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Research article

Results on optimal control of impulsive Hilfer fractional stochastic integro-differential equations

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Abstract: This paper addresses a class of Hilfer fractional stochastic nonlinear integro-differential equations incorporating impulsive effects and optimal control in Hilbert spaces. We first establish the existence of mild solutions, ensuring the solvability of the system through the application of fractional calculus, stochastic analysis, and fixed-point techniques. The analytical framework effectively manages the combined difficulties arising from nonlocal operators, stochastic perturbations, and impulsive dynamics. Subsequently, we formulate the associated optimal control problem and derive the necessary conditions for optimality. An illustrative example is provided to demonstrate the practicality and robustness of the theoretical results.

Keywords: fixed-point approach; Hilfer fractional operator; optimal control; impulsive term; differential equation
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1. Introduction

Fractional dynamical systems are governed by differential equations that involve derivatives of non-integer order, commonly referred to as fractional derivatives [1]. Unlike classical models, these systems inherently capture memory and hereditary effects, which are essential for representing many real-world processes. This feature makes them particularly effective in modeling phenomena such as viscoelastic materials, anomalous diffusion, and complex control mechanisms characterized by power-law behavior [2, 3]. Consequently, fractional calculus serves as a powerful and versatile tool, offering enhanced accuracy and deeper insight compared to traditional integer-order approaches.

The Hilfer fractional derivative, as a natural generalization of both the Riemann-Liouville and Caputo derivatives, offers a unified framework for the study of fractional calculus [1]. Its distinguishing feature lies in the parameter β, which enables a continuous transition between these two classical definitions, thereby providing significant flexibility in modeling a wide range of real-world phenomena [4]. This adaptability makes it particularly useful in areas such as viscoelasticity, anomalous diffusion, and control theory, where memory effects and nonlocal behavior play a central role [5, 6]. Moreover, its applicability has been broadened by recent developments, including extensions to variable-order formulations and the incorporation of different kernel functions [3].

Fractional stochastic differential equations extend the classical theory of stochastic differential equations by incorporating fractional derivatives, which allow them to account for memory effects and long-range dependencies inherent in many real-world systems. For example, Metzler and Klafter [7] studied anomalous diffusion using stochastic walk approaches in the setting of fractional operators. In a related direction, Hammad et al. [8] addressed optimal control and controllability issues for impulsive Hilfer fractional integro-differential inclusions, along with numerical applications. Moreover, Guo et al. [9–11] discussed solvability results for certain new classes of fractional differential equations subject to mild conditions. This extension provides a more accurate framework than previous equation types for modeling complex phenomena where past states significantly influence future behavior and where stochastic fluctuations and nonlocal interactions play a critical role. Consequently, researchers have found broad applications in finance, physics, engineering, and other fields that involve intricate dynamical systems. The study of these equations combines tools from fractional calculus, stochastic analysis, and advanced numerical methods, making it a mathematically rich and actively evolving area of research.

Connecting fractional calculus with stochastic differential equations has prompted extensive research [12–14] into the qualitative behavior of fractional dynamical systems, including their stability, existence, and controllability. Because stochastic fluctuations are inherent in most real-world systems, analyzing deterministic problems under a stochastic framework is essential for accurate modeling. For more details, see [15–17]. Stochastic differential equations serve as a critical tool for capturing complex dynamics in physics, engineering, biology, and finance, where noise and random disturbances play a central role. Additionally, differential inclusions allow for the study of systems where evolution is not uniquely determined by the current state, accommodating uncertainties , nondeterministic behaviors, and discontinuities. These combined methodologies offer a robust framework for analyzing, controlling, and simulating complex fractional and stochastic systems. See [18–20] and [21–23].

Controllability is a central concept in modern control theory, essential for analyzing and designing control systems, including applications in structural decomposition, observer design, engineering, and pole assignment. As a complement, optimal control theory provides a framework for determining control strategies that optimize a given performance criterion over time, making it widely applicable in scientific and engineering problems where minimizing cost functions associated with a system's state and control variables is critical r2,r3,r4. Given that noise and stochastic disturbances are inherent in many natural and engineered systems, stochastic differential equations are often the preferred models for capturing these random effects. Building on this foundation, researchers have explored optimal control strategies for fractional stochastic systems, including Caputo fractional stochastic integro-differential equations with impulses, resolvent operators in Hilbert spaces, and systems with Sobolev-type fractional derivatives (Caputo and Riemann–Liouville), employing fractional resolvent operators to implement control [27–29].

On the other hand, the Hilfer fractional derivative occupies a central role in fractional calculus due to its capacity to unify and generalize the classical Riemann-Liouville and Caputo derivatives. Its defining feature, the type parameter, enables a continuous interpolation between these two foundational forms, providing exceptional flexibility in capturing the memory and hereditary characteristics of complex systems. This adaptability allows the Hilfer derivative to model phenomena where neither standard integer-order derivatives nor conventional fractional derivatives suffice, offering a more precise representation of systems with long-range temporal dependencies or nonlocal interactions. Moreover, its mathematical structure accommodates a wider range of initial and boundary conditions, enhancing the accuracy and applicability of models in physics, engineering, control theory, and other scientific domains. Consequently, the Hilfer derivative not only extends the analytical toolkit of fractional calculus but also opens new avenues for research and applications in modeling, analysis, and control of real-world dynamical systems. For more details, see [30, 31].

Most research on the Hilfer fractional derivative has focused on type $\sigma \in [0, 1]$ and order $\ell \in (0, 1)$. To extend this framework, the present study examines the existence and optimal control of Hilfer fractional stochastic integro-differential systems with type $\sigma \in [0, 1]$ and higher order $\ell \in (1, 2)$, incorporating impulsive terms. Building on insights from prior work, this novel approach motivates the consideration of the following system:

$$\begin{cases} {}^{H}D_{0^+}^{\sigma,\ell}\rho(\varsigma) = \mathfrak{I}\rho(\varsigma) + \Lambda(\varsigma)h(\varsigma) + g(\varsigma, \rho(\varsigma)) \\ +\Omega\left(\varsigma, \rho(\varsigma), \int_0^{\varsigma} \varphi(\varsigma, \vartheta, \rho(\vartheta))\, d\vartheta\right) \frac{dW(\varsigma)}{d\varsigma}, \ \varsigma \in V' = (0, r], \ \varsigma \neq \varsigma_\nu, \\ (I_{0^+}^{2-\kappa}\rho)(0) = \rho_0, \ (I_{0^+}^{2-\kappa}\rho)'(0) = \rho_1, \\ \Delta\rho|_{\varsigma=\varsigma_\nu} = J_\nu(\rho(\varsigma_\nu)), \ \nu = 1, 2, \ldots, m, \end{cases} \tag{1.1}$$

where ${}^{H}D_{0^+}^{\sigma,\ell}$ denotes the Hilfer fractional derivative with order $\ell \in (1, 2)$ and type $\sigma \in [0, 1]$. On a separable Hilbert space U, the function $\rho(.)$ takes values in U. The impulsive functions $J_\nu : \mathfrak{R} \to \mathfrak{R}$ ($\nu = 1, 2, \ldots, m$) are defined at the points $0 = \varsigma_0 < \varsigma_1 < \cdots < \varsigma_m < \varsigma_{m+1} = r$. The jump of ρ at ς_ν is given by by $\Delta\rho(\varsigma_\nu)$ and defined by

$$\Delta\rho(\varsigma_\nu) = \rho(\varsigma_\nu^+) - \rho(\varsigma_\nu^-),$$

where $\rho(\varsigma_\nu^+)$ and $\rho(\varsigma_\nu^-)$ represent the right and left limits of ρ at ς_ν, respectively.

Moreover, $I_{0^+}^{2-\kappa}$ denotes the Riemann-Liouville of order $(2 - \kappa)$ such that $\kappa = \ell + \sigma(2 - \ell)$. On a separable Hilbert space U, $\mathfrak{I} : D(\mathfrak{I}) \subset U \to U$ is the infinitesimal generator of a strongly continuous cosine family $\{R(\varsigma)\}_{\varsigma \geq 0}$. Let $V = [0, r]$, and let Ξ be a separable reflexive Hilbert space; the control function $h(.)$ takes takes values in Ξ, and $\{\Lambda(\varsigma)\}_{\varsigma \geq 0} : \Xi \to U$ denotes a family of linear operators.

Let Ψ be another separable Hilbert space and $\{W(\varsigma)\}_{\varsigma \geq 0}$ be a $\Psi-$valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ described on the filtered complete probability space $(\mho, F, \{F_\varsigma\}_{\varsigma \geq 0}, P)$. The functions $g : V \times U \to U$, $\Omega : V \times U \times U \to L(\Psi, U)$, and $g : V \times U \to U$ satisfy appropriate hypotheses, where $L(\Psi, U)$ is the space of all bounded linear operators from Ψ into U. The initial data is $\rho_0, \rho_1 \in L_0^2(\Psi, U)$. For clarity, we denote

$$\left(\widetilde{\phi}\rho\right)(\varsigma) = \int_0^{\varsigma} \varphi(\varsigma, \vartheta, \rho(\vartheta))\, d\vartheta.$$

Our paper is organized as follows: Section 2 presents the essential background, covering key concepts in fractional calculus and fixed-point theory that underpin our analysis. Building on the foundational results established in Section 3 via a fixed-point approach, we extend the study to more complex scenarios, yielding new theoretical insights into the system's behavior. In Section 4, we focus on the existence of optimal controls for the Lagrange problem, carefully demonstrating the procedures for identifying such controls and the conditions ensuring their existence. Finally, Section 5 offers a detailed theoretical application, illustrating the practical significance of our results and highlighting the potential impact and utility of the proposed framework in a concrete setting.

2. Basic concepts

In this section, we present the fundamental definitions and properties essential to our study. We then explore various fractional integrals, followed by a comprehensive and detailed examination of the Hilfer fractional derivative, providing readers with the theoretical foundation required for the subsequent analysis. Define separable Hilbert spaces $(U, \|.\|_U, \langle ., . \rangle_U)$ and $(\Psi, \|.\|_\Psi, \langle ., . \rangle_\Psi)$ with their norms and inner products, respectively. Assume that $\{W(\varsigma)\}_{\varsigma \geq 0}$ is a Ψ-valued Q-Wiener process defined on the filtered complete probability space $(\mho, F, \{F_\varsigma\}_{\varsigma \geq 0}, P)$ with a finite trace nuclear covariance operator $Q \geq 0$ such that $Tr(Q) < \infty$. Let $\{s_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of Ψ and $\{\gamma_j\}_{j=1}^{\infty}$ be a bounded sequence of non-negative real numbers such that $Qs_j = \gamma_j s_j$ and

$$\langle W(\varsigma), s \rangle = \sum_{j=1}^{\infty} \sqrt{\gamma_j} \left\langle s_j, s \right\rangle \eta_j(\varsigma), \ s \in \Psi, \ \varsigma \geq 0,$$

where $\eta_j(\varsigma)$ is a sequence of mutually independent one-dimensional standard Wiener processes.

Moreover, we need to define the following spaces:

(i) F_ς^W represents the $\sum$-algebra induced by $\{W(\varsigma) : 0 \leq \vartheta \leq \varsigma\}$, with $F_\varsigma = F_\varsigma^W$.

(ii) $L_0^2 = L_0^2\left(Q^{\frac{1}{2}}\Psi, U\right)$ refers to the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\Psi$ to U endowed with the inner product

$$\langle \varkappa, \varkappa^* \rangle = Tr(\varkappa Q \varkappa^*) < \infty.$$

(iii) The space L_0^2 is a separable Hilbert space with the norm

$$\|\varkappa\|_{L_0^2} = Tr[\varkappa P \varkappa^*].$$

(iv) The space $L^2(\mho, U) = L^2(\mho, F_\varsigma, P, U)$ is a Banach space of all strongly measurable, square integrable, and Ψ-valued stochastic variables with the norm

$$\|\rho(.)\|_{L^2(\mho, U)} = \left(E\|\rho(., W)\|_U^2\right)^{\frac{1}{2}},$$

where E is a mathematical expectation defined by $E(\rho) = \int_\mho \rho(W)\, dP$ with

$$\left(\sup_{\varsigma \in V}\left(E\|\rho(\varsigma)\|^2\right)\right)^{\frac{1}{2}} < \infty.$$

(v) $L_0^2(\mho, U)$ represents a subspace of $L^2(\mho, U)$ defined by

$$L_0^2(\mho, U) = \{\rho \in L^2(\mho, U) : \rho \text{ is } F_0 - \text{measurable}\}.$$

(vi) $\Upsilon\left(V', L^2(\mho, U)\right)$ denotes a Banach space of continuous functions from V' to $L^2(\mho, U)$.

(vii) The space $\Phi = \Upsilon_{2-\kappa}\left(V, L^2(\mho, U)\right)$ is a Banach space and is described as

$$\Upsilon_{2-\kappa}\left(V, L^2(\mho, U)\right) = \left\{\rho \in \Upsilon\left(V', L^2(\mho, U)\right) : \lim_{\varsigma\to 0^+} \varsigma^{2-\kappa}\rho(\varsigma) \text{ exists and finite}\right\}$$

equipped with

$$\|\rho(.)\|_\Phi = \sqrt{\sup_{\varsigma\in V'}\left(E\left\|\varsigma^{2-\kappa}\rho(\varsigma)\right\|^2\right)},$$

where $2-\kappa = (1-\sigma)(2-\ell)$.

Definition 2.1. [1] For the function $z(\varsigma) \in C([0, r] : \mathbb{R})$,

- The Riemann-Liouville integral of order $\ell > 0$ can be expressed as

$$I_{0^+}^{\ell} z(\varsigma) = \frac{1}{\Gamma(\ell)} \int_0^{\varsigma} (\varsigma - \vartheta)^{\ell-1} z(\vartheta) d\vartheta, \ \varsigma > 0.$$

- The Riemann-Liouville derivative of order $\ell \in (u-1, u)$, $u \in \mathbb{Z}^+$ is defined by

$${}^{RL}D_{0^+}^{\ell} z(\varsigma) = \frac{1}{\Gamma(u-\ell)} \frac{d^u}{d\varsigma^u} \int_0^{\varsigma} (\varsigma - \vartheta)^{u-\ell-1} z(\vartheta) d\vartheta, \ \varsigma > 0.$$

Definition 2.2. [1] The Caputo fractional derivative of order $\ell \in (u-1, u)$, $u \in \mathbb{Z}^+$ can be written as

$${}^{C}D_{0^+}^{\ell} z(\varsigma) = \frac{1}{\Gamma(u-\ell)} \int_0^{\varsigma} (\varsigma - \vartheta)^{u-\ell-1} z^{(u)}(\vartheta) d\vartheta, \ \varsigma > 0,$$

where the function $z(\varsigma)$ is $u-1$ times continuously differentiable, and is absolutely continuous.

Definition 2.3. [2] For the function $z : [0, \infty) \to \mathbb{R}$, the Hilfer fractional derivative is given by

$${}^{H}D_{0^+}^{\sigma,\ell} z(\varsigma) = I^{\sigma(u-\ell)} \frac{d^u}{d\varsigma^u} I_{0^+}^{(1-\sigma)(u-\ell)} z(\varsigma), \ \varsigma > 0,$$

where $\ell \in (u-1, u)$ and $\sigma \in [0, 1]$.

Remark 2.4. [32] It should be noted that:

(i) In the case of $\sigma = 0$, and $\ell \in (u-1, u)$, the Hilfer fractional derivative converts to the Riemann-Liouville fractional derivative,

$${}^{H}D_{0^+}^{0,\ell} z(\varsigma) = \frac{d^u}{d\varsigma^u} I_{0^+}^{(u-\ell)} z(\varsigma) = {}^{RL}D_{0^+}^{\ell} z(\varsigma).$$

(ii) In the case of $\sigma = 1$ and $\ell \in (u-1, u)$, the Hilfer fractional derivative becomes CF derivative:

$$^{H}D_{0^+}^{1,\ell} z(\varsigma) = I^{(u-\ell)} \frac{d^u}{d\varsigma^u} z(\varsigma) =^{C} D_{0^+}^{\ell} z(\varsigma).$$

Definition 2.5. [4] If $\kappa \in (1,2)$, then, the equation below holds:

$$I_{0^+}^{\kappa}\left({}^{RL}D_{0^+}^{\kappa}\rho(\varsigma)\right) = \rho(\varsigma) - \frac{\left(I_{0^+}^{2-\kappa}\rho\right)(0)}{\Gamma(\kappa-1)}\varsigma^{\kappa-2} - \frac{\left(I_{0^+}^{2-\kappa}\rho\right)'(0)}{\Gamma(\kappa)}\varsigma^{\kappa-1},$$

provided that $\left(I_{0^+}^{2-\kappa}\rho\right)(\varsigma)$ is continuous and $\left(I_{0^+}^{2-\kappa}\rho\right)'(\varsigma)$ is absolutely continuous.

Definition 2.6. [33] A one-parameter operator $\{R(\varsigma)\}_{\varsigma\in\mathbb{R}} : U \to U$ is said to be a strongly continuous cosine family if and only if

1) $R(0) = I$;
2) for each $\rho \in U$, $R(\varsigma)\rho$ is strongly continuous in ς on $\mathbb{R}$;
3) for every $\varsigma, \vartheta \in U$, $R(\varsigma+\vartheta) + R(\varsigma-\vartheta) = 2R(\varsigma)R(\vartheta)$.

Describe the sine family $\{N(\varsigma)\}_{\varsigma\in\mathbb{R}}$ associated with the cosine family $\{R(\varsigma)\}_{\varsigma\in\mathbb{R}}$ as

$$N(\varsigma)\rho = \int_0^{\varsigma} R(\vartheta)\rho d\vartheta, \ \varsigma \in \mathbb{R}, \ \eta \in U.$$

Moreover, we assume that

$$\Im\rho = \frac{d^2}{d\varsigma^2}R(\varsigma)\rho\Big|_{\varsigma=0}, \ \rho \in D(\Im),$$

where $D(\Im) = \{\rho \in U : R(\varsigma)\rho \in \Upsilon^2(\mathbb{R}, U)\}$.

Lemma 2.7. *[33] Suppose that $\{R(\varsigma)\}_{\varsigma\in\mathbb{R}}$ on U satisfying*

$$\|R(\varsigma)\|_U \le K_0 e^{n|\varsigma|}, \ \varsigma \in \mathbb{R}\, for\ r \ge 0, \ n \ge 0, \ and K_0 \ge 1.$$

Additionally, for every $Re\left(\overline{\Xi}\right) > n$, $\overline{\Xi}^2 \in \beta(\Im)$, we have

$$\overline{\Xi}\psi(\overline{\Xi}^2;\Im)\rho = \int_0^{\infty} e^{-\overline{\Xi}\varsigma}R(\varsigma)\rho d\varsigma, \ \psi(\overline{\Xi}^2;\Im)\rho = \int_0^{\infty} e^{-\overline{\Xi}\varsigma}N(\varsigma)\rho d\varsigma$$

for $\rho \in U$, where $\Im$ is the infinitesimal generator of $\{R(\varsigma)\}_{\varsigma\in\mathbb{R}}$. It follows that there is a constant $K \ge 1$ such that $\|R(\varsigma)\|_{L(U)} \le M$ for $\varsigma \ge 0$, which yields $\|N(\varsigma)\|_{L(U)} \le M\varsigma$ for $\varsigma \ge 0$.

Now, to present the mild solution to the problem (1.1), we provide the Wright function $\{\beth_i(\hbar)\}$ as follows:

$$\beth_i(\hbar) = \sum_{\widetilde{n}=1}^{\infty} \frac{(-\hbar)^{\widetilde{n}-1}}{(\widetilde{n}-1)!\Gamma(1-i\widetilde{n})}, \ i \in (0,1), \ \hbar \in \mathbb{C},$$

which satisfies

$$\int_0^{\infty} \hbar^{\widetilde{n}}\beth_i(\hbar)d\hbar = \frac{\Gamma(1+\widetilde{n})}{\Gamma(1+i\widetilde{n})}, \ \hbar \ge 0.$$

Motivated by [34], we present a mild solution for System (1.1) as follows:

Definition 2.8. We say that an F_ς-adapted stochastic process $\rho(\varsigma) : V' \to U$ is a mild solution of System (1.1) if for each $\rho_0, \rho_1 \in L_0^2(\mho, U)$, and $\rho \in \Phi$, there exists $\chi \in L_0^2(\mho, U)$ such that $\chi(\varsigma,\rho) \in \Omega\left(\varsigma, \rho(\varsigma), \left(\widetilde{\phi}\rho\right)(\varsigma)\right)$ on $\varsigma \in V'$, and

$$\begin{aligned}\rho(\varsigma) &= {}^{RL}D_{0^+}^{1-\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_0\right) + I_{0^+}^{\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_1\right) + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\Lambda(\vartheta)h(\vartheta)\,d\vartheta \\ &\quad + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)g(\vartheta,\rho(\vartheta))d\vartheta + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\chi(\vartheta,\rho(\vartheta))dW(\vartheta) \\ &\quad + \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{q-1}\eth_q(\varsigma-\varsigma_\nu)J_\nu(\rho(\varsigma_\nu)),\end{aligned}$$

for $\varsigma \in V^*$, $\zeta = \sigma(2-\ell) \in (0,1)$, and $\ell = 2q$, where

$$\begin{cases} \left(\widetilde{\phi}\rho\right)(\varsigma) = \int_0^\varsigma \varphi(\varsigma,\vartheta,\rho(\vartheta))\,d\vartheta, \\ \eth_q(\varsigma) = \int_0^\infty q\hbar\beth_q(\hbar)N(\varsigma^q\hbar)\,d\hbar, \\ \beth_q(\hbar) = \frac{1}{q}\hbar^{-(1+\frac{1}{q})}\xi_q\left(\hbar^{-\frac{1}{q}}\right),\ \hbar > 0, \\ \xi_q(\hbar) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\hbar^{-qn-1}\frac{\Gamma(qn+1)}{n!}\sin(qn\pi),\ \hbar > 0, \\ \Delta\eta|_{\varsigma=\varsigma_\nu} = J_\nu(\eta(\varsigma_\nu)),\ \nu = 1,2,\ldots,m. \end{cases}$$

Lemma 2.9. *[13] The operator $\eth_q$ has the following axioms:*

i) For each fixed $\varsigma > 0$, the operator $\eth_p(\varsigma)$ is linear and bounded, that is, for each $\rho \in U$, the following inequalities hold:

$$\left\|\eth_q(\varsigma)\rho\right\| \le \frac{K\varsigma^q}{\Gamma(2q)}\|\rho\|,$$

$$\left\|{}^{RL}D_{0^+}^{1-\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho\right)\right\| \le \frac{K\varsigma^{\zeta+2q-2}}{(2q-1)\Gamma(\zeta+2q-1)}\|\rho\|,$$

and

$$\left\|I_{0^+}^{\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho\right)\right\| \le \frac{K\varsigma^{\zeta+2q-1}}{\Gamma(\zeta+2q)}\|\rho\|.$$

ii) $\eth_p(\varsigma)$ is uniformly continuous, that is, for any $\varsigma_1, \varsigma_2 \ge 0$,

$$\left\|\eth_q(\varsigma_2) - \eth_q(\varsigma_1)\right\| \to 0, \text{ as } \varsigma_2 - \varsigma_1 \to 0.$$

Lemma 2.10. *[35] A measurable function $f : V \to U$ is Bochner integrable if and only if its norm, $\|f\|$ is Lebesgue integrable.*

Lemma 2.11. *[36] Assume that the mapping $\alpha : V \times \mho \to L_0^2$ is strongly measurable in order that $\int_0^r E\|\alpha(\varsigma)\|_{L_0^2}^a d\varsigma < \infty$, for $a \ge 2$. Then,*

$$E\left\|\int_0^\varsigma \alpha(\vartheta)\,dU(\vartheta)\right\|^a \le L_\alpha \int_0^\varsigma E\|\alpha(\varsigma)\|_{L_0^2}^a d\vartheta, \text{ for } \varsigma \in V,$$

where $L_\alpha > 0$.

Theorem 2.12. *[4] Let $\wp$ be a Banach space and ∇ be a nonempty, closed, and convex subset of $\wp$. If S is a contraction mapping on ∇ (i.e., there exists a constant $0 \le k < 1$ such that $\|S(\varrho_1) - S(\varrho_2)\| \le k\|\varrho_1 - \varrho_2\|$ for all $\varrho_1, \varrho_2 \in \nabla$), then S has a unique FP in ∇.*

3. Existence of mild solutions

In this section, we investigate the existence and uniqueness of a mild solution for System (1.1) by reformulating the problem as a fixed-point problem. To achieve this, we introduce the following assumptions:

(A_1) The function $g : V \times U \to \mathbb{R}$ is continuous, and there exist $\widetilde{L}_g, \widetilde{W}_g > 0$ such that for each $\varsigma \in V$ and $\widetilde{\varrho}_1, \widetilde{\varrho}_2 \in U$, we have

$$E\left\|g\left(\varsigma, \widetilde{\varrho}_1\right)-g\left(\varsigma, \widetilde{\varrho}_2\right)\right\|^2 \leq \widetilde{W}_g \varsigma^{2(2-\ell)} E\left\|\widetilde{\varrho}_1-\widetilde{\varrho}_2\right\|^2$$

and

$$E\left\|g\left(\varsigma, \widetilde{\varrho}_1\right)\right\|^2 \leq \widetilde{L}_g\left(1+\varsigma^{2(2-\ell)} E\left\|\widetilde{\varrho}_1\right\|^2\right).$$

(A_2) The functions $\Omega : V \times U \times U \to U$ and $\varphi : V \times V \to U$ have the following properties:

(i) For each $\varsigma, \vartheta \in V$, $\Omega(\varsigma, ., .) : U \times U \to U$ is measurable in ς and $\varphi(\varsigma, \vartheta, .) : U \to U$ is measurable in ς and ϑ.

(ii) For every fixed $\rho \in U$, $\Omega(., \rho, (\widetilde{\phi}\rho)) : V \to V$ and $\varphi(., ., \rho) : U \to U$ are continuous in ρ, where

$$\left(\widetilde{\phi}\rho\right)(\varsigma)=\int_0^{\varsigma} \varphi(\varsigma, \vartheta, \rho(\vartheta))\, d\vartheta.$$

(A_3) The function $\chi : V \times U \to L(\Psi, U)$ is continuous and there exist $\widehat{L}_\chi, \widehat{W}_\chi > 0$ such that for each $\varsigma \in V, \widetilde{\varrho}_1, \widetilde{\varrho}_2 \in U$,

$$E\left\|\chi\left(\varsigma, \widetilde{\varrho}_1\right)-\chi\left(\varsigma, \widetilde{\varrho}_2\right)\right\|^2 \leq \widehat{W}_\chi \varsigma^{2(2-\ell)} E\left\|\widetilde{\varrho}_1-\widetilde{\varrho}_2\right\|^2,$$

and

$$E\left\|\chi\left(\varsigma, \widetilde{\varrho}_1\right)\right\|^2 \leq \widehat{L}_\chi\left(1+\varsigma^{2(2-\ell)} E\left\|\widetilde{\varrho}_1\right\|^2\right).$$

(A_4) The operator $\Lambda \in L_\infty(V, L(\Xi, U))$ with norm $\|\Lambda\|_\infty$ exists, and for every bounded set $B \subset \Xi$, $B(.)$ is measurable when $\varrho(.) \subseteq B$, where $\varrho(.) \to 2^{\Xi} \backslash\{\emptyset\}$ has convex, closed, and bounded values.

(A_5) For each $\rho \in U$ and $\nu = 1, 2, \ldots, m$, the impulse operators $J_\nu : U \to U$ are continuous, and there exists a constant $C_\nu > 0$ such that

$$E\left\|J_\nu(\rho)\right\|^2 \leq C_\nu\|\rho\|_U^2.$$

Here, we describe the admissible set as

$$\varrho_{ad}=\left\{\begin{array}{c} h(.) : V \times \mho \to \Xi : h(.) \text{ is an } F_\varsigma-\text{adapted stochastic} \\ \text{and } E \int_0^r\|h(\varsigma)\|^a d\varsigma<\infty. \end{array}\right\}$$

It is clear that ϱ_{ad} is nonempty and for $a \in (1, +\infty)$, $\varrho_{ad} \subset L^a(V, \Xi)$ is bounded, closed, and convex. Moreover, $\Lambda h \in L^a(V, U)$ for each $h \in \varrho_{ad}$ [37].

Theorem 3.1. *Under the Assumptions (A_1)–(A_5), for each $h \in \varrho_{ad}$, the fractional system (1.1) has a mild solution on V, provided that*

$$G=2\left(\frac{K}{\Gamma(2q)}\right)^2\left[r^{2(2q-\kappa+2)}\left[\frac{\widetilde{W}_g}{4q^2}+\frac{L_\alpha \widehat{W}_\chi}{r(4q-1)}\right]+C_\nu^2 \sum_{\varsigma_\nu \in(0, \varsigma)}\left(\varsigma-\varsigma_\nu\right)^{2(2q-1)}\right]<1.$$

Proof. Define the operator $\aleph : \Phi \to \Phi$ by

$$\begin{aligned}(\aleph\rho)(\varsigma) &= {}^{RL}D_{0^+}^{1-\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_0\right) + I_{0^+}^{\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_1\right) + \int_0^{\varsigma}(\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\Lambda(\vartheta)h(\vartheta)\,d\vartheta \\ &+ \int_0^{\varsigma}(\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)g(\vartheta,\rho(\vartheta))d\vartheta + \int_0^{\varsigma}(\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\chi(\vartheta,\rho(\vartheta))dW(\vartheta) \\ &+ \sum_{\varsigma_v\in(0,\varsigma)}(\varsigma-\varsigma_v)^{q-1}\eth_q(\varsigma-\varsigma_v)J_v(\rho(\varsigma_v)),\ \text{for } \varsigma\in V'.\end{aligned}$$

Choose any $u \in \Upsilon\left(V, L^2(\mho, U)\right)$, and assume that $\rho(\varsigma) = \varsigma^{\kappa-2}u(\varsigma) \in \Phi$, for $\varsigma \in V'$. Let the operator $\phi : \Upsilon\left(V, L^2(\mho, U)\right) \to \Upsilon\left(V, L^2(\mho, U)\right)$ be given by

$$(\phi u)(\varsigma) = \begin{cases} \varsigma^{\kappa-2}(\aleph\rho)(\varsigma), & \text{for } \varsigma \in V', \\ \frac{\rho_0}{\Gamma(\zeta+2q-1)}, & \text{for } \varsigma = 0. \end{cases}$$

To establish the existence of a mild solution for System (1.1) in $\aleph$, we show that it can be represented as a fixed-point of the operator ϕ in $\Upsilon\left(V, L^2(\mho, U)\right)$, where $\rho(\varsigma) = \varsigma^{\kappa-2}u(\varsigma)$ for $\varsigma \in V'$. The proof is structured in the following steps:

(S.1) Prove that the operator ϕ maps δ_c into itself. Indeed, for any $c > 0$, assume that $\delta_c = \{u \in \Upsilon\left(V, L^2(\mho, U)\right) : \|u\|_{\Upsilon} \le c\}$ and $\widehat{\delta_c} = \{\rho \in \Phi : \|\rho\|_{\Phi} \le c\}$. Obviously, δ_c and $\widehat{\delta_c}$ are closed, bounded, and convex subsets. For each $u \in \delta_c$, consider that $\rho(\varsigma) = \varsigma^{\kappa-2}u(\varsigma)$, for $\varsigma \in V'$ and $\rho \in \widehat{\delta_c}$. Now, there is a function $u_c(.) \in \delta_c$ such that

$$\begin{aligned}&E\|(\phi u_c)(\varsigma)\|^2 \\ =\ & E\left\|\varsigma^{\kappa-2}(\aleph\rho)(\varsigma)\right\|^2 \\ \le\ & 5\varsigma^{2(2-\kappa)}\Bigg\{E\left\|{}^{RL}D_{0^+}^{1-\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_0\right)\right\|^2 + E\left\|I_{0^+}^{\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_1\right)\right\|^2 \\ & + E\left\|\int_0^{\varsigma}(\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\Lambda(\vartheta)h(\vartheta)\,d\vartheta\right\|^2 \\ & + E\left\|\int_0^{\varsigma}(\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)g(\vartheta,\rho(\vartheta))d\vartheta\right\|^2 \\ & + E\left\|\int_0^{\varsigma}(\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\chi(\vartheta,\rho(\vartheta))dW(\vartheta)\right\|^2 \\ & + E\left\|\sum_{\varsigma_v\in(0,\varsigma)}(\varsigma-\varsigma_v)^{q-1}\eth_q(\varsigma-\varsigma_v)J_v(\rho(\varsigma_v))\right\|^2\Bigg\} \\ \le\ & 5\varsigma^{2(2-\kappa)}\Bigg\{\left(\frac{K\varsigma^{2p+\zeta-2}}{(2q-1)\Gamma(2p+\zeta-1)}\right)^2 E\|\rho_0\|^2 + \left(\frac{K\varsigma^{2p+\zeta-1}}{\Gamma(\zeta+2p)}\right)E\|\rho_1\|^2 \\ & + \left(\frac{K}{\Gamma(2q)}\right)^2\|\Lambda\|_{\infty}^2\left[\left(\int_0^{\varsigma}(\varsigma-\vartheta)^{\frac{a(2q-1)}{a-1}}d\vartheta\right)^{\frac{a-1}{a}}\left(\int_0^{\varsigma}E\|h(\vartheta)\|^a\,d\vartheta\right)^{\frac{1}{a}}\right]^2 \\ & + \left(\frac{K}{\Gamma(2q)}\right)^2\left(\frac{\varsigma^{2q}}{2q}\right)\int_0^{\varsigma}(\varsigma-\vartheta)^{2q-1}E\|g(\vartheta,\rho(\vartheta))\|^2\,d\vartheta\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{K}{\Gamma(2q)}\right)^2 L_\alpha \int_0^\varsigma (\varsigma-\vartheta)^{2(2q-1)} E\,\|\chi(\vartheta,\rho(\vartheta))\|^2\, d\vartheta \\
&+\left(\frac{K}{\Gamma(2q)}\right)^2 \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{2(2q-1)} E\,\|J_\nu(\eta(\varsigma_\nu))\|^2 \Bigg\} \\
\leq\;& 5\varsigma^{2(2-\kappa)} \Bigg\{ \left(\frac{K\varsigma^{2p+\zeta-2}}{(2q-1)\Gamma(2p+\zeta-1)}\right)^2 E\,\|\rho_0\|^2 + \left(\frac{K\varsigma^{2p+\zeta-1}}{\Gamma(\zeta+2p)}\right) E\,\|\rho_1\|^2 \\
&+\left(\frac{K}{\Gamma(2q)}\right)^2 \|\Lambda\|_\infty^2 \|h(\vartheta)\|^2_{L^a(\Xi,V)} \varsigma^{\frac{2(2qa-1)}{a}} \left(\frac{a-1}{2qa-1}\right)^{\frac{2(a-1)}{a}} \\
&+\left(\frac{K}{\Gamma(2q)}\right)^2 \left(\frac{\varsigma^{2q}}{2q}\right) \widetilde{L}_g \left(1+\|u\|^2_\Upsilon\right) + \left(\frac{K}{\Gamma(2q)}\right)^2 \left(\frac{\varsigma^{4q-1}}{4q-1}\right) \widehat{L}_\chi \left(1+\|u\|^2_\Upsilon\right) \\
&+\left(\frac{K}{\Gamma(2q)}\right)^2 C_\nu^2 \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{2(2q-1)} \Bigg\} \\
\leq\;& 5\left(\frac{K}{(2q-1)\Gamma(2p+\zeta-1)}\right)^2 E\,\|\rho_0\|^2 + 5\left(\frac{Ks}{\Gamma(\zeta+2p)}\right) E\,\|\rho_1\|^2 \\
&+5\left(\frac{K}{\Gamma(2q)}\right)^2 \|\Lambda\|_\infty^2 \|h(\vartheta)\|^2_{L^a(\Xi,V)} \varsigma^{2(2-\kappa+2q-\frac{1}{q})} \left(\frac{a-1}{2qa-1}\right)^{\frac{2(a-1)}{a}} \\
&+5\left(\frac{K}{\Gamma(2q)}\right)^2 \left(\frac{\varsigma^{2(2q+2-\kappa)}}{4q^2}\right) \widetilde{L}_g (1+c) + 5\left(\frac{K}{\Gamma(2q)}\right)^2 \left(\frac{\varsigma^{4q-1+4-2\kappa}}{4q-1}\right) \widehat{L}_\chi (1+s) \\
&+5\left(\frac{K}{\Gamma(2q)}\right)^2 C_\nu^2 \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{2(2q-1)} \\
\leq\;& c.
\end{aligned}
$$

Therefore, for $c > 0$, $\phi(\delta_c) \subset \delta_c$.

(S.2) Prove that ϕ is contraction on δ_c. Indeed, for each $u \in \delta_c$ and the set $\rho(\varsigma) = \varsigma^{\kappa-2} u(\varsigma)$, for $\varsigma \in V'$ then $\rho \in \widehat{\delta_c}$. Let us consider $u, \widetilde{u} \in \delta_c$, we have

$$
\begin{aligned}
&E\,\|(\phi u)(\varsigma) - (\phi\widetilde{u})(\varsigma)\|^2 \\
=\;& E\left\|\varsigma^{\kappa-2}(\aleph\rho)(\varsigma) - \varsigma^{\kappa-2}(\aleph\widetilde{\rho})(\varsigma)\right\|^2 \\
\leq\;& 2\varsigma^{2(2-\kappa)} \Bigg\{ E\left\| \int_0^\varsigma (\varsigma-\vartheta)^{q-1} \eth_q(\varsigma-\vartheta) \left[g(\vartheta,\rho(\vartheta)) - g(\vartheta,\widetilde{\rho}(\vartheta))\right] d\vartheta \right\|^2 \\
&+E\left\| \int_0^\varsigma (\varsigma-\vartheta)^{q-1} \eth_q(\varsigma-\vartheta) \left[\chi(\vartheta,\rho(\vartheta)) - \chi(\vartheta,\widetilde{\rho}(\vartheta))\right] dW(\vartheta) \right\|^2 \\
&+E\left\| \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{q-1} \eth_q(\varsigma-\varsigma_\nu) \left[J_\nu(\rho(\varsigma_\nu)) - J_\nu(\widetilde{\rho}(\varsigma_\nu))\right] \right\|^2 \Bigg\} \\
\leq\;& 2\varsigma^{2(2-\kappa)} \Bigg\{ \left(\frac{K}{\Gamma(2q)}\right)^2 \left(\frac{\varsigma^{2q}}{2q}\right) \int_0^\varsigma (\varsigma-\vartheta)^{2q-1} \widetilde{W}_g \left(\vartheta^{2(2-\kappa)} E\,\|\rho(\vartheta) - \widetilde{\rho}(\vartheta)\|^2\right) d\vartheta
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{K}{\Gamma(2q)}\right)^2 L_\alpha \int_0^\varsigma (\varsigma-\vartheta)^{2(2q-1)} \widehat{W}_\chi \left(\vartheta^{2(2-\kappa)} E\left\|\rho(\vartheta)-\widetilde{\rho}(\vartheta)\right\|^2\right) d\vartheta \\
&+\left(\frac{KC_\nu}{\Gamma(2q)}\right)^2 \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{2(2q-1)} E\left\|\rho(\varsigma_\nu)-\widetilde{\rho}(\varsigma_\nu)\right\|^2 \\
\leq\ & 2\left(\frac{K}{\Gamma(2q)}\right)^2 \widetilde{W}_g \left(\frac{\varsigma^{2(4q-\kappa+2)}}{4q^2}\right) \sup_{\varsigma\in V'} \vartheta^{2(2-\kappa)} E\left\|\rho(\vartheta)-\widetilde{\rho}(\vartheta)\right\|^2 \\
&+2\left(\frac{K}{\Gamma(2q)}\right)^2 L_\alpha \widehat{W}_\chi \left(\frac{\varsigma^{4q-1+2(2-\kappa)}}{4q-1}\right) \sup_{\varsigma\in V'} \vartheta^{2(2-\kappa)} E\left\|\rho(\vartheta)-\widetilde{\rho}(\vartheta)\right\|^2 \\
&+2\left(\frac{K\widehat{C}_\nu}{\Gamma(2q)}\right)^2 \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{2(2q-1)} E\left\|\rho(\varsigma_\nu)-\widetilde{\rho}(\varsigma_\nu)\right\|^2 \\
\leq\ & 2\left(\frac{K}{\Gamma(2q)}\right)^2 \left[r^{2(2q-\kappa+2)} \left[\frac{\widetilde{W}_g}{4q^2} + \frac{L_\alpha \widehat{W}_\chi}{r(4q-1)}\right] + C_\nu^2 \sum_{\varsigma_\nu\in(0,\varsigma)} (\varsigma-\varsigma_\nu)^{2(2q-1)} \right] \\
&\times \left\|\rho-\widetilde{\rho}\right\|_\Phi^2 \\
=\ & G\left\|\rho-\widetilde{\rho}\right\|_\Phi^2 .
\end{aligned}
$$

Because $G < 1$, then ϕ is a contraction mapping, so Theorem 2.12 guarantees a unique fixed-point $u \in \delta_c$. The relation between ϕ and $\aleph$ implies that $\aleph$ also possesses a unique fixed point $\rho \in \widehat{\delta_c}$. This fixed point ρ constitutes a mild solution to the problem (1.1).

□

4. Existence of optimal controls

This section is devoted to determining the optimal controls for System (1.1). In essence, optimal control theory seeks to identify the input that produces the best possible outcome. Here, our goal is to find a control that maximizes (or minimizes) a given performance criterion while satisfying the constraints imposed by System (1.1).

Consider the Lagrange problem as follows: Find the pair $\left(\rho^0, h^0\right) \in \Phi \times \varrho_{ad}$ such that

$$
\Delta\left(\rho^0, h^0\right) \leq \Delta\left(\rho^h, h\right) \text{ for } (\rho, h) \in \Phi \times \varrho_{ad},
$$

where

$$
\Delta\left(\rho^h, h\right) = E\left\{\int_0^r \Pi\left(\varsigma, \rho^h(\varsigma), h(\varsigma)\right) d\varsigma\right\},
$$

ρ^h is the solution of (1.1) with control $h \in \varrho_{ad}$, and Π satisfies the following assertion below:

(A_6) The function Π fulfills the following:

- i) $\Pi : V \times U \times \Xi \to \mathbb{R} \cup \{0\}$ is an F_ς-measurable;
- ii) For each $\varsigma \in V$ and every $\rho \in U$, $\Pi(\varsigma, \rho, .)$ is convex on Ξ;
- iii) For almost all $\varsigma \in V$, $\Pi(\varsigma, ., .)$ is sequentially lower semicontinuous continuous on $U \times \Xi$;

iv) There exist constants $W_1 \geq 0$, $W_2 > 0$, and a non-negative function $W_3 \in L^1(V, \mathbb{R})$ such that

$$\Pi(\varsigma, \rho, h) \leq W_3(\varsigma) + W_1 E \|\rho\|_U^2 + W_2 E \|h\|_\Xi^2 .$$

Theorem 4.1. *Under the assumptions of Theorem 3.1 and the hypothesis (A$_6$), if Λ is a strongly continuous operator, then the Lagrange problem has at least one optimal pair $\left(\rho^0, h^0\right) \in \Phi \times \varrho_{ad}$ such that*

$$\Delta\left(\rho^0, h^0\right) = E\left\{\int_0^r \Pi\left(\varsigma, \rho^0(\varsigma), h^0(\varsigma)\right) d\varsigma\right\} \leq \Delta\left(\rho^h, h\right), \textit{ for } \left(\rho^h, h\right) \in \Phi \times \varrho_{ad}.$$

Proof. In the case of $\inf\left\{\Delta\left(\rho^h, h\right) : \left(\rho^h, h\right) \in \Phi \times \varrho_{ad}\right\} = +\infty$, there is a nothing proof. So, without loss the generality, we assume that

$$\inf\left\{\Delta\left(\rho^h, h\right) : \left(\rho^h, h\right) \in \Phi \times \varrho_{ad}\right\} = \epsilon < +\infty.$$

According to (A$_6$), we get $\epsilon > -\infty$. By the definition of the infimum, there exists a minimizing sequence of feasible pairs $\Delta\left(\rho^{\widetilde{n}}, h^{\widetilde{n}}\right)$ in ϱ_{ad},
where

$$\varrho_{ad} = \{(\rho, h) : \rho \text{ is a mild solution of (1.1), and } h \in \varrho_{ad}\}$$

such that $\Delta\left(\rho^{\widetilde{n}}, h^{\widetilde{n}}\right) \to 0$ as $n \to +\infty$. Because $\{h^{\widetilde{n}}\} \subseteq \varrho_{ad}$, $\widetilde{n} = 1, 2, \ldots$, and $\{h^{\widetilde{n}}\}$ is a bounded subset of separable reflexive Banach space $L^a(V, \Xi)$, there exists a subsequence $\{h^{\widetilde{n}}\}$ and $\{h^0\}$ in $L^a(V, \Xi)$ so that

$$h^{\widetilde{n}} \overset{weakly}{\longrightarrow} h^0 \in L^a(V, \Xi).$$

As ϱ_{ad} is closed and convex, based on Mazur's lemma, $h^0 \in \varrho_{ad}$. Let $\{\rho^{\widetilde{n}}\}$ be the sequence of solutions to (1.1) corresponding to the control sequence $\{h^{\widetilde{n}}\}$, and let ρ^0 be the solution to (1.1) corresponding to the control h^0. Then, $\rho^{\widetilde{n}}$ and ρ^0 satisfy their respective integral equations:

$$\begin{aligned}
\rho^{\widetilde{n}}(\varsigma) \;=\;& {}^{RL}D_{0^+}^{1-\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_0\right) + I_{0^+}^{\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_1\right) + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\Lambda(\vartheta)h^{\widetilde{n}}(\vartheta)\, d\vartheta \\
&+ \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)g(\vartheta, \rho^{\widetilde{n}}(\vartheta))d\vartheta + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\chi(\vartheta, \rho^{\widetilde{n}}(\vartheta))dW(\vartheta) \\
&+ \sum_{\varsigma_\nu \in (0,\varsigma)} (\varsigma-\varsigma_\nu)^{q-1}\eth_q(\varsigma-\varsigma_\nu)J_\nu\left(\rho^{\widetilde{n}}(\varsigma_\nu)\right),
\end{aligned}$$

and

$$\begin{aligned}
\rho^0(\varsigma) \;=\;& {}^{RL}D_{0^+}^{1-\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_0\right) + I_{0^+}^{\zeta}\left(\varsigma^{q-1}\eth_q(\varsigma)\rho_1\right) + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\Lambda(\vartheta)h^0(\vartheta)\, d\vartheta \\
&+ \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)g(\vartheta, \rho^0(\vartheta))d\vartheta + \int_0^\varsigma (\varsigma-\vartheta)^{q-1}\eth_q(\varsigma-\vartheta)\chi(\vartheta, \rho^0(\vartheta))dW(\vartheta) \\
&+ \sum_{\varsigma_\nu \in (0,\varsigma)} (\varsigma-\varsigma_\nu)^{q-1}\eth_q(\varsigma-\varsigma_\nu)J_\nu\left(\rho^0(\varsigma_\nu)\right).
\end{aligned}$$

Because $\{h^{\widetilde{n}}\}$ and h^0 are bounded, and due to Theorem 3.1, there exists a positive constant W_4 such that $E\left\|\rho^{\widetilde{n}}\right\|^2 \leq W_4$ and $E\left\|\rho^0\right\|^2 \leq W_4$. Consider $\rho(\varsigma) = \varsigma^{\kappa-2}u(\varsigma) \in \Phi$. For $\varsigma \in V'$, one has

$$E\left\|u^{\widetilde{n}}(\varsigma) - u^0(\varsigma)\right\|^2$$

$$
\begin{aligned}
&= E\left\|\varsigma^{2-\kappa}\rho^{\widetilde{n}}(\varsigma)-\varsigma^{2-\kappa}\rho^{0}(\varsigma)\right\|^{2}\\
&\leq 3\varsigma^{2(2-\kappa)}\left\{E\left\|\int_{0}^{\varsigma}(\varsigma-\vartheta)^{q-1}\partial_{q}(\varsigma-\vartheta)\left[\Lambda(\vartheta)h^{\widetilde{n}}(\vartheta)-\Lambda(\vartheta)h^{0}(\vartheta)\right]d\vartheta\right\|^{2}\right.\\
&\quad+E\left\|\int_{0}^{\varsigma}(\varsigma-\vartheta)^{q-1}\partial_{q}(\varsigma-\vartheta)\left[g(\vartheta,\rho^{\widetilde{n}}(\vartheta))-g(\vartheta,\rho^{0}(\vartheta))\right]d\vartheta\right\|^{2}\\
&\quad+E\left\|\int_{0}^{\varsigma}(\varsigma-\vartheta)^{q-1}\partial_{q}(\varsigma-\vartheta)\left[\chi(\vartheta,\rho^{\widetilde{n}}(\vartheta))-\chi(\vartheta,\rho^{0}(\vartheta))\right]dW(\vartheta)\right\|^{2}\\
&\quad\left.+E\left\|\sum_{\varsigma_{\nu}\in(0,\varsigma)}(\varsigma-\varsigma_{\nu})^{q-1}\partial_{q}(\varsigma-\varsigma_{\nu})\left[J_{\nu}\left(\rho^{\widetilde{n}}(\varsigma_{\nu})\right)-J_{\nu}\left(\rho^{0}(\varsigma_{\nu})\right)\right]\right\|^{2}\right\}\\
&\leq 3r^{2(2-\kappa+2q-\frac{1}{a})}\left(\frac{K}{\Gamma(2q)}\right)^{2}\left(\frac{a-1}{2qa-1}\right)^{\frac{2(a-1)}{a}}\left(\int_{0}^{\varsigma}E\left\|\Lambda(\vartheta)h^{\widetilde{n}}(\vartheta)-\Lambda(\vartheta)h^{0}(\vartheta)\right\|^{a}d\vartheta\right)^{\frac{2}{a}}\\
&\quad+3r^{2(2-\kappa)}\left(\frac{K}{\Gamma(2q)}\right)^{2}\left(\frac{r^{2q}}{2q}\right)\int_{0}^{\varsigma}(\varsigma-\vartheta)^{2q-1}\widetilde{W}_{g}\left(\vartheta^{2(2-\kappa)}\left(\left\|\rho^{\widetilde{n}}(\vartheta)-\rho^{0}(\vartheta)\right\|^{2}d\vartheta\right)\right)\\
&\quad+3r^{2(2-\kappa)}\left(\frac{K}{\Gamma(2q)}\right)^{2}L_{\alpha}\int_{0}^{\varsigma}(\varsigma-\vartheta)^{2q-1}\widehat{W}_{\chi}\left(\vartheta^{2(2-\kappa)}\left(\left\|\rho^{\widetilde{n}}(\vartheta)-\rho^{0}(\vartheta)\right\|^{2}d\vartheta\right)\right)\\
&\quad+3r^{2(2-\kappa)}\left(\frac{K}{\Gamma(2q)}\right)^{2}\sum_{\varsigma_{\nu}\in(0,\varsigma)}(\varsigma-\varsigma_{\nu})^{2(q-1)}\left\|J_{\nu}\left(\rho^{\widetilde{n}}(\varsigma_{\nu})\right)-J_{\nu}\left(\rho^{0}(\varsigma_{\nu})\right)\right\|^{2}\\
&\leq 3r^{2(2-\kappa+2q-\frac{1}{a})}\left(\frac{K}{\Gamma(2q)}\right)^{2}\left(\frac{a-1}{2qa-1}\right)^{\frac{2(a-1)}{a}}\left(\int_{0}^{\varsigma}E\left\|\Lambda(\vartheta)h^{\widetilde{n}}(\vartheta)-\Lambda(\vartheta)h^{0}(\vartheta)\right\|^{a}d\vartheta\right)^{\frac{2}{a}}\\
&\quad+3r^{2(2-\kappa)}\left(\frac{K}{\Gamma(2q)}\right)^{2}\left(\frac{r^{2q}}{2q}\right)^{2}\widetilde{W}_{g}\left\|\rho^{\widetilde{n}}-\rho^{0}\right\|^{2}\\
&\quad+3\left(\frac{r^{(4q-2\kappa+3)}}{4q-1}\right)\left(\frac{K}{\Gamma(2q)}\right)^{2}L_{\alpha}\widehat{W}_{\chi}\left\|\rho^{\widetilde{n}}-\rho^{0}\right\|^{2}\\
&\quad+3r^{2(2-\kappa)}\left(\frac{K}{\Gamma(2q)}\right)^{2}\left(\sum_{\varsigma_{\nu}\in(0,\varsigma)}C_{\nu}^{2}(\varsigma-\varsigma_{\nu})^{2(q-1)}\right)\left\|\rho^{\widetilde{n}}-\rho^{0}\right\|^{2},
\end{aligned}
$$

which implies that, there is $\varphi^{*}>0$ such that

$$\sup_{\varsigma\in V'}E\left\|\varsigma^{2-\kappa}\rho^{\widetilde{n}}(\varsigma)-\varsigma^{2-\kappa}\rho^{0}(\varsigma)\right\|^{2}\leq\varphi^{*}\left\|\Lambda h^{\widetilde{n}}-\Lambda h^{0}\right\|_{L^{a}(V,\Xi)}. \tag{4.1}$$

Because, Λ is strongly continuous, we get

$$\left\|\Lambda h^{\widetilde{n}}-\Lambda h^{0}\right\|_{L^{a}(V,\Xi)}\to 0 \text{ strongly as } \widetilde{n}\to\infty. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\sup_{\varsigma\in V'}E\left\|\varsigma^{2-\kappa}\rho^{\widetilde{n}}(\varsigma)-\varsigma^{2-\kappa}\rho^{0}(\varsigma)\right\|^{2}\to 0 \text{ strongly as } \widetilde{n}\to\infty,$$

which implies that

$$\rho^{\widetilde{n}}\to\rho^{0}\in\Phi \text{ as } \widetilde{n}\to\infty.$$

Because (A_6) fulfills the requirements for Balder's theorem [38], we can conclude that

$$(\rho, h) \to E\left\{\int_0^r \Pi\left(\varsigma, \rho(\varsigma), h(\varsigma)\right) d\varsigma\right\}$$

is sequentially lower semicontinuous when $L^a(V, \Xi)$ (embedded in $L^1(V, \Xi)$) carries the weak topology, and $L^1(V, U)$ carries the strong topology. Because Δ is weakly lower semicontinuous on $L^a(V, \Xi)$ (by (A_6)(iv), where $\Delta > -\infty$) and ϱ_{ad} is a closed, convex, and bounded subset of $L^a(V, \Xi)$, Δ attains its infimum at some $h^0 \in \varrho_{ad}$, that is,

$$\epsilon = \lim_{\widetilde{n}\to\infty} E\left\{\int_0^r \Pi\left(\varsigma, \rho^{\widetilde{n}}(\varsigma), h^{\widetilde{n}}(\varsigma)\right) d\varsigma\right\} \le E\left\{\int_0^r \Pi\left(\varsigma, \rho^0(\varsigma), h^0(\varsigma)\right) d\varsigma\right\} = \Delta\left(\rho^0, h^0\right).$$

Hence,

$$\epsilon \le \Delta\left(\rho^0, h^0\right).$$

Because $\Delta\left(\rho^0, h^0\right) \ge \epsilon$, we have $\Delta\left(\rho^0, h^0\right) = \epsilon$, and this completes the proof. □

5. An illustrative example

This section provides an illustrative example to reinforce the theoretical results.

Example 5.1. Consider the following control system:

$$\begin{cases} \partial_\varsigma^{\sigma,\ell}\rho(\varsigma,\vartheta) = \frac{\partial^2}{\partial\varpi^2}\rho(\varsigma,\vartheta) + \int_0^1 \psi(\varsigma,\vartheta)\, h(\varsigma,\vartheta)\, d\varsigma + \frac{\varsigma \sin\rho(\varsigma,\vartheta)}{5(1+|\rho(\varsigma,\vartheta)|)} \\ +\frac{\sqrt{\varsigma}\cos\rho(\varsigma,\vartheta)}{3(1+|\rho(\varsigma,\vartheta)|)} \int_0^1 e^{-3\varsigma}\rho(\varsigma,\vartheta)\frac{dW(\varsigma)}{d\varsigma},\ \varsigma \in (0,1],\ \vartheta \in [0,\pi],\ \varsigma \ne \varsigma_\sigma, \\ \rho(\varsigma, 0) = \rho(\varsigma,\pi) = 0,\ \varsigma \in (0,1], \\ (I_{0^+}^{2-\kappa}\rho)(0,\vartheta) = \rho_0(\vartheta),\ (_{0^+}^{2-\kappa}\rho)'(0,\vartheta) = \rho_1(\vartheta),\ \vartheta \in [0,\pi], \\ \rho(\varsigma_\nu^+,\vartheta) - \rho(\varsigma_\nu^-,\vartheta) = J_\nu(\rho(\varsigma_\nu)),\ \nu = 1,2,\dots,m,\ \vartheta \in [0,\pi], \end{cases} \tag{5.1}$$

where $\partial_\varsigma^{\sigma,\ell}$ refers to Hilfer fractional partial derivative of order $\ell \in (1,2)$ and type $\sigma \in [0,1]$, $I_{0^+}^{2-\kappa}$ is the Riemann-Liouville integral of order $2-\kappa$ such that $\kappa = \ell + \sigma(2-\ell)$, and $W(\varsigma)$ represents a one-dimensional standard Brownian motion defined on a complete complete probability space $(\mho, F, \{F_\varsigma\}_{\varsigma\ge 0}, Q)$. Assume that $U = \Xi = \Psi = L_0^2([0,\pi], \mathbb{R}^+)$ and $\Im\rho = d^2\rho/d\varpi^2$ with the domain

$$D(\Im) = \{\rho \in U : \rho, \rho' \text{ are absolutely continuous}, \rho'' \in U \text{ and } \rho(0) = \rho(\pi) = 0\}.$$

where, $\Im$ is the infinitesimal generator for a uniformly bounded strongly continuous cosine family $\{R(\varsigma)\}_{\varsigma\ge 0}$. Therefore, $\Im$ generates

$$\Im\hbar = -\sum_{u=1}^{\infty} u^2\langle \hbar, e_u\rangle e_u,\ \hbar \in D(\Im),$$

where $e_u(\vartheta) = \sqrt{2/\pi}\sin(u\pi\vartheta)$, leading to $\{-u^2 : u \in \mathbb{N}\}$ being eigenvalues of $\Im$, and $\{e_u : u \in \mathbb{N}\}$ represents an orthonormal basis of U. From the results in [33], one can define

$$R(\varsigma)\hbar = \sum_{u=1}^{\infty} \cos(u\pi\varsigma)\langle \hbar, e_u\rangle e_u,\ \hbar \in U$$

and

$$N(\varsigma)\hbar = \sum_{u=1}^{\infty} \frac{1}{u} \sin(u\pi\varsigma) \langle \hbar, e_u \rangle e_u, \ \hbar \in U,$$

where $\langle .,. \rangle$ is the inner product in U. Again, according to the results in [33], we have

$$\eth_q(\varsigma)\hbar = \sum_{u=1}^{\infty} \varsigma^{\frac{\ell}{2}} E_{\ell,\ell}\left(-u^2\varsigma^{\ell}\right) \langle \hbar, e_u \rangle e_u, \ q = \frac{\ell}{2},$$

where $E_{\ell,\ell}(\hbar) = \sum\limits_{u=0}^{\infty} \hbar^u/\Gamma(\ell(u+1))$ is the Mittag-Leffler function. Assuming that $\rho(\varsigma)(\vartheta) = \rho(\varsigma,\vartheta)$, $h(\varsigma)(\vartheta) = h(\varsigma,\vartheta)$,

$$\begin{cases} g(\varsigma, \rho(\varsigma)(\vartheta)) = \frac{\varsigma \sin \rho(\varsigma,\vartheta)}{5(1+|\rho(\varsigma,\vartheta)|)}, \\ \Lambda(\varsigma) h(\varsigma)(\vartheta) = \int_0^1 \psi(\varsigma,\vartheta) h(\varsigma,\vartheta) \, d\vartheta, \\ \Omega(\varsigma, \rho(\varsigma), (\phi\rho)(\vartheta)) = \frac{\sqrt{\varsigma}\cos\rho(\varsigma,\vartheta)}{3(1+|\rho(\varsigma,\vartheta)|)} \int_0^1 e^{-3\varsigma} \rho(\varsigma,\vartheta) \frac{dW(\varsigma)}{d\varsigma}, \\ (\phi\rho)(\vartheta) = \int_0^1 e^{-3\varsigma}\rho(\varsigma,\vartheta) d\vartheta, \\ \rho(\varsigma_\nu^+, \vartheta) - \rho(\varsigma_\nu^-, \vartheta) = J_\nu(\rho(\varsigma_\nu)), \ \nu = 1, 2, \ldots, m. \end{cases}$$

Then, the system (5.1) represents the abstract form of the system (1.1).

To satisfy the requirements of Theorem 3.1, for any $c > 0$ such that $\|\rho\|_U \le c$, we present the following points:

- From the definition of the function g, for $\rho_1(\varsigma), \rho_2(\varsigma) \in U$, we have

$$\begin{aligned} \|g(\varsigma,\rho_1) - g(\varsigma,\rho_2)\|^2 &\le \left\| \frac{\varsigma \sin \rho_1(\varsigma,\vartheta)}{5(1+|\rho_1(\varsigma,\vartheta)|)} - \frac{\varsigma \sin \rho_2(\varsigma,\vartheta)}{5(1+|\rho_2(\varsigma,\vartheta)|)} \right\|^2 \\ &\le \frac{\varsigma^2}{25} \|\sin\rho_1(\varsigma,\vartheta) - \sin\rho_2(\varsigma,\vartheta)\|^2 \\ &\le \frac{\varsigma^2}{25} \|\rho_1 - \rho_2\|^2, \end{aligned}$$

  and for $\rho(\varsigma) \in U$, one has

$$\begin{aligned} \|g(\varsigma,\rho)\|^2 &= \left\| \frac{\varsigma \sin\rho(\varsigma,\vartheta)}{5(1+|\rho(\varsigma,\vartheta)|)} \right\|^2 \le \frac{\varsigma^2}{25} \|\sin\rho(\varsigma,\vartheta)\|^2 \\ &\le \frac{1}{20} + \frac{1}{20}\|\rho(\varsigma,\vartheta)\|^2 = \frac{1}{20}\left(1 + \|\rho\|^2\right). \end{aligned}$$

  Therefore, the assumption (A_1) is satisfied with $\widetilde{W}_g = 1/25$ and $\widetilde{L}_g = 1/20$.
- From the definition of the functions Ω and φ, it follows readily that both Ω and φ are continuous and measurable. This fulfills the assumption (A_2).
- Assume that the function $\chi : [0,1] \times U \to L(\Psi, U)$ is defined by

$$\chi(\varsigma,\rho) = \frac{\sqrt{\varsigma}\cos\rho(\varsigma,\vartheta)}{3(1+|\rho(\varsigma,\vartheta)|)}.$$

Then, $\chi \in \Omega$. Moreover, for $\rho_1(\varsigma), \rho_2(\varsigma) \in U$, we can write

$$\begin{aligned} \|\chi(\varsigma,\rho_1) - \chi(\varsigma,\rho_2)\|^2 &\le \left\| \frac{\sqrt{\varsigma}\cos\rho(\varsigma,\vartheta)}{3(1+|\rho(\varsigma,\vartheta)|)} - \frac{\sqrt{\varsigma}\cos\rho(\varsigma,\vartheta)}{3(1+|\rho(\varsigma,\vartheta)|)} \right\|^2 \\ &\le \frac{\varsigma}{9}\|\cos\rho_1(\varsigma,\vartheta) - \cos\rho_2(\varsigma,\vartheta)\|^2 \\ &\le \frac{\varsigma}{9}\|\rho_1 - \rho_2\|^2, \end{aligned}$$

and for $\rho(\varsigma) \in U$, we have

$$\begin{aligned} \|\chi(\varsigma,\rho)\|^2 &= \left\| \frac{\sqrt{\varsigma}\cos\rho(\varsigma,\vartheta)}{3(1+|\rho(\varsigma,\vartheta)|)} \right\|^2 \le \frac{\varsigma}{9}\|\cos\rho(\varsigma,\vartheta)\|^2 \\ &\le \frac{1}{8} + \frac{1}{8}\|\rho(\varsigma,\vartheta)\|^2 = \frac{1}{8}\left(1 + \|\rho\|^2\right). \end{aligned}$$

Hence, the assumption (A_3) is fulfilled with $\widehat{W}_\chi = 1/9$ and $\widehat{L}_\chi = 1/8$.

- From the definition of Λ, it is clear that $\Lambda \in L_\infty([0,1], L(\Xi, U))$. Define admissible set as

$$\begin{aligned} \varrho_{ad} = \Big\{ & h(.)(\vartheta) : [0,1] \times \mho \to \Xi : h(.)(\vartheta) \text{ is an } F_\varsigma - \text{adapted stochastic} \\ & \text{and } E\int_0^1 \|h(\varsigma,\vartheta)\|\, d\varsigma < \infty \Big\}, \end{aligned}$$

  such that $a = 1$. Therefore, $\varrho(.) \to 2^\Xi$ has convex, closed, and bounded values and for every subset Λ of Ξ; $\Lambda(.)$ is measurable; and $\varrho(.) \subseteq \Lambda$. Thus, the hypothesis ($A_4$) is true.
- Assumption (A_5) is readily satisfied by noting that $\hbar(\varsigma_\nu^+, \vartheta) - \hbar(\varsigma_\nu^-, \vartheta) = J_\nu(\hbar(\varsigma_\nu))$ with $C_\nu = 1$. and $\nu = 1, 2, ..., m$

  Hence, all the conditions of Theorem 3.1 are satisfied, and therefore, the problem (5.1) admits a mild solution.

Now, define the cost function as follows:

$$\Delta(h(\varsigma,\vartheta)) = E\left\{ \int_0^1 \Pi(\varsigma, h(\varsigma,\vartheta), h(\varsigma,\vartheta))\, d\varsigma \right\},$$

where

$$\begin{aligned} & \Pi(\varsigma, \rho(\varsigma,\vartheta), h(\varsigma,\vartheta))(\varpi) \\ = \; & \frac{e^{-\varsigma}}{3(1+\varsigma^2)} + \int_0^1 \int_0^\pi \frac{\varsigma}{1+\varsigma}\|\rho(\varsigma,\vartheta)\|^2\, d\vartheta d\varsigma + \int_0^1 \int_0^\pi \frac{\varsigma}{3}\|h(\varsigma,\vartheta)\|^2\, d\vartheta d\varsigma. \end{aligned}$$

Clearly, all conditions of hypothesis (A_6) are satisfied with $W_1 = \ln(2)$, $W_2 = 1/6$, and $W_3(\zeta) = e^{-\varsigma}/3\left(1+\varsigma^2\right)$. Therefore, by Theorem 4.1, System (5.1) has at least one optimal pair.

6. Conclusions and future work

This article develops a comprehensive framework for the optimal control of Hilfer fractional stochastic integro-differential systems of order $\ell \in (1,2)$ in Hilbert spaces, incorporating impulsive effects. By integrating tools from fractional calculus, cosine family theory, stochastic analysis, and fixed-point theory, we rigorously construct the mild solution for System (1.1) and establish the existence of an optimal control pair, supported by a concrete example to illustrate its applicability. Nevertheless, the study has certain limitations. The framework is confined to systems with fractional orders between 1 and 2, which may not capture all real-world phenomena exhibiting fractional dynamics. Additionally, the analysis focuses on optimal control in Hilbert spaces without explicitly addressing state-dependent delays, which are common in complex systems and can significantly influence control strategies. The treatment of impulsive effects is general, leaving specific impulse types such as those with variable magnitudes or durations unexplored, which may limit applicability to systems with specialized impulsive behaviors. Finally, although the fixed-point approach is effective for establishing existence, it provides limited insight into the uniqueness or stability of the optimal control, suggesting the need for further investigation using alternative methods.

Author contributions

Doha A. Kattan: Validation, Funding acquisition, Writing-review & editing. Hasanen A. Hammad: Formal analysis, Methodology, Writing-original draft, Writing-review & editing.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The author declare that they have no conflicts of interest.

References

1. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, 1998.
2. R. Hilfer, *Applications of fractional calculus in physics*, 2000. https://doi.org/10.1142/3779
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, 2006, Elsevier.
4. Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2014. https://doi.org/10.1142/9069
5. J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in fractional calculus: Theoretical developments and applications in physics and engineering*, Springer Dordrecht, 2007. https://doi.org/10.1007/978-1-4020-6042-7
6. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

7. R. Metzler, J. Klafter, The stochastic walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3
8. H. A. Hammad, T. Aboelenen, M. E. M. Abdalla, Optimal control and controllability theorems for impulsive Hilfer fractional integro-differential inclusions with numerical applications, *Ain Shams Eng. J.*, **16** (2025) 103741. https://doi.org/10.1016/j.asej.2025.103741
9. L. Guo, U. Riaz, A. Zada, M. Alam, On implicit coupled Hadamard fractional differential equations with generalized Hadamard fractional integro-differential boundary conditions, *Fractal Fract.*, **7** (2023), 13. https://doi.org/10.3390/fractalfract7010013
10. D. Min, L. Guo, Existence and nonexistence of nontrivial solutions for fractional Advection-Dispersion equation with instantaneous and non-instantaneous impulses, *Fractal Fract.*, **9** (2025), 571. https://doi.org/10.3390/fractalfract9090571
11. Y. Wang, L. Guo, Solvability of the Caputo fractional differential system with Riemann-Stieltjes boundary conditions, *J. Appl. Anal. Comput.*, **16** (2026), 34–44. https://doi.org/10.11948/20250114
12. M. M. Raja, V. Vijayakumar, R. Udhayakumar, A new approach on the approximate controllability of fractional evolution inclusion of order $1 < r < 2$ with infinite delay, *Chaos Soliton. Fract.*, **141** (2020), 110343. https://doi.org/10.1016/j.chaos.2020.110343
13. Q. Li, Y. Zhou, The existence of mild solutions for Hilfer fractional stochastic evolution equations with order $\mu \in (1, 2)$, *Fractal Fract.*, **7** (2023), 525. https://doi.org/10.3390/fractalfract7070525
14. J. Pradeesh, V. Vijayakumar, Investigating the existence results for Hilfer fractional stochastic evolution inclusions of order $1 < \mu < 2$, *Qual. Theory Dyn. Syst.*, **23** (2024), 46. https://doi.org/10.1007/s12346-023-00899-5
15. J. Pradeesh, S. K. Panda, V. Vijayakumar, Y. K. Ma , A note concerning the optimal control results for Hilfer fractional stochastic differential equations of order $1 < \mu < 2$, *J. Appl. Math. Comput.,* **71** (2025), 2803–2820. https://doi.org/10.1007/s12190-024-02330-x
16. H. A. Hammad, H. Aydi, D. A. Kattan, Further investigation of stochastic nonlinear Hilfer-fractional integro-differential inclusions using almost sectorial operators, *J. Pseudo-Differ. Oper. Appl.*, **15** (2024), 5. https://doi.org/10.1007/s11868-023-00577-9
17. H. A. Hammad, M. G. Alshehri, Application of the Mittag-Leffler kernel in stochastic differential systems for approximating the controllability of nonlocal fractional derivatives, *Chaos Soliton. Fract.*, **182** (2024), 114775. https://doi.org/10.1016/j.chaos.2024.114775
18. X. Ma, X. B. Shu, J. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, *Stoch. Dyn.*, **20** (2020), 2050003. https://doi.org/10.1142/S0219493720500033
19. R. Sakthivel, Y. Ren, A. Debbouche, N. I. Mahmudov, Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions, *Appl. Anal.*, **95** (2016), 2361–2382. https://doi.org/10.1080/00036811.2015.1090562

20. Y. Guo, X. B. Shu, Y. Li, F. Xu, The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, *Bound. Value Probl.*, **2019** (2019), 59. https://doi.org/10.1186/s13661-019-1172-6

21. P. Balasubramaniam, P. Tamilalagan, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Appl. Math. Comput.*, **256** (2015), 232–246. https://doi.org/10.1016/j.amc.2015.01.035

22. C. Dineshkumar, K. S. Nisar, R. Udhayakumar, V. Vijayakumar, A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions, *Asian J. Control*, **24** (2022), 2378–2394. https://doi.org/10.1002/asjc.2650

23. X. Pang, X. Li , Z. Liu, Decay mild solutions of Hilfer fractional differential variational–hemivariational inequalities, *Nonlinear Anal.-Real*, **71** (2023), 103834. https://doi.org/10.1016/j.nonrwa.2023.103834

24. Y. Liu, Z. Liu, N. S. Papageorgiou, Sensitivity analysis of optimal control problems driven by dynamic history-dependent variational-hemivariational inequalities, *J. Differ. Equ.*, **342** (2023), 559–595. https://doi.org/10.1016/j.jde.2022.10.009

25. Y. Liu, Z. Liu, S. Peng, C. Wen, Optimal feedback control for a class of fractional evolution equations with history-dependent operators, *Fract. Cal. Appl. Anal.*, **25** (2022), 1108–1130. https://doi.org/10.1007/s13540-022-00054-y

26. H. A. Hammad, R. A. Rashwan, A. Nafea, M. E. Samei, M. De la Sen, Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations, *Symmetry*, **14** (2022), 2579. https://doi.org/10.3390/sym14122579

27. G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge: Cambridge University Press, 1992. https://doi.org/10.1017/CBO9780511666223

28. X. Mao, *Stochastic differential equations and applications*, Horwood Publishing Limited, 2007.

29. Y. K. Chang, Y. Pei, R. Ponce, Existence and optimal controls for fractional stochastic evolution equations of Sobolev type via fractional resolvent operators, *J. Optim. Theory Appl.*, **182** (2019), 558–572. https://doi.org/10.1007/s10957-018-1314-5

30. C. Ran, X. Xu, C. Hou, X. Zhang, Numerical approximation of fourth-order fractional diffusion-wave systems using finite difference and discontinuous Galerkin method, *Netw. Heterog. Media*, **20** (2025), 1346–1366. https://doi.org/10.3934/nhm.2025058

31. X. Zhang, H. Wang, Z. Luo, L. Wei, A high-accuracy compact finite difference scheme for time-fractional diffusion equations, *Revista de la Unión Matemática Argentina*, **68** (2025), 589–609. https://doi.org/10.33044/revuma.4665

32. H.B. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **257** (2015), 344–354. https://doi.org/10.1016/j.amc.2014.10.083

33. C. C. Travis, G. F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hung.*, **32** (1978), 75–96. https://doi.org/10.1007/BF01902205

34. Y. Zhou, J. W. He, A Cauchy problem for fractional evolution equations with Hilfer's fractional derivative on semi-infinite interval, *Fract. Calc. Appl. Anal.*, **25** (2022), 924–961. https://doi.org/10.1007/s13540-022-00057-9

35. C. M. Marle, *Measures et Probabilities*, Paris: Hermann, 1974.

36. N. I. Mahmudov, A. Denker, On controllability of linear stochastic systems, *Int. J. Control*, **73** (2000), 144–151. https://doi.org/10.1080/002071700219849

37. R. W. H. Sargent, Optimal control, *J. Comput. Appl. Math.*, **124** (2000), 361–371. https://doi.org/10.1016/S0377-0427(00)00418-0

38. E. J. Balder, Necessary and sufficient conditions for L1-strong-weak lower semicontinuity of integral functionals, *Nonlinear Anal.-Theor.*, **11** (1987), 1399–1404. https://doi.org/10.1016/0362-546X(87)90092-7

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