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Research article

On constant-level set constrained robust multi-dimensional controlled models

Savin Treanţă[1,2,3,*] **and Cristina-Florentina Pîrje**[1]

[1] Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

[2] Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania

[3] Fundamental Sciences Applied in Engineering-Research Center, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

* **Correspondence:** Email: savin.treanta@upb.ro.

Abstract: This paper investigated a new family of robust multidimensional controlled models with constant-level set constraints. More precisely, we considered the extremization of a functional, given by a controlled multiple integral involving an uncertain parameter, subject to a finite set of constraints determined by path-independent curvilinear integral functionals. Necessary and sufficient optimality criteria were established for a robust feasible point. In addition, a saddle-point-type characterization of robust optimal points was studied in the second part of the paper.

Keywords: constant-level set constraints; robust optimal point; robust optimality criteria; convex functional; pseudoconvex functional
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1. Introduction

Many practical applications in the engineering field can be turned into optimization models. As the uncertain data encountered in the modeling process is often obtained by measurement or estimation, this is accompanied by the generation of estimation errors, truncation errors, etc. The accumulation of errors may render the results meaningless. The robust method is crucial for addressing optimization problems under data uncertainty in such situations. This approach seeks to minimize the impact of the highest possible uncertainty in the problem.

Extremization problems with isoperimetric-type restrictions have attracted significant attention from researchers in various fields due to their relevance in applied sciences. Notable contributions include the research by Schmitendorf [1], Forster and Long [2], and Benner et al. [3]. Specifically, Schmitendorf [1] used Pontryagin's principle to establish necessary optimality conditions for a class of isoperimetric constrained control problems with inequality constraints at the terminal time. Forster and Long [2], building on Schmitendorf's work [1], investigated the same control problem and formulated necessary optimality conditions using an alternative transformation technique. Recently, Benner et al. [3] tackled a control problem related to nonlinear chemical reactions, focusing on bang-bang control strategies for periodic trajectories under isoperimetric constraints. Numerous researchers (referencing works such as [4–6]) have explored specific controlled processes through the utilization of functionals with ordinary differential equation/partial differential equation (in short, ODE/PDE) or mixed constraints. Specifically, they have introduced and studied various categories of optimization problems controlled by multiple and path-independent curvilinear integral functionals, which incorporate mixed constraints comprising first-order PDEs of m-flow type and partial differential inequalities. Various studies on variational problems have concentrated on finding optimal solution procedures in robust control and interval analysis (see, for example, [7–10]). Recent contributions on uncertain nonlinear triangular impulsive systems and multi-agent consensus tracking control can be read in [11, 12].

On the other hand, the saddle-point technique has been used to solve various types of extremization problems. Thus, Antczak [13] established the optimality of feasible solutions for multi-objective programming problems using saddle-point criteria. Jayswal et al. [14] applied the robust penalty function method to determine the optimal solution for uncertain control optimization problems, revealing relationships between the associated uncertain Lagrange function and the penalized problem. Furthermore, the authors explored optimal solutions for generalized convex multidimensional control optimization problems using a specific class of penalty function methods. More recently, Treanţă [15] introduced a modified multidimensional variational control problem with mixed constraints and achieved optimality through the saddle-point technique associated with the corresponding Lagrange-type functional.

Motivated and inspired by the research works conducted by Hestenes [16], Lee [17], and Treanţă [18, 19], in the current study, we investigate the robust multidimensional controlled models with constant level set constraints. More precisely, by establishing the robust necessary optimality conditions associated with the considered problem (P), we further demonstrate that these are also robust sufficient optimality conditions of (P) under suitable hypotheses. Thereafter, by considering the notions of convexity, pseudoconvexity, and robust saddle-point associated with a Lagrange-type function, we establish some connections between robust optimal points to the problem (P) and robust saddle-points. Compared to the studies done until now, the present work is totally original in the following aspects: the class of optimization problems defined in this article is new due to the simultaneous presence of both multiple integral type functionals and curvilinear integral type functionals; the presence of an uncertainty parameter within the objective functionals; and the analysis of solutions of the considered problem by using the saddle-point technique associated with a Lagrange-type functional. For example, Treanţă [18] used a Lagrange-Hamilton approach (without saddle-point technique and without uncertain parameters) to investigate some classes of optimization problems with isoperimetric type constraints. In addition, Treanţă [18] recently provided necessary and sufficient optimality conditions for some robust variational problems but without constant-level set constraints.

The article is structured as follows. Section 2 introduces important preliminaries and addresses the formulation of the variational controlled model (P) with constant-level set constraints. Section 3 includes the robust necessary optimality conditions for (P). In addition, we demonstrate that robust necessary optimality conditions also serve as robust sufficient optimality conditions under appropriate hypotheses. In Section 4, by using the concept of robust saddle-point for a Lagrange-type functional associated with (P), we establish connections between robust optimal points of the problem (P) and the robust saddle-points of the corresponding uncertain Lagrange-type functional. Section 5 provides the conclusions of this paper.

2. Notations, problem formulation, and preliminaries

In this section, some basic notations, problem formulation, and certain preliminaries are considered. Thus, we use the mathematical objects:

- $\mathbb{R}^n$ is the finite-dimensional Euclidean space with dimension n;
- $t = (t^\sigma) = (t^1, t^2, \cdots, t^m) \in D_{t_0,t_1} \subset \mathbb{R}^m_+,\ \sigma = \overline{1,m}$, is the multi-time or multiple parameter of evolution, with D_{t_0,t_1} as a hyper-parallelepiped determined by the diagonally opposite corners t_0 and t_1;
- $d\omega := dt^1 \cdots dt^m$ is the volume element in $\mathbb{R}^m$;
- M is the family of all functions $a(t) = (a^\iota(t)) : D_{t_0,t_1} \mapsto \mathbb{R}^n,\ \iota = \overline{1,n}$ (C^2-class state variables), equipped with the uniform norm; the differentiability of second order for a is a hypothesis requested in the next section, where we establish the Euler-Lagrange system of equations (see relation (3.1), for instance);
- N is the family of all functions $c(t) = (c^s(t)) : D_{t_0,t_1} \mapsto \mathbb{R}^k,\ s = \overline{1,k}$ (piecewise continuous control variables), endowed with the uniform norm, as well.

For simplicity, we use the notations given as follows:

$$(t, a(t), a_\sigma(t), c(t)) = \phi_{(a,c)}, \quad (t, a^0(t), a^0_\sigma(t), c^0(t)) = \phi_{(a^0,c^0)},$$

$$(t, a(t), a_\sigma(t), c(t), x(t), x_\gamma(t), \nu(t)) = \kappa_{(a,x,\nu)}(t),$$

where $a_\sigma(t) := \frac{\partial a}{\partial t^\sigma}(t),\ x_\gamma(t) := \frac{\partial x}{\partial t^\gamma},\ \sigma, \gamma \in \{1, 2, \cdots, m\}$.

Considering the above mathematical tools, we formulate the following variational controlled model with constant-level set constraints:

$$(P) \quad \min_{(a(\cdot),c(\cdot))} \int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u\right) d\omega$$

subject to

$$\int_{\Upsilon_{t_0,t_1}} g^\beta_\gamma \left(\phi_{(a,c)}\right) dt^\gamma = I^\beta, \quad \beta = \overline{1,q}, \ \ a(t_0) = a_{t_0}, \ \ a\left(t_1\right) = a_{t_1}, \quad t \in D_{t_0,t_1},$$

where $I^\beta \in \mathbb{R}, \beta = \overline{1,q}, u \in B \subset \mathbb{R}$ is an uncertain parameter, the functions $f : J^1(D_{t_0,t_1}, M) \times N \times B \mapsto \mathbb{R},\ g^\beta_\gamma : J^1(D_{t_0,t_1}, M) \times N \mapsto \mathbb{R}^m,\ \beta = \overline{1,q}$, are C^2-class functionals, Υ_{t_0,t_1} is a differentiable curve, contained in D_{t_0,t_1}, joining the points $t_0,\ t_1 \in \mathbb{R}^m_+$, and

$$\begin{aligned} & g^\beta_\gamma \left(t, a(t), a_\sigma(t), c(t)\right) dt^\gamma \\ := {} & g^\beta_1 \left(t, a(t), a_\sigma(t), c(t)\right) dt^1 + \cdots + g^\beta_m \left(t, a(t), a_\sigma(t), c(t)\right) dt^m, \quad \beta = 1, 2, \cdots, q \end{aligned}$$

are complete integrable differential 1-forms (C^1-class functions), that is, $D_\gamma g_\zeta = D_\zeta g_\gamma$, $\gamma, \zeta \in \{1, \cdots, m\}$, $\gamma \neq \zeta$, with $D_\gamma := \dfrac{\partial}{\partial t^\gamma}$, $\gamma \in \{1, 2, \cdots, m\}$ (see $J^1(A, B)$ as the jet bundle of first order associated to A and B).

The robust counterpart, that reduces the possible uncertainties in the cost functional, associated with the problem (P), is given by

$$\text{(RP)} \quad \min_{(a(\cdot),c(\cdot))} \int_{D_{t_0,t_1}} \max_{u\in B} f\left(\phi_{(a,c)}, u\right) d\omega$$

subject to

$$\int_{\Upsilon_{t_0,t_1}} g_\gamma\left(\phi_{(a,c)}\right) dt^\gamma = I,\ a\left(t_0\right) = a_{t_0},\ a\left(t_1\right) = a_{t_1},\ t \in D_{t_0,t_1},$$

with $f, g = \left(g_\gamma^\beta\right), I = \left(I^\beta\right)$ as defined in (P).

Let

$$\mathcal{D} = \left\{(a, c) \in M \times N \mid \int_{\Upsilon_{t_0,t_1}} g_\gamma\left(\phi_{(a,c)}\right) dt^\gamma = I, a\left(t_0\right) = a_{t_0}, a\left(t_1\right) = a_{t_1}, t \in D_{t_0,t_1}\right\}$$

be the set of all feasible points to the problem (P).

Definition 2.1. A feasible point $\left(a^0, c^0\right)$ is said to be a robust optimal point of (P) if the inequality

$$\int_{D_{t_0,t_1}} \max_{u\in B} f\left(\phi_{(a^0,c^0)}, u\right) d\omega \leq \int_{D_{t_0,t_1}} \max_{u\in B} f\left(\phi_{(a,c)}, u\right) d\omega$$

holds, for all $(a, c) \in \mathcal{D}$.

Further, we introduce the curve $\Upsilon_{t_0,t} \subset \Upsilon_{t_0,t_1}$ determined by the following variables:

$$x^\beta(t) = \int_{\Upsilon_{t_0,t}} g_\gamma^\beta\left(y, a(y), a_\sigma(y), c(y)\right) dy^\gamma, \quad \beta = 1, 2, \cdots, q,$$

satisfying $x^\beta(t_0) = 0$, $x^\beta(t_1) = I^\beta$. We obtain that x^β fulfills the complete integrable PDEs of first-order

$$\frac{\partial x^\beta}{\partial t^\gamma}(t) = g_\gamma^\beta\left(t, a(t), a_\sigma(t), c(t)\right), \quad x^\beta(t_1) = I^\beta.$$

Now, by defining the multiplier $\nu = \left(\nu_\beta^\gamma(t)\right) \in \mathbb{R}_+^{qm}$ and denoting $x = (x^\beta(t))$, we introduce the following controlled Lagrangian (see summation over repeated indices):

$$\begin{aligned} &F\left(t, a(t), a_\sigma(t), c(t), x(t), x_\gamma(t), \nu(t), u\right) \\ &= f\left(t, a(t), a_\sigma(t), c(t), u\right) + \nu_\beta^\gamma(t)\left(g_\gamma^\beta\left(t, a(t), a_\sigma(t), c(t)\right) - \frac{\partial x^\beta}{\partial t^\gamma}(t)\right). \end{aligned}$$

Thus, we changed the original constrained extremization problem (with constant-level set constraints given by controlled curvilinear integrals) into a free-constraint optimization problem

$$\text{(RP)'} \quad \min_{(a(\cdot),\, c(\cdot),\, x(\cdot),\, \nu(\cdot))} \int_{D_{t_0,t_1}} F\left(t, a(t), a_\sigma(t), c(t), x(t), x_\gamma(t), \nu(t), u^0\right) d\omega$$

$$a(t_0) = a_{t_0}, \quad a(t_1) = a_{t_1},$$

$$x(t_0) = 0, \quad x(t_1) = I,$$

where $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$.

Following Lagrange's theory, it is well-known that an optimal point of (RP) (defined above) is found between the optimal points of (RP)'.

3. Robust optimality criteria for (P)

In this section, we present the robust necessary optimality criteria for the considered problem (P) on the line of Treanţă [6]. Moreover, we show that the mentioned robust necessary optimality conditions are robust sufficient criteria under some suitable hypotheses.

Theorem 3.1. If $\left(a^0(t), c^0(t), x(t), \nu(t)\right)$ is a robust optimal point to the problem (RP)', with $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$, then $\left(a^0(t), c^0(t), x(t), \nu(t)\right)$ solves the Euler-Lagrange system of equations

$$F_a\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) - D_\sigma F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) = 0, \tag{3.1}$$

$$F_c\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) = 0, \tag{3.2}$$

$$F_x\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) - D_\gamma F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) = 0, \tag{3.3}$$

$$F_\nu\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) = 0, \tag{3.4}$$

for all $t \in \Omega_{t_0,t_1}$, except at discontinuity points.

Proof. Let $\left(a^0(t), c^0(t), x(t), \nu(t)\right)$ be a robust optimal point of (RP)'. We consider an admissible control variation $c(t,\pi) = c^0(t) + \pi r(t)$, for sufficiently small $\pi > 0$, where $r(t) = (r^s(t)), s = \overline{1,k}$, is a C^1-class function with $r(t_0) = r(t_1) = 0$, and $a(t,\pi) = a^0(t) + \pi h(t)$ is the associated state variation, with $h(t) = (h^\iota(t)), \iota = \overline{1,n}$, a C^1-class function with $h(t_0) = h(t_1) = 0$. Also, let $\nu(t,\pi) = \nu(t) + \pi l(t)$ be a C^1-class variation of $\nu(t)$, and $x(t,\pi) = x(t) + \pi\zeta(t)$ be a C^1-class variation of $x(t)$ with $l(t_0) = l(t_1) = \zeta(t_0) = \zeta(t_1) = 0$. We obtain, by using these variations in F, a controlled integral function $I(\pi)$, given as follows:

$$I(\pi) = \int_{D_{t_0,t_1}} F\left(\kappa_{(a,x,\nu)}(t,\pi), u\right) d\omega,$$

and since $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$, we get

$$I(\pi) = \int_{D_{t_0,t_1}} F\left(\kappa_{(a,x,\nu)}(t,\pi), u^0\right) d\omega.$$

Differentiating with respect to π and taking $\pi = 0$, we get

$$\begin{aligned} \dot{I}(0) = &\int_{D_{t_0,t_1}} F_a\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) a_\pi(t,0) d\omega + \int_{D_{t_0,t_1}} F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) a_{\sigma_\pi}(t,0) d\omega \\ &+ \int_{D_{t_0,t_1}} F_c\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) c_\pi(t,0) d\omega + \int_{D_{t_0,t_1}} F_x\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) x_\pi(t,0) d\omega \end{aligned}$$

$$+\int_{D_{t_0,t_1}} F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)x_{\gamma_\pi}(t,0)d\omega+\int_{D_{t_0,t_1}} F_{\nu}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\nu_\pi(t,0)d\omega$$

$$=\int_{D_{t_0,t_1}} F_a\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)h(t)d\omega+\int_{D_{t_0,t_1}} F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)h_\sigma(t)d\omega \tag{3.5}$$

$$+\int_{D_{t_0,t_1}} F_c\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)r(t)d\omega+\int_{D_{t_0,t_1}} F_x\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\zeta(t)d\omega$$

$$+\int_{D_{t_0,t_1}} F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\zeta_\gamma(t)d\omega+\int_{D_{t_0,t_1}} F_{\nu}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)l(t)d\omega.$$

Now, by using the integration by parts, we obtain the following relations:

$$\int_{D_{t_0,t_1}} F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)h_\sigma(t)d\omega=\left[h(t)F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\right]_{t_0}^{t_1}-\int_{D_{t_0,t_1}} D_\sigma F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)h(t)d\omega,$$

which, by assumption $h(t_0)=h\left(t_1\right)=0$, yields

$$\int_{D_{t_0,t_1}} F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)h_\sigma(t)d\omega=-\int_{D_{t_0,t_1}} D_\sigma F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)h(t)d\omega.$$

Similarly, we have

$$\int_{D_{t_0,t_1}} F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\zeta_\gamma(t)d\omega=-\int_{D_{t_0,t_1}} D_\gamma F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\zeta(t)d\omega.$$

Given the abovementioned equalities, relation in (3.5) can be rewritten as

$$\dot{I}(0)=\int_{D_{t_0,t_1}}\left(F_a\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)-D_\sigma F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\right)h(t)d\omega+\int_{D_{t_0,t_1}} F_c\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)r(t)d\omega$$
$$+\int_{D_{t_0,t_1}}\left(F_x\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)-D_\gamma F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\right)\zeta(t)d\omega+\int_{D_{t_0,t_1}} F_\nu\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)l(t)d\omega.$$

Since $\left(a^0(t),c^0(t),x(t),\nu(t)\right)$ is a robust optimal point of (RP)', so, $\pi=0$ is an optimal point for $I(\pi)$, involving

$$\begin{aligned}\dot{I}(0)=&\int_{D_{t_0,t_1}}\left(F_a\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)-D_\sigma F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\right)h(t)d\omega\\&+\int_{D_{t_0,t_1}} F_c\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)r(t)d\omega\\&+\int_{D_{t_0,t_1}}\left(F_x\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)-D_\gamma F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)\right)\zeta(t)d\omega\\&+\int_{D_{t_0,t_1}} F_\nu\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)l(t)d\omega=0.\end{aligned}$$

Now, since $h(t)$, $r(t)$, $\zeta(t)$, and $l(t)$ are continuous functions and vanish at boundary points, we obtain

$$F_a\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)-D_\sigma F_{a_\sigma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)=0,\quad F_c\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)=0,$$

$$F_x\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)-D_\gamma F_{x_\gamma}\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)=0,\ \ F_\nu\left(\kappa_{(a^0,x,\nu)}(t),u^0\right)=0,$$

by using a fundamental lemma of the calculus of variation (see, for instance, Jayswal et al. [20]), and the proof is complete.

Remark 3.1. (i) Equations (3.1)–(3.4) are called robust necessary optimality conditions for (P).

(ii) Equations (3.1)–(3.4) of Euler-Lagrange type can be written as

$$f_a\left(\phi_{(a^0,c^0)},u^0\right)-D_\sigma f_{a_\sigma}\left(\phi_{(a^0,c^0)},u^0\right)+\nu(t)\left(g_a\left(\phi_{(a^0,c^0)}\right)-D_\sigma g_{a_\sigma}\left(\phi_{(a^0,c^0)}\right)\right)=0,\quad t\in D_{t_0,t_1},\tag{3.6}$$

$$f_c\left(\phi_{(a^0,c^0)},u^0\right)+\nu(t)g_c\left(\phi_{(a^0,c^0)}\right)=0,\quad t\in D_{t_0,t_1},\tag{3.7}$$

$$\nu_\gamma(t)=0,\quad t\in D_{t_0,t_1},\tag{3.8}$$

$$x^\beta_\gamma(t)=g^\beta_\gamma\left(\phi_{(a^0,c^0)}\right),\quad t\in D_{t_0,t_1},\tag{3.9}$$

called, as well, robust necessary optimality conditions for (P). More important is that the Lagrange multiplier ν is a constant (see (3.8)).

(iii) The application of curvilinear integrals (independent of the path) in the computation of mechanical work is well known. Also, the application of multiple integrals is in the computation of areas (Riemann-Green formula) or volumes (Gauss-Ostrogradski formula). In this article, we combine these types of integrals and, thus, obtain multiple applications in engineering and mechanics.

Illustrative application. Let us find the extremals of the following constrained double integral functional:

$$\frac{1}{2}\int_{D_{t_0,t_1}}\left(a^2_{t^1}+a^2_{t^2}+u\right)dt^1dt^2$$

subject to $\int_{\Upsilon_{t_0,t_1}}(a^2_{t^1}+c)dt^1+(a^2_{t^2}+c)dt^2=1$, $u\in[0,1]$, and the boundary conditions $a(0,0)=0$, $a(1,1)=1$, where $D_{t_0,t_1}=[0,1]^2$ and Υ_{t_0,t_1} is a curve of C^1-class joining the points $t_0=(0,0)$, $t_1=(1,1)$.

The auxiliary Lagrangian is

$$F=\frac{1}{2}\left(a^2_{t^1}+a^2_{t^2}+u\right)+\frac{1}{2}p(t^1,t^2)\left(\frac{\partial x}{\partial t^1}-a^2_{t^1}-c\right)+\frac{1}{2}q(t^1,t^2)\left(\frac{\partial x}{\partial t^2}-a^2_{t^2}-c\right).$$

We impose $p,q=const$. Since

$$\frac{\partial F}{\partial a}=0,\ \frac{\partial F}{\partial a_{t^1}}=a_{t^1}-pa_{t^1},\ \frac{\partial F}{\partial a_{t^2}}=a_{t^2}-pa_{t^2},\ \frac{\partial F}{\partial c}=-\frac{1}{2}p-\frac{1}{2}q,$$

the extremals are described by the following system of PDEs:

$$\frac{\partial}{\partial t^1}\left(a_{t^1}-pa_{t^1}\right)+\frac{\partial}{\partial t^2}\left(a_{t^2}-qu_{t^2}\right)=0,\ p,q=const.,$$

$$\frac{\partial x}{\partial t^1}-a^2_{t^1}-c=0,\ \frac{\partial x}{\partial t^2}-a^2_{t^2}-c=0,\ p+q=0.$$

The path independent condition for the curvilinear integral gives us the PDE $c_{t^1}=c_{t^2}$, and the multipliers are given by the following relations:

$$p+q=0,\ p,q=const.$$

Definition 3.1. (Jayswal et al. [20]) A multiple integral-type functional $\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u\right) d\omega$ is named convex at $\left(a^0, c^0\right)$ if

$$\begin{aligned}&\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega - \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega \\ &\geq \int_{D_{t_0,t_1}} \Big[f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t) - a^0(t)\right) + f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t) - a^0_\sigma(t)\right) \\ &\quad + f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t) - c^0(t)\right)\Big] d\omega\end{aligned}$$

is satisfied, for all $(a, c) \in M \times N$, with $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$.

Definition 3.2. (Jayswal et al. [20]) A functional $\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u\right) d\omega$ is said to be pseudoconvex at $\left(a^0, c^0\right)$ if the inequality

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega - \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega < 0$$

implies

$$\begin{aligned}&\int_{D_{t_0,t_1}} \Big[f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t) - a^0(t)\right) + f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t) - a^0_\sigma(t)\right) \\ &+ f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t) - c^0(t)\right)\Big] d\omega < 0,\end{aligned}$$

for all $(a, c) \in M \times N$, with $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$, or, equivalently,

$$\begin{aligned}&\int_{D_{t_0,t_1}} \Big[f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t) - a^0(t)\right) + f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t) - a^0_\sigma(t)\right) \\ &+ f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t) - c^0(t)\right)\Big] d\omega \geq 0\end{aligned}$$

implies

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega - \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega \geq 0,$$

for all $(a, c) \in M \times N$, with $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$.

The next theorem states robust sufficient optimality criteria that guarantee the optimality of a feasible point in (P) under the hypothesis of convexity for the involved functionals.

Theorem 3.2. Let $\left(a^0, c^0\right)$ be a feasible point to (P) and there exist the Lagrange multipliers $\nu^0(t) = \left(\nu^{\gamma 0}_\beta(t)\right) \in R^{qm}_+$ and the auxiliary variables $x^\beta_\gamma(t)$ such that the robust necessary optimality conditions (3.6)–(3.9) hold, for all $t \in D_{t_0,t_1}$, with $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$. Further, assume that the integral functionals

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega$$

and

$$\int_{D_{t_0,t_1}} \nu_\beta^{\gamma 0}(t)\left[g_\gamma^\beta\left(\phi_{(a,c)}\right)-x_\gamma^\beta(t)\right]d\omega$$

are convex on $\mathcal{D}$. Then, $\left(a^0, c^0\right)$ is a robust optimal point of (P).

Proof. We proceed by contradiction and assume that $\left(a^0, c^0\right)$ is not a robust optimal point of (P). Then, there exists $(a, c) \in \mathcal{D}$ such that

$$\int_{D_{t_0,t_1}} \max_{u\in B} f\left(\phi_{(a,c)}, u\right)d\omega < \int_{D_{t_0,t_1}} \max_{u\in B} f\left(\phi_{\left(a^0,c^0\right)}, u\right)d\omega,$$

and, since $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$, we get

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right)d\omega < \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right)d\omega. \tag{3.10}$$

Since $\left(a^0, c^0\right)$ satisfies the robust necessary optimality conditions given in (3.6)–(3.9), therefore, by multiplying the equalities (3.6) and (3.7) with $\left(a-a^0\right)$ and $\left(c-c^0\right)$, respectively, and combining them, we obtain

$$\begin{aligned}
&\int_{D_{t_0,t_1}} \Big\{f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t)-a^0(t)\right)+f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\\
&+\nu^0(t)\left(g_a\left(\phi_{\left(a^0,c^0\right)}\right)\left(a(t)-a^0(t)\right)+g_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\right)\Big\}d\omega\\
&+\int_{D_{t_0,t_1}} \Big\{f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t)-c^0(t)\right)+\nu^0(t)g_c\left(\phi_{\left(a^0,c^0\right)}\right)\left(c(t)-c^0(t)\right)\Big\}d\omega=0.
\end{aligned} \tag{3.11}$$

On the other hand, using the hypothesis that

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right)d\omega$$

is convex on $\mathcal{D}$, we get

$$\begin{aligned}
&\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right)d\omega-\int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right)d\omega\\
\geq&\int_{D_{t_0,t_1}} \Big\{f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t)-a^0(t)\right)+f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\\
&+f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t)-c^0(t)\right)\Big\}d\omega,
\end{aligned}$$

which, together with the inequality in (3.10), yields

$$\begin{aligned}
&\int_{D_{t_0,t_1}} \Big\{f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t)-a^0(t)\right)+f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\\
&+f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t)-c^0(t)\right)\Big\}d\omega<0.
\end{aligned} \tag{3.12}$$

Again, by the assumption that the integral functional

$$\int_{D_{t_0,t_1}} \nu_\beta^{\gamma 0}(t)\left[g_\gamma^\beta\left(\phi_{(a,c)}\right)-x_\gamma^\beta(t)\right] d\omega$$

is convex on $\mathcal{D}$, we get

$$\begin{aligned}
&\int_{D_{t_0,t_1}} \nu_\beta^{\gamma 0}(t)\left[g_\gamma^\beta\left(\phi_{(a,c)}\right)-x_\gamma^\beta(t)\right] d\omega-\int_{D_{t_0,t_1}} \nu_\beta^{\gamma 0}(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right)-x_\gamma^\beta(t)\right] d\omega \\
&\geq \int_{D_{t_0,t_1}}\left\{\nu^0(t)\left(g_a\left(\phi_{\left(a^0,c^0\right)}\right)\left(a(t)-a^0(t)\right)+g_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\right)\right. \\
&\left.+\nu^0(t) g_c\left(\phi_{\left(a^0,c^0\right)}\right)\left(c(t)-c^0(t)\right)\right\} d\omega .
\end{aligned}$$

The above inequality, together with the feasibility of (a, c) in (P) and the robust necessary optimality conditions given in (3.8) and (3.9), implies

$$\begin{aligned}
&\int_{D_{t_0,t_1}}\left\{\nu^0(t)\left(g_a\left(\phi_{\left(a^0,c^0\right)}\right)\left(a(t)-a^0(t)\right)+g_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\right)\right. \\
&\left.+\nu^0(t) g_c\left(\phi_{\left(a^0,c^0\right)}\right)\left(c(t)-c^0(t)\right)\right\} d\omega \leq 0 . \qquad (3.13)
\end{aligned}$$

On adding the inequalities (3.12) and (3.13), we obtain

$$\begin{aligned}
&\int_{D_{t_0,t_1}}\left\{f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t)-a^0(t)\right)+f_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\right. \\
&\left.+\nu^0(t)\left(g_a\left(\phi_{\left(a^0,c^0\right)}\right)\left(a(t)-a^0(t)\right)+g_{a_\sigma}\left(\phi_{\left(a^0,c^0\right)}\right)\left(a_\sigma(t)-a_\sigma^0(t)\right)\right)\right\} d\omega \\
&+\int_{D_{t_0,t_1}}\left\{f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t)-c^0(t)\right)+\nu^0(t) g_c\left(\phi_{\left(a^0,c^0\right)}\right)\left(c(t)-c^0(t)\right)\right\} d\omega<0,
\end{aligned}$$

which contradicts the relation in (3.11). This completes the proof.

4. Robust saddle-point technique for (P)

In this section, we define the notion of robust saddle-point associated with the Lagrange functional $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$, introduced in the previous section, and then establish its connections with the robust optimal point of (P).

Definition 4.1. A point $\left(a^0(t), c^0(t), \bar{x}(t), \bar{\nu}(t)\right)$ is said to be a robust saddle-point of the Lagrange functional $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$, defined for the problem (P), if the following inequalities:

$$(i)\ F\left(\kappa_{\left(a^0,x,\nu\right)}(t), u^0\right) \leq F\left(\kappa_{\left(a^0,\bar{x},\bar{\nu}\right)}(t), u^0\right),$$

$$(ii)\ F\left(\kappa_{\left(a^0,\bar{x},\bar{\nu}\right)}(t), u^0\right) \leq F\left(\kappa_{(a,\bar{x},\bar{\nu})}(t), u^0\right),$$

are satisfied $\forall (a, c) \in \mathcal{D},\ \nu \in \mathbb{R}_+^{qm},\ x \in \mathbb{R}^{qm},\ \max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$.

The following result, by using the notion of robust saddle-point associated with a Lagrange-type functional, formulates a characterization of robust optimal points in the considered extremization problem (P). Most importantly, the next theorem does not require the well-known assumption of pseudoconvexity (see Definition 3.2) for the objective (cost) functional in (P).

Theorem 4.1. Let $\left(a^0(t), c^0(t), \bar{x}(t), \bar{\nu}(t)\right)$ be a robust saddle-point of $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$, with $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$. Then, $\left(a^0(t), c^0(t)\right)$ is a robust optimal point to the problem (P).

Proof. Suppose, contrary to the result, that $\left(a^0(t), c^0(t)\right)$ is not a robust optimal point of (P). Therefore, we have a feasible point $(a(t), c(t)) \in \mathcal{D}$, fulfilling

$$\int_{D_{t_0,t_1}} \max f\left(\phi_{(a,c)}, u\right) d\omega < \int_{D_{t_0,t_1}} \max f\left(\phi_{\left(a^0,c^0\right)}, u\right) d\omega,$$

and, by considering $\max_{u\in B} f(\cdot, u) = f\left(\cdot, u^0\right)$, we have

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega < \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega. \tag{4.1}$$

Since $\left(a^0(t), c^0(t), \bar{x}(t), \bar{\nu}(t)\right)$ is a saddle-point of $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$, then the relation (i) gives

$$F\left(\kappa_{\left(a^0,x,\nu\right)}(t), u^0\right) \le F\left(\kappa_{\left(a^0,\bar{x},\bar{\nu}\right)}(t), u^0\right),$$

or, equivalently,

$$\int_{D_{t_0,t_1}} \left\{f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) + \nu_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - x_\gamma^\beta(t)\right]\right\} d\omega$$
$$\le \int_{D_{t_0,t_1}} \left\{f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) + \bar{\nu}_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}_\gamma^\beta(t)\right]\right\} d\omega,$$

implying the inequality

$$\int_{D_{t_0,t_1}} \nu_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - x_\gamma^\beta(t)\right] d\omega \le \int_{D_{t_0,t_1}} \bar{\nu}_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}_\gamma^\beta(t)\right] d\omega. \tag{4.2}$$

For $\nu(t) = 0$, then the inequality (4.2) gives

$$\int_{D_{t_0,t_1}} \bar{\nu}_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}_\gamma^\beta(t)\right] d\omega \ge 0. \tag{4.3}$$

By using the relations given in (4.1) and (4.3), we get

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega + \int_{D_{t_0,t_1}} \bar{\nu}_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}_\gamma^\beta(t)\right] d\omega$$
$$< \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega + \int_{D_{t_0,t_1}} \bar{\nu}_\beta^\gamma(t)\left[g_\gamma^\beta\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}_\gamma^\beta(t)\right] d\omega. \tag{4.4}$$

Now, since $\left(a^0(t), c^0(t), \bar{x}(t), \bar{\nu}(t)\right)$ is a saddle-point of $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$, then the relation (ii) yields

$$F\left(\kappa_{\left(a^0,\bar{x},\bar{\nu}\right)}(t), u^0\right) \le F\left(\kappa_{(a,\bar{x},\bar{\nu})}(t), u^0\right)$$

or, equivalently,

$$\int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega + \int_{D_{t_0,t_1}} \bar{\nu}^{\gamma}_{\beta}(t)\left[g^{\beta}_{\gamma}\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}^{\beta}_{\gamma}(t)\right] d\omega$$

$$\leq \int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega + \int_{D_{t_0,t_1}} \bar{\nu}^{\gamma}_{\beta}(t)\left[g^{\beta}_{\gamma}\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}^{\beta}_{\gamma}(t)\right] d\omega,$$

which contradicts the inequality (4.4). This completes the proof.

The next result provides a characterization of robust saddle-points in the Lagrange function $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$ by considering robust optimal points in (P) and auxiliary hypotheses of the implied tools.

Theorem 4.2. Consider $\left(a^0(t), c^0(t)\right)$ is a robust optimal point of (P) and assume that there exists $\bar{\nu}^{\gamma}_{\beta}(t) \in R^{qm}_{+}$ such that the following conditions:

$$f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right) - D_{\sigma} f_{a_{\sigma}}\left(\phi_{\left(a^0,c^0\right)}, u^0\right) + \bar{\nu}(t)\left(g_a\left(\phi_{\left(a^0,c^0\right)}\right) - D_{\sigma} g_{a_{\sigma}}\left(\phi_{\left(a^0,c^0\right)}\right)\right) = 0, \quad t \in D_{t_0,t_1}, \tag{4.5}$$

$$f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right) + \bar{\nu}(t) g_c\left(\phi_{\left(a^0,c^0\right)}\right) = 0, \quad t \in D_{t_0,t_1}, \tag{4.6}$$

$$\bar{\nu}_{\gamma}(t) = 0, \quad t \in D_{t_0,t_1}, \tag{4.7}$$

$$\bar{x}^{\beta}_{\gamma}(t) = g^{\beta}_{\gamma}\left(\phi_{\left(a^0,c^0\right)}\right), \quad t \in D_{t_0,t_1}, \tag{4.8}$$

are fulfilled, and the functionals

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega \quad and \quad \int_{D_{t_0,t_1}} \bar{\nu}^{\gamma}_{\beta}(t)\left[g^{\beta}_{\gamma}\left(\phi_{(a,c)}\right) - \bar{x}^{\beta}_{\gamma}(t)\right] d\omega$$

are convex at $\left(a^0(t), c^0(t)\right)$. Then, $\left(a^0(t), c^0(t), \bar{x}(t), \bar{\nu}(t)\right)$ is a robust saddle-point for the Lagrange function $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$.

Proof. From the convexity hypothesis of the following integral functionals:

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega \quad and \quad \int_{D_{t_0,t_1}} \bar{\nu}^{\gamma}_{\beta}(t)\left[g^{\beta}_{\gamma}\left(\phi_{(a,c)}\right) - \bar{x}^{\beta}_{\gamma}(t)\right] d\omega$$

at $\left(a^0(t), c^0(t)\right)$, we have

$$\int_{D_{t_0,t_1}} f\left(\phi_{(a,c)}, u^0\right) d\omega - \int_{D_{t_0,t_1}} f\left(\phi_{\left(a^0,c^0\right)}, u^0\right) d\omega$$

$$\geq \int_{D_{t_0,t_1}} \left\{f_a\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a(t) - a^0(t)\right) + f_{a_{\sigma}}\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(a_{\sigma}(t) - a^0_{\sigma}(t)\right)\right.$$
$$\left. + f_c\left(\phi_{\left(a^0,c^0\right)}, u^0\right)\left(c(t) - c^0(t)\right)\right\} d\omega$$

and

$$\int_{D_{t_0,t_1}} \bar{\nu}^{\gamma}_{\beta}(t)\left[g^{\beta}_{\gamma}\left(\phi_{(a,c)}\right) - \bar{x}^{\beta}_{\gamma}(t)\right] d\omega - \int_{D_{t_0,t_1}} \bar{\nu}^{\gamma}_{\beta}(t)\left[g^{\beta}_{\gamma}\left(\phi_{\left(a^0,c^0\right)}\right) - \bar{x}^{\beta}_{\gamma}(t)\right] d\omega$$

$$\geq \int_{D_{t_0,t_1}} \left\{ \bar{\nu}(t) \left(g_a \left(\phi_{(a^0,c^0)} \right) \left(a(t) - a^0(t) \right) + g_{a_\sigma} \left(\phi_{(a^0,c^0)} \right) (a_\sigma(t) \right.\right.$$
$$\left.\left. - a^0_\sigma(t) \right) \right) + \bar{\nu}(t) g_c \left(\phi_{(a^0,c^0)} \right) \left(c(t) - c^0(t) \right) \Big\} \, d\omega.$$

On adding the above two inequalities and using the conditions (4.5) and (4.6), we obtain

$$\int_{D_{t_0,t_1}} \left\{ f\left(\phi_{(a,c)}, u^0\right) + \bar{\nu}^\gamma_\beta(t) \left[g^\beta_\gamma \left(\phi_{(a,c)}\right) - \bar{x}^\beta_\gamma(t) \right] \right\} d\omega$$
$$\geq \int_{D_{t_0,t_1}} \left\{ f\left(\phi_{(a^0,c^0)}, u^0\right) + \bar{\nu}^\gamma_\beta(t) \left[g^\beta_\gamma \left(\phi_{(a^0,c^0)}\right) - \bar{x}^\beta_\gamma(t) \right] \right\} d\omega,$$

or, equivalently,

$$F\left(\kappa_{(a,\bar{x},\bar{\nu})}(t), u^0\right) \geq F\left(\kappa_{(a^0,\bar{x},\bar{\nu})}(t), u^0\right). \tag{4.9}$$

Now, since $\nu(t) \in R^{qm}_+$ and $\left(a^0(t), c^0(t)\right)$ is a feasible point to the problem (P), we get

$$\int_{D_{t_0,t_1}} \nu^\gamma_\beta(t) \left[g^\beta_\gamma \left(\phi_{(a^0,c^0)}\right) - x^\beta_\gamma(t) \right] d\omega = 0.$$

The above equation together with (4.8) implies

$$\int_{D_{t_0,t_1}} \left\{ f\left(\phi_{(a^0,c^0)}, u^0\right) + \nu^\gamma_\beta(t) \left[g^\beta_\gamma \left(\phi_{(a^0,c^0)}\right) - x^\beta_\gamma(t) \right] \right\} d\omega$$

$$\leq \int_{D_{t_0,t_1}} \left\{ f\left(\phi_{(a^0,c^0)}, u^0\right) + \bar{\nu}^\gamma_\beta(t) \left[g^\beta_\gamma \left(\phi_{(a^0,c^0)}\right) - \bar{x}^\beta_\gamma(t) \right] \right\} d\omega,$$

or, equivalently,

$$F\left(\kappa_{(a^0,x,\nu)}(t), u^0\right) \leq F\left(\kappa_{(a^0,\bar{x},\bar{\nu})}(t), u^0\right).$$

In conclusion, the above inequality together with the inequality given in (4.9) concludes that $\left(a^0(t), c^0(t), \bar{x}(t), \bar{\nu}(t)\right)$ is a saddle-point of Lagrange functional $F\left(\kappa_{(a,x,\nu)}(t), u^0\right)$.

5. Conclusions

In this paper, we investigated a class of robust controlled extremization problems determined by multiple integral cost functionals involving an uncertain parameter, subject to constraints generated by path-independent curvilinear integral functionals, formulated as constant-level sets. Also, we stated and proved necessary and sufficient criteria of optimality for a robust feasible point in the considered model (P). Moreover, a saddle-point analysis associated with robust optimal points is included at the end of this study. For other different ideas on isoperimetric problems or fuzzy optimization, the reader can consult Bildhauer [21], Curtis [22], Demyanov and Tamasyan [23], and Shi et al. [24]. Further theoretical developments of the results derived in this paper could be the study of such classes of extermization problems by using the modified objective function approach and, moreover, by considering a generalized convexity.

Author contributions

Savin Treanţă and Cristina-Florentina Pîrje: Conceptualization, Software, Validation, Formal analysis, Investigation, Writing–original draft preparation, Writing–review and editing, Visualization; Cristina-Florentina Pîrje: Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. W. E. Schmitendorf, Pontryagin's principle for problems with isoperimetric constraints and for problems with inequality terminal constraints, *J. Optim. Theory Appl.*, **18** (1976), 561–567. https://doi.org/10.1007/BF00932663
2. B. A. Forster, N. V. Long, Pontryagin's principle for problems with isoperimetric constraints and for problems with inequality terminal constraints: comment, *J. Optim. Theory Appl.*, **25** (1978), 317–322. https://doi.org/10.1007/BF00933221
3. P. Benner, A. Seidel-Morgenstern, A. Zuyev, Periodic switching strategies for an isoperimetric control problem with application to nonlinear chemical reactions, *Appl. Math. Model.*, **69** (2019), 287–300. https://doi.org/10.1016/j.apm.2018.12.005
4. A. Baranwal, A. Jayswal, Preeti, Robust duality for the uncertain multitime control optimization problems, *Int. J. Robust Nonlinear Control*, **32** (2022), 5837–5847. https://doi.org/10.1002/rnc.6113
5. S. Treanţă, Saddle-point optimality criteria in modified variational control problems with partial differential equation constraints, *Optim. Control Appl. Methods*, **41** (2020), 1160–1175. https://doi.org/10.1002/oca.2594
6. S. Treanţă, On a class of isoperimetric constrained controlled optimization problems, *Axioms*, **10** (2021), 112. https://doi.org/10.3390/axioms10020112
7. I. Ahmad, A. Jayswal, S. Al-Homidan, J. Banerjee, Sufficiency and duality in interval-valued variational programming, *Neural Comput. Appl.*, **31** (2019), 4423–4433. https://doi.org/10.1007/s00521-017-3307-y
8. A. Beck, A. Ben-Tal, Duality in robust optimization: primal worst equals dual best, *Oper. Res. Lett.*, **37** (2009), 1–6. https://doi.org/10.1016/j.orl.2008.09.010
9. H. Ishibuch, H. Tanaka, Multiobjective programming in optimization of the interval objective function, *Eur. J. Oper. Res.*, **48** (1990), 219–225. https://doi.org/10.1016/0377-2217(90)90375-L

10. R. E. Moore, *Methods and applications of interval analysis*, Society for Industrial and Applied Mathematics, 1979. https://doi.org/10.1137/1.9781611970906
11. D. B. Fan, X. F. Zhang, C. Y. Wen, Exponential regulation of uncertain nonlinear triangular impulsive systems: a logic-based switching gain approach, *IEEE Trans. Autom. Control*, **70** (2025), 4881–4888. https://doi.org/10.1109/TAC.2025.3545697
12. W. H. Li, S. M. Yan, L. Shi, J. F. Yue, M. J. Shi, B. X. Lin, Multiagent consensus tracking control over asynchronous cooperation-competition networks, *IEEE Trans. Cybernetics*, **55** (2025), 4347–4360. https://doi.org/10.1109/TCYB.2025.3583387
13. T. Antczak, Saddle points criteria in nondifferentiable multiobjective programming with V-invex functions via an η-approximation method, *Comput. Math. Appl.*, **60** (2010), 2689–2700. https://doi.org/10.1016/j.camwa.2010.09.013
14. A. Jayswal, Preeti, M. Arana-Jimenez, An exact l_1 penalty function method for a multitime control optimization problem with data uncertainty, *Optim. Control Appl. Methods*, **41** (2020), 1705–1717. https://doi.org/10.1002/oca.2634
15. S. Treanţă, Robust saddle-point criterion in second-order partial differential equation and partial differential inequation constrained control problems, *Int. J. Robust Nonlinear Control*, **31** (2021), 9282–9293. https://doi.org/10.1002/rnc.5767
16. M. R. Hestenes, *Calculus of variations and optimal control theory*, New York: Wiley, 1966.
17. E. Lee, Linear optimal control problems with isoperimetric constraints, *IEEE Trans. Autom. Control*, **12** (1967), 87–90. https://doi.org/10.1109/TAC.1967.1098488
18. S. Treanţă, Lagrange-Hamilton approach in optimization problems with isoperimetric type constraints, *J. Optim. Theory Appl.*, **194** (2022), 508–520. https://doi.org/10.1007/s10957-022-02036-9
19. S. Treanţă, Necessary and sufficient optimality conditions for some robust variational problems, *Optim. Control Appl. Methods*, **44** (2023), 81–90. https://doi.org/10.1002/oca.2931
20. A. Jayswal, Preeti, S. Treanţă, *Multi-dimensional control problems: Robust approach*, Singapore: Springer, 2022. https://doi.org/10.1007/978-981-19-6561-6
21. M. Bildhauer, M. Fuchs, J. Muller, A reciprocity principle for constrained isoperimetric problems and existence of isoperimetric subregions in convex sets, *Calc. Var. Partial Differ. Equ.*, **57** (2018), 60. https://doi.org/10.1007/s00526-018-1325-y
22. J. P. Curtis, Complementary extremum principles for isoperimetric optimization problems, *Optim. Eng.*, **5** (2004), 417–430. https://doi.org/10.1023/B:OPTE.0000042033.33845.4c
23. V. F. Demyanov, G. S. Tamasyan, Exact penalty functions in isoperimetric problems, *Optimization*, **60** (2011), 153–177. https://doi.org/10.1080/02331934.2010.534166
24. F. F. Shi, G. J. Ye, W. Liu, D. F. Zhao, S. Treanţă, Lagrangian dual theory and stability analysis for fuzzy optimization problems, *Inform. Sci.*, **657** (2024), 119953. https://doi.org/10.1016/j.ins.2023.119953