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*Research article*

## Finite-time Stepanov almost periodic synchronization for fractional-order stochastic high-order Hopfield neural networks

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**Abstract:** This paper investigates the dynamics of high-order Hopfield neural networks incorporating fractional-order derivatives, stochastic disturbances, and time-varying delays. To address the more realistic scenario of discontinuous or weakly regular time-varying parameters, the analysis is conducted within the framework of Stepanov almost-periodicity. First, sufficient criteria for the existence and uniqueness of a Stepanov almost periodic solution in distribution for the considered network are established using Banach's fixed point theorem and inequality techniques. Subsequently, by treating the studied network as a drive system, a corresponding response system is constructed. Effective control strategies are designed to achieve finite-time synchronization between these two systems. Finally, a numerical example is provided to illustrate the validity of the theoretical results. This study offers new theoretical insights for analyzing almost periodic oscillations in complex fractional-order stochastic systems with delays and has potential applications in fields requiring precise temporal coordination, such as secure communication and cooperative control.

**Keywords:** fractional-order stochastic neural networks; high-order Hopfield neural networks;  $p$ -th Stepanov periodic solution in distribution; Finite-time synchronization

**Mathematics Subject Classification:** 34K14, 34K50, 92B20

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### 1. Introduction

The study of neural network dynamics, serving as the theoretical core connecting model architecture with functional realization, has evolved from traditional integer-order ordinary differential equation frameworks toward more complex and physically realistic directions. The introduction of fractional-order calculus marks a critical advancement in this field [1–3]. Through nonlocal memory kernel functions, it endows neural network models with inherent advantages in characterizing phenomena such as long-range dependence, hereditary properties, and anomalous diffusion [4–6]. This provides a more precise mathematical language for describing non-Markovian

processes and complex spatiotemporal dynamics in biological neural systems, while significantly enhancing modeling capabilities in fields such as system control and signal processing. Investigating neural networks as dynamic systems aims to transcend static parameter analysis and delve into the mechanisms underlying their internal state evolution, stability, bifurcations, and other nonlinear behaviors. This not only provides theoretical foundations for optimizing algorithm design and interpreting training dynamics but also deepens our understanding of the computational principles underlying biological intelligence.

However, a gap between theory and practice is reflected in the discrepancies between the operational environments of real neural networks and idealized models. Time delays represent an inherent physical constraint [7, 8], whether in the chemical processes of biological synaptic transmission or signal propagation in hardware circuits [9–11]. Simultaneously, inherent random disturbances from both internal and external environments (e.g., thermal noise, channel interference) are ubiquitous. These two non-ideal factors are not merely minor perturbations; they can fundamentally reshape a system's dynamics [12–14], inducing oscillations, altering stability patterns, or triggering stochastic resonance [15–17]. Consequently, studies of neural network dynamics that neglect time delays and stochasticity often yield conclusions that deviate from real-world behavior. This necessitates an investigation of the true dynamics of neural networks within a complex framework that couples fractional-order derivatives, stochastic disturbances, and time delays [18, 19].

Furthermore, when neural network parameters, such as connection weights or external inputs, vary with time—particularly exhibiting periodic [20] or nearly periodic fluctuations—the study of their dynamics naturally focuses on the existence and stability of periodic and almost periodic solutions [21–23]; in this regard, the same holds true for ecological systems [24, 25]. Such solutions characterize the system's ability to dynamically adapt to and lock onto externally imposed regular or quasi-regular time-varying environments, which is crucial for understanding biological rhythms and designing adaptive information processing systems. Among these, almost periodic solutions, particularly the more generalized Stepanov-type almost periodic solutions, provide powerful analytical tools for handling discontinuous or weakly regular time-varying parameters. Recently, Huo and Li [4] investigated the  $S^p$  almost periodic dynamics of deterministic fractional-order neural networks, while others [26–28] studied the  $S^p$  almost periodic dynamics of integer-order stochastic neural networks. However, to date, research on the Stepanov almost periodic dynamics of neural networks that simultaneously incorporate fractional-order derivatives, stochastic disturbances, and time-varying delays remains largely unexplored and urgently requires in-depth investigation.

Regarding model selection, high-order Hopfield neural networks [29–31], owing to their richer interconnection structures and stronger function approximation capabilities, demonstrate superior performance compared with traditional first-order models in domains such as associative memory, optimization computation, image encryption, and data security, as well as in protecting sensitive image data in scenarios like the internet of vehicles and telemedicine [32, 33]. Analyzing their complex dynamics thus holds significant application value [34–36]. Given the above mentioned considerations, this paper aims to investigate a class of high-order Hopfield neural networks that simultaneously incorporate time-varying delays, stochastic disturbances, and fractional-order derivatives as follows:

$${}^c D_t^\alpha x_i(t) = -a_i x_i(t) + \sum_{j=1}^n a_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(t) g_j(x_j(t - \sigma_{ijk}(t)))$$

$$\times g_k(x_k(t - \nu_{ijk}(t))) + I_i(t) + \sum_{j=1}^n \delta_{ij}(x_j(t - \gamma_{ij}(t))) \frac{d\omega_j(t)}{dt}, \quad t > t_0, \quad (1.1)$$

where  $\alpha \in (\frac{1}{2}, 1)$ ,  $i = 1, 2, \dots, n$ , where  $n$  denotes the number of units in the network;  $x_i(t)$  is the state of the  $i$ th unit at time  $t$ ;  $a_i > 0$  is the self feedback connection weight;  $a_{ij}(t)$  and  $b_{ijk}(t)$  are the connection weights of the network;  $\tau_{ij}(t) \geq 0$ ,  $\sigma_{ijk}(t) \geq 0$ ,  $\nu_{ijk}(t) \geq 0$ ,  $\gamma_{ij}(t) \geq 0$  are the transmission delays at time  $t$ ;  $I_i(t)$  represents the external inputs at time  $t$ ;  $f_j, g_j : \mathbb{R} \rightarrow \mathbb{R}$  are the activation functions of signal transmission;  $\delta_{ij} : \mathbb{R} \rightarrow \mathbb{R}$  is a Borel measurable function; and  $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))^T$  indicates the  $n$ -dimensional two-sided Brownian motion defined on a complete probability space.

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$  be a complete probability space with a natural filtration  $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$  satisfying the usual conditions. Let  $BC_{\mathcal{F}_0}([t_0 - \rho, t_0], \mathbb{R}^n)$  denote the collection of all bounded,  $\mathcal{F}_0$ -measurable, and  $C([t_0 - \rho, t_0], \mathbb{R}^n)$ -valued random variables.

The initial condition of the system (1.1) is given by

$$x_i(s) = \psi_i(s), \quad s \in [t_0 - \rho, t_0], \quad (1.2)$$

where  $i = 1, 2, \dots, n$ ,  $\psi_i \in BC_{\mathcal{F}_0}([t_0 - \rho, t_0], \mathbb{R}^n)$ , and

$$\rho = \max_{1 \leq i, j, k \leq n} \left\{ \sup_{t \in \mathbb{R}} \{\tau_{ij}(t)\}, \sup_{t \in \mathbb{R}} \{\sigma_{ijk}(t)\}, \sup_{t \in \mathbb{R}} \{\nu_{ijk}(t)\}, \sup_{t \in \mathbb{R}} \{\gamma_{ij}(t)\} \right\}.$$

The specific research contents are as follows. First, within the more general framework of Stepanov almost-periodicity, sufficient criteria for the existence and uniqueness of almost periodic solutions for this network will be established. Second, treating the studied network as a drive system, a corresponding response system will be constructed, and effective control strategies will be designed to achieve finite-time synchronization between the two systems. This study is expected to provide, theoretically, new ideas and methods for the analysis of almost periodic oscillations in complex time-varying fractional-order stochastic systems. In application, it aims to offer a theoretical basis for problems requiring strict time constraints, such as neural computing, secure communication, and multi-agent cooperative control.

The arrangement of the remaining sections of this paper is as follows: In Section 2, we review some related concepts, introduce several fundamental lemmas that will be used in the proofs of the main results, and state the basic assumptions of this paper. In Section 3, we establish the existence and uniqueness of a Stepanov almost periodic solution in the distribution sense for the system (1.1) by employing Banach's fixed point theorem and inequality techniques. In Section 4, we construct a corresponding response system with System (1.1) as the drive system to investigate the finite-time synchronization problem. In Section 5, we provide a numerical example to illustrate the effectiveness of the theoretical results presented in this paper. In Section 6, we offer a brief conclusion.

## 2. Preliminaries

Let  $\Gamma$  be the classical gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt,$$

where  $\alpha > 0$ .

**Definition 2.1.** [37] The Mittag–Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where  $\alpha > 0$ ,  $\beta > 0$ , and  $z \in \mathbb{C}$ . When  $\beta = 1$ , we have  $E_{\alpha,\beta}(z) = E_{\alpha}(z)$ ; furthermore,  $E_{1,1}(z) = e^z$ .

**Definition 2.2.** [37] The fractional integral of order  $\alpha$  for a function  $f \in \mathcal{L}^1([a, b], \mathbb{R})$  is defined as

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t \in [a, b],$$

where  $\alpha > 0$ , and  $a$  and  $b$  may take  $-\infty$  and  $+\infty$  as their values.

**Definition 2.3.** [37] The Caputo derivative of order  $\alpha$  for a function  $f \in C^{n-1}([a, b], \mathbb{R})$  with  $f^{(n)} \in \mathcal{L}^1([a, b], \mathbb{R})$  is defined by

$${}^C D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t \in [a, b],$$

where  $n$  is a natural number such that  $n-1 < \alpha < n$ , and  $a, b$  may take  $-\infty$  or  $+\infty$  as their values.

Let  $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$  be a separable Banach space. For a random variable  $X : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{Y}$ , we use  $\mu(X) = P \circ X^{-1}$  to denote its law and  $\mathbb{E}X$  as its expectation. Let  $\mathcal{L}^p(\Omega, \mathbb{Y})$  be the space of all  $\mathbb{Y}$ -valued random variables  $X$  such that  $\mathbb{E}\|X\|_{\mathbb{Y}}^p = \int_{\Omega} \|X\|_{\mathbb{Y}}^p dP < \infty$ . Let  $C(\mathbb{R}, \mathbb{Y})$  be the set of continuous functions from  $\mathbb{R}$  to  $\mathbb{Y}$ . We use  $\mathcal{L}^{\infty}(\mathbb{R}, \mathbb{Y})$  to denote the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{Y}$  which are measurable and essentially bounded. The space  $\mathcal{L}^{\infty}(\mathbb{R}, \mathbb{Y})$  is a Banach space when it is endowed with the norm  $\|f\|_{\mathcal{L}^{\infty}} := \text{ess sup}_{t \in \mathbb{R}} \|f(t)\|_{\mathbb{Y}}$ .

**Definition 2.4.** [38] A function  $f \in C(\mathbb{R}, \mathbb{Y})$  is called almost periodic if, for every  $\varepsilon > 0$ , a positive number  $l = l(\varepsilon)$  exists such that in every interval of length  $l$ , there is a  $\tau$  satisfying

$$\|f(t+\tau) - f(t)\|_{\mathbb{Y}} < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted  $AP(\mathbb{R}, \mathbb{Y})$ .

For  $1 < p < \infty$ , we use  $\mathcal{L}_{loc}^p(\mathbb{R}, \mathbb{Y})$  to denote the space of all functions from  $\mathbb{R}$  to  $\mathbb{Y}$  which are locally  $p$ -integrable. For  $g \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathbb{Y})$ , we consider the following Stepanov norm:

$$\|g\|_{S^p} = \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|g(s)\|_{\mathbb{Y}}^p ds \right)^{\frac{1}{p}}.$$

**Definition 2.5.** [39] A function  $g \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathbb{Y})$  is called  $p$ -th Stepanov almost periodic if, for any  $\varepsilon > 0$ ,  $l > 0$  exists such that every interval of length  $l$  has a number  $\tau$  such that

$$\|g(\cdot + \tau) - g(\cdot)\|_{S^p} < \varepsilon.$$

The space of all such functions is denoted  $S^p AP(\mathbb{R}, \mathbb{Y})$ .

**Remark 2.1.** From Definitions 2.4 and 2.5, one can easily see that  $AP(\mathbb{R}, \mathbb{Y}) \subset S^p AP(\mathbb{R}, \mathbb{Y})$ .

**Definition 2.6.** [27] A stochastic process  $X \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{Y}))$  is called  $S^p$ -bounded if

$$\|X\|_{S^p} := \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \mathbb{E} \|X(s)\|_{\mathbb{Y}}^p ds \right)^{\frac{1}{p}} < \infty \quad \text{for all } t \in \mathbb{R}.$$

The space of all such stochastic processes will be denoted  $BS_s^p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{Y}))$ .

**Definition 2.7.** [27] A stochastic process  $X \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{Y}))$  is said to be  $S^p$ -continuous if, for every  $h \in \mathbb{R}$

$$\limsup_{h \rightarrow 0} \sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|X(t+h) - X(t)\|_{\mathbb{Y}}^p dt = 0.$$

**Definition 2.8.** [27] A stochastic process  $Z \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{Y}))$  is said to be  $p$ -th mean Stepanov almost periodic if for every  $\varepsilon > 0$ , there is a positive number  $l > 0$  such that every interval of length  $l$  contains a number  $\tau$  such that

$$\|Z(\cdot + \tau) - Z(\cdot)\|_{S^p} < \varepsilon.$$

Let  $(E, d)$  be a separable and complete metric space. We use  $\mathcal{P}(E)$  to denote the set of all probability measures defined on the  $\sigma$ -algebra of the Borel sets of  $E$ . Let  $BC(E)$  be the space of all bounded and continuous functions  $f: E \rightarrow \mathbb{R}$  with  $\|f\|_{\infty} := \sup_{x \in E} |f(x)| < \infty$ .

For any  $f \in BC(E)$  and  $\mu, \nu \in \mathcal{P}(E)$ , define

$$\|f\|_L = \sup_{a \neq b} \frac{|f(a) - f(b)|}{d(a, b)}, \quad \|f\|_{BL} = \max\{\|f\|_{\infty}, \|f\|_L\}, \quad d_{BL}(\mu, \nu) = \sup_{\|f\|_{BL} \leq 1} \left| \int_E f d(\mu - \nu) \right|.$$

Then, the space  $(\mathcal{P}(E), d_{BL})$  is a separable, complete metric space.

**Definition 2.9.** [27] A stochastic process  $Z: \mathbb{R} \rightarrow \mathcal{L}^p(\Omega, \mathbb{Y})$  is called  $p$ -th Stepanov almost periodic in the distribution sense if, for every  $\varepsilon > 0$ , a positive number  $l$  exists such that every interval of length  $l$  contains a number  $\tau$  such that

$$\sup_{a \in \mathbb{R}} \left( \int_a^{a+1} d_{BL}^p(P \circ [Z(t+\tau)]^{-1}, P \circ [Z(t)]^{-1}) dt \right)^{\frac{1}{p}} < \varepsilon.$$

**Lemma 2.1.** [39] Each function  $f \in S^pAP(\mathbb{R}, \mathbb{Y})$  is continuous with respect to the Stepanov norm.

**Lemma 2.2.** [40](Burkholder–Davis–Gundy inequality) Let  $f \in \mathcal{L}^2([t_0, T], \mathbb{R})$ ,  $p > 2$ , and let  $\omega(t)$  be a Brownian motion. Then

$$\mathbb{E} \left[ \sup_{t \in [t_0, T]} \left| \int_{t_0}^t f(s) d\omega(s) \right|^p \right] \leq C_p \mathbb{E} \left[ \int_{t_0}^T |f(s)|^2 ds \right]^{p/2},$$

where the optimal constant is given by  $C_p = \left( \frac{p^{p+1}}{2(p-1)^{p-1}} \right)^{p/2}$ .

**Lemma 2.3.** [41] Let  $u, v$  be two non-negative integrable functions defined on the interval  $[a, b]$ , where  $v(t)$  is a nondecreasing function. Assume that  $\psi \in C^1[a, b]$  is an increasing function such that  $\psi'(t) \neq 0$  for all  $t \in [a, b]$ . If

$$u(t) \leq v(t) + \sum_{i=1}^2 g_i(t) \int_a^t \psi'(\tau) (\psi(t) - \psi(\tau))^{p_i-1} u(\tau) d\tau,$$

where the functions  $g_1$  and  $g_2$  are the bounded and monotonic functions on  $[a, b]$ , and the constants  $\rho_i > 0$  ( $i = 1, 2$ ). Then, the following holds:

$$u(t) \leq v(t) \left[ E_{\rho_1} (g_1(t)\Gamma(\rho_1)(\psi(t) - \psi(a))^{\rho_1}) + E_{\rho_2} (g_2(t)\Gamma(\rho_2)(\psi(t) - \psi(a))^{\rho_2}) - 1 \right].$$

For convenience, we use the following notations:

$$\begin{aligned} a_{ij}^+ &= \operatorname{ess\,sup}_{t \in \mathbb{R}} \{|a_{ij}(t)|\}, & b_{ijk}^+ &= \operatorname{ess\,sup}_{t \in \mathbb{R}} \{|b_{ijk}(t)|\}, & c_{ij}^+ &= \operatorname{ess\,sup}_{t \in \mathbb{R}} \{|c_{ij}(t)|\}, & \tau_{ij}^+ &= \sup_{t \in \mathbb{R}} \{\tau_{ij}(t)\}, \\ \sigma_{ijk}^+ &= \sup_{t \in \mathbb{R}} \{\sigma_{ijk}(t)\}, & \nu_{ijk}^+ &= \sup_{t \in \mathbb{R}} \{\nu_{ijk}(t)\}, & \gamma_{ij}^+ &= \sup_{t \in \mathbb{R}} \{\gamma_{ij}(t)\}, & \dot{\tau}_{ij}^+ &= \sup_{t \in \mathbb{R}} \{|\dot{\tau}_{ij}(t)|\}, \\ \dot{\sigma}_{ijk}^+ &= \sup_{t \in \mathbb{R}} \{|\dot{\sigma}_{ijk}(t)|\}, & \dot{\nu}_{ijk}^+ &= \sup_{t \in \mathbb{R}} \{|\dot{\nu}_{ijk}(t)|\}, & \dot{\gamma}_{ij}^+ &= \sup_{t \in \mathbb{R}} \{|\dot{\gamma}_{ij}(t)|\}. \end{aligned}$$

To obtain the main results of this paper, we need the following assumptions:

- (H<sub>1</sub>) For  $i, j, k = 1, 2, \dots, n$ ,  $\tau_{ij}, \sigma_{ijk}, \nu_{ijk}, \gamma_{ij} \in AP(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$  satisfying  $1 - \dot{\tau}_{ij}^+ > 0$ ,  $1 - \dot{\sigma}_{ijk}^+ > 0$ ,  $1 - \dot{\nu}_{ijk}^+ > 0$ ,  $1 - \dot{\gamma}_{ij}^+ > 0$ ;  $a_{ij}, b_{ijk} \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$ , and  $I_i \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathbb{R})$  are  $p$ -th Stepanov almost periodic.
- (H<sub>2</sub>) For  $i, j = 1, 2, \dots, n$ ,  $f_j, g_j, \delta_{ij} \in C(\mathbb{R}, \mathbb{R})$  satisfying the Lipschitz condition, that is, the constants  $L_j^f, L_j^g, L_{ij}^\delta, M_j^f, M_j^g > 0$  exist such that for all  $x, y \in \mathbb{R}$ ,

$$\begin{aligned} |f_j(x) - f_j(y)| &\leq L_j^f |x - y|, & |g_j(x) - g_j(y)| &\leq L_j^g |x - y|, \\ |\delta_{ij}(x) - \delta_{ij}(y)| &\leq L_{ij}^\delta |x - y|, & |f_j(x)| &\leq M_j^f, & |g_j(x)| &\leq M_j^g. \end{aligned}$$

Furthermore,  $f_j(0) = g_j(0) = \delta_{ij}(0) = 0$ .

- (H<sub>3</sub>) For  $p > 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$\begin{aligned} \pi_1 &:= 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} + \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \right. \right. \\ &\quad \times \left. \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \frac{1}{1 - \dot{\sigma}_{ijk}^+} + \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \right. \right. \\ &\quad \left. \left. \times \frac{1}{1 - \dot{\nu}_{ijk}^+} \right] + C_p n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2 - \frac{1}{\alpha}}\Gamma(2 - \alpha)} \right]^{\frac{p}{2}} \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} < 1, \end{aligned}$$

and for  $p = q = 2$ , we have

$$\begin{aligned} \pi_2 &:= 11 \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^2} \left[ \sum_{j=1}^n (a_{ij}^+)^2 \sum_{j=1}^n (L_j^f)^2 \frac{1}{1 - \dot{\tau}_{ij}^+} + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (L_j^g)^2 \right) \sum_{j=1}^n (M_j^g)^2 \right. \right. \\ &\quad \times \frac{1}{1 - \dot{\sigma}_{ijk}^+} + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (M_j^g)^2 \right) \sum_{j=1}^n (L_j^g)^2 \frac{1}{1 - \dot{\nu}_{ijk}^+} \left. \right] + \sum_{j=1}^n (L_{ij}^\delta)^2 \frac{1}{1 - \dot{\gamma}_{ij}^+} \\ &\quad \left. \frac{n\Gamma(2 - \frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2 - \frac{1}{\alpha}}\Gamma(2 - \alpha)} \right\} < 1. \end{aligned}$$

### 3. Stepanov almost periodic solution in the distribution sense

In this section, we use Banach's fixed point theorem and inequality techniques to establish the existence and uniqueness of a Stepanov almost periodic solution in the distribution sense for system (1.1), and prove that the solution is synchronized in finite time.

**Definition 3.1.** An  $\mathcal{F}_t$ -progressively measurable stochastic process  $x(t)$  is called a mild solution of System (1.1) if  $x(t)$  satisfies the following stochastic integral equation:

$$\begin{aligned} x_i(t) = & U_i(t - t_0)x_0 + \int_{t_0}^t (t - s)^{\alpha-1} \varphi_i(t - s) [\Theta_i(s, x) + I_i(s)] ds \\ & + \int_{t_0}^t (t - s)^{\alpha-1} \varphi_i(t - s) \sum_{j=1}^n \delta_{ij}(x_j(s - \gamma_{ij}(s))) d\omega_j(s), t > t_0, i = 1, 2, \dots, n, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \Theta_i(s, x) = & \sum_{j=1}^n a_{ij}(s) f_j(x_j(s - \tau_{ij}(s))) + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) g_j(x_j(s - \sigma_{ijk}(s))) g_k(x_k(s - \nu_{ijk}(s))), \\ x_i(t_0) = & \psi_i(t_0) = x_0, \varrho_\alpha(r) = \frac{1}{\alpha} r^{-1-\frac{1}{\alpha}} \vartheta_\alpha(r^{\frac{1}{\alpha}}), U_i(t) = \int_0^{+\infty} \varrho_\alpha(r) e^{-a_i t r} dr, \\ \varphi_i(t) = & \alpha \int_0^{+\infty} r \varrho_\alpha(r) e^{-a_i t r} dr, \vartheta_\alpha(r) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} r^{-n\alpha-1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), r \in (0, +\infty). \end{aligned}$$

By [42], we have

$$\begin{aligned} \varrho_\alpha(r) \geq 0, r \in (0, +\infty), \int_0^{+\infty} \varrho_\alpha(r) dr = 1, \int_0^{+\infty} r \varrho_\alpha(r) dr = \frac{1}{\Gamma(\alpha + 1)}, \\ \int_0^{+\infty} r^\eta \varrho_\alpha(r) dr = \frac{\Gamma(1 + \eta)}{\Gamma(1 + \alpha\eta)}, \eta \in [0, 1]. \end{aligned}$$

Letting  $t_0 \rightarrow -\infty$ , from (3.1), we obtain

$$\begin{aligned} x_i(t) = & \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) [\Theta_i(s, x) + I_i(s)] ds \\ & + \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \sum_{j=1}^n \delta_{ij}(x_j(s - \gamma_{ij}(s))) d\omega_j(s), \end{aligned} \quad (3.2)$$

which is also a solution of System (1.1).

Let  $w = t - s$ . Invoking Lemma 2.2 in [43], we get

$$\int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) ds = \int_0^{+\infty} w^{\alpha-1} \varphi_i(w) dw = \frac{1}{a_i}, \quad i = 1, 2, \dots, n.$$

For  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , we write  $\|x\|_0 = \max_{1 \leq i \leq n} \{|x_i|\}$ . Let  $X$  be the space of all  $S^p$ -bounded and  $S^p$ -continuous functions from  $\mathbb{R}$  to  $\mathcal{L}^p(\Omega, \mathbb{R}^n)$ . For  $\phi \in X$ , we define  $\|\phi\|_X = \sup_{a \in \mathbb{R}} \left( \int_a^{a+1} \mathbb{E} \|\phi(t)\|_0^p dt \right)^{\frac{1}{p}}$ . Then the space  $(X, \|\cdot\|_X)$  is a Banach space.

Set  $\phi^0(t) = (\phi_1^0(t), \phi_2^0(t), \dots, \phi_n^0(t))^T$ , where  $\phi_i^0(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) I_i(s) ds$ ,  $t \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ , and then  $\phi^0$  is well defined. Indeed, since  $I_i$  is  $S^p$ -bounded, the positive constants  $M_i$  exist such that  $\|I_i\|_{S^p} \leq M_i$ ,  $i = 1, 2, \dots, n$ . Consequently, according to the Hölder inequality, we can take a constant  $\kappa$  such that  $\|\phi^0\|_X \leq \kappa$ .

**Theorem 3.1.** *Assume that  $(H_1) - (H_3)$  hold. Then System (1.1) has a unique  $p$ -th Stepanov almost periodic solution in the distribution sense in  $X^* = \{\phi \in X : \|\phi - \phi^0\|_X \leq \kappa\}$ .*

*Proof.* Define a mapping  $\Phi : X^* \rightarrow X^*$  by

$$(\Phi\phi)(t) = ((\Phi_1\phi)(t), (\Phi_2\phi)(t), \dots, (\Phi_n\phi)(t))^T,$$

where  $(\phi_1, \phi_2, \dots, \phi_n)^T \in X^*$ ,  $t \in \mathbb{R}$  and

$$\begin{aligned} & (\Phi_i\phi)(t) \\ &= \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \left[ \sum_{j=1}^n a_{ij}(s) f_j(\phi_j(s - \tau_{ij}(s))) + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) g_j(\phi_j(s - \sigma_{ijk}(s))) \right. \\ & \quad \left. \times g_k(\phi_k(s - \nu_{ijk}(s))) + I_i(s) \right] ds + \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \delta_{ij}(\phi_j(s - \gamma_{ij}(s))) d\omega_j(s). \end{aligned} \quad (3.3)$$

To start with, we will prove that  $\Phi$  is well defined. Noting the fact that for any  $\phi \in X^*$ , it holds that  $\|\phi\|_X \leq \|\phi^0\|_X + \|\phi - \phi^0\|_X \leq 2\kappa$ . We can then deduce that

$$\begin{aligned} & \int_a^{a+1} \mathbb{E} \|(\Phi\phi)(t) - \phi^0(t)\|_0^p dt \\ & \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n a_{ij}(s) f_j(\phi_j(s - \tau_{ij}(s))) ds \right|^p dt \right\} \\ & \quad + 3^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) \right. \right. \\ & \quad \left. \left. \times g_j(\phi_j(s - \sigma_{ijk}(s))) g_k(\phi_k(s - \nu_{ijk}(s))) ds \right|^p dt \right\} \\ & \quad + 3^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \delta_{ij}(\phi_j(s - \gamma_{ij}(s))) d\omega_j(s) \right|^p dt \right\} \\ & := A_1 + A_2 + A_3. \end{aligned} \quad (3.4)$$

By the Hölder inequality, we have

$$\begin{aligned} A_1 & \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \left[ \left( \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) ds \right)^{\frac{p}{q}} \right. \right. \\ & \quad \left. \left. \times \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \mathbb{E} \left| \sum_{j=1}^n a_{ij}(s) f_j(\phi_j(s - \tau_{ij}(s))) \right|^p ds \right] dt \right\} \\ & \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \right\} \end{aligned}$$

$$\times \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \mathbb{E} \|\phi_j(s - \tau_{ij}(s))\|^p ds dt \Big\}.$$

Let  $u = s - \tau_{ij}(s)$ . We get

$$A_1 \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right. \\ \left. \times \int_a^{a+1} \int_{-\infty}^{t-\tau_{ij}(s)} (t-u-\tau_{ij}(s))^{\alpha-1} \varphi_i(t-u-\tau_{ij}(s)) \mathbb{E} \|\phi(u)\|_0^p du dt \right\}.$$

Further, by a change of variables and the Fubini theorem, we infer that

$$A_1 \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right. \\ \left. \times \int_{-\infty}^0 \int_a^{a+1} (-u)^{\alpha-1} \varphi_i(-u) \mathbb{E} \|\phi(u+t-\tau_{ij}(s))\|_0^p dt du \right\} \\ = 3^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right. \\ \left. \times \int_{-\infty}^0 \int_{a+u-\tau_{ij}(s)}^{a+u-\tau_{ij}(s)+1} (-u)^{\alpha-1} \varphi_i(-u) \mathbb{E} \|\phi(t)\|_0^p dt du \right\} \\ = 3^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right. \\ \left. \times \int_{-\infty}^{a-\tau_{ij}(s)} \int_u^{u+1} (a-u-\tau_{ij}(s))^{\alpha-1} \varphi_i(a-u-\tau_{ij}(s)) \mathbb{E} \|\phi(t)\|_0^p dt du \right\} \\ \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right. \\ \left. \times \int_{-\infty}^{a-\tau_{ij}(s)} (a-u-\tau_{ij}(s))^{\alpha-1} \varphi_i(a-u-\tau_{ij}(s)) du \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|\phi(t)\|_0^p dt \right\} \\ \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right\} \|\phi\|_X^p. \quad (3.5)$$

Similarly, we can get

$$A_2 \leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \frac{1}{1 - \dot{\nu}_{ijk}^+} \right\} \|\phi\|_X^p. \quad (3.6)$$

When  $p > 2$ , by virtue of Lemma 2.2, the Hölder inequality, a change of variables, and the Fubini theorem, we deduce that

$$A_3 \leq 3^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \left| \sum_{j=1}^n \delta_{ij}(\phi_j(s - \gamma_{ij}(s))) \right|^2 ds \right]^{\frac{p}{2}} dt \right\}$$

$$\begin{aligned}
&\leq 3^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \left[ \left( \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) ds \right)^{\frac{p-2}{2}} \left( \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \right. \right. \right. \\
&\quad \left. \left. \left. \times \mathbb{E} \left[ \sum_{j=1}^n \delta_{ij} (\phi_j(s - \gamma_{ij}(s))) \right]^p ds \right) \right] dt \right\} \\
&\leq 3^{p-1} C_p \max_{1 \leq i \leq n} \left\{ n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \right]^{\frac{p}{2}} \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi\|_X^p. \tag{3.7}
\end{aligned}$$

When  $p = 2$ , by the Itô isometry, the Cauchy–Schwarz inequality, a change of variables and the Fubini theorem, we have

$$\begin{aligned}
A_3 &\leq 3 \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \left| \sum_{j=1}^n \delta_{ij} (\phi_j(s - \gamma_{ij}(s))) \right|^2 ds \right] dt \right\} \\
&\leq 3 \max_{1 \leq i \leq n} \left\{ \frac{n \Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi\|_X^2. \tag{3.8}
\end{aligned}$$

Therefore, when  $p > 2$ , putting (3.5)–(3.7) into (3.4), we can see that

$$\begin{aligned}
&\int_a^{a+1} \mathbb{E} \|\Phi\phi(t) - \phi^0(t)\|_0^p dt \\
&\leq 3^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} + \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \right. \right. \\
&\quad \left. \left. \sum_{j=1}^n (L_j^g)^p \frac{1}{1 - \dot{\gamma}_{ijk}^+} \right] + C_p n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \right]^{\frac{p}{2}} \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi\|_X^p \leq \kappa^p, \tag{3.9}
\end{aligned}$$

when  $p = 2$ , putting (3.5), (3.6), and (3.8) into (3.4), we can get

$$\begin{aligned}
&\int_a^{a+1} \mathbb{E} \|\Phi\phi(t) - \phi^0(t)\|_0^2 dt \\
&\leq 3 \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^2} \left[ \sum_{j=1}^n (a_{ij}^+)^2 \sum_{j=1}^n (L_j^f)^2 \frac{1}{1 - \dot{\tau}_{ij}^+} + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (M_j^g)^2 \right) \sum_{j=1}^n (L_j^g)^2 \frac{1}{1 - \dot{\gamma}_{ijk}^+} \right] \right. \\
&\quad \left. + \frac{n \Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi\|_X^2 \leq \kappa^2. \tag{3.10}
\end{aligned}$$

Consequently, from (3.9), (3.10), and  $(H_3)$ , it follows that  $\|\Phi\phi - \phi^0\|_X \leq \kappa$ .

Next, we will show that  $\Phi\phi$  is  $S^p$ -continuous. Noting that for any  $\phi \in X^*$  and  $h \in \mathbb{R}$ , without loss of generality, we can assume that  $h > 0$ ,  $\delta(\varepsilon)$  exists, and when  $|h| < \delta$ , we have  $\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|\phi(t+h) - \phi(t)\|_0^p dt < \varepsilon$ . In view of Lemma 2.1 and  $(H_1)$ , when  $|h| < \delta$ , we have

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} |a_{ij}(t+h) - a_{ij}(t)|^p dt < \varepsilon, \quad \sup_{a \in \mathbb{R}} \int_a^{a+1} |b_{ijk}(t+h) - b_{ijk}(t)|^p dt < \varepsilon,$$

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} |I_i(t+h) - I_i(t)|^p dt < \varepsilon, \quad |\tau_{ij}(t+h) - \tau_{ij}(t)| < \varepsilon = \delta,$$

$$|\sigma_{ijk}(t+h) - \sigma_{ijk}(t)| < \varepsilon = \delta, \quad |\nu_{ijk}(t+h) - \nu_{ijk}(t)| < \varepsilon = \delta, \quad |\gamma_{ij}(t+h) - \gamma_{ij}(t)| < \varepsilon = \delta.$$

We then get

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|\phi(t+h - \tau_{ij}(t+h)) - \phi(t+h - \tau_{ij}(t))\|_0^p dt < \varepsilon,$$

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|\phi(t+h - \sigma_{ijk}(t+h)) - \phi(t+h - \sigma_{ijk}(t))\|_0^p dt < \varepsilon,$$

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|\phi(t+h - \nu_{ijk}(t+h)) - \phi(t+h - \nu_{ijk}(t))\|_0^p dt < \varepsilon,$$

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|\phi(t+h - \gamma_{ij}(t+h)) - \phi(t+h - \gamma_{ij}(t))\|_0^p dt < \varepsilon.$$

By (3.3), we have

$$\begin{aligned} (\Phi_i \phi)(t+h) &= \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \left[ \sum_{j=1}^n a_{ij}(s+h) f_j(\phi_j(s+h - \tau_{ij}(s+h))) \right. \\ &\quad + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s+h) g_j(\phi_j(s+h - \sigma_{ijk}(s+h))) g_k(\phi_k(s+h - \nu_{ijk}(s+h))) \\ &\quad \left. + I_i(s+h) \right] ds + \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \delta_{ij}(\phi_j(s+h - \gamma_{ij}(s+h))) d\tilde{\omega}_j(s), \end{aligned}$$

where  $\tilde{\omega}_j(s) = \omega_j(s+h) - \omega_j(s)$  is a Brownian motion with the same distribution as  $\omega_j(s)$ .

Consider the process

$$\begin{aligned} (\Phi_i \phi)(t+h) &= \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \left[ \sum_{j=1}^n a_{ij}(s+h) f_j(\phi_j(s+h - \tau_{ij}(s+h))) \right. \\ &\quad + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s+h) g_j(\phi_j(s+h - \sigma_{ijk}(s+h))) g_k(\phi_k(s+h - \nu_{ijk}(s+h))) \\ &\quad \left. + I_i(s+h) \right] ds + \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \delta_{ij}(\phi_j(s+h - \gamma_{ij}(s+h))) d\omega_j(s). \quad (3.11) \end{aligned}$$

Then, from (3.11) and (3.3), it follows that

$$\begin{aligned} &\int_a^{a+1} \mathbb{E} \|(\Phi \phi)(t+h) - (\Phi \phi)(t)\|_0^p dt \\ &\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n a_{ij}(s+h) \right. \right. \\ &\quad \left. \left. \times [f_j(\phi_j(s+h - \tau_{ij}(s+h))) - f_j(\phi_j(s+h - \tau_{ij}(s)))] ds \right|^p dt \right\} \end{aligned}$$

$$\begin{aligned}
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n a_{ij}(s+h) \right. \right. \\
& \times \left. \left. [f_j(\phi_j(s+h-\tau_{ij}(s))) - f_j(\phi_j(s-\tau_{ij}(s)))] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [a_{ij}(s+h) - a_{ij}(s)] \right. \right. \\
& \times \left. \left. f_j(\phi_j(s-\tau_{ij}(s))) ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s+h) \right. \right. \\
& \times \left. \left. [g_j(\phi_j(s+h-\sigma_{ijk}(s+h)))g_k(\phi_k(s+h-\nu_{ijk}(s+h))) \right. \right. \\
& \left. \left. - g_j(\phi_j(s+h-\sigma_{ijk}(s+h)))g_k(\phi_k(s+h-\nu_{ijk}(s)))] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s+h) \right. \right. \\
& \times \left. \left. [g_j(\phi_j(s+h-\sigma_{ijk}(s+h)))g_k(\phi_k(s+h-\nu_{ijk}(s))) \right. \right. \\
& \left. \left. - g_j(\phi_j(s+h-\sigma_{ijk}(s+h)))g_k(\phi_k(s-\nu_{ijk}(s)))] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s+h) \right. \right. \\
& \times \left. \left. [g_j(\phi_j(s+h-\sigma_{ijk}(s+h)))g_k(\phi_k(s-\nu_{ijk}(s))) \right. \right. \\
& \left. \left. - g_j(\phi_j(s+h-\sigma_{ijk}(s)))g_k(\phi_k(s-\nu_{ijk}(s)))] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s+h) \right. \right. \\
& \times \left. \left. [g_j(\phi_j(s+h-\sigma_{ijk}(s)))g_k(\phi_k(s-\nu_{ijk}(s))) - g_j(\phi_j(s-\sigma_{ijk}(s))) \right. \right. \\
& \left. \left. \times g_k(\phi_k(s-\nu_{ijk}(s)))] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n [b_{ijk}(s+h) - b_{ijk}(s)] \right. \right. \\
& \times \left. \left. g_j(\phi_j(s-\sigma_{ijk}(s)))g_k(\phi_k(s-\nu_{ijk}(s))) ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) [I_i(s+h) - I_i(s)] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [\delta_{ij}(\phi_j(s+h-\gamma_{ij}(s+h)))] \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \delta_{ij}(\phi_j(s+h-\gamma_{ij}(s)))d\omega_j(s) \Big| dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [\delta_{ij}(\phi_j(s+h-\gamma_{ij}(s))) \right. \right. \\
& \left. \left. - \delta_{ij}(\phi_j(s-\gamma_{ij}(s)))d\omega_j(s) \right|^p dt \right\} \\
& := \sum_{i=1}^{11} J_i. \tag{3.12}
\end{aligned}$$

By the Hölder inequality, a change of variables, and the Fubini theorem, we deduce that

$$\begin{aligned}
J_1 & \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \left( \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) ds \right)^{\frac{p}{q}} \right. \right. \\
& \quad \times \left. \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \left| \sum_{j=1}^n a_{ij}(s+h) [f_j(\phi_j(s+h-\tau_{ij}(s+h))) \right. \right. \\
& \quad \left. \left. - f_j(\phi_j(s+h-\tau_{ij}(s))) \right]^p ds \right] dt \Big\} \\
& \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \right. \\
& \quad \left. \times \mathbb{E} \|\phi(s+h-\tau_{ij}(s+h)) - \phi(s+h-\tau_{ij}(s))\|_0^p ds dt \right\} \\
& = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{-\infty}^0 \int_a^{a+1} (-s)^{\alpha-1} \varphi_i(-s) \right. \\
& \quad \left. \times \mathbb{E} \|\phi(s+t+h-\tau_{ij}(s+t+h)) - \phi(s+t+h-\tau_{ij}(s+t))\|_0^p dt ds \right\} \\
& \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \right\} \\
& \quad \times \sup_{s \in \mathbb{R}} \int_s^{s+1} \mathbb{E} \|\phi(t+h-\tau_{ij}(t+h)) - \phi(t+h-\tau_{ij}(t))\|_0^p dt \\
& \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \right\} \varepsilon := \Delta_1 \varepsilon. \tag{3.13}
\end{aligned}$$

Similarly, we find

$$J_4 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \right\} \varepsilon := \Delta_4 \varepsilon, \tag{3.14}$$

$$J_6 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \right\} \varepsilon := \Delta_6 \varepsilon, \tag{3.15}$$

$$J_9 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \right\} \varepsilon := \Delta_9 \varepsilon. \quad (3.16)$$

By the Hölder inequality, we have

$$J_2 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \right. \\ \left. \times \mathbb{E} \|\phi(s+h - \tau_{ij}(s)) - \phi(s - \tau_{ij}(s))\|_0^p ds dt \right\}.$$

Moreover, via the same method used to estimate (3.5), we can get

$$J_2 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \right\} \varepsilon := \Delta_2 \varepsilon. \quad (3.17)$$

Likewise, we have

$$J_5 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \frac{1}{1 - \hat{\nu}_{ijk}^+} \right\} \varepsilon := \Delta_5 \varepsilon, \quad (3.18)$$

$$J_7 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \frac{1}{1 - \hat{\sigma}_{ijk}^+} \right\} \varepsilon := \Delta_7 \varepsilon. \quad (3.19)$$

For  $J_3$ , by the Hölder inequality, a change of variables, and the Fubini theorem, we have

$$J_3 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \right. \\ \left. \times \varphi_i(t-s) \sum_{j=1}^n |a_{ij}(s+h) - a_{ij}(s)|^p ds dt \right\} \\ = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} \int_{-\infty}^a \int_s^{s+1} (a-s)^{\alpha-1} \right. \\ \left. \times \varphi_i(a-s) \sum_{j=1}^n |a_{ij}(t+h) - a_{ij}(t)|^p dt ds \right\} \\ \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} n \right\} \varepsilon := \Delta_3 \varepsilon. \quad (3.20)$$

Similarly, we can get

$$J_8 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \sum_{j=1}^n \left( \sum_{k=1}^n (M_j^g)^q \right)^{\frac{p}{q}} n^2 \right\} \varepsilon := \Delta_8 \varepsilon. \quad (3.21)$$

When  $p > 2$ , according to Lemma 2.2, the Hölder inequality, and a change of variables, we can infer that

$$\begin{aligned}
 J_{10} &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \right. \right. \\
 &\quad \left. \left. \times \left| \sum_{j=1}^n [\delta_{ij}(\phi_j(s+h-\gamma_{ij}(s+h))) - \delta_{ij}(\phi_j(s+h-\gamma_{ij}(s)))] \right|^2 ds \right]^{\frac{p}{2}} dt \right\} \\
 &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) ds \right]^{\frac{p-2}{2}} \right. \\
 &\quad \left. \times \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \mathbb{E} \left| \sum_{j=1}^n [\delta_{ij}(\phi_j(s+h-\gamma_{ij}(s+h))) \right. \right. \right. \right. \\
 &\quad \left. \left. \left. - \delta_{ij}(\phi_j(s+h-\gamma_{ij}(s)))] \right|^p ds \right] dt \right\} \\
 &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \left[ \frac{\Gamma(2-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \right]^{\frac{p-2}{2}} n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{2(\alpha-1)} \right. \\
 &\quad \left. \times \varphi_i^2(t-s) \mathbb{E} \|\phi(s+h-\gamma_{ij}(s+h)) - \phi(s+h-\gamma_{ij}(s))\|_0^p ds dt \right\}.
 \end{aligned}$$

By a change of variables and the Fubini theorem, one can find that

$$J_{10} \leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \right]^{\frac{p}{2}} \right\} \varepsilon := \Delta_{10}^1 \varepsilon. \quad (3.22)$$

With the same method used to estimate (3.7), we can get

$$J_{11} \leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \right]^{\frac{p}{2}} \frac{1}{1-\dot{\gamma}_{ij}^+} \right\} \varepsilon := \Delta_{11}^1 \varepsilon. \quad (3.23)$$

When  $p = 2$ , by the Itô isometry, the Cauchy–Schwarz inequality and a change of variables, we have

$$\begin{aligned}
 J_{10} &\leq 11 \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \right. \right. \\
 &\quad \left. \left. \times \left| \sum_{j=1}^n [\delta_{ij}(\phi_j(s+h-\gamma_{ij}(s+h))) - \delta_{ij}(\phi_j(s-\gamma_{ij}(s)))] \right|^2 ds \right] dt \right\} \\
 &\leq 11 \max_{1 \leq i \leq n} \left\{ \frac{n\Gamma(2-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \right\} \varepsilon := \Delta_{10}^2 \varepsilon. \quad (3.24)
 \end{aligned}$$

Using the same method applied to estimate (3.8), we obtain

$$J_{11} \leq 11 \max_{1 \leq i \leq n} \left\{ \frac{n\Gamma(2-\frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \frac{1}{1-\dot{\gamma}_{ij}^+} \right\} \varepsilon := \Delta_{11}^2 \varepsilon. \quad (3.25)$$

Substituting (3.13)–(3.25) into (3.12), we obtain

$$\int_a^{a+1} \mathbb{E} \|(\Phi\phi)(t+h) - (\Phi\phi)(t)\|_0^p dt \leq \Delta^1 \varepsilon, \quad (p > 2), \quad (3.26)$$

where  $\Delta^1 = \sum_{i=1}^9 \Delta_i + \Delta_{10}^1 + \Delta_{11}^1$ , and

$$\int_a^{a+1} \mathbb{E} \|(\Phi\phi)(t+h) - (\Phi\phi)(t)\|_0^2 dt \leq \Delta^2 \varepsilon, \quad (p = 2), \quad (3.27)$$

where  $\Delta^2 = \sum_{i=1}^9 \Delta_i + \Delta_{10}^2 + \Delta_{11}^2$ .

Then, from (3.26) and (3.27) we find that  $\Phi\phi$  is  $S^p$ -continuous. Therefore, we have proved  $\Phi(X^*) \subset X^*$ .

Subsequently, we will show that  $\Phi$  is a contraction mapping. Indeed, for any  $\phi, \phi' \in X^*$ , we have

$$\begin{aligned} & \int_a^{a+1} \mathbb{E} \|(\Phi\phi)(t) - (\Phi\phi')(t)\|_0^p dt \\ & \leq 4^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n a_{ij}(s) [f_j(\phi_j(s - \tau_{ij}(s))) \right. \right. \\ & \quad \left. \left. - f_j(\phi'_j(s - \tau_{ij}(s)))] ds \right|^p dt \right\} + 4^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) \right. \right. \\ & \quad \left. \left. \times [g_j(\phi_j(s - \sigma_{ijk}(s)))g_k(\phi_k(s - \nu_{ijk}(s))) - g_j(\phi'_j(s - \sigma_{ijk}(s)))g_k(\phi_k(s - \nu_{ijk}(s)))] ds \right|^p dt \right\} \\ & \quad + 4^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) [g_j(\phi'_j(s - \sigma_{ijk}(s))) \right. \right. \\ & \quad \left. \left. \times g_k(\phi_k(s - \nu_{ijk}(s))) - g_j(\phi'_j(s - \sigma_{ijk}(s)))g_k(\phi'_k(s - \nu_{ijk}(s)))] ds \right|^p dt \right\} \\ & \quad + 4^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [\delta_{ij}(\phi_j(s - \gamma_{ij}(s))) \right. \right. \\ & \quad \left. \left. - \delta_{ij}(\phi'_j(s - \gamma_{ij}(s)))] d\omega_j(s) \right|^p dt \right\} := N_1 + N_2 + N_3 + N_4. \end{aligned} \quad (3.28)$$

Using an analogous method to that used in estimating (3.5) yields

$$N_1 \leq 4^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} \right\} \|\phi - \phi'\|_X^p, \quad (3.29)$$

$$N_2 \leq 4^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \frac{1}{1 - \dot{\sigma}_{ijk}^+} \right\} \|\phi - \phi'\|_X^p, \quad (3.30)$$

$$N_3 \leq 4^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \frac{1}{1 - \dot{\nu}_{ijk}^+} \right\} \|\phi - \phi'\|_X^p. \quad (3.31)$$

When  $p > 2$ , applying a similar approach to the one used for (3.7) gives

$$N_4 \leq 4^{p-1} C_p \max_{1 \leq i \leq n} \left\{ n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \right]^{\frac{p}{2}} \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi - \phi'\|_X^p. \quad (3.32)$$

When  $p = 2$ , following the estimation procedure for (3.8), we find

$$N_4 \leq 4 \max_{1 \leq i \leq n} \left\{ \frac{n\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi - \phi'\|_X^2. \quad (3.33)$$

Substituting (3.29)–(3.33) into (3.28), we see that

$$\begin{aligned} & \int_a^{a+1} \mathbb{E} \|(\Phi\phi)(t) - (\Phi\phi')(t)\|_0^p dt \\ & \leq 4^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \dot{\tau}_{ij}^+} + \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \right. \right. \\ & \quad \times \left. \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \frac{1}{1 - \dot{\sigma}_{ijk}^+} + \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \frac{1}{1 - \dot{\nu}_{ijk}^+} \right. \\ & \quad \left. + C_p n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \right]^{\frac{p}{2}} \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi - \phi'\|_X^p, \quad (p > 2) \end{aligned}$$

and

$$\begin{aligned} & \int_a^{a+1} \mathbb{E} \|(\Phi\phi)(t) - (\Phi\phi')(t)\|_0^2 dt \\ & \leq 4 \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^2} \left[ \sum_{j=1}^n (a_{ij}^+)^2 \sum_{j=1}^n (L_j^f)^2 \frac{1}{1 - \dot{\tau}_{ij}^+} + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (L_j^g)^2 \right) \sum_{j=1}^n (M_j^g)^2 \frac{1}{1 - \dot{\sigma}_{ijk}^+} \right. \right. \\ & \quad \left. + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (M_j^g)^2 \right) \sum_{j=1}^n (L_j^g)^2 \frac{1}{1 - \dot{\nu}_{ijk}^+} \right] + \frac{n\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}} \Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \\ & \quad \left. \times \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \|\phi - \phi'\|_X^2, \quad (p = 2), \end{aligned}$$

which, combined with  $(H_3)$ , implies that  $\Phi$  is a contraction mapping. Consequently,  $\Phi$  has a unique fixed point  $x$  in  $X^*$ , i.e., System (1.1) has a unique solution  $x$  in  $X^*$ .

Finally, let us show that  $x$  is Stepanov almost periodic in the distribution sense. Since  $x \in X^*$ , it is  $S^p$ -continuous and satisfies  $\|x\|_X \leq 2\kappa$ . So, for any  $\varepsilon > 0$ , a  $\delta \in (0, \varepsilon)$  exists for any  $h \in \mathbb{R}$ , when  $|h| < \delta$ , we have  $\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t+h) - x(t)\|_0^p dt < \varepsilon$ .

Hence, according to  $(H_1)$ , a  $\tau$  exists such that

$$\begin{aligned} & \sup_{a \in \mathbb{R}} \int_a^{a+1} |a_{ij}(t+\tau) - a_{ij}(t)|^p dt < \delta, \quad \sup_{a \in \mathbb{R}} \int_a^{a+1} |b_{ijk}(t+\tau) - b_{ijk}(t)|^p dt < \delta, \\ & \sup_{a \in \mathbb{R}} \int_a^{a+1} |I_i(t+\tau) - I_i(t)|^p dt < \delta, \quad |\tau_{ij}(t+\tau) - \tau_{ij}(t)| < \delta, \quad |\sigma_{ijk}(t+\tau) - \sigma_{ijk}(t)| < \delta, \end{aligned}$$

$$|v_{ijk}(t + \tau) - v_{ijk}(t)| < \delta, \quad |\gamma_{ij}(t + \tau) - \gamma_{ij}(t)| < \delta.$$

Consequently, we obtain that

$$\begin{aligned} \sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t - \tau_{ij}(t + \tau)) - x(t - \tau_{ij}(t))\|_0^p dt &< \varepsilon, \\ \sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t - \sigma_{ijk}(t + \tau)) - x(t - \sigma_{ijk}(t))\|_0^p dt &< \varepsilon, \\ \sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t - v_{ijk}(t + \tau)) - x(t - v_{ijk}(t))\|_0^p dt &< \varepsilon, \\ \sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t - \gamma_{ij}(t + \tau)) - x(t - \gamma_{ij}(t))\|_0^p dt &< \varepsilon. \end{aligned}$$

From (3.2), we have

$$\begin{aligned} x_i(t + \tau) = & \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \left[ \sum_{j=1}^n a_{ij}(s + \tau) f_j(x_j(s + \tau - \tau_{ij}(s + \tau))) \right. \\ & + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s + \tau) g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s + \tau - v_{ijk}(s + \tau))) \\ & \left. + I_i(s + \tau) \right] ds + \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \sum_{j=1}^n \delta_{ij}(x_j(s + \tau - \gamma_{ij}(s + \tau))) d\hat{\omega}_j(s), \end{aligned}$$

where  $\hat{\omega}_j(s) = \omega_j(s + \tau) - \omega_j(s)$  is a Brownian motion with the same distribution as  $\omega_j(s)$ .

Let us consider the process

$$\begin{aligned} x_i(t + \tau) = & \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \left[ \sum_{j=1}^n a_{ij}(s + \tau) f_j(x_j(s + \tau - \tau_{ij}(s + \tau))) \right. \\ & + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s + \tau) g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s + \tau - v_{ijk}(s + \tau))) \\ & \left. + I_i(s + \tau) \right] ds + \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \sum_{j=1}^n \delta_{ij}(x_j(s + \tau - \gamma_{ij}(s + \tau))) d\omega_j(s). \quad (3.34) \end{aligned}$$

By (3.34) and (3.2), we can deduce that

$$\begin{aligned} & \int_a^{a+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^p dt \\ & \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \sum_{j=1}^n a_{ij}(s + \tau) \right. \right. \\ & \quad \times \left. \left. [f_j(x_j(s + \tau - \tau_{ij}(s + \tau))) - f_j(x_j(s - \tau_{ij}(s + \tau)))] ds \right|^p dt \right\} \\ & \quad + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t - s)^{\alpha-1} \varphi_i(t - s) \sum_{j=1}^n a_{ij}(s + \tau) \right. \right. \end{aligned}$$

$$\begin{aligned}
& \times [f_j(x_j(s - \tau_{ij}(s + \tau))) - f_j(x_j(s - \tau_{ij}(s)))] ds \Big| dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [a_{ij}(s + \tau) \right. \right. \\
& \left. \left. - a_{ij}(s)] f_j(x_j(s - \tau_{ij}(s))) ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s + \tau) \right. \right. \\
& \times [g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s + \tau - \nu_{ijk}(s + \tau))) \\
& \left. \left. - g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s - \nu_{ijk}(s + \tau))) \right] ds \right|^p dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s + \tau) \right. \right. \\
& \times [g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s - \nu_{ijk}(s + \tau))) \\
& \left. \left. - g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s - \nu_{ijk}(s))) \right] ds \right|^p dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s + \tau) \right. \right. \\
& \times [g_j(x_j(s + \tau - \sigma_{ijk}(s + \tau))) g_k(x_k(s - \nu_{ijk}(s))) \\
& \left. \left. - g_j(x_j(s - \sigma_{ijk}(s + \tau))) g_k(x_k(s - \nu_{ijk}(s))) \right] ds \right|^p dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s + \tau) \right. \right. \\
& \times [g_j(x_j(s - \sigma_{ijk}(s + \tau))) g_k(x_k(s - \nu_{ijk}(s))) - g_j(x_j(s - \sigma_{ijk}(s))) \\
& \times g_k(x_k(s - \nu_{ijk}(s)))] ds \Big| dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n \sum_{k=1}^n [b_{ijk}(s + \tau) - b_{ijk}(s)] \right. \right. \\
& \times g_j(x_j(s - \sigma_{ijk}(s))) g_k(x_k(s - \nu_{ijk}(s))) ds \Big| dt \Big\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) [I_i(s + \tau) - I_i(s)] ds \right|^p dt \right\} \\
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [\delta_{ij}(x_j(s + \tau - \gamma_{ij}(s + \tau))) \right. \right. \\
& \left. \left. - \delta_{ij}(x_j(s - \gamma_{ij}(s + \tau)))] d\omega_j(s) \right|^p dt \right\}
\end{aligned}$$

$$\begin{aligned}
& + 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left| \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n [\delta_{ij}(x_j(s-\gamma_{ij}(s+\tau))) \right. \right. \\
& \left. \left. - \delta_{ij}(x_j(s-\gamma_{ij}(s)))] d\omega_j(s) \right|^p dt \right\} := \sum_{i=1}^{11} S_i.
\end{aligned} \tag{3.35}$$

By the Hölder inequality, we have

$$\begin{aligned}
S_2 & \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \left( \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) ds \right)^{\frac{p}{q}} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \right. \right. \\
& \quad \times \left. \left. \left| \sum_{j=1}^n a_{ij}(s+\tau) [f_j(x_j(s-\tau_{ij}(s+\tau))) - f_j(x_j(s-\tau_{ij}(s)))] \right|^p ds \right] dt \right\} \\
& \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \right. \\
& \quad \times \left. \mathbb{E} |x_j(s-\tau_{ij}(s+\tau)) - x_j(s-\tau_{ij}(s))|^p ds dt \right\}.
\end{aligned}$$

By a change of variables and Fubini's theorem, we can further deduce that

$$\begin{aligned}
S_2 & \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_a^{a+1} \int_{-\infty}^0 (-s)^{\alpha-1} \varphi_i(-s) \right. \\
& \quad \times \left. \mathbb{E} |x_j(s+t-\tau_{ij}(s+t+\tau)) - x_j(s+t-\tau_{ij}(s+t))|^p ds dt \right\} \\
& = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{-\infty}^0 \int_a^{a+1} (-s)^{\alpha-1} \varphi_i(-s) \right. \\
& \quad \times \left. \mathbb{E} |x_j(s+t-\tau_{ij}(s+t+\tau)) - x_j(s+t-\tau_{ij}(s+t))|^p dt ds \right\} \\
& = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{-\infty}^0 \int_{a+s}^{a+s+1} (-s)^{\alpha-1} \varphi_i(-s) \right. \\
& \quad \times \left. \mathbb{E} |x_j(t-\tau_{ij}(t+\tau)) - x_j(t-\tau_{ij}(t))|^p dt ds \right\} \\
& = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{-\infty}^a \int_s^{s+1} (a-s)^{\alpha-1} \varphi_i(a-s) \right. \\
& \quad \times \left. \mathbb{E} |x_j(t-\tau_{ij}(t+\tau)) - x_j(t-\tau_{ij}(t))|^p dt ds \right\} \\
& \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{-\infty}^a (a-s)^{\alpha-1} \varphi_i(a-s) ds \right. \\
& \quad \times \left. \sup_{s \in \mathbb{R}} \int_s^{s+1} \mathbb{E} \|x(t-\tau_{ij}(t+\tau)) - x(t-\tau_{ij}(t))\|_0^p dt \right\} \\
& \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \right\} \varepsilon.
\end{aligned} \tag{3.36}$$

Similarly, one can obtain

$$S_5 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \right\} \varepsilon, \quad (3.37)$$

$$S_7 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \right\} \varepsilon, \quad (3.38)$$

$$S_9 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \right\} \varepsilon. \quad (3.39)$$

As for  $S_1$ , by the Hölder inequality, we find

$$S_1 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \right. \\ \left. \times \mathbb{E} |x_j(s+\tau - \tau_{ij}(s+\tau)) - x_j(s - \tau_{ij}(s+\tau))|^p ds dt \right\}.$$

Let  $u = s - \tau_{ij}(s + \tau)$ , we can obtain

$$S_1 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \right. \\ \left. \times \int_a^{a+1} \int_{-\infty}^{t - \tau_{ij}(s+\tau)} (t-u - \tau_{ij}(s+\tau))^{\alpha-1} \varphi_i(t-u - \tau_{ij}(s+\tau)) \right. \\ \left. \times \mathbb{E} |x_j(u+\tau) - x_j(u)|^p du dt \right\}.$$

Further, by a change of variables and the Fubini theorem, one can infer that

$$S_1 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \int_a^{a+1} \int_{-\infty}^0 (-u)^{\alpha-1} \varphi_i(-u) \right. \\ \left. \times \mathbb{E} |x_j(u+t - \tau_{ij}(s+\tau) + \tau) - x_j(u+t - \tau_{ij}(s+\tau))|^p du dt \right\} \\ = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \int_{-\infty}^0 \int_a^{a+1} (-u)^{\alpha-1} \varphi_i(-u) \right. \\ \left. \times \mathbb{E} |x_j(u+t - \tau_{ij}(s+\tau) + \tau) - x_j(u+t - \tau_{ij}(s+\tau))|^p dt du \right\} \\ = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \right. \\ \left. \times \int_{-\infty}^0 \int_{a+u - \tau_{ij}(s+\tau)}^{a+u - \tau_{ij}(s+\tau) + 1} (-u)^{\alpha-1} \varphi_i(-u) \mathbb{E} |x_j(t+\tau) - x_j(t)|^p dt du \right\}$$

$$\begin{aligned}
&\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \right. \\
&\quad \times \left. \int_{-\infty}^a (a-u)^{\alpha-1} \varphi_i(a-u) du \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} |x_j(t+\tau) - x_j(t)|^p dt \right\} \\
&= 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \frac{1}{1 - \hat{\tau}_{ij}^+} \right\} \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t+\tau) - x(t)\|_0^p dt. \quad (3.40)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
S_4 &\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right) \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \frac{1}{1 - \hat{\nu}_{ijk}^+} \right\} \\
&\quad \times \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t+\tau) - x(t)\|_0^p dt, \quad (3.41)
\end{aligned}$$

$$\begin{aligned}
S_6 &\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \frac{1}{1 - \hat{\sigma}_{ijk}^+} \right\} \\
&\quad \times \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t+\tau) - x(t)\|_0^p dt. \quad (3.42)
\end{aligned}$$

As for  $S_3$ , by the Hölder inequality, a change of variables, and the Fubini theorem, we deduce that

$$\begin{aligned}
S_3 &\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \mathbb{E} \left| \sum_{j=1}^n [a_{ij}(s+\tau) - a_{ij}(s)] \right. \right. \\
&\quad \times \left. \left. f_j(x_j(s - \tau_{ij}(s))) \right|^p ds dt \right\} \\
&\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n |a_{ij}(s+\tau) - a_{ij}(s)|^p \right. \\
&\quad \times \left. \left( \sum_{j=1}^n \mathbb{E} |f_j(x_j(s - \tau_{ij}(s)))|^q \right)^{\frac{p}{q}} ds dt \right\} \\
&\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} \int_a^{a+1} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi_i(t-s) \sum_{j=1}^n |a_{ij}(s+\tau) \right. \\
&\quad \left. - a_{ij}(s)|^p ds dt \right\} \\
&\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} \sup_{s \in \mathbb{R}} \int_s^{s+1} \sum_{j=1}^n |a_{ij}(t+\tau) - a_{ij}(t)|^p dt \right\} \\
&\leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} n \right\} \varepsilon. \quad (3.43)
\end{aligned}$$

Similarly, we can obtain

$$S_8 \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \sum_{j=1}^n \left( \sum_{k=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} n^2 \right\} \varepsilon. \quad (3.44)$$

When  $p > 2$ , according to Lemma 2.2, the Hölder inequality, and a change of variables, we infer that

$$\begin{aligned} S_{11} &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \left| \sum_{j=1}^n [\delta_{ij}(x_j(s - \gamma_{ij}(s + \tau))) \right. \right. \right. \\ &\quad \left. \left. \left. - \delta_{ij}(x_j(s - \gamma_{ij}(s))) \right] \right|^2 ds \right]^{\frac{p}{2}} dt \right\} \\ &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) ds \right]^{\frac{p-2}{2}} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \right. \right. \\ &\quad \left. \left. \times \mathbb{E} \left| \sum_{j=1}^n [\delta_{ij}(x_j(s - \gamma_{ij}(s + \tau))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] \right|^p ds \right] dt \right\} \\ &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p-2}{2}} n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{2(\alpha-1)} \right. \\ &\quad \left. \times \varphi_i^2(t-s) \mathbb{E} \left| \sum_{j=1}^n [\delta_{ij}(x_j(s - \gamma_{ij}(s + \tau))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] \right|^p ds dt \right\}. \end{aligned}$$

By a change of variables and the Fubini theorem, we have

$$S_{11} \leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p}{2}} \right\} \varepsilon. \quad (3.45)$$

Similarly, we find

$$\begin{aligned} S_{10} &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p-2}{2}} n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \int_a^{a+1} \int_{-\infty}^t (t-s)^{2(\alpha-1)} \right. \\ &\quad \left. \times \varphi_i^2(t-s) \mathbb{E} |x_j(s + \tau - \gamma_{ij}(s + \tau)) - x_j(s - \gamma_{ij}(s + \tau))|^p ds dt \right\}. \end{aligned}$$

Let  $u = s - \gamma_{ij}(s + \tau)$ , we get

$$\begin{aligned} S_{10} &\leq 11^{p-1} C_p \max_{1 \leq i \leq n} \left\{ \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p-2}{2}} n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \frac{1}{1 - \gamma_{ij}^+} \right. \\ &\quad \times \int_a^{a+1} \int_{-\infty}^{t - \gamma_{ij}(s + \tau)} (t - \gamma_{ij}(s + \tau) - u)^{2(\alpha-1)} \varphi_i^2(t - \gamma_{ij}(s + \tau) - u) \\ &\quad \left. \times \mathbb{E} |x_j(u + \tau) - x_j(u)|^p du dt \right\}. \end{aligned}$$

By a change of variables and the Fubini theorem, we find

$$S_{10} \leq 11^{p-1} \max_{1 \leq i \leq n} \left\{ \left[ \frac{\Gamma(2 - \frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \right]^{\frac{p}{2}} n^{\frac{p}{2}} \sum_{j=1}^n (L_{ij}^{\delta})^p \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \\ \times \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^p dt. \quad (3.46)$$

When  $p = 2$ , by the Itô isometry and the Cauchy–Schwarz inequality, we have

$$S_{11} \leq 11 \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) \right. \right. \\ \left. \left. \times \left| \sum_{j=1}^n [\delta_{ij}(x_j(s - \gamma_{ij}(s + \tau))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] \right|^2 ds \right] dt \right\} \\ \leq 11 \max_{1 \leq i \leq n} \left\{ \int_a^{a+1} \int_{-\infty}^t (t-s)^{2(\alpha-1)} \varphi_i^2(t-s) n \sum_{j=1}^n (L_{ij}^{\delta})^2 \right. \\ \left. \times \mathbb{E} |x_j(s - \gamma_{ij}(s + \tau)) - x_j(s - \gamma_{ij}(s))|^2 ds dt \right\}.$$

By a change of variables and the Fubini theorem, one can obtain

$$S_{11} \leq 11 \max_{1 \leq i \leq n} \left\{ n \sum_{j=1}^n (L_{ij}^{\delta})^2 \frac{\Gamma(2 - \frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \right\} \varepsilon. \quad (3.47)$$

From the Itô isometry, the Cauchy–Schwarz inequality, a change of variables, and the Fubini theorem, it follows that

$$S_{10} \leq 11 \max_{1 \leq i \leq n} \left\{ \frac{n\Gamma(2 - \frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \sum_{j=1}^n (L_{ij}^{\delta})^2 \frac{1}{1 - \dot{\gamma}_{ij}^+} \right\} \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^2 dt. \quad (3.48)$$

Substituting (3.36)–(3.48) into (3.35), we have the following two cases.

Case 1. When  $p > 2$ , we have

$$\int_a^{a+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^p dt \leq \beta_1 \varepsilon + \pi_1 \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^p dt, \quad (3.49)$$

where  $\pi_1$  is given in  $(H_3)$ , and

$$\beta_1 = 11^{p-1} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^p} \left[ \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p + n \left( \sum_{j=1}^n (M_j^f)^q \right)^{\frac{p}{q}} + \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \right. \right. \right. \\ \left. \left. \times \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{p}{q}} \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p + \sum_{j=1}^n \left[ \left( \sum_{l=1}^n (b_{ijlk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{q=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \right. \\ \left. + n^2 \sum_{j=1}^n \left( \sum_{k=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (M_j^g)^q + 1 \right) + C_p n^{\frac{p}{2}} \sum_{j=1}^n (L_{ij}^{\delta})^p \times \left[ \frac{\Gamma(2 - \frac{1}{\alpha})\Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i)^{2-\frac{1}{\alpha}}\Gamma(2-\alpha)} \right]^{\frac{p}{2}} \right\}.$$

According to (3.49) and  $(H_3)$ , we can see that

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^p dt \leq \frac{\beta_1 \varepsilon}{1 - \pi_1}. \quad (3.50)$$

Case 2. When  $p = 2$ , we have

$$\int_a^{a+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^2 dt \leq \beta_2 \varepsilon + \pi_2 \sup_{u \in \mathbb{R}} \int_u^{u+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^2 dt, \quad (3.51)$$

where  $\pi_2$  is given in  $(H_3)$ , and

$$\begin{aligned} \beta_2 = & 11 \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i^2} \left[ \sum_{j=1}^n (a_{ij}^+)^2 \sum_{j=1}^n (L_j^f)^2 + n \sum_{j=1}^n (M_j^f)^2 + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{j=1}^n (M_j^g)^2 \right) \sum_{j=1}^n (L_j^g)^2 \right. \right. \\ & + \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (L_j^g)^2 \right) \sum_{j=1}^n (M_j^g)^2 + n^2 \sum_{j=1}^n \left( \sum_{k=1}^n (M_j^g)^2 \right) \sum_{j=1}^n (M_j^g)^2 + 1 \left. \right] \\ & + \frac{n \Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha) (2a_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \left. \right\}. \end{aligned}$$

According to (3.51) and  $(H_3)$ , we can see that

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^2 dt \leq \frac{\beta_2 \varepsilon}{1 - \pi_2}. \quad (3.52)$$

Note that

$$d_{BL}^p(P \circ [x(t + \tau)]^{-1}, P \circ [x(t)]^{-1}) \leq \mathbb{E} \|x(t + \tau) - x(t)\|_0^p.$$

We can then see that

$$\sup_{a \in \mathbb{R}} \left( \int_a^{a+1} d_{BL}^p(P \circ [x(t + \tau)]^{-1}, P \circ [x(t)]^{-1}) dt \right)^{\frac{1}{p}} \leq \sup_{a \in \mathbb{R}} \left( \int_a^{a+1} \mathbb{E} \|x(t + \tau) - x(t)\|_0^p dt \right)^{\frac{1}{p}}. \quad (3.53)$$

Thereupon, combining (3.50), (3.52), and (3.53), we conclude that  $x(t)$  is  $p$ -th Stepanov almost periodic in the distribution sense. This completes the proof.  $\square$

#### 4. Finite-time Stepanov almost periodic synchronization

Take System (1.1) as a drive system and consider the following response system:

$$\begin{aligned} {}^c D_t^\alpha y_i(t) = & -a_i y_i(t) + \sum_{j=1}^n a_{ij}(t) f_j(y_j(t - \tau_{ij}(t))) + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(t) g_j(y_j(t - \sigma_{ijk}(t))) \\ & \times g_k(y_k(t - \nu_{ijk}(t))) + I_i(t) + \sum_{j=1}^n \delta_{ij}(y_j(t - \gamma_{ij}(t))) \frac{d\omega_j(t)}{dt} + E_i(t), t > t_0, \end{aligned} \quad (4.1)$$

where  $y_i(t)$  denotes the state of the response system at time  $t$ ;  $E_i(t)$  is a controller, the other notations are the same as those in System (1.1).

Let the controller be in the following form:

$$E_i(t) = -d_i e_i(t) + \sum_{j=1}^n c_{ij}(t) h_j(e_j(t - \eta_{ij}(t))), i = 1, 2, \dots, n, \quad (4.2)$$

where  $d_i > 0$ .

The initial values of Systems (1.1) and (4.1) are given by

$$y_i(s) = \xi_i(s), x_i(s) = \psi_i(s), s \in [t_0 - \rho', t_0],$$

where

$$\rho' = \max_{1 \leq i, j, k \leq n} \left\{ \sup_{t \in \mathbb{R}} \{\tau_{ij}(t)\}, \sup_{t \in \mathbb{R}} \{\sigma_{ijk}(t)\}, \sup_{t \in \mathbb{R}} \{\nu_{ijk}(t)\}, \sup_{t \in \mathbb{R}} \{\gamma_{ij}(t)\}, \sup_{t \in \mathbb{R}} \{\eta_{ij}(t)\} \right\}.$$

Let  $e_i = y_i - x_i, i = 1, 2, \dots, n$ . From (1.1) and (4.1), we get the following error system:

$$\begin{aligned} {}^c D_{t_0}^\alpha e_i(t) &= -a_i e_i(t) + \sum_{j=1}^n a_{ij}(t) [f_j(y_j(t - \tau_{ij}(t))) - f_j(x_j(t - \tau_{ij}(t)))] + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(t) \\ &\times [g_j(y_j(t - \sigma_{ijk}(t))) g_k(y_k(t - \nu_{ijk}(t))) - g_j(x_j(t - \sigma_{ijk}(t))) g_k(x_k(t - \nu_{ijk}(t)))] \\ &+ \sum_{j=1}^n [\delta_{ij}(y_j(t - \gamma_{ij}(t))) - \delta_{ij}(x_j(t - \gamma_{ij}(t)))] \frac{d\omega_j(t)}{dt} + E_i(t), t > t_0. \end{aligned} \quad (4.3)$$

The initial values of System (4.3) are given by

$$e_i(s) = \xi_i(s) - \psi_i(s), s \in [t_0 - \rho', t_0], i = 1, 2, \dots, n.$$

**Definition 4.1.** The drive system (1.1) is said to be finite-time synchronized with the response system (4.1), if the constants  $\delta, \varepsilon, t_0, T$  exist such that when  $\|\xi - \psi\|_{t_0} < \delta$ , one has  $\mathbb{E}\|y(t) - x(t)\|_0^p \leq \varepsilon, \forall t \in [t_0 - \rho', t_0]$ , where  $0 < \delta < \varepsilon, T > t_0, \|\xi - \psi\|_{t_0} = \max_{1 \leq i \leq n} \sup_{t \in [t_0 - \rho', t_0]} \{\mathbb{E}|\xi_i(t) - \psi_i(t)|^p\}$ .

**Theorem 4.1.** Under the assumptions of Theorem 3.1, we assume the following:

(H<sub>4</sub>) For  $i, j = 1, 2, \dots, n, \eta_{ij} \in AP(\mathbb{R}, \mathbb{R}^+), c_{ij} \in \mathcal{L}_{loc}^p(\mathbb{R}, \mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$  is  $p$ -th Stepanov almost periodic.

(H<sub>5</sub>) For  $j = 1, 2, \dots, n, h_j \in C(\mathbb{R}, \mathbb{R})$ , the constants  $L_j^h > 0$  exist such that for all  $x, y \in \mathbb{R}$ ,

$$|h_j(x) - h_j(y)| \leq L_j^h |x - y|;$$

moreover,  $h_j(0) = 0$ .

(H<sub>6</sub>) For  $p > 2, \frac{1}{p} + \frac{1}{q} = 1$ , there are positive numbers  $\delta$  and  $\varepsilon \in \mathbb{R}^+$  such that

$$\delta [E_\alpha (M_1 \Gamma(\alpha) T^\alpha) + E_{2\alpha-1} (M_2 \Gamma(2\alpha - 1) T^{2\alpha-1}) - 1] \leq \frac{\varepsilon}{5^{p-1}},$$

where

$$\begin{aligned}
 M_1 &= \frac{5^{p-1}}{\Gamma(\alpha)} \max_{1 \leq i \leq n} \left\{ \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left[ \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p + 2^{p-1} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \right. \right. \right. \\
 &\quad \times \left. \sum_{j=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} + 2^{p-1} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \right. \\
 &\quad \times \left. \left. \sum_{j=1}^n (L_j^g)^p + \left( \sum_{j=1}^n (c_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^h)^p \right] \right\}, \\
 M_2 &= \frac{5^{p-1}}{\Gamma^2(\alpha)} C_p \max_{1 \leq i \leq n} \left\{ n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha) (2a_i + 2d_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p-2}{2}} \right\};
 \end{aligned}$$

for  $p = q = 2$ , there are positive numbers  $\hat{\delta}$  and  $\hat{\varepsilon}$  such that

$$\hat{\delta} [E_\alpha (F_1 \Gamma(\alpha) T^\alpha) + E_{2\alpha-1} (F_2 \Gamma(2\alpha - 1) T^{2\alpha-1}) - 1] \leq \frac{\hat{\varepsilon}}{5},$$

where

$$\begin{aligned}
 F_1 &= \frac{5}{\Gamma(\alpha)} \max_{1 \leq i \leq n} \left\{ \frac{1}{a_i + d_i} \left[ \sum_{j=1}^n (a_{ij}^+)^2 \sum_{j=1}^n (L_j^f)^2 + 2 \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (L_j^g)^2 \right) \right. \right. \\
 &\quad \times \left. \sum_{j=1}^n (M_j^g)^2 + 2 \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^2 \sum_{k=1}^n (M_j^g)^2 \right) \sum_{j=1}^n (L_j^g)^2 + \sum_{j=1}^n (c_{ij}^+)^2 \sum_{j=1}^n (L_j^h)^2 \right\}, \\
 F_2 &= \frac{5n}{\Gamma^2(\alpha)} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n (L_{ij}^\delta)^2 \right\}.
 \end{aligned}$$

Then the drive system (1.1) and the response system (4.1) are finite-time synchronized with respect to  $\{\delta, \varepsilon, t_0, T\}$ .

*Proof.* From (4.2) and (4.3), we can see that

$$\begin{aligned}
 {}^c D_t^\alpha e_i(t) &= -(a_i + d_i)e_i(t) + \sum_{j=1}^n a_{ij}(t) [f_j(y_j(t - \tau_{ij}(t))) - f_j(x_j(t - \tau_{ij}(t)))] + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(t) \\
 &\quad \times [g_j(y_j(t - \sigma_{ijk}(t)))g_k(y_k(t - \nu_{ijk}(t))) - g_j(x_j(t - \sigma_{ijk}(t)))g_k(x_k(t - \nu_{ijk}(t)))] \\
 &\quad + \sum_{j=1}^n [\delta_{ij}(y_j(t - \gamma_{ij}(t))) - \delta_{ij}(x_j(t - \gamma_{ij}(t)))] \frac{d\omega_j(t)}{dt} + \sum_{j=1}^n c_{ij}(t)h_j(e_j(t - \eta_{ij}(t))). \quad (4.4)
 \end{aligned}$$

According to Definition 3.1, if  $e = (e_1, e_2, \dots, e_n)^T$  is a mild solution to the system (4.4), then it satisfies the following integral equation:

$$e_i(t) = \tilde{U}_i(t - t_0)(\xi_i(t_0) - \psi_i(t_0)) + \int_{t_0}^t (t - s)^{\alpha-1} \tilde{\varphi}_i(t - s) \left[ \sum_{j=1}^n a_{ij}(s) [f_j(y_j(s - \tau_{ij}(s)))] \right.$$

$$\begin{aligned}
& - f_j(x_j(s - \tau_{ij}(s))) + \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) [g_j(y_j(s - \sigma_{ijk}(s))) g_k(y_k(s - \nu_{ijk}(s))) \\
& - g_j(x_j(s - \sigma_{ijk}(s))) g_k(x_k(s - \nu_{ijk}(s)))] + \sum_{j=1}^n c_{ij}(s) h_j(e_j(s - \eta_{ij}(s))) \Big] ds \\
& + \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n [\delta_{ij}(y_j(s - \gamma_{ij}(s))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] d\omega_j(s), \quad (4.5)
\end{aligned}$$

where  $\tilde{U}_i(t) = \int_0^{+\infty} \varrho_\alpha(r) e^{-(a_i+d_i)t^\alpha r} dr$ ,  $\tilde{\varphi}_i(t) = \alpha \int_0^{+\infty} r \varrho_\alpha(r) e^{-(a_i+d_i)t^\alpha r} dr$ .

By calculation, it can be seen that

$$\int_0^{+\infty} (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) ds = \frac{1}{a_i + d_i}.$$

For  $t \geq t_0$ , in view of the Hölder inequality, we have

$$\begin{aligned}
& \mathbb{E} |\tilde{U}_i(t-t_0)(\xi_i(t_0) - \psi_i(t_0))|^p \\
& = \mathbb{E} \left| \int_0^{+\infty} \varrho_\alpha(r) e^{-(a_i+d_i)(t-t_0)^\alpha r} (\xi_i(t_0) - \psi_i(t_0)) dr \right|^p \\
& \leq \left( \int_0^{+\infty} \varrho_\alpha(r) e^{-(a_i+d_i)(t-t_0)^\alpha r} dr \right)^{\frac{p}{q}} \left( \int_0^{+\infty} \varrho_\alpha(r) e^{-(a_i+d_i)(t-t_0)^\alpha r} \mathbb{E} |\xi_i(t_0) - \psi_i(t_0)|^p dr \right) \\
& \leq \int_0^{+\infty} \varrho_\alpha(r) e^{-(a_i+d_i)(t-t_0)^\alpha r} \mathbb{E} |\xi_i(t_0) - \psi_i(t_0)|^p dr \\
& \leq \|\xi - \psi\|_{t_0}. \quad (4.6)
\end{aligned}$$

Let  $u(t) = \max_{1 \leq i \leq n} \{u_i(t)\}$ , where  $u_i(t) = \sup_{\theta \in [t_0 - \rho', t]} \{\mathbb{E} |e_i(\theta)|^p\}$ ,  $t \in [t_0, T]$ . Hence, we have

$$\begin{aligned}
& \mathbb{E} |e_i(t)|^p \leq 5^{p-1} \mathbb{E} |\tilde{U}_i(t-t_0)(\xi_i(t_0) - \psi_i(t_0))|^p \\
& + 5^{p-1} \mathbb{E} \left| \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n a_{ij}(s) [f_j(y_j(s - \tau_{ij}(s))) - f_j(x_j(s - \tau_{ij}(s)))] ds \right|^p \\
& + 5^{p-1} \mathbb{E} \left| \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) [g_j(y_j(s - \sigma_{ijk}(s))) g_k(y_k(s - \nu_{ijk}(s))) \right. \\
& \left. - g_j(x_j(s - \sigma_{ijk}(s))) g_k(x_k(s - \nu_{ijk}(s)))] ds \right|^p \\
& + 5^{p-1} \mathbb{E} \left| \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n c_{ij}(s) h_j(e_j(s - \eta_{ij}(s))) ds \right|^p \\
& + 5^{p-1} \mathbb{E} \left| \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n [\delta_{ij}(y_j(s - \gamma_{ij}(s))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] d\omega_j(s) \right|^p \\
& := \sum_{i=1}^5 W_i. \quad (4.7)
\end{aligned}$$

It is easy to see that

$$W_1 = 5^{p-1} \mathbb{E} |\tilde{U}_i(t - t_0)(\xi_i(t_0) - \psi_i(t_0))|^p \leq 5^{p-1} \|\xi - \psi\|_{t_0}^p. \quad (4.8)$$

Applying the Hölder inequality, we can infer that

$$\begin{aligned} W_2 &\leq 5^{p-1} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \mathbb{E} |e_j(s - \tau_{ij}(s))|^p ds \\ &\leq 5^{p-1} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{t_0}^t (t-s)^{\alpha-1} \\ &\quad \times \left( \alpha \int_0^{+\infty} r \varrho_\alpha(r) e^{-(a_i+d_i)(t-s)r} dr \right) u(s) ds \\ &\leq 5^{p-1} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{t_0}^t (t-s)^{\alpha-1} \left( \alpha \int_0^{+\infty} r \varrho_\alpha(r) dr \right) u(s) ds \\ &\leq \frac{5^{p-1}}{\Gamma(\alpha)} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (a_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^f)^p \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds. \end{aligned} \quad (4.9)$$

Similarly, we can find that

$$\begin{aligned} W_3 &\leq 10^{p-1} \mathbb{E} \left| \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) [g_j(y_j(s - \sigma_{ijk}(s))) \right. \\ &\quad \times g_k(y_k(s - \nu_{ijk}(s))) - g_j(x_j(s - \sigma_{ijk}(s))) g_k(y_k(s - \nu_{ijk}(s)))] ds \Big|^p \\ &\quad + 10^{p-1} \mathbb{E} \left| \int_{t_0}^t (t-s)^{\alpha-1} \tilde{\varphi}_i(t-s) \sum_{j=1}^n \sum_{k=1}^n b_{ijk}(s) [g_j(x_j(s - \sigma_{ijk}(s))) \right. \\ &\quad \times g_k(y_k(s - \nu_{ijk}(s))) - g_j(x_j(s - \sigma_{ijk}(s))) g_k(x_k(s - \nu_{ijk}(s)))] ds \Big|^p \\ &\leq \frac{10^{p-1}}{\Gamma(\alpha)} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \sum_{j=1}^n \left[ \left( \sum_{k=1}^n (b_{ijk}^+)^q \right)^{\frac{p}{q}} \sum_{k=1}^n (L_j^g)^p \right] \left( \sum_{j=1}^n (M_j^g)^q \right)^{\frac{p}{q}} \\ &\quad \times \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + \frac{10^{p-1}}{\Gamma(\alpha)} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left[ \sum_{j=1}^n \left( \sum_{k=1}^n (b_{ijk}^+)^q \right) \right. \\ &\quad \times \left. \left( \sum_{k=1}^n (M_j^g)^p \right)^{\frac{q}{p}} \right]^{\frac{p}{q}} \sum_{j=1}^n (L_j^g)^p \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds, \end{aligned} \quad (4.10)$$

$$W_4 \leq \frac{5^{p-1}}{\Gamma(\alpha)} \left( \frac{1}{a_i + d_i} \right)^{\frac{p}{q}} \left( \sum_{j=1}^n (c_{ij}^+)^q \right)^{\frac{p}{q}} \sum_{j=1}^n (L_j^h)^p \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds. \quad (4.11)$$

For  $p > 2$ , according to Lemma 2.2, the Hölder inequality, and a change of variables, we have

$$W_5 \leq 5^{p-1} C_p \mathbb{E} \left[ \int_{t_0}^t (t-s)^{2(\alpha-1)} \tilde{\varphi}_i^2(t-s) \left| \sum_{j=1}^n [\delta_{ij}(y_j(s - \gamma_{ij}(s))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] \right|^2 ds \right]^{\frac{p}{2}}$$

$$\begin{aligned}
&\leq 5^{p-1} C_p \left[ \int_{t_0}^t (t-s)^{2(\alpha-1)} \tilde{\varphi}_i^2(t-s) ds \right]^{\frac{p-2}{2}} \left[ \int_{t_0}^t (t-s)^{2(\alpha-1)} \tilde{\varphi}_i^2(t-s) \right. \\
&\quad \times \mathbb{E} \left[ \sum_{j=1}^n [\delta_{ij}(y_j(s - \gamma_{ij}(s))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] \right]^p ds \Big] \\
&\leq 5^{p-1} C_p n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i + 2d_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p-2}{2}} \int_{t_0}^t (t-s)^{2(\alpha-1)} \tilde{\varphi}_i^2(t-s) u(s) ds \\
&\leq \frac{5^{p-1}}{\Gamma^2(\alpha)} C_p n^{\frac{p}{q}} \sum_{j=1}^n (L_{ij}^\delta)^p \left[ \frac{\Gamma(2 - \frac{1}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma(\alpha)(2a_i + 2d_i)^{2 - \frac{1}{\alpha}} \Gamma(2 - \alpha)} \right]^{\frac{p-2}{2}} \int_{t_0}^t (t-s)^{2(\alpha-1)} u(s) ds. \tag{4.12}
\end{aligned}$$

When  $p = 2$ , by the Itô isometry and the Cauchy–Schwarz inequality, we can infer that

$$\begin{aligned}
W_5 &\leq 5 \mathbb{E} \left[ \int_{-\infty}^t (t-s)^{2(\alpha-1)} \tilde{\varphi}_i^2(t-s) \left| \sum_{j=1}^n [\delta_{ij}(y_j(s - \gamma_{ij}(s))) - \delta_{ij}(x_j(s - \gamma_{ij}(s)))] \right|^2 ds \right] \\
&\leq \frac{5n}{\Gamma^2(\alpha)} \sum_{j=1}^n (L_{ij}^\delta)^2 \int_{t_0}^t (t-s)^{2(\alpha-1)} u(s) ds. \tag{4.13}
\end{aligned}$$

Substituting (4.8)–(4.13) into (4.7), we have the following two cases.

Case 1. When  $p > 2$ , we have

$$\mathbb{E}|e_i(t)|^p \leq 5^{p-1} \|\xi - \psi\|_{t_0} + M_1 \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + M_2 \int_{t_0}^t (t-s)^{2\alpha-2} u(s) ds,$$

where  $M_1$  and  $M_2$  are given in  $(H_6)$ .

Then, for all  $\theta \in [t_0, t]$ , we have

$$\mathbb{E}|e_i(\theta)|^p \leq 5^{p-1} \|\xi - \psi\|_{t_0} + M_1 \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + M_2 \int_{t_0}^t (t-s)^{2\alpha-2} u(s) ds.$$

Therefore, for all  $t \in [t_0, T]$ , we have

$$u(t) \leq 5^{p-1} \|\xi - \psi\|_{t_0} + M_1 \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + M_2 \int_{t_0}^t (t-s)^{2\alpha-2} u(s) ds.$$

According to Lemma 2.3 and  $(H_6)$ , we can see that

$$u(t) \leq 5^{p-1} \|\xi - \psi\|_{t_0} [E_\alpha(M_1 \Gamma(\alpha)(t - t_0)^\alpha) + E_{2\alpha-1}(M_2 \Gamma(2\alpha - 1)(t - t_0)^{2\alpha-1}) - 1].$$

Consequently, for all  $t \in [t_0, T]$ , we can deduce that

$$\mathbb{E}|e_i(t)|^p \leq u(t) \leq 5^{p-1} \delta [E_\alpha(M_1 \Gamma(\alpha) T^\alpha) + E_{2\alpha-1}(M_2 \Gamma(2\alpha - 1) T^{2\alpha-1}) - 1] \leq \varepsilon.$$

Case 2. When  $p = 2$ , we have

$$\mathbb{E}|e_i(t)|^2 \leq 5 \|\xi - \psi\|_{t_0} + F_1 \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + F_2 \int_{t_0}^t (t-s)^{2\alpha-2} u(s) ds,$$

where  $F_1$  and  $F_2$  are given in  $(H_6)$ .

Then, for all  $\theta \in [t_0, t]$ , we have

$$\mathbb{E}|e_i(\theta)|^2 \leq 5\|\xi - \psi\|_{t_0} + F_1 \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + F_2 \int_{t_0}^t (t-s)^{2\alpha-2} u(s) ds.$$

Consequently, for all  $t \in [t_0, T]$ , we have

$$u(t) \leq 5\|\xi - \psi\|_{t_0} + F_1 \int_{t_0}^t (t-s)^{\alpha-1} u(s) ds + F_2 \int_{t_0}^t (t-s)^{2\alpha-2} u(s) ds.$$

By virtue of Lemma 2.3 and  $(H_6)$ , we can see that

$$u(t) \leq 5\|\xi - \psi\|_{t_0} [E_\alpha(F_1\Gamma(\alpha)(t-t_0)^\alpha) + E_{2\alpha-1}(F_2\Gamma(2\alpha-1)(t-t_0)^{2\alpha-1}) - 1].$$

Then, for all  $t \in [t_0, T]$ , we have

$$\mathbb{E}|e_i(t)|^2 \leq u(t) \leq 5\hat{\delta} [E_\alpha(F_1\Gamma(\alpha)T^\alpha) + E_{2\alpha-1}(F_2\Gamma(2\alpha-1)T^{2\alpha-1}) - 1] \leq \hat{\varepsilon}.$$

Consequently, the drive system (1.1) and the response system (4.1) are finite-time synchronized with respect to  $\{\delta, \varepsilon, t_0, T\}$ . The proof is completed.  $\square$

**Remark 4.1.** Under the conditions of Theorem 4.1, System (1.1) possesses a unique Stepanov almost periodic solution, while System (1.1) and System (4.1) are synchronized. Therefore, we may refer to this synchronization as Stepanov almost periodic synchronization.

## 5. An example

In this section, we present one example and some simulations to illustrate the feasibility of our main results.

In System (1.1) and System (4.3), let  $n = 2$ ,  $t_0 = 0$  and take  $a_1 = 8$ ,  $a_2 = 10$ ,  $d_1 = \frac{1}{4}$ ,  $d_2 = \frac{1}{5}$ ,

$$\begin{aligned} f_1(x_1) &= 0.025 \sin(x_1), f_2(x_2) = -0.005 + 0.005 \cos(x_2), g_1(x_1) = 0.05 \tanh(x_1), \\ g_2(x_2) &= 0.02 \sin(x_2), \delta_{11}(x_1) = 0.02 \sin(x_1), \delta_{12}(x_2) = 0.02 \sin(x_2), \delta_{21}(x_1) = 0.02 \tanh(x_1), \\ \delta_{22}(x_2) &= 0.02 \cos(x_2) - 0.02, h_1(e_1) = 0.015 \sin(e_1), h_2(e_2) = 0.018 \tanh(e_2) \\ I_1(t) &= 0.5 \sin \sqrt{3}t + 0.314 \cos \sqrt{5}t, I_2(t) = 0.4 \cos \sqrt{2}t + 0.27 \sin \sqrt{3}t, \\ a_{11}(t) &= a_{12}(t) = 0.05 \sin \sqrt{2}t + 0.05 \cos t, a_{21}(t) = a_{22}(t) = 0.06 \cos \sqrt{3}t + 0.03 \sin \sqrt{7}t, \\ b_{111}(t) &= 0.06 \sin \sqrt{3}t + 0.02 \cos \sqrt{5}t, b_{112}(t) = 0.03 \sin \sqrt{2}t + 0.05 \sin \sqrt{7}t, \\ b_{121}(t) &= 0.04 \sin \sqrt{3}t + 0.04 \cos \sqrt{2}t, b_{122}(t) = 0.025 \sin \sqrt{5}t + 0.055 \cos \sqrt{2}t, \\ b_{211}(t) &= 0.025 \cos \sqrt{7}t + 0.045 \sin t, b_{212}(t) = 0.02 \sin \sqrt{5}t + 0.05 \cos \sqrt{7}t, \\ b_{221}(t) &= 0.035 \cos \sqrt{2}t + 0.035 \sin \sqrt{5}t, b_{222}(t) = 0.04 \sin \sqrt{3}t + 0.03 \cos \sqrt{7}t, \\ c_{11}(t) &= c_{12}(t) = 0.02 \sin t + 0.07 \cos \sqrt{2}t, c_{21}(t) = c_{22}(t) = 0.025 \cos \sqrt{5}t + 0.018 \sin \sqrt{3}t, \\ \tau_{ij}(t) &= 0.05 + 0.02 \sin 0.2t, \sigma_{ijk}(t) = 0.8 + 0.2 \cos 0.4t, \nu_{ijk}(t) = 0.4 + 0.2 \cos 0.1t, \end{aligned}$$

$$\gamma_{ij}(t) = 0.05 + 0.2 \cos t, \eta_{ij}(t) = 0.07 + 0.02 \cos 0.2t.$$

By a simple calculation, we have

$$\begin{aligned} a_{11}^+ &= a_{12}^+ = 0.1, a_{21}^+ = a_{22}^+ = 0.09, b_{1jk}^+ = 0.08, b_{2jk}^+ = 0.07, c_{11}^+ = c_{12}^+ = 0.09, c_{21}^+ = c_{22}^+ = 0.043, \\ \tau_{ij}^+ &= 0.07, \hat{\tau}_{ij}^+ = 0.004, \sigma_{ijk}^+ = 1, \hat{\sigma}_{ijk}^+ = 0.08, \nu_{ijk}^+ = 0.6, \hat{\nu}_{ijk}^+ = 0.02, \gamma_{ij}^+ = 0.07, \hat{\gamma}_{ij}^+ = 0.02, \\ L_1^f &= 0.025, L_2^f = 0.005, L_1^g = 0.05, L_2^g = 0.02, L_{11}^\delta = L_{12}^\delta = L_{21}^\delta = L_{22}^\delta = 0.02, L_1^h = 0.015, \\ L_2^h &= 0.018, M_1^g = 0.05, M_2^g = 0.02. \end{aligned}$$

Choose  $\alpha = 0.55$ , by calculation, when  $p = 3, q = \frac{3}{2}, \varsigma_1 \approx 0.8833 < 1$ , and when  $p = 2, q = 2, \varsigma_2 \approx 0.044 < 1$ . Let  $T = 8, \delta = 0.01, \varepsilon = 0.5, \hat{\delta} = 0.01$ , and  $\hat{\varepsilon} = 0.3$ . Using the MATLAB program, to compute the Mittag–Leffler function, we obtain

$$\begin{aligned} \delta[E_{0.55}(M_1\Gamma(0.55)T^{0.55}) + E_{0.1}(M_2\Gamma(0.1)T^{0.1}) - 1] &\approx 0.015 < \frac{\varepsilon}{25} = 0.02, \\ \hat{\delta}[E_{0.55}(F_1\Gamma(0.55)T^{0.55}) + E_{0.1}(F_2\Gamma(0.1)T^{0.1}) - 1] &\approx 0.01038 < \frac{\hat{\varepsilon}}{5} = 0.06. \end{aligned}$$

Choose  $\alpha = 0.8$ , by calculation, when  $p = 3, q = \frac{3}{2}, \varsigma_1 \approx 0.0146 < 1$ , and when  $p = 2, q = 2, \varsigma_2 \approx 0.00286 < 1$ . Let  $T = 8, \delta = 0.01, \varepsilon = 0.5, \hat{\delta} = 0.01$ , and  $\hat{\varepsilon} = 0.3$ . Using the MATLAB program, to compute the Mittag–Leffler function, we obtain

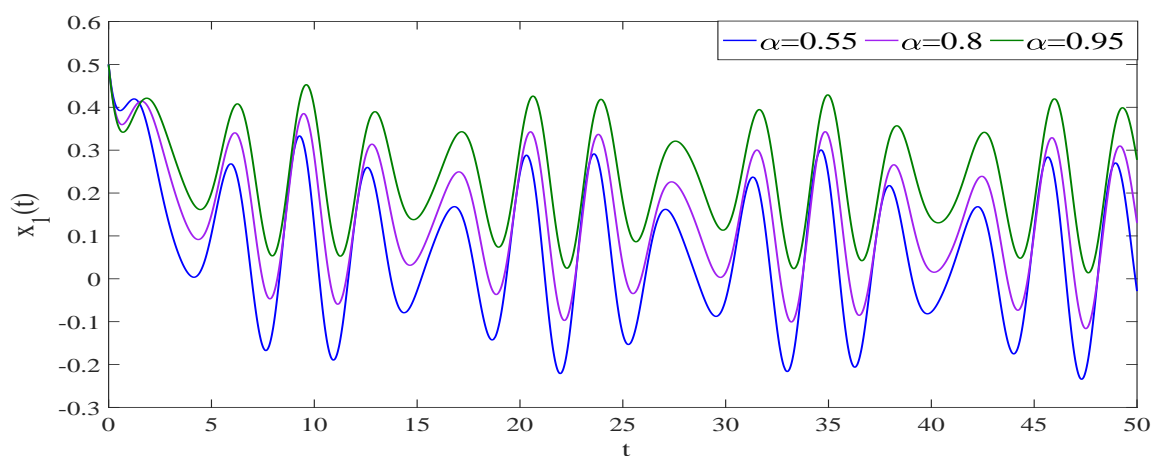
$$\begin{aligned} \delta[E_{0.8}(M_1\Gamma(0.8)T^{0.8}) + E_{0.6}(M_2\Gamma(0.6)T^{0.6}) - 1] &\approx 0.01045 < \frac{\varepsilon}{25} = 0.02, \\ \hat{\delta}[E_{0.8}(F_1\Gamma(0.8)T^{0.8}) + E_{0.6}(F_2\Gamma(0.6)T^{0.6}) - 1] &\approx 0.0102 < \frac{\hat{\varepsilon}}{5} = 0.06. \end{aligned}$$

Choose  $\alpha = 0.95$ , by calculation, when  $p = 3, q = \frac{3}{2}, \varsigma_1 \approx 0.0402 < 1$ , and when  $p = 2, q = 2, \varsigma_2 \approx 0.0028 < 1$ . Let  $T = 8, \delta = 0.01, \varepsilon = 0.5, \hat{\delta} = 0.01$ , and  $\hat{\varepsilon} = 0.3$ . Using the MATLAB program, to compute the Mittag–Leffler function, we obtain

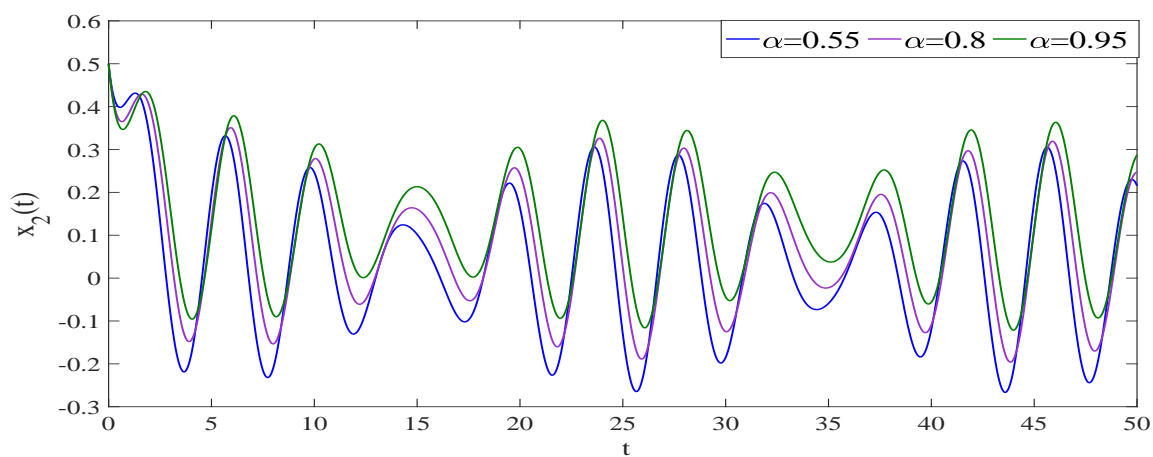
$$\begin{aligned} \delta[E_{0.95}(M_1\Gamma(0.95)T^{0.95}) + E_{0.9}(M_2\Gamma(0.9)T^{0.9}) - 1] &\approx 0.01087 < \frac{\varepsilon}{25} = 0.02, \\ \hat{\delta}[E_{0.95}(F_1\Gamma(0.95)T^{0.95}) + E_{0.9}(F_2\Gamma(0.9)T^{0.9}) - 1] &\approx 0.01051 < \frac{\hat{\varepsilon}}{5} = 0.06. \end{aligned}$$

Therefore, according to Theorem 4.1, the drive system (1.1), and the response system (4.1) are finite-time synchronized with respect to  $\{\delta, \varepsilon, t_0, T\}$  (see Figures 1–5).

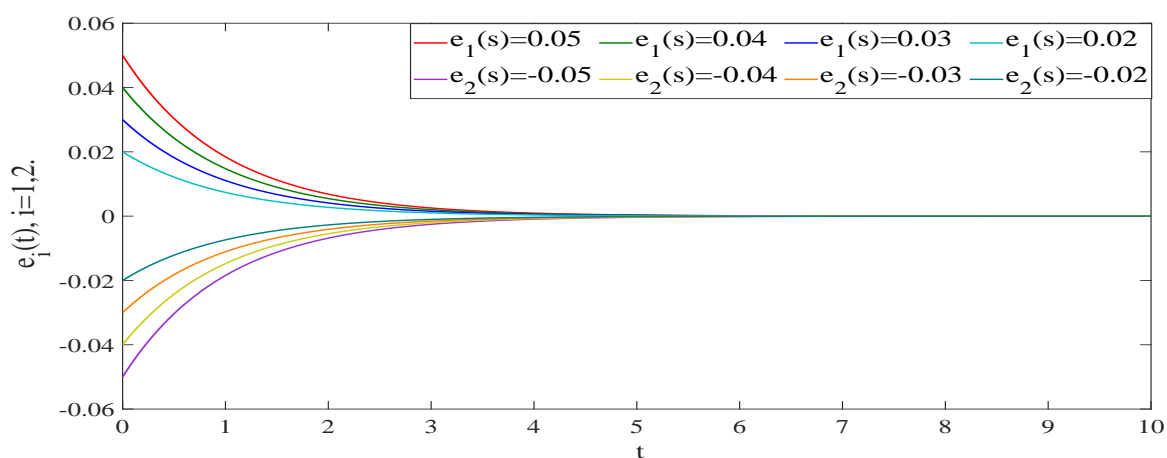
When the time delays and stochastic disturbances in System (1.1) disappear, the evolution of the system's states over time is shown in Figures 6–10. It can be observed from the figures that the changes in the system's states are not significant, which indicates, to some extent, that System (1.1) possesses good “immunity” to its non-ideal factors (time delays and noise). In other words, the system is “good” and easy to implement in modeling, as it is not highly demanding regarding parameter precision or environmental purity. However, this also implies that if one wishes to utilize time delays or stochasticity to induce new dynamic behaviors (such as coherence resonance or delay-induced oscillations), it is necessary to further increase the delay or noise intensity to surpass a certain critical threshold.



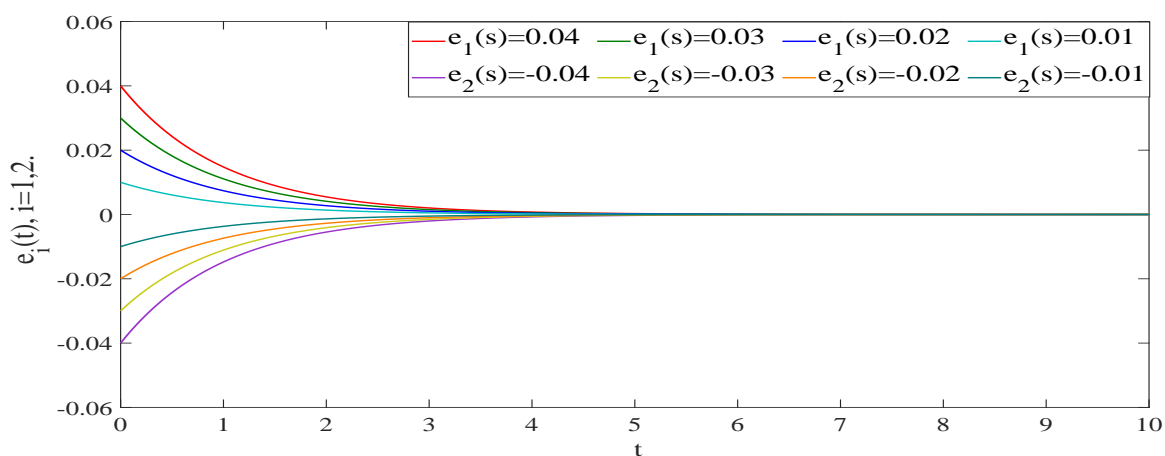
**Figure 1.** State  $x_1(t)$  of System (1.1) with different values of  $\alpha$ .



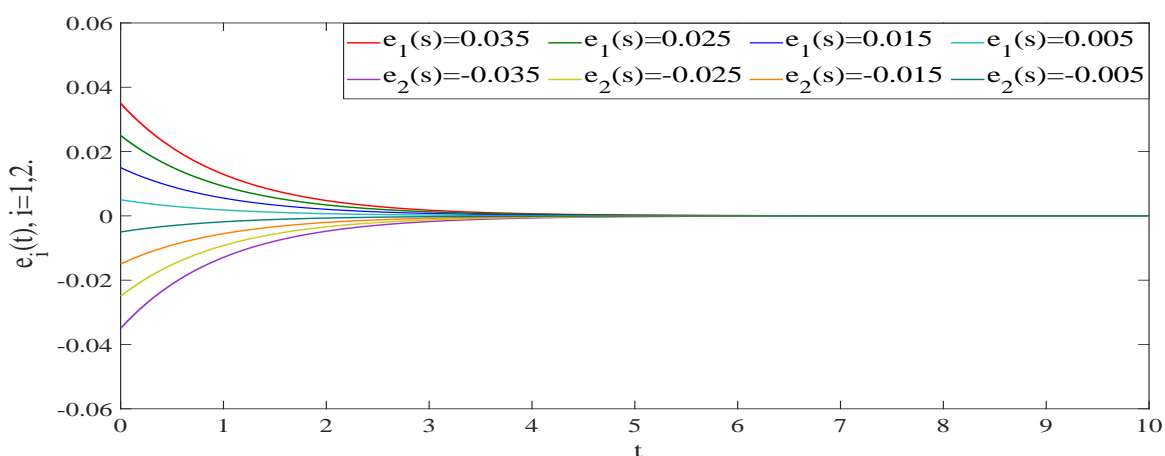
**Figure 2.** State  $x_2(t)$  of System (1.1) with different values of  $\alpha$ .



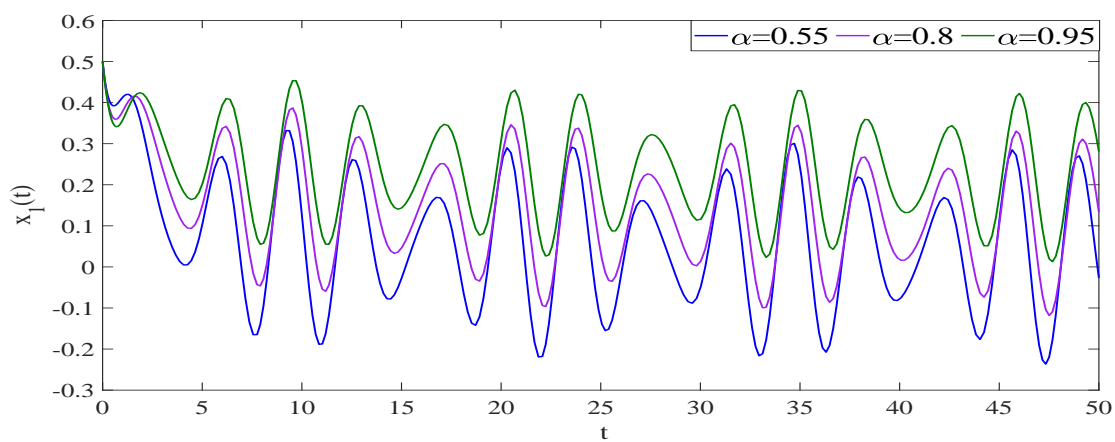
**Figure 3.** States  $e_1(t)$  and  $e_2(t)$  of System (4.3) with  $\alpha = 0.55$  and different initial values.



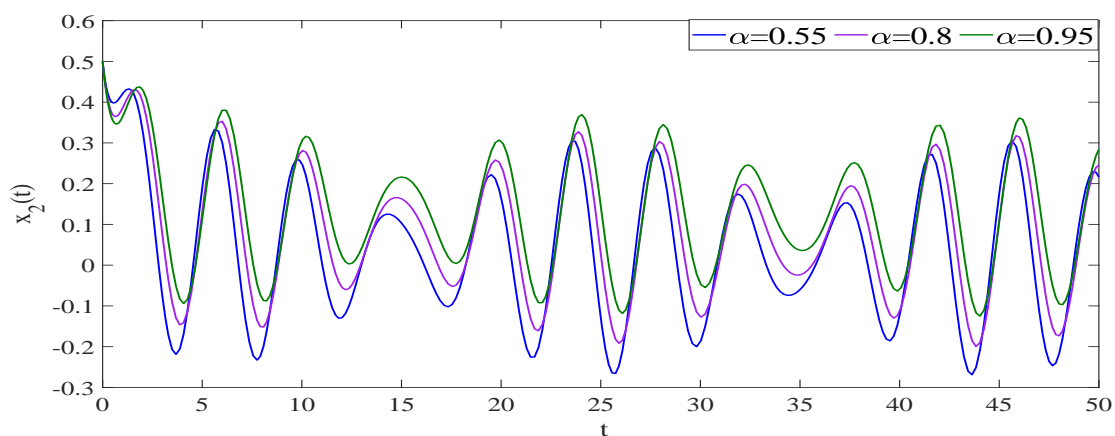
**Figure 4.** States  $e_1(t)$  and  $e_2(t)$  of System (4.3) with  $\alpha = 0.8$  and different initial values.



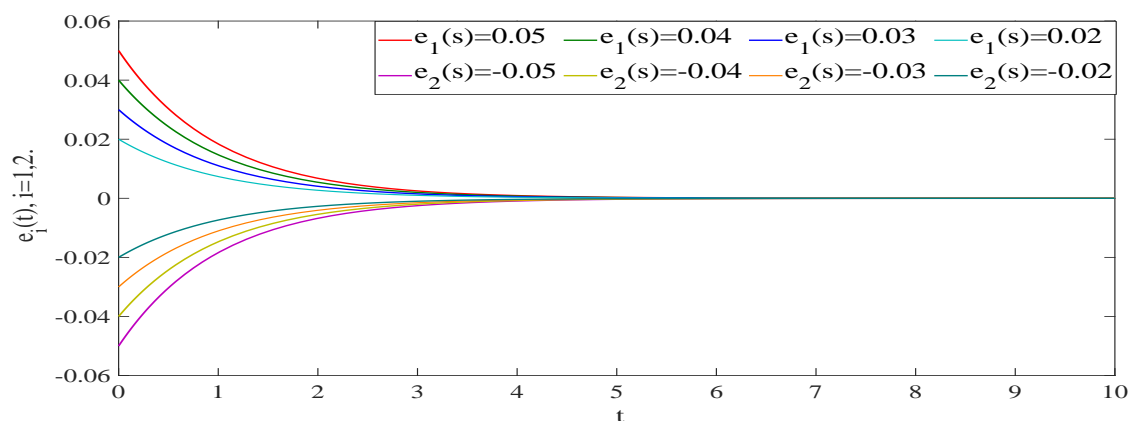
**Figure 5.** States  $e_1(t)$  and  $e_2(t)$  of System (4.3) with  $\alpha = 0.95$  and different initial values.



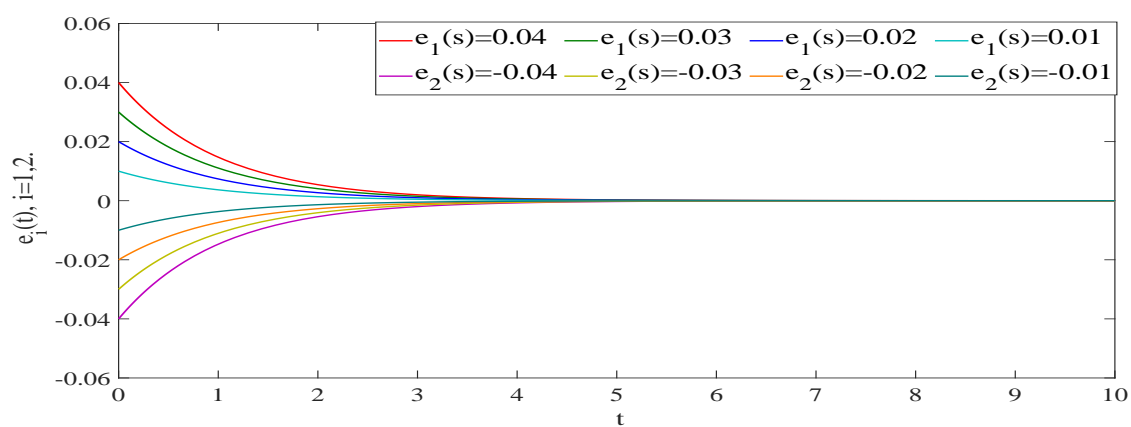
**Figure 6.** State  $x_1(t)$  of System (1.1) with different values of  $\alpha$ , and without delay and stochastic perturbation.



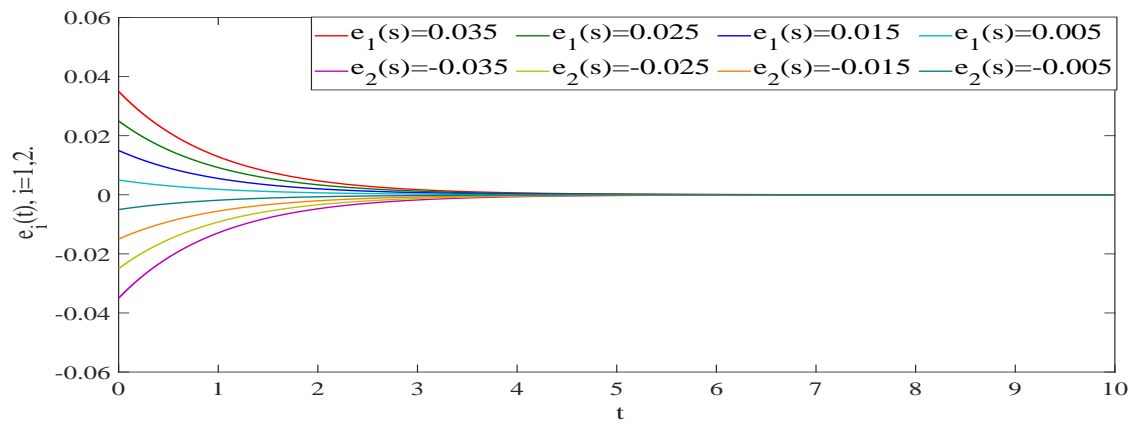
**Figure 7.** State  $x_2(t)$  of System (1.1) with different values of  $\alpha$ , and without delay and stochastic perturbation.



**Figure 8.** States  $e_1(t)$  and  $e_2(t)$  of System (4.3) with  $\alpha = 0.55$  and different initial values, and without delay and stochastic perturbation.



**Figure 9.** States  $e_1(t)$  and  $e_2(t)$  of System (4.3) with  $\alpha = 0.8$  and different initial values, and without delay and stochastic perturbation.



**Figure 10.** States  $e_1(t)$  and  $e_2(t)$  of System (4.3) with  $\alpha = 0.95$  and different initial values, and without delay and stochastic perturbation.

## 6. Conclusions

This paper has studied the dynamics of a class of high-order fractional-order Hopfield neural networks subjected to stochastic disturbances and time-varying delays. Addressing the critical yet under-explored gap in modeling complex, real-world neural systems, the research was conducted under the general framework of Stepanov-almost periodicity to handle discontinuous or irregular time-varying parameters. The main contributions are summarized as follows.

First, sufficient conditions were rigorously established to guarantee the existence and uniqueness of a Stepanov almost periodic solution in distribution for the proposed network model. By using Banach's fixed point theorem and appropriate inequality techniques, the analysis provided a solid theoretical foundation for the oscillatory behavior of such complex systems.

Second, on the basis of the almost periodic solution obtained, the finite-time synchronization problem between the drive system and a constructed response system was investigated. Effective control strategies were designed to achieve synchronization within a pre-specified finite time, which is crucial for applications with strict temporal constraints.

The theoretical results of this study not only advance the understanding of almost periodic dynamics in fractional-order stochastic delayed systems but also provide a potential theoretical framework for practical applications such as secure communication, neural information processing, and cooperative control of multi-agent systems, where robustness against delays, noise, and time-varying parameters is essential.

Future research may extend this work to other types of neural network architectures, explore different synchronization schemes (e.g., fixed-time or preassigned-time synchronization), or investigate the dynamics under more general types of stochastic processes and delay distributions.

## Author contributions

Yisen Zhou: Investigation, Writing-original draft; Yongkun Li: Conceptualization, Methodology, Investigation, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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