \\
=&{}_{a-L}I_a^{\alpha,g} f(a) + \frac{(g(s)-g(a))^1}{1!}\,{}_{a-L}I_a^{\alpha,g}\delta f(a) +_a I_s^{2,g}\,{}_{s-L}I_s^{\alpha,g}\delta^2 f(s) \\
&+_a I_s^{1,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}f(s-L)(g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right] \\
&+_a I_s^{2,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}\delta^1 f(s-L)(g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right] \\
=&... \\
=&\sum_{k=0}^{n-1}\frac{{}_{a-L}I_a^{\alpha,g}\delta^k f(a)}{k!}(g(s)-g(a))^k +_a I_s^{n,g}({}_{s-L}I_s^{\alpha,g}\delta^n f(s)) \\
&+ \sum_{k=1}^{n} {}_aI_s^{k,g}\left[\frac{(g(s)-g(s-L))^{\alpha-1}[\delta^{k-1}f(s-L)](g'(s)-g'(s-L))}{g'(s)\Gamma(\alpha)}\right].
\end{aligned}
$$

□

3. General Laplace transform for general fractional operators with FML

Definition 3.1. *[3] Let $f, g : [a, +\infty) \to R$ be real valued functions such that $g(s)$ is continuous and $g'(s) > 0$ on $[a, \infty)$. The general Laplace transform of f is defined by*

$$\mathcal{L}_g\{f(s)\}(\lambda) = \int_a^{+\infty} e^{-\lambda(g(s)-g(a))} f(s)g'(s)ds,$$

where the above integral converges for every value of λ.

Definition 3.2. *[3] Let $f : [a, +\infty) \to R$ be a real-valued function defined on $[a, +\infty)$. It is said to be of $g(s)$-exponential order provided that there exist non-negative constants M, c, T such that $|f(s)| \leq Me^{cg(s)}$ for all $s \geq T$.*

Definition 3.3. *[3] Let f and h be two functions both piecewise continuous on every interval $[0, T]$ and of $g(s)$-exponential order. The general convolution of f and h denoted by $(f *_g h)(s)$ is defined as*

$$(f *_g h)(s) = \int_a^s f(\tau)h(g^{-1}(g(s) + g(a) - g(\tau)))g'(\tau)d\tau.$$

Lemma 3.4. *[3] Let f and h be two functions both piecewise continuous on every interval $[0, T]$ and of $g(s)$-exponential order. Then $\mathcal{L}_g\{f *_g h\} = \mathcal{L}_g\{f\}\mathcal{L}_g\{h\}$.*

Lemma 3.5. *[3] $\mathcal{L}_g\{(g(s) - g(a))^{\beta-1}\}(\lambda) = \frac{\Gamma(\beta)}{\lambda^\beta}, \quad \beta > 0, \lambda > 0.$*

Theorem 3.6. *Let $\alpha > 0$, $g(s) \in C^1[a - L, b]$ with $g(a - L) \geq 0$, and $g'(s) > 0$. If $f : [a - L, b] \to R$ is a function of $g(s)$-exponential order, then ${}_{s-L}I_s^{\alpha,g}$ is of $g(s)$-exponential order. Moreover, if $g'(s) = g'(s - L)$, $s \in [a, b]$, and then*

$$\begin{aligned}\mathcal{L}_g\{{}_{s-L}I_s^{\alpha,g}\} =& \frac{\mathcal{L}_g\{f(s)\}}{\lambda^\alpha}\left[1 - \frac{\Gamma[\alpha, \lambda(g(a) - g(a - L))]}{\Gamma(\alpha)}\right] \\ &+ \frac{1}{\lambda^\alpha\Gamma(\alpha)}\int_{a-L}^a e^{\lambda(g(\tau)-g(a)} f(\tau)g'(\tau)\Gamma[\alpha, \lambda(g(a) - g(\tau)]d\tau \\ &- \frac{\Gamma[\alpha, \lambda(g(a) - g(a - L))]}{\Gamma(\alpha)\lambda^\alpha}\int_{a-L}^a e^{-\lambda(g(\tau)-g(a))} f(\tau)g'(\tau)d\tau,\end{aligned}$$

where $\Gamma[\alpha, x] = \int_x^{+\infty} s^{\alpha-1}e^{-s}ds$ is the incomplete gamma function.

Proof. Since f is a function of $g(s)$-exponential order, there exist non-negative constants $M, c, a - L \leq T \leq b$ such that $|f(s)| \leq Me^{cg(s)}$, $\forall s \geq T$.

Next, by Example 2.17, for every $s \geq T$, we have

$$\begin{aligned}|{}_{s-L}I_s^{\alpha,g} f(s)| \leq& {}_{s-L} I_s^{\alpha,g}|f(s)| \leq M {}_{s-L}I_s^{\alpha,g} e^{cg(s)} = M\sum_{n=0}^{\infty} \frac{c^n}{n!} {}_{s-L}I_s^{\alpha,g} g^n(s) \\ =& M\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c^n}{\Gamma(\alpha + k + 1)} \frac{(g(s) - g(s - L))^{\alpha+k} g^{n-k}(s - L)}{(n - k)!} \\ \leq& M(g(b) - g(a - L))^\alpha \mathbb{E}_{1,1+\alpha}(c(g(b) - g(a - L)))e^{cg(s)},\end{aligned}$$

and therefore ${}_{s-L}I_s^{\alpha,g}$ is of $g(s)$-exponential order.

In addition,

$$\begin{aligned}
{}_{s-L}I_s^{\alpha,g}f(s) =& {}_{a-L}I_a^{\alpha,g}f(s) + {}_a I_s^{\alpha,g}f(s) - {}_{a-L} I_{s-L}^{\alpha,g}f(s) \\
=& \frac{1}{\Gamma(\alpha)}\int_{a-L}^{a}(g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)d\tau + \frac{1}{\Gamma(\alpha)}\int_{a}^{s}(g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)d\tau \\
& - \frac{1}{\Gamma(\alpha)}\int_{a}^{s}(g(s)-g(\tau-L))^{\alpha-1}f(\tau-L)g'(\tau-L)d\tau.
\end{aligned}$$

Then

$$\mathcal{L}_g\{{}_{s-L}I_s^{\alpha,g}f(s)\} = I_1 + I_2 - I_3,$$

where

$$\begin{aligned}
I_1 &= \mathcal{L}_g\left\{\frac{1}{\Gamma(\alpha)}\int_{a-L}^{a}(g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)d\tau\right\}, \\
I_2 &= \mathcal{L}_g\left\{\frac{1}{\Gamma(\alpha)}\int_{a}^{s}(g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)d\tau\right\},
\end{aligned}$$

and

$$I_3 = \mathcal{L}_g\left\{\frac{1}{\Gamma(\alpha)}\int_{a}^{s}(g(s)-g(\tau-L))^{\alpha-1}f(\tau-L)g'(\tau-L)d\tau\right\}.$$

By doing the change of variable $\sigma = \lambda(g(s)-g(\tau))$, we have

$$\begin{aligned}
I_1 =& \frac{1}{\Gamma(\alpha)}\int_{a}^{+\infty}e^{-\lambda(g(s)-g(a))}\int_{a-L}^{a}(g(s)-g(\tau))^{\alpha-1}f(\tau)g'(\tau)d\tau g'(s)ds \\
=& \frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_{a-L}^{a}e^{-\lambda(g(\tau)-g(a))}f(\tau)g'(\tau)\Gamma[\alpha,\lambda(g(a)-g(\tau))]d\tau.
\end{aligned}$$

According to Theorems 3.4 and 3.5, we have

$$I_2 = \frac{1}{\Gamma(\alpha)}\mathcal{L}_g\left\{(g(s)-g(a))^{\alpha-1} *_g f(s)\right\} = \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{\lambda^{\alpha}}\mathcal{L}_g\{f(s)\} = \frac{\mathcal{L}_g\{f(s)\}}{\lambda^{\alpha}}.$$

According to $g'(s) = g'(s-L)$, we have $\int_{s_1}^{s_2} g'(s)ds = \int_{s_1}^{s_2} g'(s-L)ds$, namely,

$$g(s_2) - g(s_2-L) = g(s_1) - g(s_1-L) = g(a) - g(a-L), \forall s_1, s_2 \in [a,b].$$

Hence,

$$\begin{aligned}
I_3 =& \mathcal{L}_g\left\{\frac{1}{\Gamma(\alpha)}\int_{a}^{s}[g(s)+g(a)-g(a-L)-g(\tau)]^{\alpha-1}f(\tau-L)g'(\tau)d\tau\right\} \\
=& \frac{1}{\Gamma(\alpha)}\mathcal{L}_g\left\{[g(s)-g(a-L)]^{\alpha-1} *_g f(s-L)\right\} \\
=& \frac{1}{\Gamma(\alpha)}\mathcal{L}_g\left\{[g(s)-g(a-L)]^{\alpha-1}\right\}\mathcal{L}_g\left\{f(s-L)\right\}.
\end{aligned}$$

Making the change of variable $\sigma = g(s) - g(a-L)$, then

$$
\begin{aligned}
&\mathcal{L}_g\left\{[g(s)-g(a-L)]^{\alpha-1}\right\}(\lambda)\\
=&\int_{g(a)-g(a-L)}^{+\infty} e^{-\lambda\sigma}e^{\lambda(g(a)-g(a-L))}\sigma^{\alpha-1}d\sigma\\
=&e^{\lambda(g(a)-g(a-L))}\lambda^{-\alpha}\Gamma[\alpha,\lambda(g(a)-g(a-L))].
\end{aligned}
$$

Therefore,

$$
I_3=\frac{e^{\lambda(g(a)-g(a-L))}\Gamma[\alpha,\lambda(g(a)-g(a-L))]}{\Gamma(\alpha)\lambda^{\alpha}}\mathcal{L}_g\left\{f(s-L)\right\}.
$$

Note that

$$
\begin{aligned}
\mathcal{L}_g\{f(s-L)\}(\lambda)=&\int_{a-L}^{+\infty} e^{-\lambda(g(s+L)-g(a))}f(s)g'(s+L)ds\\
=&\int_{a-L}^{+\infty} e^{-\lambda(g(s)-g(a)+g(a)-g(a-L))}f(s)g'(s)ds\\
=&e^{-\lambda(g(a)-g(a-L))}\left[\int_{a-L}^{a} e^{-\lambda(g(s)-g(a))}f(s)g'(s)ds+\mathcal{L}_g\{f(s)\}(\lambda)\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{L}_g\{{}_{s-L}I_s^{\alpha,g}\}=&\frac{\mathcal{L}_g\{f(s)\}}{\lambda^{\alpha}}\left[1-\frac{\Gamma[\alpha,\lambda(g(a)-g(a-L))]}{\Gamma(\alpha)}\right]\\
&+\frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_{a-L}^{a} e^{\lambda(g(\tau)-g(a)}f(\tau)g'(\tau)\Gamma[\alpha,\lambda(g(a)-g(\tau)]d\tau\\
&-\frac{\Gamma[\alpha,\lambda(g(a)-g(a-L))]}{\Gamma(\alpha)\lambda^{\alpha}}\int_{a-L}^{a} e^{-\lambda(g(\tau)-g(a))}f(\tau)g'(\tau)d\tau.
\end{aligned}
$$

□

Theorem 3.7. *Let* $\alpha>0$*,* $g(s)\in C^1[a-L,b]$ *with* $g(a-L)\geq 0$*, and* $g'(s)>0$*. If* $f:[a-L,b]\to R$ *is a function of* $g(s)$*-exponential order,* $g'(s)=g'(s-L),\ s\in[a,b]$*, and* $f|_{[a-L,a]}=0$*. Then*

$$
\mathcal{L}_g\{{}_{s-L}I_s^{\alpha,g}\}=\frac{\mathcal{L}_g\{f(s)\}}{\lambda^{\alpha}}\left[1-\frac{\Gamma[\alpha,\lambda(g(a)-g(a-L))]}{\Gamma(\alpha)}\right].
$$

Proof. By assumptions, we have

$$
\int_{a-L}^{a} e^{\lambda(g(\tau)-g(a)}f(\tau)g'(\tau)\Gamma[\alpha,\lambda(g(a)-g(\tau)]d\tau=0
$$

and

$$
\int_{a-L}^{a} e^{-\lambda(g(\tau)-g(a))}f(\tau)g'(\tau)d\tau=0.
$$

Next, by Theorem 3.6, we obtain

$$
\mathcal{L}_g\{{}_{s-L}I_s^{\alpha,g}\}=\frac{\mathcal{L}_g\{f(s)\}}{\lambda^{\alpha}}\left[1-\frac{\Gamma[\alpha,\lambda(g(a)-g(a-L))]}{\Gamma(\alpha)}\right].
$$

□

Theorem 3.8. *Let $g(s) \in C^1[a-L,b]$ with $g(a-L) \geq 0$ and $g'(s) > 0$. If function $f(s) \in AC[a,+\infty)$ and of $g(s)$-exponential order such that $\delta f(s)$ is piecewise continuous in the interval $[a,+\infty)$, then the general Laplace transform of $\delta f(s)$ exists,*

$$\mathcal{L}_g\{\delta f(s)\}(\lambda) = \lambda \mathcal{L}_g\{f(s)\}(\lambda) - f(a).$$

Proof. Denote $a < s_1 < s_2 < \cdots < s_n < +\infty$ as all the discontinuity points of $\delta f(s)$ in the interval $[a,+\infty)$. Then

$$\begin{aligned}\int_a^{+\infty} e^{-\lambda(g(s)-g(a))}\delta f(s)g'(s)\,ds = &\int_a^{s_1} e^{-\lambda(g(s)-g(a))} f'(s)ds \\ &+ \sum_{i=1}^{n-1}\int_{s_i}^{s_{i+1}} e^{-\lambda(g(s)-g(a))} f'(s)ds + \int_{s_n}^{+\infty} e^{-\lambda(g(s)-g(a))} f'(s)ds.\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}\int_a^{+\infty} e^{-\lambda(g(s)-g(a))}\delta f(s)g'(s)ds =& e^{-\lambda(g(s)-g(a))} f(s)|_a^{s_1} + \sum_{i=1}^{n-1} e^{-\lambda(g(s)-g(a))} f(s)|_{s_i}^{s_{i+1}} + e^{-\lambda(g(s)-g(a))} f(s)|_{s_n}^{+\infty} \\ &+ \lambda\int_a^{s_1} e^{-\lambda(g(s)-g(a))} f(s)g'(s)ds + \lambda\sum_{i=1}^{n-1}\int_{s_i}^{s_{i+1}} e^{-\lambda(g(s)-g(a))} f(s)g'(s)ds \\ &+ \lambda\int_{s_n}^{+\infty} e^{-\lambda(g(s)-g(a))} f(s)g'(s)ds.\end{aligned}$$

Based on the above results, it further follows that

$$\begin{aligned}\int_a^{+\infty} e^{-\lambda(g(s)-g(a))}\delta f(s)g'(s)\,ds =& \lim_{T\to+\infty} e^{-\lambda(g(T)-g(a))} f(T) - f(a) + \lambda\int_a^{+\infty} e^{-\lambda(g(s)-g(a))} f(s)g'(s)ds \\ =& \lambda\mathcal{L}_g\{f(s)\}(\lambda) - f(a).\end{aligned}$$

□

Theorem 3.9. *Let $g(s) \in C^1[a-L,b]$ with $g(a-L) \geq 0$ and $g'(s) > 0$. If function $f(s) \in AC_\delta^n[a,+\infty]$ and of $g(s)$-exponential order such that $\delta^i f(s), i = 1,2,\cdots,n-1$, is piecewise continuous in the interval $[a,+\infty]$, then the general Laplace transform of $\delta^n f(s)$ exists,*

$$\mathcal{L}_g\{\delta^n f(s)\}(\lambda) = \lambda^n \mathcal{L}_g\{f(s)\}(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1}\delta^k (f(s))|_{s=a}. \tag{3.1}$$

Proof. When $n = 1$, according to Theorem 3.8, we have $\mathcal{L}_g\{\delta f(s)\}(\lambda) = \lambda\mathcal{L}_g\{f(s)\}(\lambda) - f(a)$, that is, Eq (3.1) holds for $n = 1$. Assume that (3.1) holds for $n = k-1$, namely,

$$\mathcal{L}_g\{\delta^{k-1} f(s)\}(\lambda) = \lambda^{k-1}\mathcal{L}_g\{f(s)\}(\lambda) - \sum_{j=0}^{k-2} \lambda^{k-j-2}\delta^j (f(s))|_{s=a}.$$

Then we show that Eq (3.1) also holds for $n = k$. That is,

$$\begin{aligned}\mathcal{L}_g\{\delta^k f(s)\}(\lambda) =&\mathcal{L}_g\{\delta[\delta^{k-1} f(s)]\}(\lambda) = \lambda\mathcal{L}_g\{\delta^{k-1} f(s)\}(\lambda) - \mathcal{L}_g\{\delta^{k-1} f(a)\}\\ =&\lambda \cdot [\lambda^{k-1}\mathcal{L}_g\{f(s)\}(\lambda) - \sum_{j=0}^{k-2} \lambda^{k-j-2}\delta^j (f(s))\,|_{s=a}] - \mathcal{L}_g\{\delta^{k-1} f(a)\}\\ =&\lambda^k\mathcal{L}_g\{f(s)\}(\lambda) - \sum_{j=0}^{k-1} \lambda^{k-j-1}\delta^j (f(s))\,|_{s=a}.\end{aligned}$$

Therefore, Eq (3.1) holds. □

Theorem 3.10. *Let,* $\forall n \in Z^+$*,* $g(s) \in C^1[a-L, b]$ *with* $g(a-L) \geq 0$ *and* $g'(s) > 0$*. If* $f \in AC^n_\delta[a-L, b]$ *is of* $g(s)$*-exponential order and* $g'(s) = g'(s-L), \forall s \in [a, b]$*, then* $\forall\alpha > 0$*,*

$$\mathcal{L}_g\{\delta^n\,{}_{s-L}I^{\alpha,g}_s f(s)\} = \mathcal{L}_g\{{}_{s-L}I^{\alpha,g}_s \delta^n f(s)\} = \lambda^n\mathcal{L}_g\left\{{}_{s-L}I^{\alpha,g}_s f(s)\right\}(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1}\left(\delta^k\,{}_{s-L}I^{\alpha,g}_s f(s)\right)|_{s=a}.$$

In particular, if $n = [\alpha] + 1$*, we have*

$$\mathcal{L}_g\{{}_{s-L}D^{\alpha,g}_s f(s)\} = \mathcal{L}_g\{{}_{s-L}{}^CD^{\alpha,g}_s f(s)\} = \lambda^n\mathcal{L}_g\left\{{}_{s-L}I^{n-\alpha,g}_s f(s)\right\}(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1}\left(\delta^k\,{}_{s-L}I^{n-\alpha,g}_s f(s)\right)|_{s=a}.$$

Proof. By Theorem 2.19, we have

$$\begin{aligned}{}_{s-L}I^{\alpha,g}_s f(s) =& \sum_{k=0}^{n-1} \frac{{}_{a-L}I^{\alpha,g}_a \delta^k f(a)}{k!}(g(s) - g(a))^k +_a I^{n,g}_s({}_{s-L}I^{\alpha,g}_s \delta^n f(s)(s))\\ &+ \sum_{k=1}^{n} {}_aI^{k,g}_s\left[\frac{(g(s) - g(s-L))^{\alpha-1}[\delta^{n-1}]f(s-L)(g'(s) - g'(s-L))}{g'(s)\Gamma(\alpha)}\right](s).\end{aligned}$$

In addition, $g'(s) = g'(s-L), \forall s \in [a, b]$, hence

$${}_{s-L}I^{\alpha,g}_s f(s) = \sum_{k=0}^{n-1} \frac{{}_{a-L}I^{\alpha,g}_a \delta^k f(a)}{k!}(g(s) - g(a))^k +_a I^{n,g}_s({}_{s-L}I^{\alpha,g}_s \delta^n f(s)(s)).$$

Next, applying the operator δ^n and by the fundamental theorem of calculus, we obtain

$$\delta^n\,{}_{s-L}I^{\alpha,g}_s f(s) = \sum_{k=0}^{n-1} \frac{{}_{a-L}I^{\alpha,g}_a \delta^k f(a)}{k!}\delta^n(g(s) - g(a))^k + \delta^n_a I^{n,g}_s({}_{s-L}I^{\alpha,g}_s \delta^n f(s)) =_{s-L} I^{\alpha,g}_s \delta^n f(s).$$

Hence,

$$\mathcal{L}_g\{\delta^n\,{}_{s-L}I^{\alpha,g}_s f(s)\} = \mathcal{L}_g\{{}_{s-L}I^{\alpha,g}_s \delta^n f(s)\}.$$

According to Theorem 3.9, we have

$$\mathcal{L}_g\left\{\delta^n\,{}_{s-L}I^{\alpha,g}_s f(s)\right\}(\lambda) = \lambda^n\mathcal{L}_g\left\{{}_{s-L}I^{\alpha,g}_s f(s)\right\}(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1}\left(\delta^k\,{}_{s-L}I^{\alpha,g}_s f(s)\right)|_{s=a}.$$

Therefore, if $n = [\alpha] + 1$, we have

$$\mathcal{L}_g\{{}_{s-L}D_s^{\alpha,g} f(s)\} = \mathcal{L}_g\{{}_{s-L}{}^C D_s^{\alpha,g} f(s)\} = \mathcal{L}_g\left\{\delta^n{}_{s-L}I_s^{n-\alpha,g} f(s)\right\}(\lambda)$$
$$= \lambda^n \mathcal{L}_g\left\{{}_{s-L}I_s^{n-\alpha,g} f(s)\right\}(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1}\left(\delta^k{}_{s-L}I_s^{n-\alpha,g} f(s)\right)|_{s=a}.$$

□

4. Differential equations involving the general fractional derivative with FML

In this section, we consider the following general FDEs with FML:

$$\begin{cases} {}_{s-L}D_s{}^{\alpha,g} x(s) = f(s, x(s)), \quad a \leq s \leq b, \\ x(s)|_{[a-L,a]} = 0, \end{cases} \tag{4.1}$$

where $\alpha \in (0, 1)$, $a > L > 0$, $b > a + L$, $g(s) \in C^1[a - L, b]$ with $g(a - L) \geq 0$ and $g'(s) = g'(s - L) > 0$, and $f : [a, b] \times R \to R$ is a continuous function.

By Lemma 2.10, we can transform Eq (4.1) into an equivalent FDE, that is,

$$\begin{cases} {}_aD_s{}^{\alpha,g} x(s) = f(s, x(s)) + \frac{1}{\Gamma(-\alpha)} \int_a^{s-L} (g(s) - g(\tau))^{-\alpha-1} f(\tau) g'(\tau) d\tau \\ \qquad\qquad + \frac{(g(s)-g(s-L))^{-\alpha} x(s-L)}{\Gamma(1-\alpha)}, \quad a \leq s \leq b, \\ x(s)|_{[a-L,a]} = 0. \end{cases} \tag{4.2}$$

Lemma 4.1. *$x \in C[a - L, b]$ is a solution of Eq (4.2), if and only if it satisfies the fractional integral equation*

$$x(s) = \begin{cases} {}_aI_s{}^{\alpha,g} f(s, x(s)) + \frac{1}{\Gamma(-\alpha)} {}_aI_s{}^{\alpha,g} \int_a^{s-L} (g(s) - g(\tau))^{-\alpha-1} x(\tau) g'(\tau) d\tau \\ \qquad\qquad + {}_aI_s{}^{\alpha,g} \frac{(g(s)-g(s-L))^{-\alpha} x(s-L)}{\Gamma(1-\alpha)}, \quad a \leq s \leq b, \\ 0, \quad a - L \leq s \leq a. \end{cases} \tag{4.3}$$

Proof. On the one hand, when x satisfies Eq (4.2), we get

$${}_aI_s{}^{1-\alpha,g} x(a) = 0.$$

According to Lemma 2.9, we get

$${}_aI_s{}^{\alpha,g}{}_aD_s{}^{\alpha,g} x(s) = x(s) - \frac{{}_aI_s{}^{1-\alpha,g} x(a)(g(s) - g(a))^{\alpha-1}}{\Gamma(\alpha)},$$

and then

$${}_aI_s{}^{\alpha,g}{}_aD_s{}^{\alpha,g} x(s) = x(s).$$

Applying the operator ${}_aI_s{}^{\alpha,g}$ to Eq (4.2), we obtain

$$x(s) = {}_aI_s{}^{\alpha,g} f(s, x(s)) + \frac{1}{\Gamma(-\alpha)} {}_aI_s{}^{\alpha,g} \int_a^{s-L} (g(s) - g(\tau))^{-\alpha-1} x(\tau) g'(\tau) d\tau$$
$$+ {}_aI_s{}^{\alpha,g} \frac{(g(s) - g(s - L))^{-\alpha} x(s - L)}{\Gamma(1 - \alpha)}, \quad a \leq s \leq b.$$

Hence x satisfies Eq (4.3).

On the other hand, if x satisfies Eq (4.3), then

$$x|_{[a-L,a]} = 0.$$

Besides, by Lemma 2.8, we get

$$\begin{aligned} {}_aD_s^{\alpha,g}x(s) =& f(s,x(s)) + \frac{1}{\Gamma(-\alpha)}\int_a^{s-L}(g(s)-g(\tau))^{-\alpha-1}x(\tau)g'(\tau)d\tau \\ &+ \frac{(g(s)-g(s-L))^{-\alpha}x(s-L)}{\Gamma(1-\alpha)}. \end{aligned}$$

□

In the following, we will employ the Banach fixed-point theorem to investigate the existence and uniqueness of solutions to Eq (4.2). To this end, we define the space X as

$$X = \{x \in C[a-L,b] : x(s)|_{[a-L,a]} = 0\} \subset C[a-L,b],$$

with norm

$$\|x\| = \sup_{s\in[a-L,b]} |x(s)| = \sup_{s\in[a,b]} |x(s)|.$$

Adopting the method from [9, Lemma 4.3], we can prove that X is a Banach space.

Subsequently, we let $f \in C([a,b]\times R, R)$ satisfy:

(H_1) There exist $A > 0$ and $0 \le \sigma < \alpha$ such that

$$(g(s)-g(a))^{\sigma}|f(s,\nu)-f(s,\psi)| \le A|\nu-\psi|, \quad a \le s \le b, \quad \forall \nu,\ \psi \in R.$$

(H_2) Let $\alpha \in (0,1)$, $L > 0$, $A > 0$, and $0 \le \sigma < \alpha$ such that

$$\sup_{s\in[a,b]} (g(s)-g(s-L))^{-\alpha} < \frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{4(g(b)-g(a))^{\alpha}}$$

and

$$\frac{(g(b)-g(a))^{\alpha-\sigma}}{\Gamma(1-\sigma+\alpha)} < \frac{1}{2A\Gamma(1-\sigma)}.$$

Lemma 4.2. *Suppose that* (H_1) *and* (H_2) *hold. Then the operator* $F : X \to X$ *defined as*

$$F(x(s)) = \begin{cases} {}_aI_s^{\alpha,g} f(s,x(s)) + \frac{1}{\Gamma(-\alpha)} {}_aI_s^{\alpha,g} \int_a^{s-L}(g(s)-g(\tau))^{-\alpha-1}x(\tau)g'(\tau)d\tau \\ \qquad + {}_aI_s^{\alpha,g}\frac{(g(s)-g(s-L))^{-\alpha}x(s-L)}{\Gamma(1-\alpha)}, \quad a \le s \le b, \\ 0, \quad a-L \le s \le a, \end{cases}$$

is a contraction operator.

Proof. $[F_1]$ Consider the map $F_1 : X \to X$ defined as:

$$F_1(x(s)) = \begin{cases} {}_aI_s^{\alpha,g}\frac{(g(s)-g(s-L))^{-\alpha}x(s-L)}{\Gamma(1-\alpha)}, \quad a \le s \le b, \\ 0, \quad a-L \le s \le a. \end{cases}$$

Clearly, we have $F_1x \in X$. Now, for any $x, y \in X$, $a - L \le s \le a$, we obtain

$$|F_1x - F_1y| = 0.$$

Besides, for $a \le s \le b$, we have

$$\begin{aligned}|F_1x(s) - F_1y(s)| \le& \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^s (g(s) - g(\tau))^{\alpha-1}(g(\tau) - g(\tau - L))^{-\alpha}|x(\tau - L) - y(\tau - L)|g'(\tau)d\tau \\ \le& \frac{(g(b) - g(a))^{\alpha}\|x - y\|}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \cdot \sup_{s\in[a,b]} (g(s) - g(s - L))^{-\alpha}.\end{aligned}$$

Thus from (H_2), we have

$$\|F_1x - F_1y\| \le \frac{(g(b) - g(a))^{\alpha}\|x - y\|}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \cdot \sup_{s\in[a,b]} (g(s) - g(s - L))^{-\alpha} < \frac{1}{4}\|x - y\|, \quad \forall x, y \in X.$$

$[F_2]$ Consider the map $F_2 : X \to X$ defined as:

$$F_2(x(s)) = \begin{cases} \frac{1}{\Gamma(-\alpha)} {}_aI_s^{\alpha,g} \int_a^{s-L} (g(s) - g(\tau))^{-\alpha-1} x(\tau)g'(\tau)d\tau, & a \le s \le b, \\ 0, \quad a - L \le s \le a. \end{cases}$$

Clearly we have $F_2x \in X$. Now, $\forall x, y \in X$:
(1) If $a - L \le s \le a$, then

$$|F_2x - F_2y| = 0.$$

(2) If $a \le s \le a + L$, then $\forall w \in X$, we have

$$\begin{aligned}F_2w(s) =& \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)} \int_a^s (g(s) - g(\tau))^{\alpha-1} \int_a^{\tau-L} (g(\tau) - g(\tau))^{-\alpha-1} w(\tau)g'(\tau)d\tau g'(\tau)d\tau \\ =& -\frac{1}{\Gamma(\alpha)\Gamma(-\alpha)} \int_a^s (g(s) - g(\tau))^{\alpha-1} \int_{\tau-L}^a (g(\tau) - g(\tau))^{-\alpha-1} w(\tau)g'(\tau)d\tau g'(\tau)d\tau.\end{aligned}$$

As $w|_{[a-L,a]} = 0$, and $a - L \le \tau - L \le a$ for all $a \le \tau \le a + L$, then $w|_{[\tau-L,a]} = 0$ for $a \le \tau \le a + L$. Hence

$$F_2w(s) = 0,$$

and consequently

$$|F_2x(s) - F_2y(s)| = 0.$$

(3) If $a + L \le s \le b$, then $\forall w \in X$, we obtain

$$\begin{aligned}F_2w(s) =& \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)} \int_a^s (g(s) - g(\tau))^{\alpha-1} \int_a^{\tau-L} (g(\tau) - g(\tau))^{-\alpha-1} w(\tau)g'(\tau)d\tau g'(\tau)d\tau \\ =& \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)} \int_a^{a+L} (g(s) - g(\tau))^{\alpha-1} \int_a^{\tau-L} (g(\tau) - g(\upsilon))^{-\alpha-1} w(\upsilon)g'(\upsilon)d\upsilon g'(\tau)d\tau \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)} \int_{a+L}^s (g(s) - g(\tau))^{\alpha-1} \int_a^{\tau-L} (g(\tau) - g(\upsilon))^{-\alpha-1} w(\upsilon)g'(\upsilon)d\upsilon g'(\tau)d\tau.\end{aligned}$$

For all $a \leq \tau \leq a+L$, we have $a-L \leq \tau - L \leq a$. By using the fact that $w|_{[a-L,a]}=0$, we have

$$-\frac{1}{\Gamma(\alpha)\Gamma(-\alpha)}\int_a^{a+L}(g(s)-g(\tau))^{\alpha-1}\int_{\tau-L}^{a}(g(\tau)-g(\upsilon))^{-\alpha-1}w(\upsilon)g'(\upsilon)d\upsilon g'(\tau)d\tau = 0.$$

Moreover, for each $a+L \leq \tau \leq s$, $a \leq \tau - L$, we have

$$F_2w(s) = \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)}\int_{a+L}^{s}(g(s)-g(\tau))^{\alpha-1}\int_{a}^{\tau-L}(g(\tau)-g(\upsilon))^{-\alpha-1}w(\upsilon)g'(\upsilon)d\upsilon g'(\tau)d\tau.$$

Consequently, we have

$$|F_2x(s)-F_2y(s)| \leq \frac{\|x-y\|}{\Gamma(\alpha)(-\Gamma(-\alpha))}\int_{a+L}^{s}(g(s)-g(\tau))^{\alpha-1}\int_{\tau-L}^{a}(g(\tau)-g(\upsilon))^{-\alpha-1}g'(\upsilon)d\upsilon g'(\tau)d\tau.$$

Note that

$$\begin{aligned}
&\int_{a+L}^{s}(g(s)-g(\tau))^{\alpha-1}\int_{\tau-L}^{a}(g(\tau)-g(\tau))^{-\alpha-1}g'(\tau)d\tau g'(\tau)d\tau\\
=&\frac{1}{\alpha}\int_{a+L}^{s}(g(s)-g(\tau))^{\alpha-1}(g(\tau)-g(\tau-L))^{-\alpha}g'(\tau)d\tau\\
&-\frac{1}{\alpha}\int_{a+L}^{s}(g(s)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\alpha}g'(\tau)d\tau\\
\leq&\frac{1}{\alpha}\int_{a}^{s}(g(s)-g(\tau))^{\alpha-1}(g(\tau)-g(\tau-L))^{-\alpha}g'(\tau)d\tau\\
\leq&\frac{(g(b)-g(a))^{\alpha}}{\alpha^2}\cdot\sup_{s\in[a,b]}(g(s)-g(s-L)^{-\alpha}.
\end{aligned}$$

Hence from (H_2), we have

$$\begin{aligned}
|F_2x(s)-F_2y(s)| \leq& \frac{(g(b)-g(a))^{\alpha}\|x-y\|}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\cdot\sup_{s\in[a,b]}(g(s)-g(s-L)^{-\alpha}\\
<& \frac{1}{4}\|x-y\|, \quad \forall x,y\in X.
\end{aligned}$$

Therefore

$$\|F_2x(s)-F_2y(s)\| < \frac{1}{4}\|x-y\|, \quad \forall x,y\in X.$$

$[F_3]$ Consider the map $F_3 : X \to X$ defined as:

$$F_3(x(s)) = \begin{cases} {}_aI_s^{\alpha,g}f(s,x(s)), & a\leq s\leq b,\\ 0, & a-L\leq s\leq a.\end{cases}$$

Hence $F_3x \in X$. For all $x,y\in X$, $a-L\leq s\leq a$, we obtain

$$|F_3x(s)-F_3y(s)| = 0.$$

Besides, when $a \leq s \leq b$, by using (H_1), we get

$$\begin{aligned}|F_3x(s)-F_3y(s)| \leq & \frac{1}{\Gamma(\alpha)}\int_a^s (g(s)-g(\tau))^{\alpha-1}|f(\tau,x(\tau))-f(\tau,y(\tau)|g'(\tau)d\tau \\ \leq & \frac{1}{\Gamma(\alpha)}\int_a^s (g(s)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\sigma}A|x(\tau)-y(\tau)|g'(\tau)d\tau \\ \leq & \frac{A\|x-y\|}{\Gamma(\alpha)}\int_a^s (g(s)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\sigma}g'(\tau)d\tau.\end{aligned}$$

Making the change of variable $\tau = \frac{g(\tau)-g(a)}{g(s)-g(a)}$, then

$$\begin{aligned}&\int_a^s (g(s)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\sigma}g'(\tau)d\tau \\ &= \int_0^1 [(1-\tau)(g(s)-g(a))]^{\alpha-1}[\tau(g(s)-g(a))]^{-\sigma}(g(s)-g(a)d\tau \\ &\leq \frac{(g(b)-g(a))^{\alpha-\sigma}\Gamma(1-\sigma)\Gamma(\alpha)}{\Gamma(1-\sigma+\alpha)}.\end{aligned}$$

Hence from (H_2), we have

$$\|F_3x(s)-F_3x(s)\| \leq \frac{A(g(b)-g(a))^{\alpha-\sigma}\Gamma(1-\sigma)\|x-y\|}{\Gamma(1-\sigma+\alpha)} < \frac{1}{2}\|x-y\|, \quad \forall x,y \in X.$$

$[F]$ Finally, due to

$$\begin{aligned}\|F_1x(s)-F_1x(s)\| &< \frac{1}{4}\|x-y\|, \quad \forall x,y \in X, \\ \|F_2x(s)-F_2x(s)\| &< \frac{1}{4}\|x-y\|, \quad \forall x,y \in X, \\ \|F_3x(s)-F_3x(s)\| &< \frac{1}{2}\|x-y\|, \quad \forall x,y \in X,\end{aligned}$$

there exists a constant $0 < K < 1$ such that

$$\|Fx-Fy\| \leq K\|x-y\|, \quad \forall x,y \in X.$$

Therefore, F is a contraction. □

Theorem 4.3. *Suppose that* (H_1) *and* (H_2) *hold. Then, Eq (4.2) has a unique solution and as a consequence, Eq (4.1) has a unique solution.*

Proof. Define the mapping $F : X \to X$ as described in Lemma 4.2, where X is a Banach space. Thus, F is a contraction map. By the Banach fixed-point theorem, there is a unique $\nu \in X$ that satisfies $F\nu = \nu$. Therefore, given the equivalence between Eqs (4.2) and (4.1), we can conclude that Eq (4.1) has a unique solution. □

Example 4.4. *Consider the following problem:*

$$\begin{cases} {}_{s-2\pi}D_s^{0.9,sins+1.2s}x(s) = 4s + 0.05x(s), \quad 2.5\pi \leq s \leq 5\pi, \\ x(s)|_{[0.5\pi,2.5\pi]} = 0, \end{cases} \tag{4.4}$$

where $\alpha = 0.9$, $L = 2\pi$, $a = 2.5\pi$, $b = 5\pi$, $g(s) = \sin s + 1.2s$.

Let $f \in C([2.5\pi, 5\pi] \times R, R)$ *be defined as*

$$f(s, v) = 4s + 0.05v, \quad 2.5\pi \le s \le 5\pi, \ v \in \mathbb{R}.$$

Hence, we have that f *is continuous and*

$$\left|f(s, v) - f(s, w)\right| = 0.05\left|v - w\right|, \quad \forall v, w \in \mathbb{R}.$$

Then $A = 0.05$ *and* $\sigma = 0$. *Moreover, we have*

$$\sup_{s\in[a,b]} (g(s) - g(s - L))^{-\alpha} \approx 0.1623, \ \frac{\Gamma(1 + \alpha)\Gamma(1 - \alpha)}{4(g(b) - g(a))^{\alpha}} \approx 0.3360,$$

and

$$\frac{(g(b) - g(a))^{\alpha-\sigma}}{\Gamma(1 - \sigma + \alpha)} \approx 7.0784, \ \frac{1}{2A\Gamma(1 - \sigma)} = 10.$$

Hence

$$\sup_{s\in[a,b]} (g(s) - g(s - L))^{-\alpha} < \frac{\Gamma(1 + \alpha)\Gamma(1 - \alpha)}{4(g(b) - g(a))^{\alpha}}$$

and

$$\frac{(g(b) - g(a))^{\alpha-\sigma}}{\Gamma(1 - \sigma + \alpha)} < \frac{1}{2A\Gamma(1 - \sigma)}.$$

That is, (H_1) *and* (H_2) *are satisfied. Therefore, by Theorem 4.3, problem (4.4) has a unique solution, see Figure 1.*

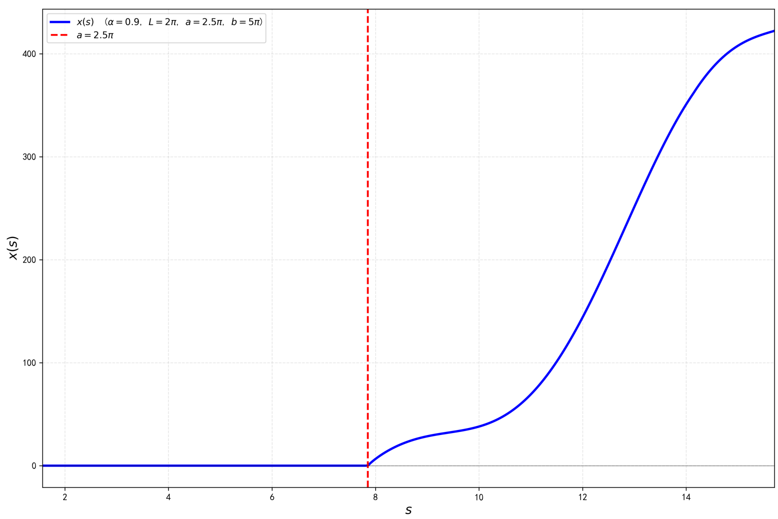

Figure 1. The solution $x(s)$ of problem (4.4).

5. Conclusions

In this paper, we initially investigate the properties of general fractional operators with FML, followed by an analysis of their general Laplace transform. Additionally, we consider a class of general FDEs with FML, successfully deriving their solutions via integral methods and establishing the uniqueness of the solutions through the Banach fixed-point theorem. Although this study has made progress in the theoretical construction and solution of general FDEs with FML, it still has the following limitations:

(a) The current study mainly focuses on low-order ($0 < \alpha < 1$) FDEs with FML, but does not involve high-order FDEs (e.g., coupled systems with multiple fractional derivatives where $\alpha > 1$). This makes it difficult to characterize the complex behaviors of multi-scale dynamic coupling in practical systems.

(b) The model assumptions ignore time delay (e.g., reaction delays in biological systems, signal transmission lags in control systems). While the factor is prevalent in practical fractional systems (e.g., chaotic systems with time-delayed feedback control), it may lead to deviations between the qualitative behaviors of solutions (e.g., stability, periodicity) and theoretical analyses.

To address the above limitations, future research can be extended in the following directions:

(I) High-order FDEs with FML: Extend the results to high-order FDEs, and study the existence and uniqueness of their solutions.

(II) Time-delayed FDEs with FML: Introduce a time-delay term $\tau > 0$, establish a theoretical framework for time-delayed FDEs with FML, and analyze the influence of time delay on the stability of periodic solutions.

Author contributions

Rong-Fu Wang: writing-original draft, methodology, software; Chuan-Yun Gu: methodology, supervision, writing-review & editing, funding acquisition, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no generative Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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