



Research article

Geometric interpretation of Caputo's fractional derivative and its relation to initial value problems of integer and fractional order

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Abstract: In this paper, we give a new geometric interpretation to Caputo's fractional derivative, based on the fundamental theorem of plane curves and the fractional curvature of plane curves introduced by Rubio-López et al. in 2023. Furthermore, our results are related to initial value problems of integer and fractional order. Finally, some examples related to differential geometry and viscoelasticity theory are given.

Keywords: curvature of curve; fractional curvature; Caputo's fractional derivative; geometric interpretation; Frenet-Serret equations

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1. Introduction

The problem with the geometrical and physical interpretation of the derivative and integral of fractional order, has been present from the beginnings of fractional calculus, as can be seen in the correspondence, in 1695, between Leibniz and L'Hospital [1, 2].

More than 3 centuries have passed, and during this period several articles have been presented with different points of view. Thus, in the area of mathematics, Gorenflo gives an interpretation on fractional derivatives and integrals [3], Ben Adda F. bases his interpretation on contact theory [4, 5], Bullock uses the Stieltjes integral [6], Machado gives an interpretation for the Grünwal-Letnikov derivative [7], Stanislavsky bases his interpretation on the basis of probability [8], and Tarasov bases his interpretation on Jet bundle geometry [9, 10]. Other interpretations are given by Tavassoli [11] and Hilfer [12].

Some physical interpretations are given by Nigmatullin [13], Rutman [14], Moshrefi-Torbati [15], and Ciol [16]. Podlubny proposed other interpretations on the integral, derivative, and initial conditions

of fractional order [17, 18].

In this paper, we present a new geometric interpretation of Caputo's fractional derivative, related to the geometric interpretation of the integer-order derivative, with important differences from the geometric interpretation given by Podlubny [2].

Our interpretation is based on the fundamental theorem of plane curves in classical differential geometry and on the fractional curvature of plane curves in fractional differential geometry.

Rubio et al. [19], introduced a new definition of fractional curvature of plane curves in 2023, and in 2024, they extended their results to regular curves and surfaces in \mathbb{R}^3 [20], obtaining a geometric interpretation for Caputo's fractional derivative, related to the fractional curvature and fractional torsion of the helix.

Here, we relate our results to initial value problems of integer and fractional order, and give some examples in geometry and in viscoelasticity theory.

2. Preliminaries

2.1. Regular plane curves

Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length. The Frenet-Serret frame of the curve $\alpha(s) = (x(s), y(s))$ is given by the orthonormal basis $\{t(s), n(s)\}$, $\forall s \in I$, where

$$t(s) = \alpha'(s) = (x'(s), y'(s)), \quad (2.1)$$

$$n(s) = (-y'(s), x'(s)). \quad (2.2)$$

The Frenet-Serret equations of the curve (Tenenblat K, 2008), are given by

$$\begin{cases} t'(s) = k(s)n(s), \\ n'(s) = -k(s)t(s), \end{cases} \quad (2.3)$$

where $k(s)$ is the integer or classical curvature of $\alpha(s)$ at point s and is given by

$$k(s) = \langle t'(s), n(s) \rangle, \quad \forall s \in I. \quad (2.4)$$

Theorem 1. *Fundamental theorem of plane curves [21].*

- Given a differentiable function $k(s)$, $\forall s \in I$, there exists a regular curve $\alpha(s)$, parameterized by arc length, whose curvature is $k(s)$.
- The $\alpha(s)$ curve is unique if we fix $\alpha(s_0) = p_0$, $\alpha'(s_0) = v_0$, and $\|v_0\| = 1$, $v_0 \in \mathbb{R}^2$.
- If two curves $\alpha(s)$ and $\beta(s)$ have the same curvature, then they differ only by their position in the plane; that is, there exist a rotation L and a translation T in \mathbb{R}^2 , such that

$$\alpha(s) = (L \circ T)(\beta(s)). \quad (2.5)$$

2.2. Caputo fractional derivative

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function of class $C^1([a, b])$. The Caputo fractional derivative of order λ is defined by [22–24].

$${}^c D^\lambda f(t) = \frac{1}{\Gamma(1-\lambda)} \int_a^t \frac{1}{(t-u)^\lambda} f'(u) du, \quad (2.6)$$

where $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, and Γ is Euler's gamma function.

2.3. Fractional curvature of plane curves

Definition 1. [19, 20] Let $\alpha : [a, b] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length s . The fractional derivative vector of order λ of α at s , is given by

$${}^c D^\lambda \alpha(s) = ({}^c D^\lambda x(s), {}^c D^\lambda y(s)). \quad (2.7)$$

In the Frenet-Serret frame, we have

$${}^c D^\lambda \alpha(s) = a^\lambda(s)t(s) + b^\lambda(s)n(s), \quad \forall s \in [a, b], \quad (2.8)$$

where $\lim_{\lambda \rightarrow 1} a^\lambda(s) = 1$ and $\lim_{\lambda \rightarrow 1} b^\lambda(s) = 0$.

Definition 2. [19, 20] Let $\alpha : [a, b] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length and ${}^c D^\lambda \alpha(s)$ is the fractional derivative vector of order λ . The fractional curvature of order λ , and $0 < \lambda < 1$, of the curve α at point s is given by

$$k^\lambda(s) = a^\lambda(s)k(s) + \frac{d}{ds}b^\lambda(s), \quad \forall s \in (a, b), \quad (2.9)$$

where $k(s)$ is the integer curvature of α at point s , where $a^\lambda(s)$ and $b^\lambda(s)$ are functions given by

$$a^\lambda(s) = x'(s){}^c D^\lambda x(s) + y'(s){}^c D^\lambda y(s), \quad b^\lambda(s) = x'(s){}^c D^\lambda y(s) - y'(s){}^c D^\lambda x(s).$$

Now, consider the fractional-order initial value problem (FIVP)

$$\begin{cases} {}^c D^\lambda y(t) = f(t, y(t)), \\ y(0) = y_0, \end{cases} \quad (2.10)$$

where $\lambda \in (0, 1)$, $t \in [0, T]$, and $T > 0$.

Next, two theorems on the existence and uniqueness of the solution of the fractional IVP (2.10) are presented.

Theorem 2 (Existence). [25, 26] Assume that $M = [0, \chi^*] \times [y_0 - \delta, y_0 + \delta]$, $\chi^* > 0$, and $\delta > 0$, and let the function $f : M \rightarrow \mathbb{R}$ be continuous. Furthermore, define $\chi = \min \left\{ \chi^*, \left(\frac{\delta \Gamma(\lambda+1)}{\|f\|_\infty} \right)^{\frac{1}{\lambda}} \right\}$.

Then, there exists a function $y : [0, \chi] \rightarrow \mathbb{R}$ solving the initial value problem (2.10).

Theorem 3 (Uniqueness). [25, 26] Assume that $M = [0, \chi^*] \times [y_0 - \delta, y_0 + \delta]$, with $\chi^* > 0$ and $\delta > 0$. Furthermore, let the function $f : M \rightarrow \mathbb{R}$ be bounded on M and fulfil a Lipschitz condition with respect to the second variable, i.e.,

$$|f(t, x) - f(t, y)| \leq L|x - y|,$$

with constant $L > 0$ independent of t, x , and y . Then, denoting χ as in Theorem 2, there exists at most one function $y : [0, \chi] \rightarrow \mathbb{R}$ solving the initial value problem (2.10).

Now, let $g : [a, b] \rightarrow \mathbb{R}$ be a function defined on the finite interval $[a, b]$.

Theorem 4. [27] If $g \in L(a, b)$, then,

- (i) $G(x) = \int_a^x g(t)dt$ is a continuous function on $[a, b]$.
- (ii) $G \in BV[a, b]$.

Theorem 5. [27] Let $[a, b]$ be a finite interval, and let $g \in L(a, b)$ with indefinite integral G ; then $G' = g$ almost everywhere (a.e) in $[a, b]$.

3. Theory development

Theorem 6. Let $\alpha : [0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length. Then,

$${}^c D^\lambda \alpha(s) = A^\lambda(s)t(s), \quad (3.1)$$

where

$$A^\lambda(s) = \begin{pmatrix} a^\lambda(s) & -b^\lambda(s) \\ b^\lambda(s) & a^\lambda(s) \end{pmatrix}, \quad (3.2)$$

and

$$\det(A^\lambda(s)) > 0, \quad \forall s \in (0, a].$$

Proof. Using the Frenet-Serret reference frame of the curve in (2.8), we have:

$$\begin{aligned} {}^c D^\lambda \alpha(s) &= a^\lambda(s) (x'(s), y'(s)) + b^\lambda(s) (-y'(s), x'(s)), \\ {}^c D^\lambda \alpha(s) &= ({}^c D^\lambda x(s), {}^c D^\lambda y(s)) = (a^\lambda(s)x'(s) - b^\lambda(s)y'(s), a^\lambda(s)y'(s) + b^\lambda(s)x'(s)). \end{aligned}$$

Then,

$$\begin{cases} {}^c D^\lambda x(s) = a^\lambda(s)x'(s) - b^\lambda(s)y'(s), \\ {}^c D^\lambda y(s) = a^\lambda(s)y'(s) + b^\lambda(s)x'(s). \end{cases} \quad (3.3)$$

Expressing (3.3) in matrix form

$$\begin{pmatrix} {}^c D^\lambda x(s) \\ {}^c D^\lambda y(s) \end{pmatrix} = \begin{pmatrix} a^\lambda(s) & -b^\lambda(s) \\ b^\lambda(s) & a^\lambda(s) \end{pmatrix} \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = A^\lambda(s)t(s).$$

Furthermore, we have

$$\det(A^\lambda(s)) = (a^\lambda(s))^2 + (b^\lambda(s))^2 > 0, \quad \forall s \in (0, a].$$

□

Theorem 7. Let $\alpha : [0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length. Then,

$${}^c D^\lambda \alpha(s) = \|{}^c D^\lambda \alpha(s)\| R^\lambda(s)t(s), \quad (3.4)$$

where

$$R^\lambda(s) = \begin{pmatrix} \cos(\theta^\lambda(s)) & -\sin(\theta^\lambda(s)) \\ \sin(\theta^\lambda(s)) & \cos(\theta^\lambda(s)) \end{pmatrix}, \quad (3.5)$$

and $\theta^\lambda(s) = \angle(t(s), {}^c D^\lambda \alpha(s))$.

Proof. Let $\theta^\lambda(s) = \angle(t(s), {}^c D^\lambda \alpha(s))$ be the smallest angle determined by vectors $t(s)$ and ${}^c D^\lambda \alpha(s)$, $\forall s \in (0, a]$.

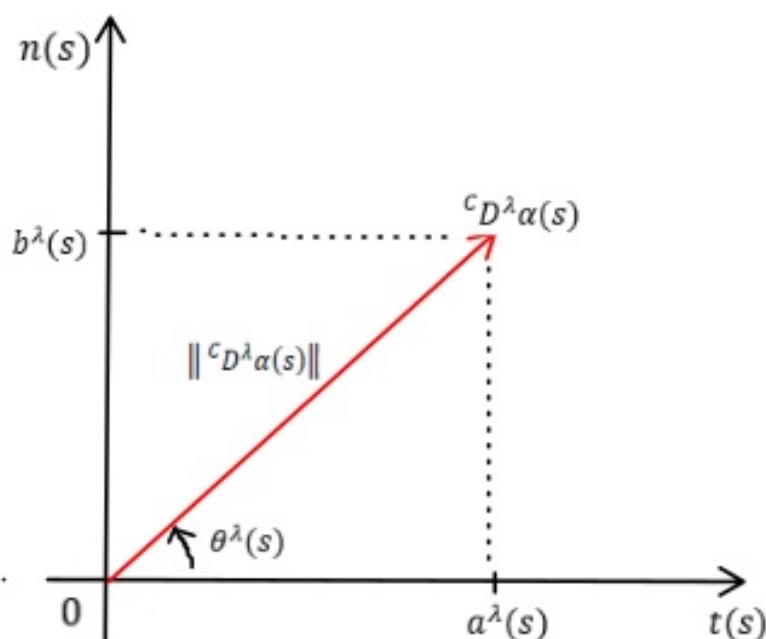


Figure 1. Coordinates of the Caputo fractional derivative vector ${}^c D^\lambda \alpha(s)$ in the Frenet-Serret frame.

By Figure 1, we have

$$\sin(\theta^\lambda(s)) = \frac{b^\lambda(s)}{\|{}^c D^\lambda \alpha(s)\|}, \quad \text{then } b^\lambda(s) = \|{}^c D^\lambda \alpha(s)\| \sin(\theta^\lambda(s)), \quad (3.6)$$

$$\cos(\theta^\lambda(s)) = \frac{a^\lambda(s)}{\|{}^c D^\lambda \alpha(s)\|}, \quad \text{then } a^\lambda(s) = \|{}^c D^\lambda \alpha(s)\| \cos(\theta^\lambda(s)). \quad (3.7)$$

Using (3.6) and (3.7) in the matrix $A^\lambda(s)$ we obtain

$$A^\lambda(s) = \begin{pmatrix} \|{}^c D^\lambda \alpha(s)\| \cos(\theta^\lambda(s)) & -\|{}^c D^\lambda \alpha(s)\| \sin(\theta^\lambda(s)) \\ \|{}^c D^\lambda \alpha(s)\| \sin(\theta^\lambda(s)) & \|{}^c D^\lambda \alpha(s)\| \cos(\theta^\lambda(s)) \end{pmatrix} = \|{}^c D^\lambda \alpha(s)\| \begin{pmatrix} \cos(\theta^\lambda(s)) & -\sin(\theta^\lambda(s)) \\ \sin(\theta^\lambda(s)) & \cos(\theta^\lambda(s)) \end{pmatrix}$$

$$A^\lambda(s) = \|{}^c D^\lambda \alpha(s)\| R^\lambda(s), \quad (3.8)$$

where

$$R^\lambda(s) = \begin{pmatrix} \cos(\theta^\lambda(s)) & -\sin(\theta^\lambda(s)) \\ \sin(\theta^\lambda(s)) & \cos(\theta^\lambda(s)) \end{pmatrix}.$$

Substituting (3.8) into (3.1), we obtain

$${}^c D^\lambda \alpha(s) = \|{}^c D^\lambda \alpha(s)\| R^\lambda(s) t(s).$$

□

Theorem 8. Let $\alpha : [0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length. Then,

$${}^c D^\lambda y(s) = \frac{x'(s) \sin(\theta^\lambda(s)) + y'(s) \cos(\theta^\lambda(s))}{x'(s) \cos(\theta^\lambda(s)) - y'(s) \sin(\theta^\lambda(s))} {}^c D^\lambda x(s). \quad (3.9)$$

Proof. From (3.4), we get

$${}^c D^\lambda \alpha(s) = \begin{pmatrix} {}^c D^\lambda x(s) \\ {}^c D^\lambda y(s) \end{pmatrix} = \|{}^c D^\lambda \alpha(s)\| \begin{pmatrix} \cos(\theta^\lambda(s)) & -\sin(\theta^\lambda(s)) \\ \sin(\theta^\lambda(s)) & \cos(\theta^\lambda(s)) \end{pmatrix} \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix}.$$

Then,

$${}^c D^\lambda x(s) = \|{}^c D^\lambda \alpha(s)\| (x'(s) \cos(\theta^\lambda(s)) - y'(s) \sin(\theta^\lambda(s))), \quad (3.10)$$

$${}^c D^\lambda y(s) = \|{}^c D^\lambda \alpha(s)\| (x'(s) \sin(\theta^\lambda(s)) + y'(s) \cos(\theta^\lambda(s))). \quad (3.11)$$

From (3.10) and (3.11), we get

$$\frac{{}^c D^\lambda y(s)}{x'(s) \sin(\theta^\lambda(s)) + y'(s) \cos(\theta^\lambda(s))} = \frac{{}^c D^\lambda x(s)}{x'(s) \cos(\theta^\lambda(s)) - y'(s) \sin(\theta^\lambda(s))}.$$

Therefore,

$${}^c D^\lambda y(s) = \frac{x'(s) \sin(\theta^\lambda(s)) + y'(s) \cos(\theta^\lambda(s))}{x'(s) \cos(\theta^\lambda(s)) - y'(s) \sin(\theta^\lambda(s))} {}^c D^\lambda x(s).$$

□

Theorem 9. Let $\alpha : [0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length. Then,

$${}^c D^\lambda y(s) = \tan(\theta^\lambda(s) + \phi(s)) {}^c D^\lambda x(s), \quad (3.12)$$

where $\phi(s) = \angle(e_1, t(s))$.

Proof. Let $\phi(s) = \angle(e_1, t(s))$ be the angle determined by vectors $e_1 = (1, 0)$ and $t(s)$. (See Figure 2)

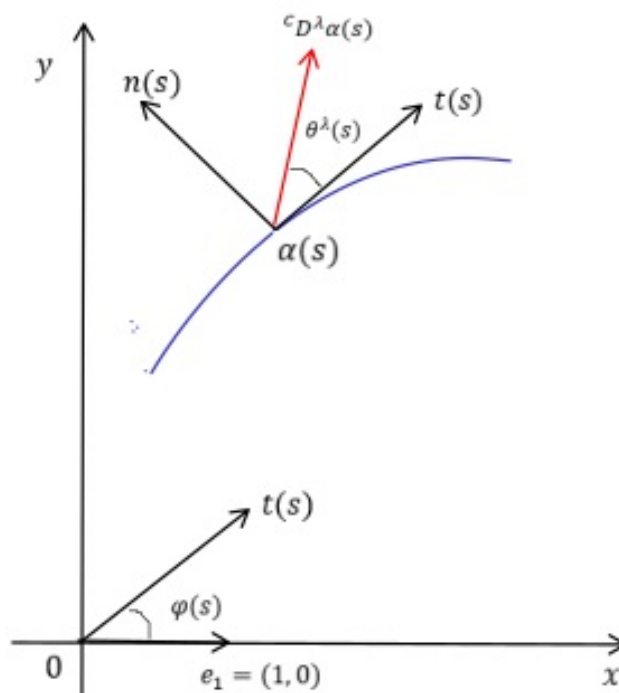


Figure 2. The angles $\theta^\lambda(s) = \angle(t(s), {}^c D^\lambda \alpha(s))$ and $\phi(s) = \angle(e_1, t(s))$ in the bases $\{t(s), n(s)\}$ and $\{e_1, e_2\}$, respectively.

Consider the following rotation matrix:

$$A(s) = \begin{pmatrix} \cos(\phi(s)) & -\sin(\phi(s)) \\ \sin(\phi(s)) & \cos(\phi(s)) \end{pmatrix}. \quad (3.13)$$

Then,

$$t(s) = A(s)e_1. \quad (3.14)$$

Using (3.14) in (3.4), we get

$${}^c D^\lambda \alpha(s) = \|{}^c D^\lambda \alpha(s)\| R^\lambda(s) A(s) e_1.$$

$$\begin{aligned} {}^c D^\lambda \alpha(s) &= \|{}^c D^\lambda \alpha(s)\| \begin{pmatrix} \cos(\theta^\lambda(s)) & -\sin(\theta^\lambda(s)) \\ \sin(\theta^\lambda(s)) & \cos(\theta^\lambda(s)) \end{pmatrix} \begin{pmatrix} \cos(\phi(s)) & -\sin(\phi(s)) \\ \sin(\phi(s)) & \cos(\phi(s)) \end{pmatrix} e_1 \\ &= \|{}^c D^\lambda \alpha(s)\| \begin{pmatrix} \cos(\theta^\lambda(s) + \phi(s)) & -\sin(\theta^\lambda(s) + \phi(s)) \\ \sin(\theta^\lambda(s) + \phi(s)) & \cos(\theta^\lambda(s) + \phi(s)) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \|{}^c D^\lambda \alpha(s)\| \begin{pmatrix} \cos(\theta^\lambda(s) + \phi(s)) \\ \sin(\theta^\lambda(s) + \phi(s)) \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{cases} {}^c D^\lambda x(s) = \|{}^c D^\lambda \alpha(s)\| \cos(\theta^\lambda(s) + \phi(s)), \\ {}^c D^\lambda y(s) = \|{}^c D^\lambda \alpha(s)\| \sin(\theta^\lambda(s) + \phi(s)). \end{cases} \quad (3.15)$$

From (3.15), we get

$$\frac{{}^c D^\lambda y(s)}{{}^c D^\lambda x(s)} = \frac{\|{}^c D^\lambda \alpha(s)\| \sin(\theta^\lambda(s) + \phi(s))}{\|{}^c D^\lambda \alpha(s)\| \cos(\theta^\lambda(s) + \phi(s))} = \frac{\sin(\theta^\lambda(s) + \phi(s))}{\cos(\theta^\lambda(s) + \phi(s))} = \tan(\theta^\lambda(s) + \phi(s)).$$

Hence,

$${}^c D^\lambda y(s) = \tan(\theta^\lambda(s) + \phi(s)) {}^c D^\lambda x(s).$$

□

Now, consider the straight line passing through the point $\alpha(s)$ in the direction of the vector ${}^c D^\lambda \alpha(s)$. (See Figure 3)

$$L_s^\lambda = \left\{ \alpha(s) + t {}^c D^\lambda \alpha(s) \mid t \in \mathbb{R} \right\}.$$

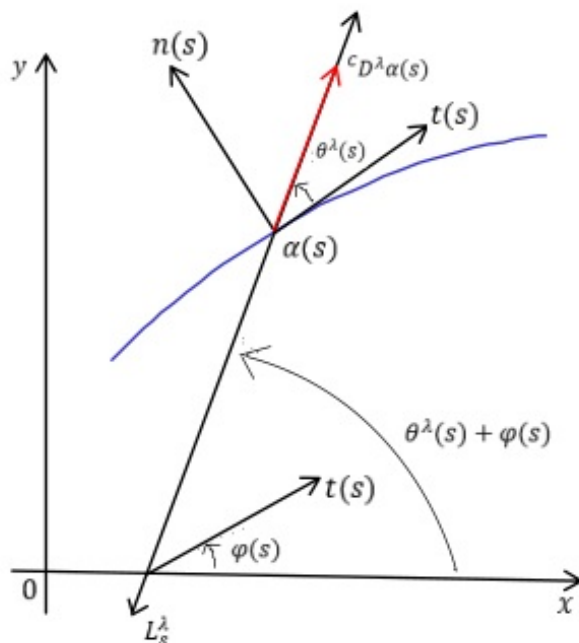


Figure 3. The straight line L_s^λ passing through $\alpha(s)$ in the direction of vector ${}^cD^\lambda\alpha(s)$.

Let m_s^λ be the slope of L_s^λ ; then

$$m_s^\lambda = \tan(\theta^\lambda(s) + \phi(s)) = \frac{{}^cD^\lambda y(s)}{{}^cD^\lambda x(s)}. \tag{3.16}$$

Remark 1. 1) For an arbitrary parameter t , the theory is analogous for a regular curve parameterized $\alpha(t) = (x(t), y(t))$, $t \in [0, a]$.

2) Let $f : [0, a] \rightarrow \mathbb{R}$ be a function of class $C^\infty([0, a])$, and consider the regular curve $\alpha(t) = (t, f(t))$, $t \in [0, a]$. According to Rubio-López et al. [19, 20],

$$\begin{aligned} {}^cD^\lambda x(t) &= \frac{1}{\Gamma(1-\lambda)} \int_0^t \frac{1}{(t-u)^\lambda} du = \frac{1}{\Gamma(1-\lambda)} \lim_{r \rightarrow 0} \int_0^{t-r} \frac{1}{(t-u)^\lambda} du = -\frac{1}{\Gamma(1-\lambda)} \lim_{r \rightarrow 0} \int_0^{t-r} (t-u)^{-\lambda} d(t-u) \\ &= -\frac{1}{(1-\lambda)\Gamma(1-\lambda)} \lim_{r \rightarrow 0} [r^{1-\lambda} - t^{1-\lambda}] = \frac{1}{(1-\lambda)\Gamma(1-\lambda)} t^{1-\lambda} = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}; \text{ then,} \end{aligned}$$

$${}^cD^\lambda \alpha(t) = ({}^cD^\lambda(t), {}^cD^\lambda f(t)) = \left(\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}, {}^cD^\lambda f(t) \right).$$

Therefore, by (3.12), we get

$${}^cD^\lambda f(t) = \tan(\theta^\lambda(s) + \phi(s)) \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}. \tag{3.17}$$

The following theorem gives the conditions for proving that fractional curvature is a differentiable function.

Theorem 10. Let $\alpha : [0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length, such that ${}^cD^\lambda x(s)$ and ${}^cD^\lambda y(s)$ are differentiable functions, and $k^\lambda(s)$ is the fractional curvature of order λ , where $0 < \lambda < 1$. For each $s_0 \in (0, a)$, consider $\alpha(s_0)$, $e^\lambda(s_0) = \frac{{}^cD^\lambda \alpha(s_0)}{\|{}^cD^\lambda \alpha(s_0)\|}$. Then, there exists a unique regular curve $\beta_0 : (0, a) \rightarrow \mathbb{R}^2$ parameterized by arc length, such that

$$k_{\beta_0}(s) = k^\lambda(s), \quad \beta_0(s_0) = \alpha(s_0), \quad \beta_0'(s_0) = e^\lambda(s_0), \tag{3.18}$$

where $k_{\beta_0}(s)$ is the integer or classic curvature of β_0 at s .

Proof. Since $\alpha(s) = (x(s), y(s))$ is differentiable and by hypothesis, ${}^c D^\lambda x(s)$ and ${}^c D^\lambda y(s)$ are differentiable functions, we have

$$a^\lambda(s) = x'(s){}^c D^\lambda x(s) + y'(s){}^c D^\lambda y(s), \quad b^\lambda(s) = x'(s){}^c D^\lambda y(s) - y'(s){}^c D^\lambda x(s)$$

are differentiable functions. Therefore, $k(s)$ is a differentiable function.

Let $s_0 \in (0, a]$, and consider

$$\alpha(s_0) = (x(s_0), y(s_0)), \quad e^\lambda(s_0) = \frac{{}^c D^\lambda \alpha(s_0)}{\|{}^c D^\lambda \alpha(s_0)\|} = (w_1, w_2) \in \mathbb{R}^2. \quad (3.19)$$

By the fundamental theorem of plane curves 1, with the initial conditions given in (3.19), there exists a unique regular curve $\beta_0(s) = (u(s), v(s))$ that is the solution to the initial value problem

$$\left\{ \begin{array}{l} u''(s) = -k^\lambda(s)v'(s), \\ v''(s) = k^\lambda(s)u'(s), \\ u(s_0) = x(s_0) \quad , \quad v(s_0) = y(s_0), \\ u'(s_0) = w_1 \quad , \quad v'(s_0) = w_2, \end{array} \right. \quad (3.20)$$

such that $k_{\beta_0}(s) = k^\lambda(s)$, $\forall s \in (0, a]$, (see Figure 4)

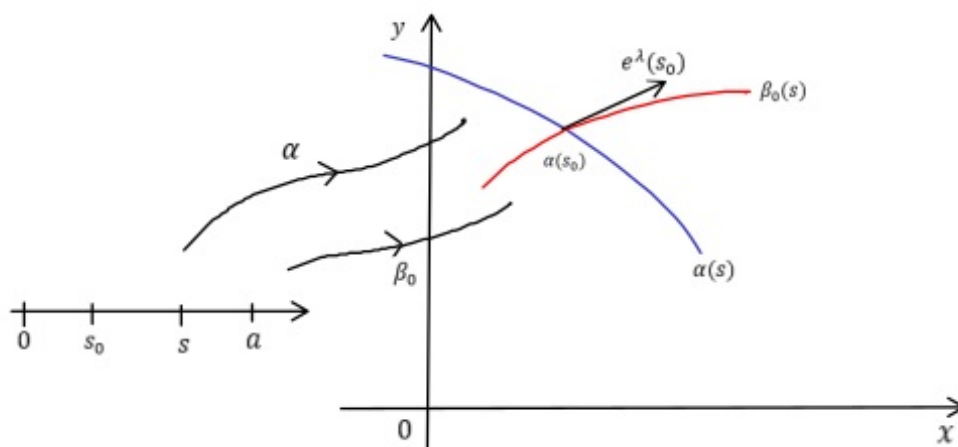


Figure 4. Curve $\beta_0(s)$ tangent to vector $e^\lambda(s_0)$ at s_0 .

□

The following theorem allows us to consider the fractional derivative vector field ${}^c D^\lambda \alpha(s)$ along the curve α as the tangent vector field along the curve $\beta_0(s)$, the solution of the IVP (3.20).

Theorem 11. Let $\alpha : [0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length, $k^\lambda(s)$ its fractional curvature of order λ , $0 < \lambda < 1$, $s_0 \in (0, a]$, and $\beta_0 : (0, a] \rightarrow \mathbb{R}^2$ the regular curve parameterized by arc length given by theorem 10.

Then, $\forall s \in (0, a]$, there exists a curve $\beta : (0, a] \rightarrow \mathbb{R}^2$ parameterized by arc length given by theorem 10, and an isometry $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\beta(s) = (F \circ \beta_0)(s), \quad \forall s \in (0, a]. \quad (3.21)$$

In addition, there is

$$e^\lambda(s) = \beta'(s) = dF_{\beta_0(s)}(\beta_0'(s)), \quad \forall s \in (0, a]. \quad (3.22)$$

Proof. For each $\bar{s} \in (0, a]$, by Theorem 10, there exists a curve $\beta : (0, a] \rightarrow \mathbb{R}^2$ such that

$$\beta(\bar{s}) = \alpha(\bar{s}), \quad \beta'(\bar{s}) = e^\lambda(\bar{s}) = \frac{{}^c D^\lambda \alpha(\bar{s})}{\|{}^c D^\lambda \alpha(\bar{s})\|}, \quad (3.23)$$

and

$$k_{\beta_0}(s) = k^\lambda(s) = k_\beta(s), \quad s \in (0, a].$$

Then, by the fundamental theorem of plane curves 1, there exists a rotation L and a translation T such that $F = L \circ T$ is an isometry, and $\beta(s) = (F \circ \beta_0)(s)$, $\forall s \in (0, a]$. Hence,

$$e^\lambda(s) = \beta'(s) = dF_{\beta_0(s)}(\beta_0'(s)).$$

□

Theorem 12. Let $s_0 \in (0, a)$, $\beta_0 : (0, a] \rightarrow \mathbb{R}^2$ be a regular curve parameterized by arc length given by theorem 10. Then, there exists a neighborhood $V_{s_0} \subset [0, a]$ of s_0 , such that $\beta_0(V_{s_0})$ is the graph of a differentiable function that has one of the forms: $v = \varphi(u)$ or $u = \psi(v)$.

Furthermore, we have

$$\frac{{}^c D^\lambda y(s_0)}{{}^c D^\lambda x(s_0)} = \varphi'(u_0) = \varphi'(u(s_0)) \quad \acute{o} \quad \frac{{}^c D^\lambda x(s_0)}{{}^c D^\lambda y(s_0)} = \psi'(v_0) = \psi'(v(s_0)). \quad (3.24)$$

Proof. The proof of this theorem is analogous to the proof of proposition 3, chapter 2, on regular surfaces in [28].

Since $\beta_0(s) = (u(s), v(s))$ is a regular curve, then we have

$$\beta_0'(s_0) = (u'(s_0), v'(s_0)) \neq 0.$$

Now, assume that $u'(s_0) \neq 0$, and consider the map $\pi \circ \beta_0 : [0, a] \rightarrow \mathbb{R}$, where $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection map onto the first component.

Then $(\pi \circ \beta_0)(s) = u(s)$, and since $u'(s_0) \neq 0$, by the inverse map theorem, there exist neighborhoods V_1 of s_0 and V_2 of $(\pi \circ \beta_0)(s_0) = u(s_0)$, such that $\pi \circ \beta_0 : V_1 \rightarrow V_2$ is a diffeomorphism; furthermore $(\pi \circ \beta_0)^{-1} : V_2 \rightarrow V_1$ is a diffeomorphism, and $(\pi \circ \beta_0)^{-1}(u) = s(u)$.

Then,

$$v \circ (\pi \circ \beta_0)^{-1}(u) = v(s(u)) = \varphi(u). \quad (3.25)$$

From (3.25), it is concluded $v = \varphi(u)$.

By Theorem 11, we have that $e^\lambda(s_0)$ is tangent to the graph of φ at s_0 . Hence,

$$\varphi'(u_0) = \varphi'(u(s_0)) = \frac{{}^c D^\lambda y(s_0)}{{}^c D^\lambda x(s_0)}.$$

Similarly, the other case can be proved. □

4. Relation with integer and fractional order initial value problems

In this section, we relate our results to fractional-order initial value problems.

Consider the fractional initial value problem (2.10):

$$\begin{cases} {}^c D^\lambda y(t) = f(t, y(t)), & t > 0, 0 < \lambda < 1, \\ y(0) = y_0, \end{cases}$$

where $\lambda \in (0, 1)$, $t \in [0, a]$, and $a > 0$.

Now, suppose that f satisfies the hypothesis of Theorems 4 and 5; then, there exists a unique solution $y : [0, T] \rightarrow \mathbb{R}$ of (2.10), where

$$T = \min \left\{ \chi^*, \left(\frac{\delta \Gamma(\lambda + 1)}{\|f\|_\infty} \right)^{\frac{1}{\lambda}} \right\}.$$

Using the solution $y(t)$ of the fractional initial value problem (2.10), we obtain the curve $\alpha(t) = (x(t), y(t)) = (t, y(t))$, whose fractional derivative vector is given by:

$${}^c D^\lambda \alpha(t) = ({}^c D^\lambda t, {}^c D^\lambda y(t)) = \left(\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}, {}^c D^\lambda y(t) \right).$$

Using the Theorems 11 and 12, we obtain the integer-order initial value problem given by

$$\begin{cases} z'(t) = \frac{\Gamma(2-\lambda)f(t, y(t))}{t^{1-\lambda}}, & t > 0, 0 < \lambda < 1, \\ z(0) = y_0, \end{cases} \quad (4.1)$$

where the function $z : [0, a] \rightarrow \mathbb{R}$ is the solution of (4.1).

Given $\varepsilon > 0$, consider the set

$$M_\varepsilon = [\varepsilon, T] \times [y_0 - \delta, y_0 + \delta], \quad \delta > 0, T > 0. \quad (4.2)$$

Theorem 13. Let $F : M_\varepsilon \rightarrow \mathbb{R}$ be defined by

$$F(t, y) = \frac{\Gamma(2-\lambda)f(t, y)}{t^{1-\lambda}}, \quad (4.3)$$

where the function f satisfies the hypothesis of Theorems 4 and 5. Then,

- 1) F is a bounded function on M_ε .
- 2) F is a Lipschitz function with respect to the second variable.
- 3) F is a continuous function on M_ε .

Proof. (1) We have $\varepsilon \leq t \leq T$, so then

$$\varepsilon^{1-\lambda} \leq t^{1-\lambda} \leq T^{1-\lambda}. \quad (4.4)$$

From (4.4), we obtain

$$\frac{1}{T^{1-\lambda}} \leq \frac{1}{t^{1-\lambda}} \leq \frac{1}{\varepsilon^{1-\lambda}}. \quad (4.5)$$

Moreover,

$$|F(t, y)| = \left| \frac{\Gamma(2-\lambda)f(t, y)}{t^{1-\lambda}} \right| = \frac{\Gamma(2-\lambda)}{t^{1-\lambda}} |f(t, y)|. \quad (4.6)$$

By hypothesis, f is bounded; therefore, exists $C \in \mathbb{R}$ such that

$$|f(t, y)| \leq C, \quad \forall (t, y) \in M_\varepsilon. \quad (4.7)$$

Using (4.5) and (4.7) in (4.6), we get

$$|F(t, y)| \leq \frac{\Gamma(2-\lambda)}{\varepsilon^{1-\lambda}} C. \quad (4.8)$$

Since $\Gamma(1) = \Gamma(2) = 1$, and the gamma function is log-convex on the interval $(1, 2)$, we have $\Gamma(2-\lambda) \leq 1$, $\forall \lambda \in (0, 1)$.

Therefore,

$$|F(t, y)| \leq \frac{C}{\varepsilon^{1-\lambda}}. \quad (4.9)$$

(2) Let $(t, y_1), (t, y_2) \in M_\varepsilon$, then,

$$\begin{aligned} |F(t, y_1) - F(t, y_2)| &\leq \frac{\Gamma(2-\lambda)}{t^{1-\lambda}} |f(t, y_1) - f(t, y_2)| \\ &\leq \frac{\Gamma(2-\lambda)}{t^{1-\lambda}} L |y_1 - y_2| \\ &\leq L_\varepsilon |y_1 - y_2|, \quad \forall (t, y_1), (t, y_2) \in M_\varepsilon, \end{aligned}$$

where $L_\varepsilon = \frac{L}{\varepsilon^{1-\lambda}}$.

(3) Since $f(t, y)$ is continuous, it follows immediately that $F(t, y)$ is continuous on M_ε . \square

Theorem 14. Let $g : (0, T] \rightarrow \mathbb{R}$ be a function defined by $g(t) = F(t, y(t))$. Then,

$$g \in L^1((0, T]). \quad (4.10)$$

Proof.

$$\begin{aligned} \int_0^T |g(t)| dt &= \int_0^T |F(t, y(t))| dt = \int_0^T \left| \frac{\Gamma(2-\lambda)f(t, y(t))}{t^{1-\lambda}} \right| dt \\ &\leq \Gamma(2-\lambda) C \int_0^T t^{\lambda-1} dt, \quad \text{since } f \text{ bounded.} \\ &= \frac{\Gamma(2-\lambda) C}{\lambda} T^\lambda < \infty. \end{aligned}$$

Hence, $g \in L^1((0, T])$. \square

Theorem 15. Let $g : (0, T] \rightarrow \mathbb{R}$ be a function defined by $g(t) = F(t, y(t))$. Then,

- 1) $G(t) = \int_0^t g(\tau) d\tau$ is a continuous function on $[0, T]$.
- 2) $G'(t) = g(t)$ a.e in $[0, T]$.

Proof. (1) By Theorem 14, $g \in L^1((0, T])$. Then using Theorem 4, we obtain that

$$G(t) = \int_0^t g(\tau) d\tau$$

is a continuous function on $[0, T]$.

(2) By Theorem 5, we have that $G'(t) = g(t)$ a.e in $[0, T]$. □

Using Theorem 15, in the IVP (4.1), we get

$$\begin{cases} z'(t) = \frac{\Gamma(2-\lambda)f(t, y(t))}{t^{1-\lambda}} = g(t), & t > 0, 0 < \lambda < 1, \\ z(0) = y_0. \end{cases}$$

It follows that the solution of this initial value problem

$$z(t) = y_0 + \int_0^t \frac{\Gamma(2-\lambda)f(\tau, y(\tau))}{\tau^{1-\lambda}} d\tau \quad (4.11)$$

is a continuous function on $[0, T]$.

Let $y = y(t)$ and $z = z(t)$ be the solutions of the initial value problems (2.10) and (4.1), respectively, and consider the curves:

$$\alpha(t) = (t, y(t)), \quad t \in [0, T], \quad (4.12)$$

$$\beta(t) = (t, z(t)), \quad t \in [0, T]. \quad (4.13)$$

Theorem 16. Let $y = y(t)$ and $z = z(t)$ be the solutions of the initial value problems (2.10) and (4.1), respectively. Then,

$${}^c D^\lambda \alpha(t) = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} \beta'(t), \quad \forall t > 0. \quad (4.14)$$

Proof. Using (4.1) in (4.13), we obtain

$$\beta'(t) = (1, z'(t)) = \left(1, \frac{\Gamma(2-\lambda)f(t, y(t))}{t^{1-\lambda}}\right), \quad t > 0.$$

Then,

$$t^{1-\lambda} \beta'(t) = (t^{1-\lambda}, \Gamma(2-\lambda)f(t, y(t))),$$

and

$$\frac{t^{1-\lambda}}{\Gamma(2-\lambda)} \beta'(t) = \left(\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}, f(t, y(t))\right) = \left(\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}, {}^c D^\lambda y(t)\right).$$

Hence,

$${}^c D^\lambda \alpha(t) = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} \beta'(t).$$

□

Note that, for $t > 0$, Theorem 16 shows that the vector fields ${}^cD^\lambda \alpha(t)$ and $\beta'(t)$ along the curves $\alpha(t)$ and $\beta(t)$, respectively, are parallel (see Figure 5).

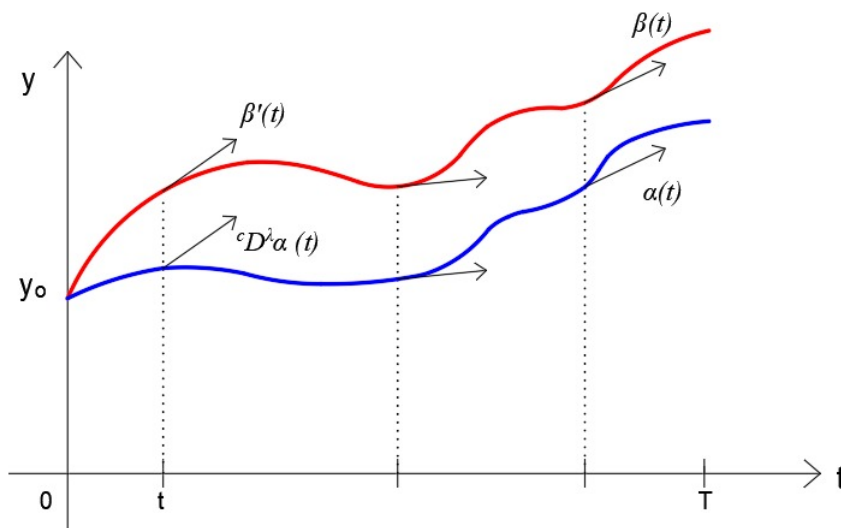


Figure 5. Vector fields along the curves $\alpha(t)$ and $\beta(t)$, respectively.

Now, consider the integer-order initial value problem:

$$\begin{cases} w'(t) = h(t, w(t)), & t > 0, \\ w(0) = w_0, \end{cases} \quad (4.15)$$

where $h : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function defined on the domain U , continuous, Lipschitz with respect to the second variable, and $(0, w_0) \in U$. The Picard–Lindelöf theorem [29], guarantees the existence and uniqueness of the solution $w(t)$ on the interval $[0, T^*]$.

Multiplying Eq (4.15) by $\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}$:

$$w'(t) \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} h(t, w(t)). \quad (4.16)$$

Using (3.24), we have

$${}^cD^\lambda y(t) = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} h(t, w(t)), \quad t > 0.$$

Therefore, we obtain a fractional-order initial value problem:

$$\begin{cases} {}^cD^\lambda y(t) = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} h(t, w(t)), & t > 0, \\ y(0) = w_0. \end{cases} \quad (4.17)$$

Let $G(t, w) = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}h(t, w)$. It is verified that G is a continuous function and satisfies the Lipschitz condition. Therefore, (4.17) has a unique solution.

Let the curves be

$$\begin{aligned}\alpha(t) &= (t, y(t)), \quad t \in [0, T^*], \\ \beta(t) &= (t, w(t)), \quad t \in [0, T^*],\end{aligned}$$

where $y = y(t)$ and $w = w(t)$ are the solutions of the initial value problems (4.17) and (4.15), respectively. Analogous to the previous case, the vector fields ${}^cD^\lambda\alpha(t)$ and $\beta'(t)$, along the curves $\alpha(t)$ and $\beta(t)$, respectively, are parallel.

5. Numerical examples

Example 1. Consider the unit circle

$$\alpha(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]. \quad (5.1)$$

Then

$${}^cD^\lambda\alpha(t) = ({}^cD^\lambda(\cos t), {}^cD^\lambda(\sin t)) = (S_2(t), S_1(t)), \quad (5.2)$$

where

$$S_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1-\lambda}}{\Gamma(2k+2-\lambda)}, \quad S_2(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} t^{2k+2-\lambda}}{\Gamma(2k+3-\lambda)}.$$

For $t = \frac{\pi}{4}$ consider the following

$$\alpha\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad {}^cD^\lambda\alpha\left(\frac{\pi}{4}\right), \quad \text{and the unit vector } v = (v_1, v_2) = \frac{{}^cD^\lambda\alpha\left(\frac{\pi}{4}\right)}{\|{}^cD^\lambda\alpha\left(\frac{\pi}{4}\right)\|}. \quad (5.3)$$

By Rubio-López et al. [19, 20], the fractional curvature of the curve α at t is

$$k^\lambda(t) = -\sin(t)S'_1(t) - \cos(t)S'_2(t), \quad (5.4)$$

where $S'_1(t)$ and $S'_2(t)$ are the integer-order derivatives of $S_1(t)$ and $S_2(t)$, respectively, given by

$$S'_1(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k-\lambda}}{\Gamma(2k+1-\lambda)}, \quad S'_2(t) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{t^{2k+1-\lambda}}{\Gamma(2k+2-\lambda)}.$$

Using the fundamental theorem of plane curves 1, (5.3), and (5.4), we obtain the initial value problem:

$$\begin{cases} x''(t) = -k^\lambda(t)y'(t), \\ y''(t) = k^\lambda(t)x'(t), \\ x\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \\ x'\left(\frac{\pi}{4}\right) = v_1, \quad y'\left(\frac{\pi}{4}\right) = v_2. \end{cases} \quad (5.5)$$

Using the fourth-order Runge–Kutta method for systems of ordinary differential equations of integer order, numerical solutions of the IVP (5.5) are obtained for different values of λ .

In Figure 6(a), the numerical solution and the field of its tangent vectors (shown in blue) of the initial value problem (5.5) are plotted, for $\lambda = 0.5$, $\alpha\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, and the direction given by the vector $v = (v_1, v_2) = \frac{{}^c D^\lambda \alpha\left(\frac{\pi}{4}\right)}{\left\| {}^c D^\lambda \alpha\left(\frac{\pi}{4}\right) \right\|}$. The curve $\alpha(t)$ and the vector field given by the Caputo fractional derivative ${}^c D^\lambda \alpha(t)$, along of the curve $\alpha(t)$ (shown in red), are also plotted. In Figure 6(b), analogous to the previous case, the plots correspond to $\lambda = 0.75$.

As can be seen in Figures 6(a) and 6(b), the fractional derivative vector of Caputo ${}^c D^\lambda \alpha\left(\frac{\pi}{4}\right)$ is tangent to the curve $\beta(t) = (x(t), y(t))$, solution of the IVP (5.5); at point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, the integer curvature $k_\beta(t)$ of the curve $\beta(t)$ coincides with the fractional curvature $k^\lambda(t)$.

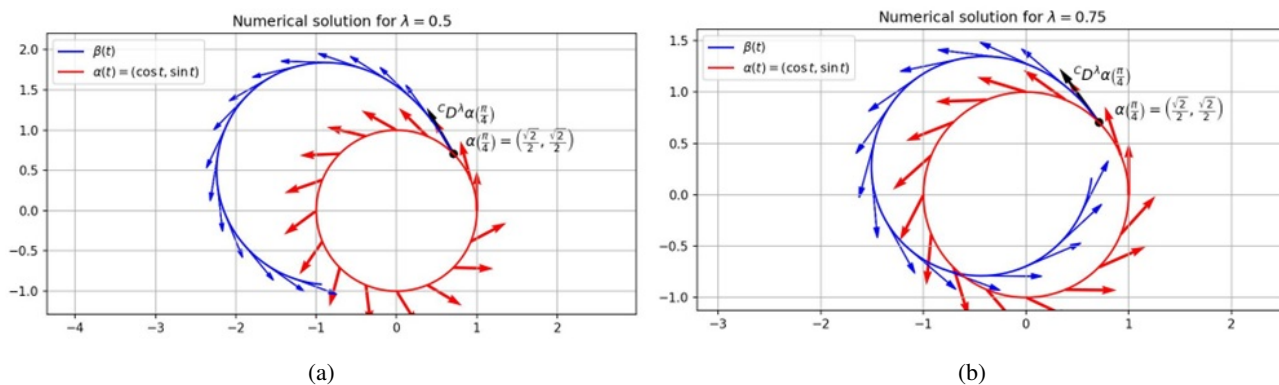


Figure 6. The red curves and vectors correspond to the curve $\alpha(t)$ and ${}^c D^\lambda \alpha(t)$, respectively, whereas the blue curves and vectors correspond to the curve $\beta(t)$ and $\beta'(t)$, respectively. (a) For $\lambda = 0.5$; (b) For $\lambda = 0.75$.

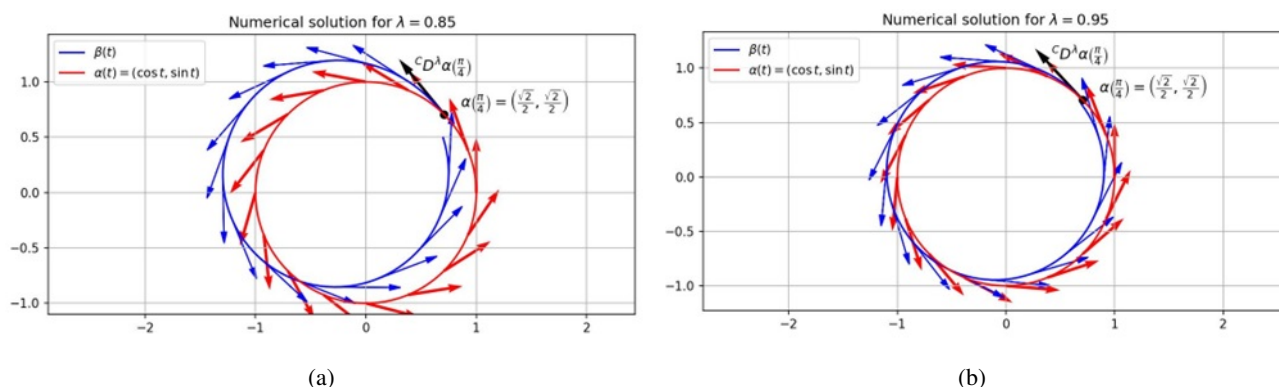


Figure 7. The red curves and vectors correspond to the curve $\alpha(t)$ and ${}^c D^\lambda \alpha(t)$, respectively, whereas the blue curves and vectors correspond to the curve $\beta(t)$ and $\beta'(t)$, respectively. (a) For $\lambda = 0.85$; (b) For $\lambda = 0.95$.

It is observed in Figure 7 that if λ approaches 1, then the curves solving the initial value problem

(5.5) approach the circumference.

Example 2. *Fractional viscoelastic Kelvin–Voigt model*

In viscoelastic materials, the Kelvin–Voigt model is given by the ordinary differential equation of integer order:

$$\sigma(t) = E_0 \varepsilon(t) + E_1 \frac{d\varepsilon(t)}{dt}, \quad (5.6)$$

where E_0 is the modulus of elasticity, E_1 is the viscosity coefficient, $\sigma(t)$ is the stress, and $\varepsilon(t)$ is the strain.

If the stress is a constant equal to σ_0 , the solution of the ordinary differential equation (5.6) is given by

$$\varepsilon(t) = \frac{\sigma_0}{E_0} \left[1 - e^{-\left(\frac{E_0}{E_1}\right)t} \right]. \quad (5.7)$$

Next, we present two methods for obtaining mathematical models using the Caputo fractional derivative.

Method 1.

In the study of viscoelastic materials, from the perspective of fractional calculus, it is common to replace the integer-order derivative in (5.6) by the fractional derivative in dimensionless units, thereby obtaining a fractional-order ordinary differential equation.

Alternatively, we use our results to obtain a fractional-order ordinary differential equation by employing the Caputo fractional derivative, thus formulating a fractional initial value problem.

From (5.6), we have

$$\frac{d\varepsilon(t)}{dt} = \frac{\sigma_0}{E_1} - \frac{E_0}{E_1} \varepsilon(t). \quad (5.8)$$

Using (3.24) and (5.8), we get

$${}^c D^\lambda y(t) = \frac{d\varepsilon(t)}{dt} {}^c D^\lambda x(t) = \frac{d\varepsilon(t)}{dt} {}^c D^\lambda(t) = \frac{d\varepsilon(t)}{dt} \frac{t^{1-\lambda}}{\Gamma(2-\lambda)} = \left(\frac{\sigma_0}{E_1} - \frac{E_0}{E_1} \varepsilon(t) \right) \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}.$$

Using the solution $\varepsilon(t)$ given in (5.7), we have

$${}^c D^\lambda y(t) = \frac{\sigma_0}{E_1} e^{-\frac{E_0}{E_1}t} \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}.$$

This allows us to consider the fractional-order initial value problem:

$$\begin{cases} {}^c D^\lambda y(t) = \frac{\sigma_0}{E_1} e^{-\frac{E_0}{E_1}t} \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}, & t > 0, 0 < \lambda < 1, \\ y(0) = 0. \end{cases} \quad (5.9)$$

In the literature, several numerical methods exist to obtain a numerical solution of the fractional-order initial value problem (5.9) [30, 31]. Here, we will use the method of Odibat et al. [30].

The interval $[0, 5]$ was considered, with $E_0 = E_1 = \sigma_0 = 1$ and $N = 100$ for the partition of the interval. For the parameter λ , the following values were used: $\lambda = 0.2$, $\lambda = 0.4$, $\lambda = 0.5$, and $\lambda = 0.6$; and the corresponding solutions are plotted in Figure 8.

Figure 8 shows that for these values of the parameter λ , the strain reaches a maximum and then decreases.

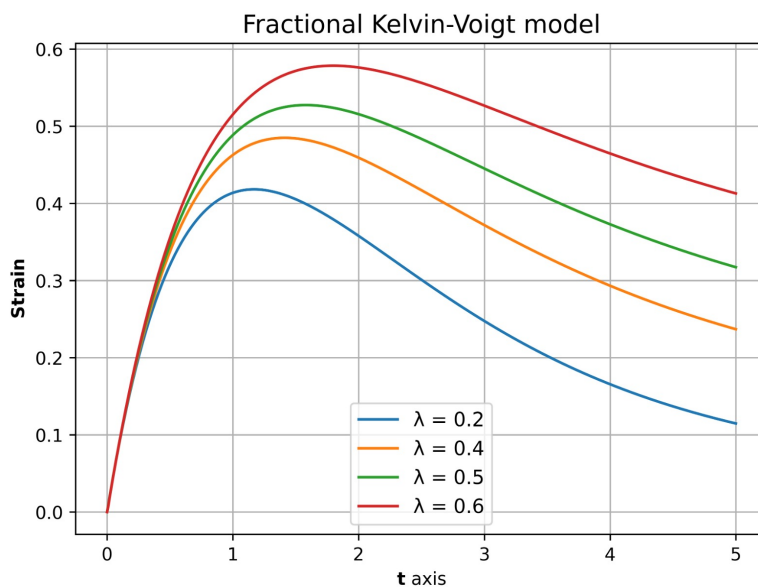


Figure 8. Graphs of the deformation for different values of λ .

Figure 9 shows the different solutions for $\lambda = 0.7$, $\lambda = 0.8$, $\lambda = 0.95$, and $\lambda = 1.0$; it is observed that as the parameter λ tends to 1, the solutions approach the integer-order case.

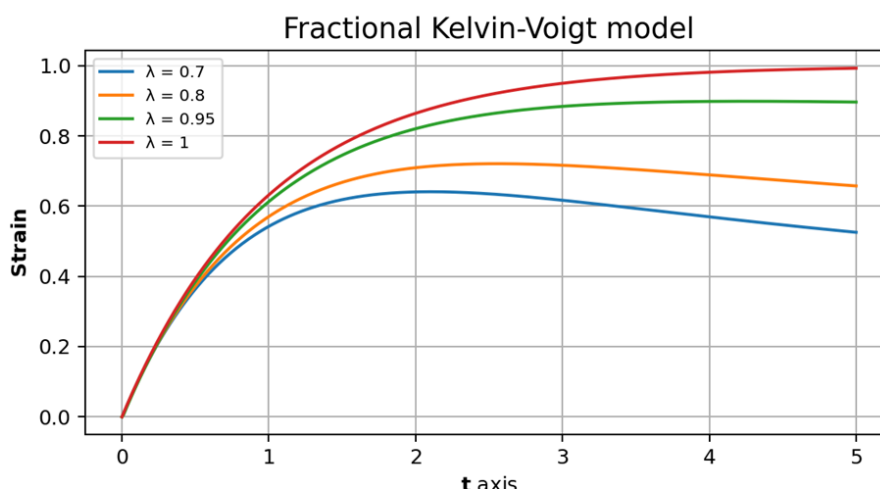


Figure 9. The deformations approach the integer case if λ approaches 1.

Figure 10 shows the solutions for $\lambda = 0.5$, $\lambda = 0.6$, $\lambda = 0.8$, and $\lambda = 0.9$, as well as the fractional derivative vectors of the FIVP (5.9) solutions for $t = 3$. As can be seen, the vectors are parallel to the tangent vector at $t = 3$, of the integer case.

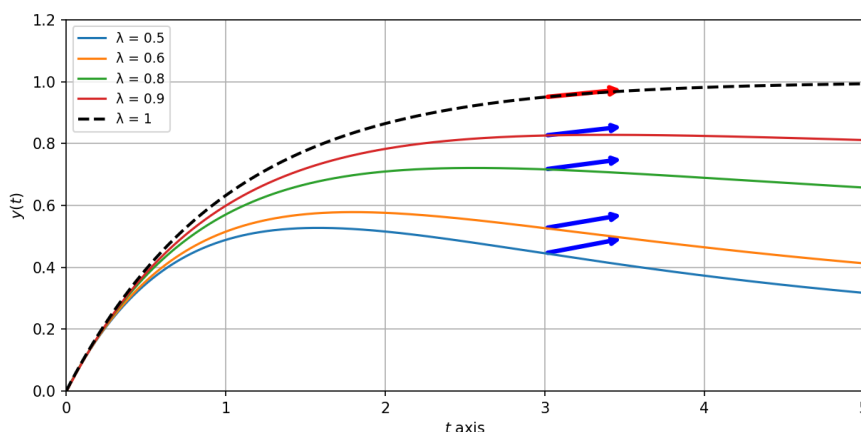


Figure 10. Fractional derivative vectors (blue) parallel to the tangent vector (red) at the point $t = 3$.

Method 2.

By replacing the integer-order derivative in (5.8) with the Caputo fractional derivative, we obtain a fractional-order ordinary differential equation.

$${}^c D^\lambda \varepsilon(t) = \frac{\sigma_0}{E_1} - \frac{E_0}{E_1} \varepsilon(t), \quad 0 < \lambda < 1. \quad (5.10)$$

Using (5.10), we obtain the following fractional-order initial value problem:

$$\begin{cases} {}^c D^\lambda \varepsilon(t) = \frac{\sigma_0}{E_1} - \frac{E_0}{E_1} \varepsilon(t), & t > 0, 0 < \lambda < 1, \\ \varepsilon(0) = 0. \end{cases} \quad (5.11)$$

The interval $[0, 2]$ was considered, with $E_0 = E_1 = \sigma_0 = 1$ and $N = 100$ for the partition of the interval. For the parameter λ , the following values were used: $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, and $\lambda = 0.95$; and the corresponding solutions are plotted in Figure 11.

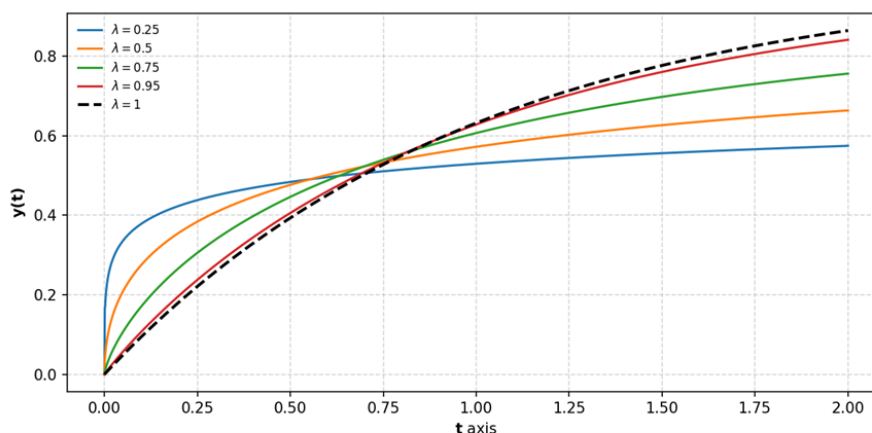


Figure 11. Graphs of the deformation for different values of the parameter λ .

Now, using (3.24), we get

$$z'(t) = \frac{{}^c D^\lambda \varepsilon(t)}{{}^c D^\lambda(t)} = \frac{{}^c D^\lambda \varepsilon(t)}{\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}} = \frac{\Gamma(2-\lambda)}{t^{1-\lambda}} {}^c D^\lambda \varepsilon(t),$$

and by FIVP (5.11), we obtain

$$z'(t) = \frac{\Gamma(2-\lambda)}{t^{1-\lambda}} \left(\frac{\sigma_0}{E_1} - \frac{E_0}{E_1} \varepsilon(t) \right).$$

We obtain an initial value problem of integer order:

$$\begin{cases} z'(t) = \frac{\Gamma(2-\lambda)}{t^{1-\lambda}} \left(\frac{\sigma_0}{E_1} - \frac{E_0}{E_1} \varepsilon(t) \right), & t > 0, \quad 0 < \lambda < 1, \\ z(0) = \varepsilon(0) = 0. \end{cases} \quad (5.12)$$

We use the solution of IVP (5.11) in IVP (5.12), and we obtain a numerical solution of IVP (5.12), using the predictor-corrector method.

The interval $[0, 2]$ was considered, with $E_0 = E_1 = \sigma_0 = 1$ and $N = 100$ for the partition of the interval. For the parameter λ , the following values were considered: $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, and $\lambda = 0.95$; in Figure 12, we plot the different solutions.

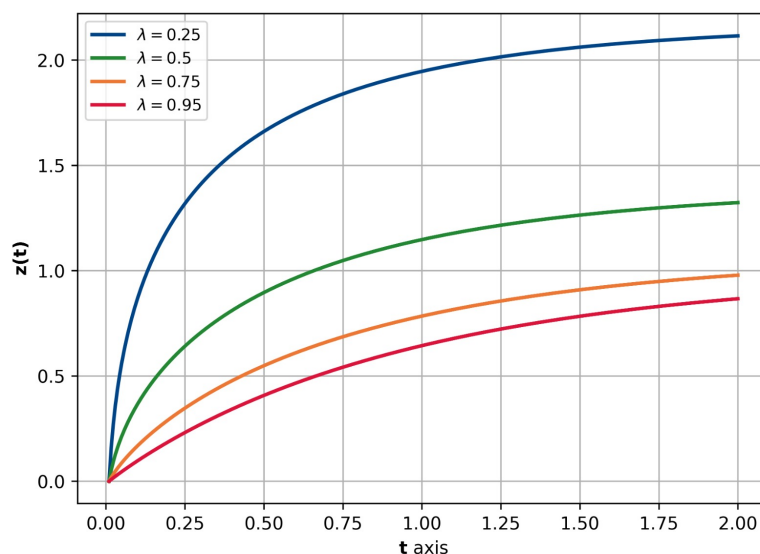


Figure 12. Behavior of the solutions of the FIVP for different values of λ .

Example 3. Fractional relaxation equation

We consider the integer-order ordinary differential equation that governs the relaxation phenomenon, as seen in Mainardi [32–34]. In dimensionless units, the initial value problem is given by:

$$\begin{cases} \frac{dz(t)}{dt} = -z(t), & t > 0, \\ z(0^+) = 1. \end{cases} \quad (5.13)$$

The solution of IVP (5.13) is given by

$$z(t) = e^{-t}. \quad (5.14)$$

Mainardi [32–34] considers a new problem, replacing the integer derivative in (5.13) by the Caputo fractional derivative, and obtaining a fractional-order initial value problem for the fractional relaxation process:

$$\begin{cases} {}^c D^\lambda z(t) = -z(t), & t > 0, 0 < \lambda < 1, \\ z(0^+) = 1. \end{cases} \quad (5.15)$$

By applying the Laplace transform, the solution to the fractional initial value problem (5.15) can be expressed in terms of the Mittag–Leffler function:

$$z(t) = E_\lambda(-t^\lambda), \quad (5.16)$$

where

$$E_\lambda(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\lambda k + 1)}.$$

Figure 13 shows the plots of the solution of the fractional initial value problem (5.15) for different values of the parameter $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, and $\lambda = 1.0$.

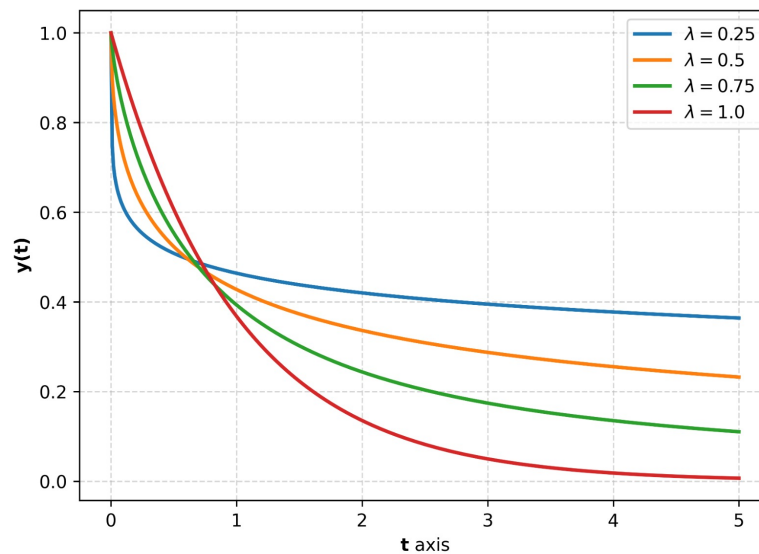


Figure 13. Plots of the Mittag-Leffler function $E_\lambda(-t^\lambda)$ for $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, and $\lambda = 1.0$.

Now, using our result (3.24) and Eqs (5.13) and (5.14), the following fractional-order initial value problem is obtained:

$$\begin{cases} {}^c D^\lambda y(t) = -\frac{t^{1-\lambda} e^{-t}}{\Gamma(2-\lambda)}, & t > 0, 0 < \lambda < 1, \\ y(0^+) = 1. \end{cases} \quad (5.17)$$

Using Odibat's fractional modified trapezoidal numerical method [30], we obtain approximate solutions of (5.17) over the interval $[0, 5]$, with the points of a uniform partition given by $t_j = jh$, $\forall j = 0, 1, \dots, N$, and $h = \frac{5}{N}$, $N = 500$.

In Figure 14, the solutions are plotted for different values of the parameter: $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, and $\lambda = 1.0$.

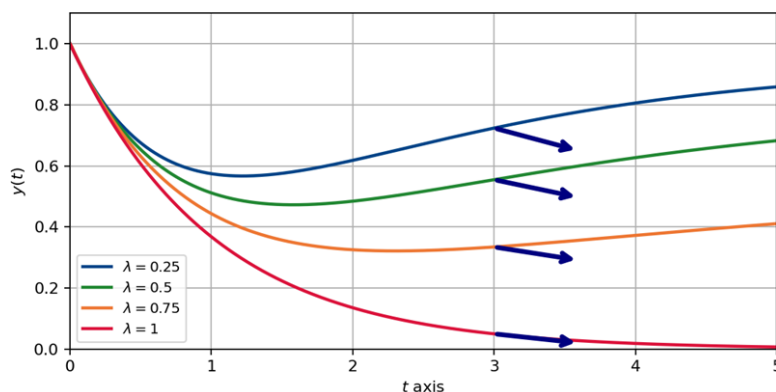


Figure 14. Plots of the solution $y(t)$ for $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, and $\lambda = 1.0$.

In Figure 15, the solutions are plotted for different values of the parameter: $\lambda = 0.65$, $\lambda = 0.75$, $\lambda = 0.85$, and $\lambda = 1.0$.

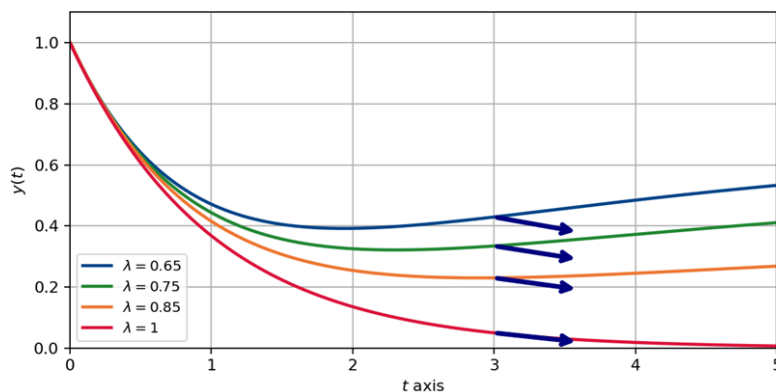


Figure 15. Plots of the solution $y(t)$ for $\lambda = 0.65$, $\lambda = 0.75$, $\lambda = 0.85$, and $\lambda = 1.0$.

In Figures 14 and 15, for $t = 3$, the vectors ${}^c D^\lambda \alpha(3)$ and $\alpha'(3)$ are plotted, respectively.

Moreover, it can be observed in both figures that the vectors ${}^c D^\lambda \alpha(3)$ and $\alpha'(3)$ are parallel, which confirms that ${}^c D^\lambda y(t)$ represents the slope of the tangent line to the function $z(t)$, at $t = 3$, up to a multiplicative factor.

Tables 1 and 2 present the results of evaluating the derivatives $z'(t)$ and ${}^c D^\lambda y(t)$ at the points $t = 3$ and $t = 4.8$ for different values of λ : 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0. The fractional factor FF and the error $|{}^c D^\lambda y(t) - z'(t)|$ are also evaluated.

As can be observed in Tables 1 and 2, as λ approaches to 1, the error tends to zero, and therefore, ${}^c D^\lambda y(t)$ approaches the integer order derivative $z'(t)$.

Table 1. Computation for $t = 3$.

λ	$FF = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}$	$z'(t)$	${}^c D^\lambda y(t)$	$ {}^c D^\lambda y(t) - z'(t) $
0.100000	2.794730	-0.049787	-0.139141	0.089354
0.200000	2.585642	-0.049787	-0.128732	0.078944
0.300000	2.374617	-0.049787	-0.118225	0.068438
0.400000	2.163569	-0.049787	-0.107718	0.057931
0.500000	1.954410	-0.049787	-0.097304	0.047517
0.600000	1.749024	-0.049787	-0.087079	0.037292
0.700000	1.549231	-0.049787	-0.077132	0.027345
0.800000	1.356756	-0.049787	-0.067549	0.017762
0.900000	1.173198	-0.049787	-0.058410	0.008623
1.000000	1.000000	-0.049787	-0.049787	0.000000

Table 2. Computation for $t = 4.8$.

λ	$FF = \frac{t^{1-\lambda}}{\Gamma(2-\lambda)}$	$z'(t)$	${}^c D^\lambda y(t)$	$ {}^c D^\lambda y(t) - z'(t) $
0.100000	4.266264	-0.008230	-0.035110	0.026881
0.200000	3.765861	-0.008230	-0.030992	0.022762
0.300000	3.299724	-0.008230	-0.027156	0.018926
0.400000	2.868420	-0.008230	-0.023606	0.015377
0.500000	2.472155	-0.008230	-0.020345	0.012115
0.600000	2.110784	-0.008230	-0.017371	0.009141
0.700000	1.783824	-0.008230	-0.014680	0.006451
0.800000	1.490479	-0.008230	-0.012266	0.004037
0.900000	1.229655	-0.008230	-0.010120	0.001890
1.000000	1.000000	-0.008230	-0.008230	0.000000

Now, we consider the integer-order initial value problem:

$$\begin{cases} z'(t) = -\frac{\Gamma(2-\lambda)E_\lambda(-t^\lambda)}{t^{1-\lambda}}, & t > 0, 0 < \lambda < 1, \\ z(0^+) = 1. \end{cases} \quad (5.18)$$

Using the predictor-corrector method for integer-order initial value problems, the numerical solutions for $\lambda = 0.75$, $\lambda = 0.85$, $\lambda = 0.95$, and $\lambda = 1.0$, are obtained, as shown in Figure 16.

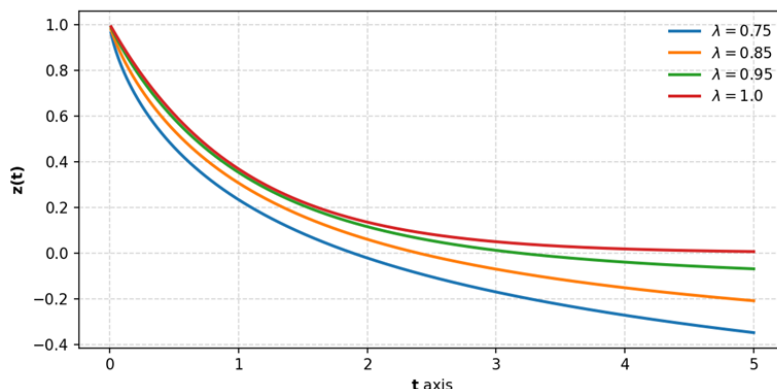


Figure 16. Solutions of the integer-order initial value problem for different values of the parameter λ .

6. Conclusions

In this paper, we obtain a new geometric interpretation of the Caputo fractional derivative, based on the fundamental theorem of plane curves and the fractional curvature of plane curves, introduced by Rubio-López et al., 2023. We show that the vector field ${}^c D^\lambda \alpha(s)$ along the curve $\alpha(s)$ can be interpreted as the vector field tangent to the curve $\beta_0(s)$ (Theorem 10). Furthermore, Methods 1 and 2 allow us to obtain functions $y(t)$, solutions to integer and fractional-order initial value problems, such that the fractional derivative ${}^c D^\lambda y(t)$ is a multiple of the integer order derivative $y'(t)$.

Author contributions

Franco Rubio-López: Conceptualization, Methodology, Software, Visualization, Writing original draft preparation, Writing–review and editing; Dennis Quispe-Sánchez: Methodology, Software, Visualization, Writing–review and editing; Obidio Rubio: Conceptualization, Methodology, Writing original draft preparation, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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