Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 10518–10532.
DOI:10.3934/math.2026433
Received: 16 February 2026
Revised: 28 March 2026
Accepted: 02 April 2026
Published: 17 April 2026

Research article

On geometric properties of a class of generalized Bazilevič harmonic functions

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Abstract: In this paper, we introduce and study a class of generalized harmonic functions related to the Bazilevič function. We establish necessary and sufficient coefficient conditions, growth estimates, and convex combination properties for this class. Several new results are obtained, extending known properties from analytic to harmonic mappings.

Keywords: harmonic univalent function; Bazilevič harmonic functions; coefficient estimates; growth estimates
Mathematics Subject Classification: 30C45, 30C50, 30C80

1. Introduction

For the sake of simplicity, we specify the following parameters throughout the article:

$$\mu > 0, \eta \geq 0, 0 \leq \lambda < 1,$$

and

$$k \geq 1, k \in \mathbb{N}, |z| = r, r \in (0, 1].$$

Let $\mathcal{H}_k(a)(k \geq 1, a \in \mathbb{R})$ be the class of analytic functions p defined on the unit disk $\mathcal{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and have the form

$$p(z) = a + \sum_{n=k+1}^{\infty} p_n z^n. \tag{1.1}$$

Let $\mathcal{A}_k(k \geq 1)$ denote the class of analytic functions ϕ in $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ having the expansion

$$\phi(z) = z + \sum_{n=k+1}^{\infty} a_n z^n. \tag{1.2}$$

The set of univalent functions in $\mathcal{A}_k$ is denoted by $\mathcal{S}_k$.

Definition 1.1. (analytic Bazilevič function [1]) A function $\phi(z) \in \mathcal{A}_k$ belongs to the class $\mathfrak{L}_k(\mu, \lambda)$ if it satisfies the geometric condition

$$\mathrm{Re}\left\{\phi'(z)\left(\frac{\phi(z)}{z}\right)^{\mu-1}\right\} > \lambda,$$

where the powers are taken as principal values and $\mu > 0, 0 \leq \lambda < 1$.

It is known that $\mathfrak{L}_k(\mu, \lambda) \subset \mathcal{S}_k$ (see [1]). For research results on the class $\mathfrak{L}_k(\mu, \lambda)$, the reader may refer to the following works. Singh [2] established fundamental properties of Bazilevič functions. Deng [3] provided estimates for adjacent coefficients. An overview of univalent function theory, including Bazilevič-type classes, is given by Thomas et. al [4]. Niu and Li [5] investigated Milin coefficient estimation for Bazilevič function. Additionally, Marjono et al. [6] computed the fifth and sixth coefficients for a subclass of Bazilevič functions.

In 1977, Chichra [7] introduced the class $G(\eta; r)$ for $\eta \geq 0$, consisting of regular functions $\phi(z)$ satisfying

$$\mathrm{Re}\left\{(1-\eta)\frac{\phi(z)}{z} + \eta\phi'(z)\right\} > 0,$$

for $|z| < r$ with $0 < r \leq 1$.

Let $\phi(z) \in \mathcal{A}_k$ be given by (1.2). Then, $\phi(z) \in \mathfrak{L}_k(\mu, 0)$ if and only if

$$z\left(\frac{\phi(z)}{z}\right)^{\mu} \in G(\frac{1}{\mu}; r). \tag{1.3}$$

Definition 1.2. Let $\phi(z) \in \mathcal{A}_k$ be given by (1.2). Then, $\phi(z) \in \mathfrak{L}_k^1(\eta, \lambda)$ if and only if

$$\mathrm{Re}\left\{\frac{\phi(z)}{z} + \eta z\left(\frac{\phi(z)}{z}\right)'\right\} > \lambda,$$

where $\eta > 0$ and $0 \leq \lambda < 1$.

It is straightforward to verify that $\phi(z) \in \mathfrak{L}_k(\mu, \lambda)$ $(\mu > 0)$ if and only if

$$z\left(\frac{\phi(z)}{z}\right)^{\mu} \in \mathfrak{L}_k^1(\frac{1}{\mu}, \lambda),$$

where the powers are taken as principal values. The function $\left(\frac{\phi(z)}{z}\right)^{\mu}$ can be expressed as

$$\left(\frac{\phi(z)}{z}\right)^{\mu} = 1 + \sum_{n=k+1}^{\infty} \gamma_{n-1} z^{n-1}, \tag{1.4}$$

where $\gamma_{n-1} := \gamma_{n-1}(a_{k+1}, a_{k+2}, \ldots, a_n; \mu)$ represents a Faber polynomial of degree $n-1$ (see [8]). Clearly, for $\mu = 1$, we have $\gamma_{n-1} = a_n$.

The class H consists of all complex-valued harmonic functions f in $\mathcal{D}$ that possess a unique representation $f(z) = \phi(z) + \overline{\psi(z)}$, where ϕ and ψ are analytic in $\mathcal{D}$ satisfying

$$\phi(0) = 0, \phi'(0) = 1, \psi(0) = 0.$$

The subclass $\mathcal{S}_H \subset H$ consists of those functions that are univalent and sense-preserving in $\mathcal{D}$. Such functions can be expressed in the series form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} \quad (|b_2| < 1).$$

For $k \geq 1$, we define the subclass H_0^k consisting of harmonic functions whose analytic and co-analytic parts vanish to order at the origin:

$$H_0^k = \left\{ f = \phi + \overline{\psi} \in H : \phi^{(\nu)}(0) = \psi^{(\nu)}(0) = 0 \text{ for } 1 \leq \nu \leq k \right\}.$$

Each $f \in H_0^k$ can be expressed as

$$f(z) = \phi(z) + \overline{\psi(z)} = z + \sum_{n=k+1}^{\infty} a_n z^n + \sum_{n=k+1}^{\infty} \overline{b_n z^n}, \quad k \geq 1. \tag{1.5}$$

As established in [9, 10], a harmonic function $f(z) = \phi(z) + \overline{\psi(z)}$ is locally univalent and sense-preserving in $\mathcal{D}$ precisely when $|\phi'(z)| > |\psi'(z)|$.

In 2019, Liu and Yang [11] introduced the subclass $G_H^k(\eta; r) \subset H_0^k$, consisting of functions $f = \phi + \overline{\psi}$ satisfying

$$\operatorname{Re}\left(\frac{\phi(z)}{z} + \eta z \left(\frac{\phi(z)}{z} \right)' \right) > \left| \frac{\psi(z)}{z} + \eta z \left(\frac{\psi(z)}{z} \right)' \right|, |z| < r,$$

where $\eta \geq 0, k \geq 1, r \in (0, 1]$.

For $\mu > 0$ and $f = \phi + \overline{\psi}$ as in (1.5), we define

$$\Phi(z) = z \left(\frac{\phi(z)}{z} \right)^{\mu} = z + \sum_{n=k+1}^{\infty} \gamma_{n-1} z^n, \quad \Psi(z) = z \left(\frac{\psi(z)}{z} \right)^{\mu} = \sum_{n=k+1}^{\infty} \delta_{n-1} z^n, \tag{1.6}$$

where γ_{n-1} and δ_{n-1} are determined via binomial expansion (taking principal values). Clearly, for $\mu = 1$, we have $\gamma_{n-1} = a_n$ and $\delta_{n-1} = b_n$.

Inspired by the above, and in conjunction with Definitions 1.1 and 1.2, we will now proceed to construct a class of generalized Bazilevič harmonic functions.

Definition 1.3. (generalized Bazilevič harmonic functions) A function $f = \phi + \overline{\psi} \in H_0^k$ belongs to the class $\mathfrak{L}_H^k(\mu, \eta, \lambda; r)$ if and only if

$$\operatorname{Re}\left\{ \frac{\Phi(z)}{z} + \eta z \left(\frac{\Phi(z)}{z} \right)' \right\} > \left| \frac{\Psi(z)}{z} + \eta z \left(\frac{\Psi(z)}{z} \right)' \right| + \lambda \tag{1.7}$$

for all $z \in \mathcal{D}$ with $|z| = r$, where $\Phi(z)$ and $\Psi(z)$ are given by (1.6), and all powers are taken as principal values.

According to the above definition, we can deduce special cases and relations as follows.

Remark 1.1. Let $\eta = \frac{1}{\mu}$ in Definition 1.3. We have $f = \phi + \overline{\psi} \in \mathfrak{L}_H^k(\mu, \frac{1}{\mu}, \lambda; r)$ if and only if

$$\operatorname{Re}\left\{ \phi'(z) \left(\frac{\phi(z)}{z} \right)^{\mu-1} \right\} > \left| \psi'(z) \left(\frac{\psi(z)}{z} \right)^{\mu-1} \right| + \lambda. \tag{1.8}$$

Especially, note

$$\mathfrak{L}_H^k(\mu, \frac{1}{\mu}, \lambda; r) := \mathfrak{L}_H^k(\mu, \lambda; r).$$

For $\psi(z) = 0$, we note $\mathfrak{L}_H^k(\mu, \lambda; r) = \mathfrak{L}_k(\mu, \lambda)$.

Remark 1.2. Let $\mu = 1$ in Definition 1.3. We have $f = \phi + \overline{\psi} \in \mathfrak{L}_H^k(1, \eta, \lambda; r)$ if and only if

$$\mathrm{Re}\left\{\frac{\phi(z)}{z} + \eta z\left(\frac{\phi(z)}{z}\right)'\right\} > \left|\frac{\psi(z)}{z} + \eta z\left(\frac{\psi(z)}{z}\right)'\right| + \lambda.$$

For $\eta = 1$, $\mathfrak{L}_H^k(1, 1, \lambda; r) = \mathcal{P}_H^0(\lambda)$, introduced by Li and Ponnusamy [12] in 2013.

For $\eta = 0$, $\mathfrak{L}_H^k(1, 0, \lambda; r) = g_H^0(\lambda)$, introduced by Li and Ponnusamy [13] in 2016.

For $\lambda = 0$, $\mathfrak{L}_H^k(1, \eta, 0; r) = G_H^k(\eta; r)$, introduced by Liu and Yang [11] in 2019.

Especially, for $\psi(z) = 0$, $\mathfrak{L}_H^k(1, \eta, \lambda; r) := Q_\eta(\lambda)$, introduced by Ding et al. [14] in 1995, and we note

$$\mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r) = \mathfrak{L}_k^1(\frac{1}{\mu}, \lambda).$$

2. Preliminary lemmas

In this section, we collect several lemmas that will be used in the proof of our main results.

Lemma 2.1. For $f(z) = \phi(z) + \overline{\psi(z)}$, a necessary and sufficient condition for f to be in H_0^k is that $F(z) = \Phi(z) + \overline{\Psi(z)} \in H_0^k$, with $\Phi(z)$ and $\Psi(z)$ given by (1.6) (taking principal values).

Proof. H_0^k consists of functions $f(z) = \phi(z) + \overline{\psi(z)}$ with expansions

$$\phi(z) = z + \sum_{n=k+1}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=k+1}^{\infty} b_n z^n.$$

i.e., $\phi(0) = 0, \phi'(0) = 1, \psi(0) = 0$, and $\phi^{(\nu)}(0) = \psi^{(\nu)}(0) = 0$ for $1 \leq \nu \leq k$.

($\Rightarrow$) If $f \in H_0^k$, then by the binomial series,

$$\left(\frac{\phi(z)}{z}\right)^{\mu} = 1 + \sum_{n=k+1}^{\infty} \gamma_{n-1} z^{n-1}, \quad \left(\frac{\psi(z)}{z}\right)^{\mu} = \sum_{n=k+1}^{\infty} \delta_{n-1} z^{n-1},$$

so, $\Phi(z) = z + \sum_{n=k+1}^{\infty} \gamma_{n-1} z^n$ and $\Psi(z) = \sum_{n=k+1}^{\infty} \delta_{n-1} z^n$. Hence, $F \in H_0^k$.

($\Leftarrow$) If $F \in H_0^k$, then

$$\frac{\phi(z)}{z} = \left(\frac{\Phi(z)}{z}\right)^{1/\mu} = 1 + \sum_{n=k+1}^{\infty} \alpha_n z^{n-1}, \quad \frac{\psi(z)}{z} = \left(\frac{\Psi(z)}{z}\right)^{1/\mu} = \sum_{n=k+1}^{\infty} \beta_n z^{n-1}$$

by the binomial series. Thus, $\phi(z) = z + \sum_{n=k+1}^{\infty} \alpha_n z^n$ and $\psi(z) = \sum_{n=k+1}^{\infty} \beta_n z^n$, so $f \in H_0^k$. □

The following lemmas are immediate consequences.

Lemma 2.2. Let $f = \phi + \overline{\psi} \in H_0^k$. Then, $f \in \mathfrak{L}_H^k(\mu, \eta, \lambda; r)$ if and only if $F = \Phi + \overline{\Psi} \in \mathfrak{L}_H^k(1, \eta, \lambda; r)$, where all powers are taken as principal values.

Lemma 2.3. [15] Suppose that χ is convex in $\mathcal{D}$, satisfying $\chi(0) = a, \nu \neq 0$, and $\mathrm{Re}(\nu) \geq 0$. If $\varphi(z) \in \mathcal{H}_k(a)$ and

$$\varphi(z) + \frac{1}{\nu} z\varphi'(z) \prec \chi(z),$$

then

$$\varphi(z) \prec \varpi(z) \prec \chi(z),$$

where

$$\varpi(z) = \frac{\nu}{kz^{\frac{\nu}{k}}} \int_0^z \chi(t) t^{\frac{\nu}{k}-1} \mathrm{d}t.$$

Lemma 2.4. [16] A sufficient condition for a harmonic mapping $f = \phi + \overline{\psi}$ to be close-to-convex and univalent in $\mathcal{D}$ is that $|\psi'(0)| < |\phi'(0)|$, and the analytic function $f_\varepsilon(z) = \phi(z) + \varepsilon\psi(z)$ is close-to-convex for every complex number ε ($|\varepsilon| = 1$).

The following coefficient conditions for harmonic mappings are established in [17].

Lemma 2.5. [17] Let $f(z) = \phi(z) + \overline{\psi(z)}$ be of the form (1.5).

(i) $f(z)$ is starlike in $\mathcal{D}$ if $\sum_{n=k+1}^{\infty} n(|a_n| + |b_n|) \leq 1$;

(ii) $f(z)$ is convex in $\mathcal{D}$ if $\sum_{n=k+1}^{\infty} n^2(|a_n| + |b_n|) \leq 1$.

Lemma 2.6. [11] If a function p is analytic in $\mathcal{D}$ satisfying $\mathrm{Re}h(z) > 0$ in $\mathcal{D}$ with the series expansion $p(z) = 1 + p_k z^k + p_{k+1} z^{k+1} + \cdots$, then for all $z \in \mathcal{D}$,

$$\mathrm{Re}\, p(z) \geq \frac{1 - |z|^k}{1 + |z|^k}.$$

3. Main results

In this section, we present the main results of this paper, including coefficient estimates, growth estimates, and convex combination properties for the class $\mathfrak{L}_H^k(\mu, \lambda; r)$.

Theorem 3.1. Let $f(z) = \phi(z) + \overline{\psi(z)} \in H_0^k$, and let Φ, Ψ be defined as in (1.6). Then, $f(z) \in \mathfrak{L}_H^k(\mu, \lambda; r)$ if and only if for every complex number ε ($|\varepsilon| = 1$), the rotated analytic combination $F_\varepsilon(z) = \Phi(z) + \varepsilon\Psi(z)$ belongs to $\mathfrak{L}_k^1(\frac{1}{\mu}, \lambda)$.

Proof. Assume that $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$, by Lemma 2.2, which is equivalent to

$$F(z) = \Phi(z) + \overline{\Psi(z)} \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r).$$

In turn, we only need to show

$$F_\varepsilon(z) = \Phi(z) + \varepsilon\Psi(z) \in \mathfrak{L}_k^1(\frac{1}{\mu}, \lambda),$$

for every complex number $\varepsilon(|\varepsilon| = 1)$.

According to Definitions 1.2 and 1.3, a direct computation confirms that for such ε,

$$\begin{aligned}\operatorname{Re}\left\{\frac{F_\varepsilon(z)}{z}+\frac{z}{\mu}\left(\frac{F_\varepsilon(z)}{z}\right)'\right\} &= \operatorname{Re}\left\{\left(\frac{\Phi(z)+\varepsilon\Psi(z)}{z}\right)+\frac{z}{\mu}\left(\frac{\Phi(z)+\varepsilon\Psi(z)}{z}\right)'\right\}\\ &> \operatorname{Re}\left\{\frac{\Phi(z)}{z}+\frac{z}{\mu}\left(\frac{\Phi(z)}{z}\right)'\right\}-\left|\frac{\Psi(z)}{z}+\frac{z}{\mu}\left(\frac{\Psi(z)}{z}\right)'\right|\\ &> \lambda\ (z\in\mathcal{D}).\end{aligned}$$

Thus, $F_\varepsilon(z)\in\mathfrak{L}_k^1(\frac{1}{\mu},\lambda)$.

Conversely, suppose $F_\varepsilon(z)\in\mathfrak{L}_k^1(\frac{1}{\mu},\lambda)$ for each $|\varepsilon|=1$. Then for any $z\in\mathcal{D}$,

$$\operatorname{Re}\left\{\frac{\Phi(z)}{z}+\frac{z}{\mu}\left(\frac{\Phi(z)}{z}\right)'\right\}>\operatorname{Re}\left[-\varepsilon\left(\frac{\Psi(z)}{z}+\frac{z}{\mu}\left(\frac{\Psi(z)}{z}\right)'\right)\right]+\lambda.$$

Since ε can be chosen with arbitrary argument, we obtain

$$\operatorname{Re}\left\{\frac{\Phi(z)}{z}+\frac{z}{\mu}\left(\frac{\Phi(z)}{z}\right)'\right\}>\left|\frac{\Psi(z)}{z}+\frac{z}{\mu}\left(\frac{\Psi(z)}{z}\right)'\right|+\lambda\ (z\in\mathcal{D}),$$

which is exactly the condition for $F(z)=\Phi(z)+\overline{\Psi(z)}\in\mathfrak{L}_H^k(1,\frac{1}{\mu},\lambda;r)$. Applying Lemma 2.2 once more gives $f\in\mathfrak{L}_H^k(\mu,\lambda;r)$. Therefore, the proof is completed. □

Theorem 3.2. Let $f(z)=\phi(z)+\overline{\psi(z)}\in\mathfrak{L}_H^k(\mu,\lambda;r)$. Then,

$$\left(\frac{\phi(z)}{z}\right)^\mu+\varepsilon\left(\frac{\psi(z)}{z}\right)^\mu\prec\varpi(z)\prec\frac{1+(1-2\lambda)z}{1-z},$$

where

$$\varpi(z)=\frac{\mu}{kz^{\frac{\mu}{k}}}\int_0^z\frac{1+(1-2\lambda)t}{1-t}t^{\frac{\mu}{k}-1}\mathrm{d}t.$$

Proof. Suppose $f(z)=\phi(z)+\overline{\psi(z)}\in\mathfrak{L}_H^k(\mu,\lambda;r)$. By Theorem 3.1, for $z\in\mathcal{D}$, it yields

$$\operatorname{Re}\left\{\frac{F_\varepsilon(z)}{z}+\frac{z}{\mu}\left(\frac{F_\varepsilon(z)}{z}\right)'\right\}>\lambda,$$

where

$$F_\varepsilon(z)=z\left(\frac{\phi(z)}{z}\right)^\mu+\varepsilon z\left(\frac{\psi(z)}{z}\right)^\mu.$$

Applying Lemma 2.3, we can deduce the desired subordination, so the proof is completed. □

The following theorem provides a sharp coefficient bound for functions in the class $\mathfrak{L}_H^k(\mu,\lambda;r)$.

Theorem 3.3. For any $f\in\mathfrak{L}_H^k(\mu,\lambda;r)$, the coefficient δ_{n-1} appearing in (1.6) satisfies

$$|\delta_{n-1}|\le\frac{\mu(1-\lambda)}{\mu+(n-1)},\quad n\ge k+1. \tag{3.1}$$

The estimate is sharp, and equality holds exactly for the function

$$f(z) = \left(z^{\mu} + \mu(1-\lambda)\int_0^z \frac{t^{k+\mu-1}}{1-t}\mathrm{d}t\right)^{\frac{1}{\mu}} \tag{3.2}$$

and its rotations.

Proof. Suppose $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$. By Theorem 3.1, $F_\varepsilon(z) = \Phi(z) + \varepsilon\Psi(z) \in \mathfrak{L}_k^1(\frac{1}{\mu}, \lambda)$ for all $|\varepsilon| = 1$. From (1.5), we have

$$\frac{\Psi(z)}{z} + \frac{1}{\mu}z\left(\frac{\Psi(z)}{z}\right)' = \sum_{n=k+1}^{\infty}\left(1 + \frac{n-1}{\mu}\right)\delta_{n-1}z^{n-1}. \tag{3.3}$$

Using Cauchy's integral formula, we deduce

$$\begin{aligned}\left(1 + \frac{n-1}{\mu}\right)|\delta_{n-1}| &= \left|\frac{1}{2\pi i}\int_{|z|=r}\left(\frac{\Psi(z)}{z} + \frac{z}{\mu}\left(\frac{\Psi(z)}{z}\right)'\right)\frac{\mathrm{d}z}{z^n}\right| \\ &\leq \frac{1}{2\pi r^{n-1}}\int_0^{2\pi}\left|\frac{\Psi(re^{i\theta})}{re^{i\theta}} + \frac{re^{i\theta}}{\mu}\left(\frac{\Psi(re^{i\theta})}{re^{i\theta}}\right)'\right|\mathrm{d}\theta.\end{aligned}$$

Since $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$, we have

$$\mathrm{Re}\left\{\frac{\Phi(z)}{z} + \frac{z}{\mu}\left(\frac{\Phi(z)}{z}\right)'\right\} > \left|\frac{\Psi(z)}{z} + \frac{z}{\mu}\left(\frac{\Psi(z)}{z}\right)'\right| + \lambda, \quad z \in \mathcal{D}.$$

For $z = re^{i\theta}$,

$$\left|\frac{\Psi(re^{i\theta})}{re^{i\theta}} + \frac{re^{i\theta}}{\mu}\left(\frac{\Psi(re^{i\theta})}{re^{i\theta}}\right)'\right| < \mathrm{Re}\left\{\frac{\Phi(re^{i\theta})}{re^{i\theta}} + \frac{re^{i\theta}}{\mu}\left(\frac{\Phi(re^{i\theta})}{re^{i\theta}}\right)'\right\} - \lambda.$$

Integrating over $\theta \in [0, 2\pi]$ and using the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left\{\frac{\Phi(re^{i\theta})}{re^{i\theta}} + \frac{re^{i\theta}}{\mu}\left(\frac{\Phi(re^{i\theta})}{re^{i\theta}}\right)'\right\}\mathrm{d}\theta = 1,$$

we get

$$\frac{(\mu + n - 1)}{\mu}|\delta_{n-1}| \leq (1-\lambda)r^{1-n}.$$

Since this inequality holds for every $r \in (0, 1)$, we may let $r \to 1^-$. Taking the limit yields

$$\frac{(\mu + n - 1)}{\mu}|\delta_{n-1}| \leq 1 - \lambda.$$

Consequently,

$$|\delta_{n-1}| \leq \frac{\mu(1-\lambda)}{n + \mu - 1}.$$

Thus, the desired coefficient estimate for $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$ is established. Equality holds when the coefficients satisfy

$$|\delta_{n-1}(f)| := |\delta_{n-1}(b_{k+1}, b_{k+2}, \ldots, b_n; \mu)(f)| = \frac{\mu(1-\lambda)}{n+\mu-1}.$$

Let

$$\left(\frac{f(z)}{z}\right)^{\mu} = 1 + \sum_{n=k+1}^{\infty} \frac{\mu(1-\lambda)}{n+\mu-1} z^{n-1}.$$

After computation, it is not difficult to obtain the extremal function in (3.2). This completes the proof. □

Theorem 3.4. For $f(z) \in \mathfrak{L}_H^k(\mu, \lambda; r)$, the coefficients defined in (1.6) obey

(i) $|\gamma_{n-1}| + |\delta_{n-1}| \leq \dfrac{2\mu(1-\lambda)}{n+\mu-1}$;

(ii) $|\gamma_{n-1}| - |\delta_{n-1}| \leq \dfrac{2\mu(1-\lambda)}{n+\mu-1}$;

(iii) $|\gamma_{n-1}| \leq \dfrac{2\mu(1-\lambda)}{n+\mu-1}$.

The bounds are sharp, and equalities occur precisely for the function

$$f(z) = \left(z^{\mu} + 2\mu(1-\lambda)\int_0^z \frac{t^{k+\mu-1}}{1-t}\mathrm{d}t\right)^{\frac{1}{\mu}}, \tag{3.4}$$

and its rotations.

Proof. Since $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$, by Theorem 3.1, for all $|\varepsilon| = 1$,

$$\operatorname{Re}\left\{\frac{F_{\varepsilon}(z)}{z} + \frac{z}{\mu}\left(\frac{F_{\varepsilon}(z)}{z}\right)'\right\} > \lambda \ (z \in \mathcal{D}),$$

that is,

$$\frac{F_{\varepsilon}(z)}{z} + \frac{z}{\mu}\left(\frac{F_{\varepsilon}(z)}{z}\right)' \prec \frac{1+(1-2\lambda)z}{1-z}. \tag{3.5}$$

Thus, an analytic function $p(z) = 1 + \sum_{n=k+1}^{\infty} p_{n-1}z^{n-1}$ can be constructed with $|p_{n-1}| \leq 2(1-\lambda)$ for $n \geq k+1$, satisfying

$$\frac{F_{\varepsilon}(z)}{z} + \frac{z}{\mu}\left(\frac{F_{\varepsilon}(z)}{z}\right)' = p(z). \tag{3.6}$$

Comparing coefficients in (3.6) yields

$$\gamma_{n-1} + \varepsilon\delta_{n-1} = \frac{\mu}{n+\mu-1} p_{n-1}, \quad n \geq k+1.$$

Take the absolute values and use the triangle inequalities

$$|\gamma_{n-1}| + |\delta_{n-1}| \leq \frac{\mu}{n+\mu-1}|p_{n-1}| \leq \frac{2\mu(1-\lambda)}{n+\mu-1},$$
$$|\gamma_{n-1}| - |\delta_{n-1}| \leq \frac{\mu}{n+\mu-1}|p_{n-1}| \leq \frac{2\mu(1-\lambda)}{n+\mu-1}.$$

Therefore, from the above two inequalities, we can easily derive inequality (iii). This completes the proof. □

Especially, in Theorem 3.4, for $\mu = 1$ and $k = 1$, we obtain the following results.

Corollary 3.1. Let $f(z)$ of the form (1.5) be in $\mathfrak{L}_H^1(1, \lambda; r)$. Then,

(i) $|\gamma_n| + |\delta_n| \leq \frac{2(1-\lambda)}{n}$;

(ii) $|\gamma_n| - |\delta_n| \leq \frac{2(1-\lambda)}{n}$;

(iii) $|\gamma_n| \leq \frac{2(1-\lambda)}{n}$.

The bounds are sharp, and equalities occur precisely for the function

$$f(z) = z - 2(1-\lambda)(z + \log(1-z))$$

and its rotations.

Growth estimates for the class $\mathfrak{L}_H^k(\mu, \lambda; r)$ are established in the next theorem.

Theorem 3.5. Let $f(z) = \phi(z) + \overline{\psi(z)} \in \mathfrak{L}_H^k(\mu, \lambda; r)$, and let $F(z) = \Phi(z) + \overline{\Psi(z)}$ be given by (1.6). Then for $z \in \mathcal{D}$,

$$|z| - 2\mu(1-\lambda)|z|^{1-\mu} \int_0^{|z|} \frac{\varsigma^{k+\mu-1}}{1+\varsigma^k} d\varsigma \leq |F(z)| \leq |z| + 2\mu(1-\lambda)|z|^{1-\mu} \int_0^{|z|} \frac{\varsigma^{k+\mu-1}}{1-\varsigma^k} d\varsigma. \tag{3.7}$$

The bounds are sharp, and equalities occur precisely for the function

$$F(z) = z + 2\mu(1-\lambda)z^{1-\mu} \int_0^{z} \frac{\varsigma^{k+\mu-1}}{1+\varsigma^k} d\varsigma$$

and its rotations.

Proof. Let $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$. By Theorem 3.1, for every $\varepsilon(|\varepsilon| = 1)$, we have $F_\varepsilon(z) = \Phi(z) + \varepsilon\Psi(z) \in \mathfrak{L}_k^1(\frac{1}{\mu}, \lambda)(|\varepsilon| = 1)$, which means

$$\mathrm{Re}\left\{\frac{F_\varepsilon(z)}{z} + \frac{z}{\mu}\left(\frac{F_\varepsilon(z)}{z}\right)'\right\} > \lambda \ (z \in \mathcal{D}).$$

Suppose that

$$\frac{F_\varepsilon(z)}{z} + \frac{z}{\mu}\left(\frac{F_\varepsilon(z)}{z}\right)' = (1-\lambda)p(z) + \lambda,$$

where p is analytic in $\mathcal{D}$, satisfies $\mathrm{Re}p(z) > 0, p(0) = 1$, and $p^{(j)}(0) = 0$ for $1 \leq j \leq k-1$. From the representation, such a function p can be expressed in terms of a Schwarz function ν in $\mathcal{D}$ satisfying $|\nu(z)| < 1$ and $\nu^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k-1$. Specifically,

$$p(z) = \frac{1 + (1-2\lambda)\nu(z)}{1 - \nu(z)}.$$

Thus,

$$\frac{F_\varepsilon(z)}{z} + \frac{z}{\mu}\left(\frac{F_\varepsilon(z)}{z}\right)' = \frac{1 + (1-2\lambda)\nu(z)}{1 - \nu(z)}.$$

This implies that

$$(1 - \frac{1}{\mu})\frac{F_\varepsilon(z)}{z} + \frac{1}{\mu}F_\varepsilon'(z) = \frac{1 + (1-2\lambda)\nu(z)}{1 - \nu(z)}.$$

Multiplying by $z^{\mu-1}$ yields

$$(1-\frac{1}{\mu})z^{\mu-2}F_{\varepsilon}(z)+\frac{1}{\mu}z^{\mu-1}F'_{\varepsilon}(z)=z^{\mu-1}\frac{1+(1-2\lambda)\nu(z)}{1-\nu(z)},$$

that is,

$$\left(\frac{1}{\mu}z^{\mu-1}F_{\varepsilon}(z)\right)'=z^{\mu-1}\frac{1+(1-2\lambda)\nu(z)}{1-\nu(z)}.$$

By integrating both sides above, it is straightforward to obtain

$$\frac{1}{\mu}z^{\mu-1}F_{\varepsilon}(z)=\int_0^z\zeta^{\mu-1}\frac{1+(1-2\lambda)\nu(\zeta)}{1-\nu(\zeta)}\mathrm{d}\zeta.$$

Taking moduli and applying the elementary bounds

$$\left|\frac{1+(1-2\lambda)t}{1-t}\right|\le\frac{1+(1-2\lambda)|t|}{1-|t|}\text{ and }\mathrm{Re}\frac{1+(1-2\lambda)t}{1-t}\ge\frac{1-(1-2\lambda)|t|}{1+|t|},$$

the latter via Lemma 2.6 together with the vanishing derivative conditions on ν, we arrive at

$$\begin{aligned}\left|z^{\mu-1}F_{\varepsilon}(z)\right|&=\left|\mu\int_0^{|z|}\frac{1+(1-2\lambda)\nu(te^{i\theta})}{1-\nu(te^{i\theta})}\left(te^{i\theta}\right)^{\mu-1}e^{i\theta}\mathrm{d}t\right|\\&\le\mu\int_0^{|z|}\frac{1+(1-2\lambda)t^k}{1-t^k}t^{\mu-1}\mathrm{d}t\\&=|z|^{\mu}+2\mu(1-\lambda)\sum_{j=1}^{\infty}\frac{1}{jk+\mu}|z|^{jk+\mu}\\&=|z|^{\mu}+2\mu(1-\lambda)\int_0^{|z|}\frac{\varsigma^{k+\mu-1}}{1-\varsigma^k}\mathrm{d}\varsigma,\end{aligned}$$

and consequently,

$$|F_{\varepsilon}(z)|\le|z|+2\mu(1-\lambda)|z|^{1-\mu}\int_0^{|z|}\frac{\varsigma^{k+\mu-1}}{1-\varsigma^k}\mathrm{d}\varsigma.$$

In the same way, using the lower bound on the real part,

$$\begin{aligned}\left|z^{\mu-1}F_{\varepsilon}(z)\right|&=\left|\mu\int_0^{|z|}\frac{1+(1-2\lambda)\nu(te^{i\theta})}{1-\nu(te^{i\theta})}\left(te^{i\theta}\right)^{\mu-1}e^{i\theta}\mathrm{d}t\right|\\&\ge\mu\int_0^{|z|}\mathrm{Re}\frac{1+(1-2\lambda)\nu(te^{i\theta})}{1-\nu(te^{i\theta})}t^{\mu-1}\mathrm{d}t\\&\ge\mu\int_0^{|z|}\frac{1-(1-2\lambda)t^k}{1+t^k}t^{\mu-1}\mathrm{d}t\\&=|z|^{\mu}+2\mu(1-\lambda)\sum_{j=1}^{\infty}\frac{(-1)^j}{jk+\mu}|z|^{jk+\mu}\\&=|z|^{\mu}-2\mu(1-\lambda)\int_0^{|z|}\frac{\varsigma^{k+\mu-1}}{1+\varsigma^k}\mathrm{d}\varsigma,\end{aligned}$$

hence

$$|F_\varepsilon(z)| \geq |z| - 2\mu(1-\lambda)|z|^{1-\mu} \int_0^{|z|} \frac{\varsigma^{k+\mu-1}}{1+\varsigma^k} d\varsigma.$$

Because ε is arbitrary, these estimates hold for $F(z) = \Phi(z) + \overline{\Psi(z)}$ as well. Therefore, for every $\mu > 0$,

$$|z| - 2\mu(1-\lambda)|z|^{1-\mu} \int_0^{|z|} \frac{\varsigma^{k+\mu-1}}{1+\varsigma^k} d\varsigma \leq |F(z)| \leq |z| + 2\mu(1-\lambda)|z|^{1-\mu} \int_0^{|z|} \frac{\varsigma^{k+\mu-1}}{1-\varsigma^k} d\varsigma.$$

Sharpness is demonstrated by the extremal function

$$F(z) = z + 2\mu(1-\lambda)z^{1-\mu} \int_0^{z} \frac{\varsigma^{k+\mu-1}}{1+\varsigma^k} d\varsigma$$

and its rotations. This completes the proof. □

Theorem 3.6. Let $f(z) = \phi(z) + \overline{\psi(z)} \in H_0^k$, and let $F(z) = \Phi(z) + \overline{\Psi(z)}$ be given by (1.6). Let $0 < \mu \leq 1$, and denote

$$\vartheta_n = \frac{\mu + (n-1)}{\mu}, \quad n \geq k+1. \tag{3.8}$$

If

$$\sum_{n=k+1}^{\infty} \vartheta_n (|\gamma_{n-1}| + |\delta_{n-1}|) \leq 1 - \lambda, \tag{3.9}$$

then

(i) F is harmonic univalent and sense-preserving in $\mathcal{D}$ with $F \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$. Moreover, F is close-to-convex in $\mathcal{D}$.

(ii) f is harmonic univalent and sense-preserving in $\mathcal{D}$ with $f \in \mathfrak{L}_H^k(\mu, \lambda; r)$. Moreover, f is close-to-convex in $\mathcal{D}$.

Proof. (i) For $0 < |z_1| \leq |z_2| < 1$, we get

$$\begin{aligned}
|F(z_1) - F(z_2)| &\geq |\Phi(z_1) - \Phi(z_2)| - |\Psi(z_1) - \Psi(z_2)| \\
&> |z_1 - z_2| \left[1 - \sum_{n=k+1}^{\infty} n(|\gamma_{n-1}| + |\delta_{n-1}|) \right] \\
&\geq |z_1 - z_2| \left[1 - \sum_{n=k+1}^{\infty} \vartheta_n(|\gamma_{n-1}| + |\delta_{n-1}|) \right] \\
&\geq 0.
\end{aligned}$$

Note that $\vartheta_n \geq n$ for $\mu \leq 1$. Therefore, $F(z)$ is univalent in $\mathcal{D}$.

In addition, $F(z)$ is sense-preserving in $\mathcal{D}$ because

$$\begin{aligned}
|\Phi'(z)| &\geq 1 - \sum_{n=k+1}^{\infty} n|\gamma_{n-1}||z|^{n-1} \\
&> 1 - \sum_{n=k+1}^{\infty} \vartheta_n |\gamma_{n-1}|
\end{aligned}$$

$$\begin{aligned}&\geq \sum_{n=k+1}^{\infty} \vartheta_n|\delta_{n-1}|\\&> \sum_{n=k+1}^{\infty} n|\delta_{n-1}||z|^{n-1}\\&\geq |\Psi'(z)|.\end{aligned}$$

Next, we prove that $F(z) \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$. According to (3.9), we get

$$\begin{aligned}\mathrm{Re}\left\{\frac{\phi(z)}{z} + \frac{z}{\mu}\left(\frac{\phi(z)}{z}\right)'\right\} &= 1 + \mathrm{Re}\left\{\sum_{n=k+1}^{\infty} \frac{n-1+\mu}{\mu}\gamma_{n-1}z^{n-1}\right\}\\&\geq 1 - \left|\sum_{n=k+1}^{\infty} \frac{n-1+\mu}{\mu}\gamma_{n-1}z^{n-1}\right|\\&> \left|\sum_{n=k+1}^{\infty} \frac{n-1+\mu}{\mu}\delta_{n-1}z^{n-1}\right| + \lambda\\&= \left|\frac{\psi(z)}{z} + \frac{z}{\mu}\left(\frac{\psi(z)}{z}\right)'\right| + \lambda.\end{aligned}$$

Thus, we obtain $F \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$.

To establish that $F(z) = \Phi(z) + \overline{\Psi(z)}$ is close-to-convex, Lemma 2.4 reduces the problem to showing that $F_\varepsilon = \Phi(z) + \varepsilon\Psi(z)$ is close-to-convex for every $|\varepsilon| = 1$. This follows if we can verify $|F'_\varepsilon(z) - 1| < 1$ for $z \in \mathcal{D}$:

$$\left|F'_\varepsilon - 1\right| = \left|\sum_{n=k+1}^{\infty} n(\gamma_{n-1} + \varepsilon\delta_{n-1})z^{n-1}\right| < \sum_{n=k+1}^{\infty} n(|\gamma_{n-1}| + |\delta_{n-1}|) \leq \sum_{n=k+1}^{\infty} \vartheta_n(|\gamma_{n-1}| + |\delta_{n-1}|) \leq 1.$$

This leads to the conclusion that $F(z) = \Phi(z) + \overline{\Psi(z)}$ is close-to-convex in $\mathcal{D}$.

(ii) For $f(z) = \phi(z) + \overline{\psi(z)} \in \mathfrak{L}_H^k(\mu, \lambda; r)$, we define an operator I by

$$I(f) = z\left(\frac{\phi(z)}{z}\right)^\mu + \overline{z\left(\frac{\psi(z)}{z}\right)^\mu} = \Phi(z) + \overline{\Psi(z)},$$

where all powers are taken as principal values. Clearly, the operator I is a one-to-one mapping from $\mathfrak{L}_H^k(\mu, \lambda; r)$ to $\mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$.

Indeed, take any two distinct functions $f_1(z), f_2(z) \in \mathfrak{L}_H^k(\mu, \lambda; r)$ with $f_1(z) \neq f_2(z)$ for $z \in \mathcal{D}$. Then,

$$F_1(z) = I(f_1)(z) = \Phi_1(z) + \overline{\Psi_1(z)} \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r),$$

$$F_2(z) = I(f_2)(z) = \Phi_2(z) + \overline{\Psi_2(z)} \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r),$$

and, because the principal value mapping is injective, we have $F_1(z) \neq F_2(z)$.

Furthermore, if $f(z) = \phi(z) + \overline{\psi(z)}$ is close-to-convex in $\mathcal{D}$, then $I(f(z)) = \Phi(z) + \overline{\Psi(z)}$ is close-to-convex in $\mathcal{D}$.

Combining this observation with Lemma 2.1, we see that $f(z) = \phi(z) + \overline{\psi(z)}$ is close-to-convex in $\mathcal{D}$ if and only if $F(z) = \Phi(z) + \overline{\Psi(z)}$ is close-to-convex in $\mathcal{D}$. Hence, conclusion (ii) holds. This completes the proof of Theorem 3.6. □

Theorem 3.7. The classes $\mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$ and $\mathfrak{L}_H^k(\mu, \lambda; r)$ are closed under convex combinations.

Proof. (i) Let $F_m(z) = \Phi_m(z) + \overline{\Psi}_m(z) \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$ for $m = 1, 2, \cdots$, where

$$\Phi_m(z) = z + \sum_{n=k+1}^{\infty} \gamma_{n-1}^{(m)} z^n, \quad \Psi_m(z) = \sum_{n=k+1}^{\infty} \delta_{n-1}^{(m)} z^n.$$

Let $t_m \geq 0$ satisfy $\sum_{m=1}^{\infty} t_m = 1$. Define the convex combination

$$F(z) = \sum_{m=1}^{\infty} t_m F_m(z) = \Phi(z) + \overline{\Psi(z)},$$

where

$$\Phi(z) = \sum_{m=1}^{\infty} t_m \Phi_m(z), \quad \Psi(z) = \sum_{m=1}^{\infty} t_m \Psi_m(z).$$

Then,

$$\begin{aligned}
\operatorname{Re}\left\{\frac{\Phi(z)}{z} + \frac{z}{\mu}\left(\frac{\Phi(z)}{z}\right)'\right\} &= 1 + \operatorname{Re}\left\{\sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} t_m \gamma_{n-1}^{(m)} z^{n-1}\right\} \\
&\geq 1 - \sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} t_m |\gamma_{n-1}^{(m)}| \\
&\geq \sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} t_m |\delta_{n-1}^{(m)}| + \lambda \\
&\geq \left|\sum_{m=1}^{\infty}\sum_{n=k+1}^{\infty} t_m \delta_{n-1}^{(m)} z^{n-1}\right| + \lambda \\
&= \left|\frac{\Psi(z)}{z} + \frac{z}{\mu}\left(\frac{\Psi(z)}{z}\right)'\right| + \lambda.
\end{aligned}$$

Thus, $F \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$.

(ii) Let $f_m(z) = \phi_m(z) + \overline{\psi_m(z)} \in \mathfrak{L}_H^k(\mu, \lambda; r)$ for $m = 1, 2, \ldots$, and let $t_m \geq 0$ with $\sum_{m=1}^{\infty} t_m = 1$. Define

$$f(z) = \sum_{m=1}^{\infty} t_m f_m(z) = \phi(z) + \overline{\psi(z)}.$$

By Lemma 2.2, $I(f_m) \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$. Since $\mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$ is closed under convex combinations,

$$\sum_{m=1}^{\infty} t_m I(f_m) \in \mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r).$$

As I is a bijection between $\mathfrak{L}_H^k(\mu, \lambda; r)$ and $\mathfrak{L}_H^k(1, \frac{1}{\mu}, \lambda; r)$, we conclude

$$f(z) = \sum_{m=1}^{\infty} t_m f_m(z) \in \mathfrak{L}_H^k(\mu, \lambda; r).$$

Therefore, both classes are convex.

This completes the proof of Theorem 3.7. □

4. Conclusions

In this paper, we introduced and investigated a new class $\mathfrak{L}_H^k(\mu, \eta, \lambda; r)$ of generalized harmonic functions associated with the Bazilevič function. We established necessary and sufficient coefficient conditions, sharp coefficient bounds, growth estimates, and convex combination properties for this class. The results extend several classical results from analytic functions to the harmonic setting. Future work may include investigating extreme points, partial sums, and radius problems for this class.

Author contributions

Shuhai Li: Conceptualization, Methodology, Investigation, Writing–original draft preparation, Funding acquisition; Lina Ma: Writing–review and editing, Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The present investigation was supported by the Natural Science Foundation of Inner Mongolia of the People's Republic of China under Grant 2024MS01014 and 2019MS01023.

Conflict of interest

The authors declare no conflict of interest.

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