Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 10462–10477.
DOI: 10.3934/math.2026431
Received: 21 September 2025
Revised: 19 March 2026
Accepted: 30 March 2026
Published: 16 April 2026

Research article

On extreme lower approximation neighborhood systems: Application to networks for representing $X-$ray structures of some chemical compounds

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Abstract: This paper introduces a new class of topological structures on undirected simple graphs, called extreme graphical topological spaces, using the concepts of extreme systems and lower approximation neighborhood systems. We explore the conditions under which graphs produce either the indiscrete or discrete topology. Several fundamental properties related to both topology and graph theory are examined. Using this framework, along with the concept of extreme outer connectedness in geodesic paths, we define and study new types of connected graphs and discrete spaces, such as extreme maximally connected geodesic graphs and extreme maximally geodesic discrete spaces. Finally, we demonstrate applications of these structures by analyzing the connectedness and discrete characteristics of networks that represent X-ray structures of certain chemical compounds.

Keywords: extreme vertices; simple graph; connectedness; topology
Mathematics Subject Classification: 05C20, 05C99, 18F60

1. Introduction

In recent years, many researchers have explored various classes of topologies within graph theory. These studies primarily focus on defining topological structures over the vertex set of simple graphs, with many constructions depending on the types of neighborhood systems and the relations between vertices or edges. For instance, Nada et al. [1] introduced specific relations on graphs to define new topologies. Abdu and Kiliciman [2] in 2018 proposed a class of topologies derived from a sub-basis formed by the endpoints of all edges. Later, Kiliciman and Abdu [3] defined two topological classes on the edge set. Jafarian et al. [4] presented the graphical space of simple graphs $\omega = (\varphi(\omega), \xi(\omega))$, which is given by a pair $(\varphi(\omega), \tau_{\mathcal{A}_\varphi})$, where $\tau_{\mathcal{A}_\varphi}$ is a topology on $\varphi(\omega)$ generated by the set of open neighborhoods $N(x)$ of vertices in ω.

Canoy and Nianga [5, 6] investigated graph topologies through algebraic operations and hop neighborhoods.

Sari and Kopuzlu [7], based their work on that of Amiri et al. [4] to create new topologies on the vertex set, and they studied the continuity of maps. Zomam et al. [8] employed the graphical spaces in [4] to investigate the Alexandroff property by using a locally finitely property. Othman et al. [9] in 2022 introduced the concept of pathless fields s on the vertices set and presented some applications on the human heart. Othman et al. [10] used an open neighborhood C-sets system to introduce a class of topologies, called L_2-topological spaces, and explained the relations between some properties in graph theory and in general topology such as homeomorphisms, isomorphisms, and connectedness. Abu-Gdairi et al. [11] also explored topologies using neighborhood systems. In the context of rough set theory, Yao [12] defined lower and upper approximations, which were later adapted to graphs by Atik et al. [13]. Guler [14] employed ideal collections to compare approximation methods.

Ganesamoorthy and Jayanthi [15] introduced the concept of extreme outer connected geodesic graphs, which plays a central role in our study.

Timmanaikar et. al. [16], contributed numerical representations of graphs using topological indices.

Damag et al. [17, 18] constructed rough directed topological spaces using monophonic paths and applied these concepts to biological systems, including the circulatory and nervous systems.

Other notable contributions include Sergio and Anabel [19], who used the m-eccentric system to define graph topologies, and Gamorez et al. [20], who employed tensor products and edge coronas.

Blake et al. [21] studied topologies in Euler diagrams, and Atef et al. [22] investigated fuzzy topologies in fuzzy graphs. In mathematical chemistry, Ahmed et al. [23] and Deka et al. [24] explored topological properties in chemical graph structures. Damag et al. [25] examined the monophonic eccentric system's role in modeling the spread of diseases such as COVID-19.

In this paper, we propose a new class of topologies on undirected simple graphs by introducing the extreme lower approximation neighborhood system, which combines ideas from both rough approximation systems and extreme vertices. Using this system, we define a new structure called the extreme graphical topological space.

We analyze the discrete properties of several standard graphs, including the friendship graph F_n, generalized friendship graph GF_n, complete bipartite graph $\kappa_{n,m}$, complete graph κ_n, path P_n, and cycle graph C_n. In Section 3, we introduce and study extreme maximal connectedness geodesic graphs and extreme maximal geodesic discrete spaces, building upon the notion of extreme outer connectedness from [15]. Section 4 discusses applications of our framework by examining the connectedness and discrete properties of extreme graphical topological spaces in networks representing the X-ray crystallographic structures of certain chemical compounds.

A graph ω is a pair $(\varphi(\omega), \xi(\omega))$ of a vertex set $\varphi(\omega) \neq \emptyset$ and edge set $\xi(\omega) \neq \emptyset$. In this paper, we will assume all used graphs are undirected. If the vertices $x, y \in \varphi(\omega)$ join by an edge $\varsigma \in \xi(\omega)$, then we write $\varphi(\varsigma) = \{x, y\}$. The edge ς with $\varphi(\varsigma) = \{x\}$ is called a loop. If $\varphi(\varsigma) = \varphi(\varsigma')$, then ς and ς' are called multiple edges. If a graph ω is without loops and multiple edges, then it is called a simple graph. ω_{xy} denotes the edge joining x and y. $\mathbb{D}_x$ denotes the degree of x that is defined as the number of points that join x. C_n denotes a cycle graph. A friendship graph F_n with $n > 1$ is a simple graph which can be constructed by joining n copies of the cycle graphs C_3 with a common vertex. $\kappa_{n,m}$ denotes a complete bipartite graph, and κ_n denotes a complete graph. The open neighbourhood $\mathbb{N}(x)$ of a point x is the set of vertices that are adjacent to x. A path is a sequence of vertices connected by edges, with each vertex adjacent to the next in the sequence. A connected graph is a graph where there is a path between every pair of vertices. For any two vertices $x, y \in \varphi(\omega)$, $d(x, y)$ denotes the distance between points x and y, which is the length of a shortest path between x and y in ω. Any path of length $d(x, y)$ is called xy-geodesic. By $C[x, y]$, we mean the closed interval which consists of all vertices lying on some xy-geodesic of ω. A subset $A \subseteq \varphi(\omega)$ is called a geodetic in ω if $C[A] = \varphi(\omega)$, where $C[A] = \cup_{x,y \in A} C[x, y]$. A subset $A \subseteq \varphi(\omega)$ is called an outer connected geodetic set if A is a geodetic set in ω, and either $A = \varphi(\omega)$ or the subgraph induced by $\varphi(\omega) - A$ is connected graph. The outer connected geodetic number of ω is defined by the minimum cardinality of an outer connected geodetic set of ω and denoted by ω_{goc}. A vertex $x \in \varphi(\omega)$ is called an extreme vertex of a graph ω if the subgraph induced by $\mathbb{N}(x)$ is complete. For any simple graph ω, the set of all extreme vertices is denoted by $\varphi(\omega)_{ex}$, and the number of extreme vertices in ω is called an extreme order of ω and is denoted by $|\omega|_{ex}$. If $|\omega|_{ex} = \omega_{goc}$, then the graph ω is called an extreme outer connected geodesic (EOCG) graph . For any subset M of a nonempty set D and for any binary relation η on a nonempty set D, the lower approximation $\underline{\eta}(M)$ and upper approximation $\overline{\eta}(M)$ of a set M are defined by $\underline{\eta}(M) = \{x \in D : \eta_x \subseteq M\}$ and $\overline{\eta}(M) = \{x \in D : \eta_x \cap M \neq \emptyset\}$, respectively, where $\eta_x = \{y \in D : x\eta y\}$. For any subgraph Q of ω, Atik et al., [13] defined the notions of lower $\underline{N_k}(\varphi(Q))$ and upper $\overline{N_k}(\varphi(Q))$ rough approximation k-systems given by $\underline{N_k}(\varphi(Q)) = \{x \in \varphi(\omega) : N_k(x) \subseteq \varphi(Q)\}$ and $\overline{N_k}(\varphi(Q)) = \varphi(Q) \cup \{x \in \varphi(\omega) : N_k(x) \cap \varphi(Q) \neq \emptyset\}$.

2. The extreme graphical topology

In the class of simple graphs $\omega = (\varphi(\omega), \xi(\omega))$, we use the extreme system and the rough approximation j-neighborhood system to define a new neighborhood system for the elements of $\varphi(\omega)$, called an extreme lower approximated neighbourhood system. To motivate our construction, we aim to combine two important ideas: (i) extreme vertices in graphs, which capture local structural rigidity, and (ii) lower approximation operators from rough set theory, which describe certainty regions. Our goal is to define a neighborhood system that reflects both the extremal structure of vertices and approximation-based inclusion.

Let $\omega = (\phi(\omega), \xi(\omega))$ be a simple graph. We define a relation η on $\phi(\omega)$ as follows: for $x, y \in \phi(\omega)$ $x \,\eta\, y$ if and only if $x \in N(y)$, and x is an extreme vertex. The extreme neighborhood of a vertex x is defined by

$$\eta_\omega(x) = \{x\} \cup \{y \in \phi(\omega) : y \,\eta\, x\}.$$

Next, define

$$\theta_\omega(x) = \{y \in \phi(\omega) : \eta_\omega(x) \subseteq \eta_\omega(y)\}.$$

Then, the extreme lower approximation neighborhood system is given by

$$\pi_\omega(\varphi(\omega)) = \{\pi_\omega(x) : x \in \varphi(\omega)\},$$

where

$$\pi_\omega(x) = \eta_\omega(x) \cup \theta_\omega(x).$$

Intuitively, $\pi_\omega(x)$ consists of: (i) vertices directly related to x through extreme adjacency, and (ii) vertices whose extreme neighborhoods contain that of x. Thus, $\pi_\omega(x)$ captures both local extremal structure and hierarchical inclusion among neighborhoods. For example, consider a simple path P_3 with vertices $\{1, 2, 3\}$. Then, $\eta_\omega(2) = \{1, 2, 3\}$, and $\pi_\omega(2) = \{1, 2, 3\}$. This illustrates how $\pi_\omega(x)$ extends beyond immediate adjacency.

Theorem 2.1. Let ω be a simple graph. Then $\mathbb{B}_\omega = \{\pi^x_\omega : x \in \varphi(\omega)\}$ forms a basis for a topology on $\varphi(\omega)$, where $\pi^x_\omega = \bigcap\{\pi_\omega(y) : x \in \pi_\omega(y)\}$.

Proof. We verify the two basis axioms. (1) Covering condition: We show that $\bigcup_{x\in\varphi(\omega)} \pi^x_\omega = \varphi(\omega)$. Let $x \in \varphi(\omega)$. Because $x \in \eta^{ex}_\omega(x)$, it follows that $x \in \pi_\omega(x)$. Hence, $x \in \pi^x_\omega$, and so $\varphi(\omega) \subseteq \bigcup_{x\in\varphi(\omega)} \pi^x_\omega$. The reverse inclusion is obvious ; thus, the equality holds.
(2) Intersection condition: Let $\pi^x_\omega, \pi^y_\omega \in \mathbb{B}_\omega$, and let $z \in \pi^x_\omega \cap \pi^y_\omega$. Because $z \in \pi^x_\omega$, we have $z \in \pi_\omega(u)$ for all u such that $x \in \pi_\omega(u)$. Similarly, because $z \in \pi^y_\omega$, we have $z \in \pi_\omega(v)$ for all v such that $y \in \pi_\omega(v)$. Let $w \in \varphi(\omega)$ such that $z \in \pi_\omega(w)$. Then, by definition, $\pi^z_\omega \subseteq \pi_\omega(w)$, and it follows that $\pi^z_\omega \subseteq \pi^x_\omega \cap \pi^y_\omega$. Hence, for every $z \in \pi^x_\omega \cap \pi^y_\omega$, there exists $\pi^z_\omega \in \mathbb{B}_\omega$ such that $z \in \pi^z_\omega \subseteq \pi^x_\omega \cap \pi^y_\omega$. Therefore, $\pi^x_\omega \cap \pi^y_\omega = \bigcup_{z\in\pi^x_\omega\cap\pi^y_\omega} \pi^z_\omega$. Hence, $\mathbb{B}_\omega$ is a basis for a topology on $\varphi(\omega)$. □

In theorem above, the topology which is induce by $\mathbb{B}_\omega$ is denoted by $\tau_{\mathbb{B}_\omega}$ and called an extreme graphical topology of ω.

Corollary 2.2. The collection $\pi_\omega(\varphi(\omega))$ is a sub-basis for the space $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$.

Proof. It is clear that from the definition of an extreme neighborhood system $\pi_\omega(\varphi(\omega))$ and by Theorem 2.1 with a basis $\mathbb{B}_\omega$. □

Example 2.3. In Figure 1(a), $\omega = (\varphi(\omega), \xi(\omega))$ has an extreme lower approximation neighborhood system $\pi_\omega(\varphi(\omega))$ which is given by

$$\pi_\omega(1') = \pi_\omega(2') = \pi_\omega(3') = \pi_\omega(4') = \{1', 2', 3', 4'\}$$

and

$$\pi_\omega(5') = \pi_\omega(6') = \pi_\omega(7') = \pi_\omega(8') = \{5', 6', 7', 8'\}.$$

The basis $\mathbb{B}_\omega = \{\{1', 2', 3', 4'\}, \{5', 6', 7', 8'\}\}$ induces the quasi-discrete topology

$$\tau_{\mathbb{B}_\omega} = \Big\{\emptyset, \varphi(\omega), \{1', 2', 3', 4'\}, \{5', 6', 7', 8'\}\Big\}.$$

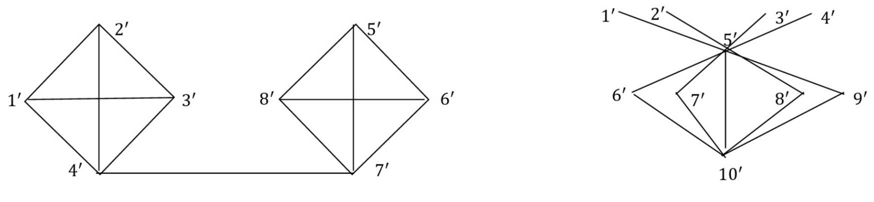
In Figure 1(b), the graph $\omega = (\varphi(\omega), \xi(\omega))$ has an extreme lower approximation neighborhood system $\pi_\omega(\varphi(\omega))$ which is given by

$$\pi_\omega(k') = \{k', 5'\}, \ \pi_\omega(j') = \{j', 5', 10'\},$$

$\pi_\omega(5') = \varphi(\omega) \setminus \{10'\}$, and $\pi_\omega(10') = \{6', 7', 8', 9', 10'\}$ for all $k = 1, 2, 3, 4$ and $j = 6, 7, 8, 9$. The basis is

$$\mathbb{B}_\omega = \Big\{\{5'\}, \{5', 10'\}, \{1', 5'\}, \{2', 5'\}, \{3', 5'\}, \{4', 5'\}, \{5', 6', 10''\}, \{5', 7', 10''\},$$

$$\{5', 8', 10''\}, \{5', 9', 10''\}\Big\}.$$



(a) Extreme I

(b) Extreme II

Figure 1. Representation of an extreme graphical topology.

Theorem 2.4. In the simple graph $\omega = (\varphi(\omega), \xi(\omega))$, if $\mathcal{F}_1, \mathcal{F}_2 \subseteq \varphi(\omega)$, and $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then $\pi_\omega(\mathcal{F}_2) \subseteq \pi_\omega(\mathcal{F}_1)$.

Proof. Let $\rho_1 \in \pi_\omega(\mathcal{F}_2)$. Then, $\rho_1 \in \eta_\omega(\mathcal{F}_2)$, or $\rho_1 \in \theta_\omega(\mathcal{F}_2)$. If $\rho_1 \in \eta_\omega(\mathcal{F}_2)$, then $\rho_1 \in \eta_\omega(\rho_2)$ for some $\rho_2 \in \mathcal{F}_2$. Because $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then $\rho_2 \in \mathcal{F}_1$, and hence, $\rho_1 \in \eta_\omega(\rho_2) \subseteq \pi_\omega(\mathcal{F}_1)$. If $\rho_1 \in \theta_\omega(\mathcal{F}_2)$, then $\eta_\omega(\rho_2) \subset \eta_\omega(\rho_1)$ for some $\rho_2 \in \mathcal{F}_2$. Because $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then $\rho_2 \in \mathcal{F}_1$, and hence, $\rho_1 \in \theta_\omega(\mathcal{F}_1) \subseteq \pi_\omega(\mathcal{F}_1)$. Hence, $\pi_\omega(\mathcal{F}_2) \subseteq \pi_\omega(\mathcal{F}_1)$. □

Theorem 2.5. If $\mathcal{F}_1, \mathcal{F}_2 \subseteq \varphi(\omega)$, then $\pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2) = \pi_\omega(\mathcal{F}_1) \cup \pi_\omega(\mathcal{F}_2)$.

Proof. Because $\mathcal{F}_1, G \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$, then from Theorem 2.4, $\pi_\omega(\mathcal{F}_1) \subseteq \pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$, and $\pi_\omega(\mathcal{F}_2) \subseteq \pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$, that is, $\pi_\omega(\mathcal{F}_1) \cup \pi_\omega(\mathcal{F}_2) \subseteq \pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$. For the other side, let $x \in \pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$. Then, $x \in \eta_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$, or $x \in \theta_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$. If $x \in \eta_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$, then $x \in \eta_\omega(y)$ for some $y \in \mathcal{F}_1 \cup \mathcal{F}_2$. If $y \in \mathcal{F}_1$, then $x \in \eta_\omega(y) \subseteq \pi_\omega(\mathcal{F}_1)$, and similar, if $y \in \mathcal{F}_2$, then $x \in \eta_\omega(y) \subseteq \pi_\omega(\mathcal{F}_2)$. If $x \in \theta_\omega(\mathcal{F}_1 \cup \mathcal{F}_2)$, then $\eta_\omega(y) \subset \eta_\omega(x)$ for some $y \in \mathcal{F}_1 \cup \mathcal{F}_2$. If $y \in \mathcal{F}_1$, then $x \in \theta_\omega(y) \subseteq \pi_\omega(\mathcal{F}_1)$, and if $y \in \mathcal{F}_2$, then $x \in \theta_\omega(y) \subseteq \pi_\omega(\mathcal{F}_2)$. Therefore, $\pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2) \subseteq \pi_\omega(\mathcal{F}_1) \cup \pi_\omega(\mathcal{F}_2)$, that is, $\pi_\omega(\mathcal{F}_1 \cup \mathcal{F}_2) = \pi_\omega(\mathcal{F}_1) \cup \pi_\omega(\mathcal{F}_2)$. □

For the intersection, we obtain that $\pi_\omega(\mathcal{F}_1 \cap \mathcal{F}_2) \subseteq \pi_\omega(\mathcal{F}_1) \cap \pi_\omega(\mathcal{F}_2)$, but it is not needed to be $\pi_\omega(\mathcal{F}_1 \cap \mathcal{F}_2) = \pi_\omega(\mathcal{F}_1) \cap \pi_\omega(\mathcal{F}_2)$; see Figure 1(b). If we take $\mathcal{F}_1 = \{5'\}$ and $\mathcal{F}_2 = \{10'\}$, then $\pi_\omega(\mathcal{F}_1 \cap \mathcal{F}_2) = \emptyset \neq \{6', 7', 8', 9'\} = [\varphi(\omega) \setminus \{10'\}] \cap \{6', 7', 8', 9', 10'\} = \pi_\omega(\mathcal{F}_1) \cap \pi_\omega(\mathcal{F}_2)$. Note that $\pi_\omega(x) = \{x\}$ for any isolated point x. Also, $\pi_\omega(x) = \pi_\omega(y) = \{x, y\}$ for an isolated edge ω_{xy}. A path $P_3 : 1' \to 2' \to 3'$ has an extreme lower approximation neighborhood system $\pi_\omega(P_3)$, which is given by $\pi_{P_3}(1') = \{1', 2'\}$, $\pi_{P_3}(2') = \{1', 2', 3'\}$ and $\pi_{P_3}(3') = \{2', 3'\}$. Similar, if we have a path $P_4 : 1' \to 2' \to 3' \to 4'$, then the space $(\varphi(P_4), \tau_{\mathbb{B}_{P_4}})$ is a quasi-discrete given by $\tau_{\mathbb{B}_{P_4}} = \{\emptyset, \varphi(P_4), \{1', 2'\}, \{3', 4'\}\}$. For the path P_n, where $n \geq 5$, we have the following theorem.

Theorem 2.6. Consider $P_r : 1' \to 2' \to \cdots \to k'$, where $k > 4$. Then,

$$\mathbb{B}_{P_r} = \Big\{\{1', 2'\}, \{(k-1)', r'\}, \{m'\} : 3 \leq m \leq k-2\Big\}.$$

Proof. Note that we have two extreme vertices which are $1'$ and k', that is, $\varphi(P_r)_{ex} = \{1', k'\}$. Then,

$$\eta_{P_r}(m') = \{m'\}, \ \eta_{P_r}(2') = \{1', 2'\}, \eta_{P_r}((k-1)') = \{(k-1)', k'\}$$

for all $m = 1, 3, 4, \cdots, k-2, k$. So, we get that $\theta_{P_r}(1') = \{2'\}$, $\theta_{P_r}(k') = \{(k-1)'\}$ and $\theta_{P_r}(m') = \emptyset$ for all $m = 2, 3, \cdots, k-1$. Hence,

$$\pi_{P_r}(1') = \pi_{P_r}(2') = \{1', 2'\}, \ \pi_{P_r}((k-1)') = \pi_{P_r}(k') = \{(k-1)', k'\},$$

and $\pi_{P_r}(m') = \{m'\}$ for all $m = 3, 4, \cdots, k-2$. From Theorem 2.1, the basis $\mathbb{B}_{P_r}$ is given by $\mathbb{B}_{P_r} = \{\{1', 2'\}, \{(k-1)', k'\}, \{m'\} : 3 \leq m \leq k-2\}$. □

If $n = m = 1$, then it is clear that the space $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is an indiscrete.

Theorem 2.7. Let $m > 1$ and $\varphi(\kappa_{1,m}) = \varphi_1 \cup \varphi_m$, where $\varphi_1 = \{0'\}$, and $\varphi_m = \{1', 2', \cdots, m'\}$. Then, the basis of $(\varphi(\kappa_{1,m}), \tau_{\mathbb{B}_{\kappa_{1,m}}})$ is given by

$$\mathbb{B}_{\kappa_{1,m}} = \Big\{\{0'\}, \{0', j'\} : j = 1, 2, \cdots, m\Big\}.$$

Proof. Note that $\varphi(\kappa_{1,m})_{ex} = \varphi_m$. So, we have $\eta_{\kappa_{1,m}}(0') = \varphi(\kappa_{1,m})$ and $\eta_{\kappa_{1,m}}(j') = \{j'\}$ for all $j = 1, 2, 3, \cdots, m$. Then, $\theta_{\kappa_{1,m}}(0') = \emptyset$ and $\theta_{\kappa_{1,m}}(j') = \{0'\}$ for all $j = 1, 2, 3, \cdots, m$. Hence, $\pi_{\kappa_{1,m}}(0') = \varphi(\kappa_{1,m})$ and $\pi_{\kappa_{1,m}}(j') = \{0', j'\}$ for all $j = 1, 2, 3, \cdots, m$. From Theorem 2.1, the basis $\mathbb{B}_{\kappa_{1,m}}$ is given by $\mathbb{B}_{\kappa_{1,m}} = \{\{0'\}, \{0', j'\} : j = 1, 2, \cdots, m\}$. □

Theorem 2.8. The space $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is discrete for every $n, m > 1$.

Proof. Consider $\varphi(\kappa_{n,m}) = \varphi_n \cup \varphi_m$, and $\rho \in \varphi(\kappa_{n,m})$. Because $\varphi_n \cap \varphi_m = \emptyset$, then $\rho \in \varphi_n$, or $\rho \in \varphi_m$. Let $\rho \in \varphi_n$. Then, $N(\rho) = \varphi_m$. Because the subgraph of ω which induced by φ_m is not complete, then $\eta_{\kappa_{n,m}}(\rho) = \{\rho\}$, and so $\theta_{\kappa_{n,m}}(\rho) = \emptyset$. Hence, $\pi_{\kappa_{n,m}}(\rho) = \{\rho\}$. Similar, if $\rho \in \varphi_m$, we have $\pi_{\kappa_{n,m}}(\rho) = \{\rho\}$. Hence, from Theorem 2.1, the basis $\mathbb{B}_{\kappa_{n,m}}$ is given by $\mathbb{B}_{\kappa_{n,m}} = \{\{\rho\} : \rho \in \varphi(\kappa_{n,m})\}$. Hence, $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ has a discrete property for any $n, m > 1$. □

Theorem 2.9. The space $(\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ has an indiscrete property for every $n > 0$.

Proof. It is clear that for all $x \in \varphi(\kappa_n)$, $\eta_{\kappa_n}(x) = \varphi(\kappa_n)$, and so we get that $\theta_{\kappa_n}(x) = \emptyset$. Hence, the extreme neighborhood system is given by

$$\pi_{\kappa_n}(x) = \eta_{\kappa_n}(x) \cup \theta_{\kappa_n}(x) = \varphi(\kappa_n)$$

for all $x \in \varphi(\kappa_n)$. Hence, $(\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ is an indiscrete space. □

In the cycle graph $C_3 : 1' \to 2' \to 3' \to 1'$, we have $\varphi(C_3)_{ex} = \{1', 2', 3'\}$. So, the space $(\varphi(C_3), \tau_{\mathbb{B}_{C_3}})$ is indiscrete.

Theorem 2.10. The space $(\varphi(C_n), \tau_{\mathbb{B}_{C_n}})$ is discrete for all $n > 3$.

Proof. Note that for all $n > 3$, $\varphi(C_n)_{ex} = \emptyset$. So, $\eta_{C_n}(m') = \{m'\}$ for all $m = 1, 2, 3, \cdots, n$. Hence, $\theta_{C_n}(m') = \emptyset$ for all $m = 1, 2, 3, \cdots, n$, and hence, $\pi_{C_n}(m') = \{m'\}$ for all $m = 1, 2, 3, \cdots, n$. From Theorem 2.1, the space $(\varphi(C_n), \tau_{\mathbb{B}_{C_n}})$ is discrete for all $n > 3$. □

Theorem 2.11. For the friendship graph F_n with $n > 1$ in Figure 2, the basis of the space $(\varphi(F_n), \tau_{\mathbb{B}_{F_n}})$ is given by

$$\mathbb{B}_{F_n} = \Big\{\{0\}, \{0, m', m''\} : 1 \leq m \leq n\Big\}.$$

Figure 2. Friendship graph F_n.

Proof. It is clear to see that for all $n > 1$, $\varphi(F_n)_{ex} = \varphi(F_n) \setminus \{0\}$. So, $\eta_{F_n}(0) = \varphi(F_n)$ and $\eta_{F_n}(m') = \eta_{F_n}(m'') = \{m', m''\}$ for all $m = 1, 2, 3, \cdots, n$. Hence, $\theta_{F_n}(0) = \emptyset$ and $\theta_{F_n}(m') = \theta_{F_n}(m'') = \{0\}$ for all $m = 1, 2, 3, \cdots, n$. Then, $\pi_{F_n}(0) = \varphi(F_n)$ and $\pi_{F_n}(m') = \pi_{F_n}(m'') = \{0, m', m''\}$ for all $m = 1, 2, 3, \cdots, n$. So, by Theorem 2.1, the basis of the space $(\varphi(F_n), \tau_{\mathbb{B}_{F_n}})$ is given by $\mathbb{B}_{F_n} = \{\{0\}, \{0, m', m''\} : 1 \leq m \leq n\}$. □

By the generalized friendship graph GF_n with $n > 3$, we mean a simple graph which can be constructed by attaching n copies of the cycle graphs C_3 with a cycle graph C_n as shown in Figure 3.

Theorem 2.12. The basis of $(\varphi(GF_n), \tau_{\mathbb{B}_{GF_n}})$ in Figure 3 is given by

$$\mathbb{B}_{F_n} = \Big\{\{m, m', m''\} : 1 \leq m \leq n\Big\}.$$

Proof. From Figure 3, GF_n has the extreme set $\varphi(GF_n)_{ex} = \varphi(GF_n) \setminus \{1, 2, \cdots, n\}$. Hence, $\eta_{GF_n}(m) = \{m, m', m''\}$, $\eta_{GF_n}(m') = \eta_{GF_n}(m'') = \{m', m''\}$, $\theta_{GF_n}(m) = \emptyset$, $\theta_{GF_n}(m') = \theta_{GF_n}(m'') = \{m\}$, and $\pi_{GF_n}(m) = \pi_{GF_n}(m') = \pi_{GF_n}(m'') = \{m, m', m''\}$ for all $m = 1, 2, 3, \cdots, n$. So, from Theorem 2.1, the basis of $(\varphi(GF_n), \tau_{\mathbb{B}_{GF_n}})$ is given by $\mathbb{B}_{F_n} = \{\{m, m', m''\} : 1 \leq m \leq n\}$. □

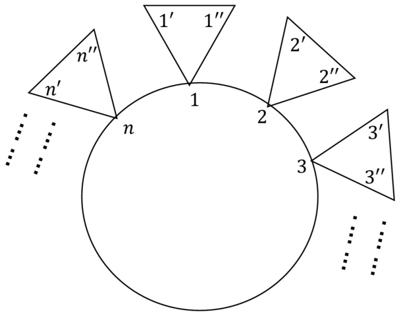


Figure 3. Generalized friendship graph GF_n.

3. On the maximality of extreme vertices

A map $\lambda : (W_1, \tau_1) \to (W_2, \tau_2)$ of a topological space (W_1, τ_1) into a topological space (W_2, τ_2) is called continuous if $\lambda^{-1}(B)$ is an open set in (W_1, τ_1) for every open set B in (W_2, τ_2). A map $\lambda : (W_1, \tau_1) \to (W_2, \tau_2)$ is called open if $\lambda(B)$ is an open set in W_2 for any open $B \subseteq W_1$. Recalling [26], a map $\lambda : (W_1, \tau_1) \to (W_2, \tau_2)$ is a homeomorphism if it is continuous, bijective, and open. By Theorem 2.8, because $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is a discrete space for any $n, m > 1$, then any map $\lambda : (\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}}) \to (\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is continuous. Also, by Theorem 2.10, because $(\varphi(C_n), \tau_{\mathbb{B}_{C_n}})$ has a discrete property for all $n \geq 4$, then any map $\lambda : (\varphi(C_n), \tau_{\mathbb{B}_{C_n}}) \to (\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is continuous. From Theorem 2.9, because $(\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ has an indiscrete property for all $n > 0$, then any map $\lambda : (\varphi(\omega), \tau_{\mathbb{B}_\omega}) \to (\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ is a continuous map.

Theorem 3.1. *Let* $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$ *and* $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ *be two simple graphs. A map* $\Theta : (\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}}) \to (\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ *is open if and only if* $\pi_{\omega_2}^{\Theta(\alpha)} \subseteq \Theta[\pi_{\omega_1}^{\alpha}]$ *for all* $\alpha \in \varphi(\omega_1)$.

Proof. Assume that Θ is an open map, and let $\alpha \in \varphi(\omega_1)$. Because $\pi_{\omega_1}^{\alpha}$ is an open set containing α, it follows that $\Theta[\pi_{\omega_1}^{\alpha}]$ is an open set in $(\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ containing $\Theta(\alpha)$. By Theorem 2.1, $\pi_{\omega_2}^{\Theta(\alpha)}$ is the smallest open set containing $\Theta(\alpha)$; hence, $\pi_{\omega_2}^{\Theta(\alpha)} \subseteq \Theta[\pi_{\omega_1}^{\alpha}]$.

Conversely, suppose that $\pi^{\Theta(\alpha)}_{\omega_2} \subseteq \Theta[\pi^{\alpha}_{\omega_1}]$ for all $\alpha \in \varphi(\omega_1)$. Let B be an open set in $(\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}})$, and let $\beta \in \Theta(B)$. Then, there exists $\alpha \in B$ such that $\Theta(\alpha) = \beta$. Because B is open, there exists $\pi^{\alpha}_{\omega_1} \in \mathbb{B}_{\omega_1}$ such that $\alpha \in \pi^{\alpha}_{\omega_1} \subseteq B$. Applying Θ, we obtain $\Theta[\pi^{\alpha}_{\omega_1}] \subseteq \Theta(B)$. By the hypothesis, $\pi^{\Theta(\alpha)}_{\omega_2} \subseteq \Theta[\pi^{\alpha}_{\omega_1}]$, hence $\pi^{\beta}_{\omega_2} \subseteq \Theta(B)$. Since $\pi^{\beta}_{\omega_2}$ is an open set containing β, and β was arbitrary, it follows that $\Theta(B)$ is open. Therefore, Θ is an open map. □

Theorem 3.2. Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$ and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be two simple graphs. A map $\Theta : (\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}}) \to (\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ is continuous if and only if $\Theta[\pi^{\alpha}_{\omega_1}] \subseteq \pi^{\Theta(\alpha)}_{\omega_2}$ for all $\alpha \in \varphi(\omega_1)$.

Proof. Assume that Θ is continuous, and let $\alpha \in \varphi(\omega_1)$. Because $\pi^{\Theta(\alpha)}_{\omega_2}$ is an open set containing $\Theta(\alpha)$, it follows that $\Theta^{-1}[\pi^{\Theta(\alpha)}_{\omega_2}]$ is an open set in $(\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}})$ containing α. By Theorem 2.1, $\pi^{\alpha}_{\omega_1}$ is the smallest open set containing α; hence, $\pi^{\alpha}_{\omega_1} \subseteq \Theta^{-1}[\pi^{\Theta(\alpha)}_{\omega_2}]$, which implies $\Theta[\pi^{\alpha}_{\omega_1}] \subseteq \pi^{\Theta(\alpha)}_{\omega_2}$.

Conversely, suppose that $\Theta[\pi^{\alpha}_{\omega_1}] \subseteq \pi^{\Theta(\alpha)}_{\omega_2}$ for all $\alpha \in \varphi(\omega_1)$. Let B be an open set in $(\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$, and let $\alpha \in \Theta^{-1}(B)$. Then, $\Theta(\alpha) \in B$. Because B is open, there exists $\pi^{\Theta(\alpha)}_{\omega_2} \in \mathbb{B}_{\omega_2}$ such that $\Theta(\alpha) \in \pi^{\Theta(\alpha)}_{\omega_2} \subseteq B$. By the hypothesis $\Theta[\pi^{\alpha}_{\omega_1}] \subseteq \pi^{\Theta(\alpha)}_{\omega_2} \subseteq B$, we get $\pi^{\alpha}_{\omega_1} \subseteq \Theta^{-1}(B)$. Because $\pi^{\alpha}_{\omega_1}$ is open, and $\alpha \in \pi^{\alpha}_{\omega_1} \subseteq \Theta^{-1}(B)$, it follows that $\Theta^{-1}(B)$ is open. Therefore, Θ is continuous. □

Corollary 3.3. Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$ and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be two simple graphs. A bijective map $\Theta : (\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}}) \to (\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ is a homeomorphism if and only if $\Theta[\pi^{\alpha}_{\omega_1}] = \pi^{\Theta(\alpha)}_{\omega_2}$ for all $\alpha \in \varphi(\omega_1)$.

Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$ and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be two simple graphs. The two graphs ω_1 and ω_2 are said to be isomorphic, $\omega_1 \cong \omega_2$, if there is a bijective map $\Theta : \varphi(\omega_1) \to \varphi(\omega_2)$ such that $(\omega_1)_{\alpha\beta} \in \xi(\omega_1)$ if and only if $(\omega_2)_{\Theta(\alpha)\Theta(\beta)} \in \xi(\omega_2)$ for all $\alpha, \beta \in \varphi(\omega_1)$.

Theorem 3.4. Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$, and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be two simple graphs. Then, $(\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}})$ and $(\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ are homeomorphic if ω_1 and ω_2 are isomorphic graphs.

Proof. Because ω_1 and ω_2 are isomorphic, there is a bijective map $\Theta : \varphi(\omega_1) \to \varphi(\omega_2)$ such that $(\omega_1)_{\alpha\beta} \in \xi(\omega_1)$ if and only if $(\omega_2)_{\Theta(\alpha)\Theta(\beta)} \in \xi(\omega_2)$ for all $\alpha, \beta \in \varphi(\omega_1)$. By Corollary 3.3, it is enough to prove that $\Theta[\pi^{\varsigma}_{\omega_1}] = \pi^{\Theta(\varsigma)}_{\omega_2}$ for all $\varsigma \in \varphi(\omega_1)$. Let $\varsigma \in \varphi(\omega_1)$ be any point, and $\varsigma' \in \Theta[\pi^{\varsigma}_{\omega_1}]$. Because Θ is an injective there is one point $\zeta \in \pi^{\varsigma}_{\omega_1}$ only such that $\varsigma' = \Theta(\zeta)$. Hence, $\zeta \in \pi^{\alpha}_{\omega_1}$ for all $\varsigma \in \pi^{\alpha}_{\omega_1}$, that is, $\zeta \in \eta_{\omega_1}(\alpha)$ or $\eta_{\omega_1}(\alpha) \subset \eta_{\omega_1}(\zeta)$ for all $\varsigma \in \pi^{\alpha}_{\omega_1}$. By an isomorphic of ω_1 and ω_2, we get that $\varsigma' = \Theta(\zeta) \in \eta_{\omega_2}(\Theta(\alpha))$ or $\eta_{\omega_2}(\Theta(\alpha)) \subset \eta_{\omega_2}(\Theta(\zeta))$ for all $\Theta(\varsigma) \in \pi^{\Theta(\alpha)}_{\omega_2}$. That is, $\varsigma' \in \pi^{\Theta(\varsigma)}_{\omega_2}$, and hence, $\Theta[\pi^{\varsigma}_{\omega_1}] \subseteq \pi^{\Theta(\varsigma)}_{\omega_2}$. For the other side, let $\varsigma' \in \pi^{\Theta(\varsigma)}_{\omega_2}$. Because Θ is an injective there is one point $\zeta \in \pi^{\varsigma}_{\omega_1}$ only such that $\varsigma' = \Theta(\zeta)$. Hence, $\varsigma' \in \pi^{\beta}_{\omega_2}$ for all $\Theta(\varsigma) \in \pi^{\beta}_{\omega_2}$, that is, $\varsigma' \in \eta_{\omega_2}(\beta)$ or $\eta_{\omega_2}(\beta) \subset \eta_{\omega_2}(\varsigma')$ for all $\Theta(\varsigma) \in \pi^{\beta}_{\omega_2}$. Because Θ is an injective there is one point $\theta \in \pi^{\varsigma}_{\omega_1}$ only such that $\beta = \Theta(\theta)$. By an isomorphism of ω_1 and ω_2, we get that $\zeta \in \eta_{\omega_1}(\theta)$ or $\eta_{\omega_1}(\theta) \subset \eta_{\omega_1}(\zeta)$ for all $\varsigma \in \pi^{\theta}_{\omega_1}$. That is, $\zeta \in \pi^{\varsigma}_{\omega_1}$, and hence, $\varsigma' = \Theta(\zeta) \in \Theta[\pi^{\varsigma}_{\omega_1}]$. We therefore get $\pi^{\Theta(\varsigma)}_{\omega_2} \subseteq \Theta[\pi^{\varsigma}_{\omega_1}]$. □

By Theorem 2.8 the space $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is a discrete space, and by Theorem 2.10, the space $(\varphi(C_{n+m}), \tau_{\mathbb{B}_{C_{n+m}}})$ is discrete for any $n + m \geq 4$. So, $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ and $(\varphi(C_{n+m}), \tau_{\mathbb{B}_{C_{n+m}}})$ are homeomorphic, but $\kappa_{n,m}$ and C_{n+m} are not isomorphic for all $n + m \geq 4$.

The connectedness property is an isomorphic property and homeomorphic property. We show the relation between the connectedness of simple undirected graphs and the connectedness for the class of the corresponding extreme graphical topological spaces.

Theorem 3.5. Let $\omega = (\varphi(\omega), \xi(\omega))$ be any simple graph. Then, ω is a connected graph if the space $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is a connected space.

Proof. Suppose that $\omega = (\varphi(\omega), \xi(\omega))$ is disconnected. Then, we can put $\lambda := \{\lambda_m : m \in L\}$ as the set of all connected components in ω, where $\lambda_m = (\varphi(\lambda_m), \xi(\lambda_m))$ for all $m \in L$. For all $m \in L$, $\varphi(\lambda_m) = \cup_{x \in \varphi(\lambda_m)} \pi_\omega(x)$. Then, $L := \varphi(\lambda_{m_o})$ is nonempty proper open in $\varphi(\omega)$, where $m_o \in L$. Then, $[\varphi(\lambda_m)]^c = \cup_{m \in M \setminus \{m_o\}} \varphi(\lambda_m)$ is also nonempty open proper in $\varphi(\omega)$. Hence, $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is disconnected, a contradiction. □

Note that by Theorem 2.8, the space $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is disconnected, whereas $\kappa_{n,m}$ is a connected graph. The compactness property is homeomorphic. The space $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ of $\omega = (\varphi(\omega), \xi(\omega))$ is compact if $\varphi(\omega)$ is finite. Note that if $\varphi(\kappa_n)$ is infinite, then by Theorem 2.9, the space $(\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ is compact.

Let $\omega = (\varphi(\omega), \xi(\omega))$ be any simple graph. A vertex $\alpha \in \varphi(\omega)$ is called an extreme maximal vertex if there is no another vertex $\beta \in \varphi(\omega)$ such that $\eta_\omega(\alpha) \subset \eta_\omega(\beta)$, that is, $\theta_\omega(\alpha) = \emptyset$. The subgraph of ω induced by the set of all maximal extreme vertices in ω is denoted by ωM_e. A simple graph $\omega = (\varphi(\omega), \xi(\omega))$ is called an extreme maximal connected geodesic (EMCG) graph if ωM_e is an EOCG, and otherwise, it is called an extreme maximal disconnected geodesic graph. If the subtopology $\tau_{\mathbb{B}_\omega}|_{\varphi(\omega M_e)}$ is a discrete on a set $\varphi(\omega M_e)$, then $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is called an extreme maximal geodesic discrete space (EMGD-S). The space $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is called an extreme maximal connected geodesic space (EMCG-S) if it is EMGD-S, and ω is EMCG. Otherwise, it is called an extreme maximal disconnected geodesic space. In the path $P_2 : 1' \to 2'$, $\eta_{P_2}(1') = \eta_{P_2}(2') = \{1', 2'\}$. So, $\theta_{P_2}(1') = \theta_{P_2}(2') = \emptyset$, that is, $\varphi(P_2 M_e) = \{1', 2'\}$. Note that $\{1', 2'\}$ is an outer connected geodetic set with the minimum cardinality in $P_2 M_e$. Then, $|P_2 M_e|_{ex} = 2 = (P_2 M_e)_{goc}$. Hence, $P_2 M_e$ is an EOCG, that is, P_2 is an EMCG. The space $(\varphi(P_2), \tau_{\mathbb{B}_{P_2}})$ is extreme maximal disconnected geodesic and not extreme maximal geodesic discrete, because $(\varphi(P_2), \tau_{\mathbb{B}_{P_2}})$ is indiscrete, and so $\tau_{\mathbb{B}_{P_2}}|_{\varphi(P_2 M_e)}$ is not discrete.

Theorem 3.6. The space $(\varphi(P_n), \tau_{\mathbb{B}_{P_n}})$ of P_n is an EMGD-S for all $n > 2$.

Proof. Let $P_n : 1' \to 2' \to \cdots \to n'$ be any path. Let $n = 3$. Because $\varphi(P_3)_{ex} = \{1', 3'\}$, $\eta_{P_3}(1') = \{1'\}$, $\eta_{P_3}(2') = \{1', 2', 3'\}$, and $\eta_{P_3}(3') = \{3'\}$. So, we get $\theta_{P_3}(1') = \{2'\}$, $\theta_{P_3}(2') = \emptyset$, and $\theta_{P_3}(3') = \{2'\}$. Hence, $\varphi(P_3 M_e) = \{2'\}$. Hence, $(\varphi(P_3 M_e), \tau_{\mathbb{B}_{P_3}}|_{\varphi(P_3 M_e)})$ is discrete, that is, $(\varphi(P_3), \tau_{\mathbb{B}_{P_3}})$ is an EMGD-S. Let $n = 4$. Because $\varphi(P_4)_{ex} = \{1', 4'\}$, then $\eta_{P_4}(1') = \{1'\}$, $\eta_{P_4}(2') = \{1', 2'\}$, $\eta_{P_4}(3') = \{3', 4'\}$, and $\eta_{P_4}(4') = \{4'\}$. So, we get $\theta_{P_4}(1') = \{2'\}$, $\theta_{P_4}(2') = \emptyset$, $\theta_{P_4}(3') = \emptyset$, and $\theta_{P_4}(4') = \{3'\}$. Hence, $\varphi(P_4 M_e) = \{2', 3'\}$. We then have $\tau_{\mathbb{B}_{P_4}} = \{\emptyset, \varphi(P_4), \{1', 2'\}, \{3', 4'\}\}$. Then,

$$\tau_{\mathbb{B}_{P_4}}|_{\varphi(P_4 M_e)} = \left\{B \cap \varphi(P_4 M_e) : B \in \tau_{\mathbb{B}_{P_4}}\right\} = \{\emptyset, \varphi(P_4 M_e), \{2'\}, \{3'\}\}.$$

Hence, $(\varphi(P_4 M_e), \tau_{\mathbb{B}_{P_4}}|_{\varphi(P_4 M_e)})$ is discrete, that is, $(\varphi(P_4), \tau_{\mathbb{B}_{P_4}})$ is an EMGD-S. Let $n > 4$. Because $\varphi(P_n)_{ex} = \{1', n'\}$, $\eta_{P_n}(2') = \{1', 2'\}$, $\eta_{P_n}((n-1)') = \{(n-1)', n'\}$, and $\eta_{P_n}(k') = \{k'\}$ for all $k = 1, 3, 4, \cdots, n-2, n$. So, we get $\theta_{P_n}(1') = \{2'\}$, $\theta_{P_n}(n') = \{(n-1)'\}$, and $\theta_{P_n}(k') = \emptyset$ for all $k = 2, 3, \cdots, n-1$. Hence, $\varphi(P_n M_e) = \{2', 3', \cdots, (n-1)'\}$. By Theorem 2.4, for all $n > 4$, the basis of $(\varphi(P_n), \tau_{\mathbb{B}_{P_n}})$ is given by

$$\mathbb{B}_{P_n} = \left\{\{1', 2'\}, \{(n-1)', n'\}, \{k'\} : 3 \le k \le n-2\right\}.$$

Then, the basis of $(\varphi(P_nM_e), \tau_{\mathbb{B}_{P_n}}|_{\varphi(P_nM_e)})$ is given by

$$\mathbb{B}_{P_nM_e} = \left\{B \cap \varphi(P_nM_e) : B \in \mathbb{B}_{P_n}\right\} = \left\{\{k'\} : 2 \le k \le n-1\right\}.$$

Hence, $(\varphi(P_nM_e), \tau_{\mathbb{B}_{P_n}}|_{\varphi(P_nM_e)})$ is discrete, that is, $(\varphi(P_n), \tau_{\mathbb{B}_{P_n}})$ is an EMGD-S for all $n > 4$. □

Theorem 3.7. The space $(\varphi(P_n), \tau_{\mathbb{B}_{P_n}})$ of a path P_n is an EMCG-S for all $n > 2$.

Proof. From the proof of Theorem 3.6, we have $\varphi(P_nM_e) = \{2', 3', \cdots, (n-1)'\}$. Because the subgraph of P_nM_e with vertices set $\varphi(P_nM_e) \setminus \{2', (n-1)'\}$ is connected, and $C(\{2', (n-1)'\}) = \varphi(P_nM_e)$, then $\{2', (n-1)'\}$ is an outer connected geodetic set with the minimum cardinality in P_nM_e. Because $\varphi(P_nM_e)_{ex} = \{2', (n-1)'\}$, then $|P_nM_e|_{ex} = 2 = (P_nM_e)_{goc}$. Hence, P_nM_e is an EOCG, that is, P_n is an EMCG for all $n > 2$. So, by Theorem 3.6, the space $(\varphi(P_n), \tau_{\mathbb{B}_{P_n}})$ is an EMCG-S for all $n > 2$. □

In Figure 2, the friendship graph F_n has the extreme vertices set $\varphi(F_n)_{ex} = \varphi(F_n) \setminus \{0\}$. Note that $\eta_{F_n}(0) = \varphi(F_n)$, $\eta_{F_n}(k') = \eta_{F_n}(k'') = \{k', k''\}$, $\theta_{F_n}(0) = \emptyset$, $\theta_{F_n}(k') = \theta_{F_n}(k'') = \{k\}$, $\pi_{F_n}(k) = \varphi(F_n)$, and $\pi_{F_n}(k') = \pi_{F_n}(k'') = \{0, k', k''\}$ for all $k = 1, 2, 3, \cdots, n$. Note that $\varphi(F_nM_e) = \{0\} = \varphi(C_n)$. Then, F_n is an EMCG, and the space $(\varphi(F_n), \tau_{\mathbb{B}_{F_n}})$ is an EMCG-S. Figure(3), the generalized friendship graph GF_n has the extreme vertices set $\varphi(GF_n)_{ex} = \varphi(GF_n) \setminus \{1, 2, \cdots, n\}$. Note that $\eta_{GF_n}(k) = \{k, k', k''\}$, $\eta_{GF_n}(k') = \eta_{GF_n}(k'') = \{k', k''\}$, $\theta_{GF_n}(k) = \emptyset$, $\theta_{GF_n}(k') = \theta_{GF_n}(k'') = \{k\}$, and $\pi_{GF_n}(k) = \pi_{GF_n}(k') = \pi_{GF_n}(k'') = \{k, k', k''\}$ for all $k = 1, 2, 3, \cdots, n$. Note that $\varphi(GF_nM_e) = \{1, 2, \cdots, n\} = \varphi(C_n)$. The cycle C_n is not an EOCG, that is, GF_n is not an EMCG, and the space $(\varphi(GF_n), \tau_{\mathbb{B}_{GF_n}})$ is not an EMCG-S. The space $(\varphi(GF_n), \tau_{\mathbb{B}_{GF_n}})$ is extreme maximal geodesic discrete, where the basis of $(\varphi(GF_n), \tau_{\mathbb{B}_{GF_n}})$ is given by $\mathbb{B}_{GF_n} = \{\{k, k', k''\} : 1 \le k \le n\}$, and the basis of the relative topology $\tau_{\mathbb{B}_{GF_n}}|_{\varphi(GF_nM_e)}$ is given by $\mathbb{B}_{GF_nM_e} = \{\{k\} : 1 \le k \le n\}$.

Theorem 3.8. The space $(\varphi(\kappa_{1,m}), \tau_{\mathbb{B}_{\kappa_{1,m}}})$ is an EMGD-S for all $m > 1$.

Proof. Let $m > 1$ and $\varphi(\kappa_{1,m}) = \varphi_1 \cup \varphi_m$, where $\varphi_1 = \{0\}$, and $\varphi_m = \{1', 2', \cdots, m'\}$. Note that $\varphi(\kappa_{1,m})_{ex} = \varphi_m$. So, we have $\eta_{\kappa_{1,m}}(0') = \varphi(\kappa_{1,m})$ and $\eta_{\kappa_{1,m}}(k') = \{k'\}$ for all $k = 1, 2, 3, \cdots, m$. Then, $\theta_{\kappa_{1,m}}(0') = \emptyset$ and $\theta_{\kappa_{1,m}}(k') = \{0'\}$ for all $k = 1, 2, 3, \cdots, m$. Hence, $\pi_{\kappa_{1,m}}(0') = \varphi(\kappa_{1,m})$ and $\pi_{\kappa_{1,m}}(k') = \{0', k'\}$ for all $k = 1, 2, 3, \cdots, m$. So, we get that $\varphi(\kappa_{1,m}M_e) = \{0'\}$. Hence, $(\varphi(\kappa_{1,m}), \tau_{\mathbb{B}_{\kappa_{1,m}}})$ is an EMGD-S for all $m > 1$. □

Theorem 3.9. The space $(\varphi(\kappa_{1,m}), \tau_{\mathbb{B}_{\kappa_{1,m}}})$ is an EMCG-S for all $n > 2$.

Proof. It is clear by the proof of Theorem 3.8 that we have $\varphi(\kappa_{1,m}M_e) = \{0'\}$. Directly and by Theorem 3.8, the space $(\varphi(\kappa_{1,m}), \tau_{\mathbb{B}_{\kappa_{1,m}}})$ is an EMCG-S for all $m > 1$. □

From Theorem 2.8, $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is a discrete space for all $n, m > 1$. So, it is an EMGD-S for all $m, n > 1$. Because $\varphi(\kappa_{n,m})_{ex} = \emptyset$ for for all $m, n > 1$, $\kappa_{n,m}$ is not an EMCG, and so $(\varphi(\kappa_{n,m}), \tau_{\mathbb{B}_{\kappa_{n,m}}})$ is not an EMCG for all $n, m > 1$. From Theorem 2.9, the space $(\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ is a discrete space for all $n > 1$. So, it is an EMGD-S for all $n > 1$. Because $\varphi(\kappa_n)_{ex} = \emptyset$ for for all $m, n > 1$, κ_n is not an EMCG-S, and so $(\varphi(\kappa_n), \tau_{\mathbb{B}_{\kappa_n}})$ is not an EMCG-S for all $n > 1$. If $n = 3$, then it is clear that the space $(\varphi(C_3), \tau_{\mathbb{B}_{C_3}})$ is an indiscrete space. Hence, $(\varphi(C_3), \tau_{\mathbb{B}_{C_3}})$ is an EMGD-S. Note that $\varphi(C_3)_{ex} = \{1', 2', 3'\}$, and $\varphi(P_nM_e) = \varphi(C_3)$. Take $\varphi(C_3)$ as a set with the minimum cardinality in P_nM_e. So, we get that C_3 is not an EMCG, and so $(\varphi(C_n), \tau_{\mathbb{B}_{C_n}})$ is an EMCG-S. From Theorem 2.10, $(\varphi(C_n), \tau_{\mathbb{B}_{C_n}})$ is a discrete space for all $n > 3$. Because $\varphi(C_3)_{ex} = \{1', 2', 3'\}$, it is an EMGD-S for all $n > 3$. Because $\varphi(C_n)_{ex} = \emptyset$ for for all $m, n > 1$, C_n is not an EMCG, and so $(\varphi(C_n), \tau_{\mathbb{B}_{C_n}})$ is not an EMCG-S for all $n > 3$.

Lemma 3.10. Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$ and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be two isomorphic simple graphs by $\Theta : \varphi(\omega_1) \to \varphi(\omega_2)$. Then, A is an outer connected geodetic set with minimum cardinality in ω_1 if and only if $\Theta(A)$ is an outer connected geodetic set with minimum cardinality in ω_2.

Proof. For an outer connected geodetic property, let A be an outer connected geodetic set in ω_1. It is clear that for any $x, y \in \varphi(\omega_1)$, $\Theta[C[x, y]] = C[\Theta(x), \Theta(y)]$. Because A is a geodetic set in ω_1, $C[A] = \varphi(\omega_1)$. Because Θ is bijective, $C[\Theta(A)] = \Theta[C[A]] = \Theta[\varphi(\omega_1)] = \varphi(\omega_2)$, that is, $\Theta(A)$ is a geodetic set in ω_2. If $A = \varphi(\omega_1)$ and because Θ is bijective, then $\Theta(A) = \Theta[\varphi(\omega_1)] = \varphi(\omega_2)$. If the subgraph induced by $\varphi(\omega_1) - A$ is a connected graph, then the subgraph induced by $\varphi(\omega_2) - \Theta(A)$ is connected. So, in two cases, $\Theta(A)$ is an outer connected geodetic set in ω_2. For the converse, it is clear by the first part Θ^{-1} is an isomorphism. Now, we prove the fact for the minimum cardinality property. Let A have minimum cardinality in ω_1 for the class of outer connected geodetic sets in ω_1. We prove that that $\Theta(A)$ has the minimum cardinality in ω_2 for the class of outer connected geodetic sets in ω_2. Suppose that there is another outer connected geodetic set B' in ω_2 such that $|B'| < |\Theta(A)|$. By the first part $B = \Theta^{-1}(B')$ is an outer connected geodetic set in ω_1. Because Θ is bijective, $|B'| = |B|$ and $|A| = |\Theta(A)|$. Hence, $|B| < |A|$, and this is a contradiction, A has minimum cardinality in ω_1 for the class of outer connected geodetic sets in ω_1. For the converse, it is clear by the first part, and Θ^{-1} is an isomorphism. □

Theorem 3.11. The EMCG is an isomorphic property.

Proof. Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$, and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be any two isomorphic, simple graphs and ω_1 be an EMCG. We prove that ω_2 is an EMCG. Because $\omega_1 \cong \omega_2$, then there is a bijective map $\Theta : \varphi(\omega_1) \to \varphi(\omega_2)$ such that $(\omega_1)_{\alpha\beta} \in \xi(\omega_1)$ if and only if $(\omega_2)_{\Theta(\alpha)\Theta(\beta)} \in \xi(\omega_2)$ for all $\alpha, \beta \in \varphi(\omega_1)$. Because ω_1 is an EMCG, $\omega_1 M_e$ is an EOCG, that is, there is an outer connected geodetic set $A \subseteq \varphi(\omega_1 M_e)$ with the minimum cardinality for the class of outer connected geodetic sets in $\omega_1 M_e$ such that $|\omega_1 M_e|_{ex} = |A| = (\omega_1 M_e)_{goc}$. Define the restriction map $\Omega = \Theta|_{\varphi(\omega_1 M_e)}$. For $\alpha \in \varphi(\omega_1 M_e)$, $\theta_{\omega_1}(\alpha) = \emptyset$, it is clear to get that $\theta_{\omega_2}(\Theta(\alpha)) = \emptyset$, that is, $\Omega(\alpha) = \Theta(\alpha) \in \varphi(\omega_2 M_e)$. Hence, we can consider Ω is a well-defined map from $\varphi(\omega_1 M_e)$ onto $\varphi(\omega_2 M_e)$. Because Θ is bijective, then Ω is bijective. Hence, $\Omega : \varphi(\omega_1 M_e) \to \varphi(\omega_2 M_e)$ is an isomorphism between graphs $\omega_1 M_e$ and $\omega_2 M_e$. Hence, by Lemma 3.10, $\Omega(A)$ is an outer connected geodetic set with the minimum cardinality for the class of outer connected geodetic sets in $\omega_2 M_e$, $|\Omega(A)| = |A|$, and $|\omega_2 M_e|_{ex} = |\omega_1 M_e|_{ex}$. Hence, $(\omega_2 M_e)_{goc} = |\Omega(A)| = |A| = |\omega_1 M_e|_{ex} = |\omega_2 M_e|_{ex}$, that is, ω_2 is an EMCG. □

Theorem 3.12. The EMGD-S is a homeomorphic property.

Proof. Let $\omega_1 = (\varphi(\omega_1), \xi(\omega_1))$ and $\omega_2 = (\varphi(\omega_2), \xi(\omega_2))$ be two simple graphs and $\Theta : \varphi(\omega_1) \to \varphi(\omega_2)$ be a homeomorphism. Let $(\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}})$ be an EMGD-S. We prove that $(\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ is an EMGD-S. Let $\alpha' \in \varphi(\omega_2 M_e)$ be any point. Because Θ is bijective, there is $\alpha \in \varphi(\omega_1 M_e)$ such that $\alpha' = \Theta(\alpha)$. Because $(\varphi(\omega_1), \tau_{\mathbb{B}_{\omega_1}})$ has EMGD-S property, $\{\alpha\}$ is open in $(\varphi(\omega_1 M_e), \tau_{\mathbb{B}_{\omega_1}}|_{\varphi(\omega_1 M_e)})$. Because Θ is a bijective and, open map, $\Theta(\{\alpha\}) = \{\Theta(\alpha)\} = \{\alpha'\}$ is an open set in $(\varphi(\omega_2 M_e), \tau_{\mathbb{B}_{\omega_2}}|_{\varphi(\omega_2 M_e)})$. That is, $(\varphi(\omega_2), \tau_{\mathbb{B}_{\omega_2}})$ is an EMGD-S. □

Corollary 3.13. In the class of extreme graphical topological spaces, the EMCG-S is a homeomorphic-isomorphic property.

Proof. The proof follows from Theorems 3.12 and 3.11. □

4. Some applications

We know that a chemical compound is a substance formed from two or more different chemical elements that are chemically bonded together in a fixed ratio; for example, see Figure 4 [27]. The $X-$ray structure refers to the three-dimensional arrangement of atoms within a chemical compound, determined using $X-$ray crystallography. This technique analyzes how a crystalline sample diffracts an $X-$ray beam, producing patterns from which a detailed picture of atomic positions and chemical bonds can be calculated. Tariq et al. [27, 28] introduced two new compounds, called 3-benzyl-6-phenylpyridazin-4(1H)-one (for short, compound 4) and 2,5-Diphenyl-7-(prop-2-ynylthio)pyrazolo[1,5-c]pyrimidine (for short, compound 2a). This section will study the connectedness and discrete properties of extreme graphical topological spaces for Figures 5(b) and 6(b) that represent the $X-$ray structures of compound 2a [27] and compound 4 [28], respectively. In the corresponding Figure 5(b), we have a graph $\omega = (\varphi(\omega), \xi(\omega))$ with the vertices set

$$\varphi(\omega) = \{k, m, n_1, n_2, n_3, s_1 : k = 1, \cdots, 5, 8, 10, 13, \cdots, 17, 19, 21$$
$$\text{and } m = 6, 7, 9, 11, 12, 18, 20\}.$$

An extreme space $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ has a basis $\mathbb{B}_\omega$ given by Table 1,

$$\mathbb{B}_\omega = \Big\{\{k, k'\}, \{m\}, \{n_1\}, \{n_2\}, \{n_3\}, \{s_1\}, \{19, 19''\} :$$
$$k = 1, \cdots, 5, 8, 10, 13, \cdots, 17, 19, 21 \text{ and } m = 6, 7, 9, 11, 12, 18, 20\Big\}.$$

Note that $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ is disconnected and nondiscrete. In the corresponding Figure 6(b), then we have a graph $\omega = (\varphi(\omega), \xi(\omega))$ with

$$\varphi(\omega) = \{k, m, n_2, o_1 : k = 1, \cdots, 5, 8, 9, 11, 13, \cdots, 17, 19, 21, n_1 \text{ and } m = 6, 7, 10, 12\}.$$

An extreme space $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ has a basis $\mathbb{B}_\omega$ given by Table 2,

$$\mathbb{B}_\omega = \Big\{\{k, k'\}, \{m\}, \{n_2\}, \{o_1, 9\}, \{11, 11''\} :$$
$$k = 1, \cdots, 5, 8, 9, 11, 13, \cdots, 17, 19, 21, n_1 \text{ and } m = 6, 7, 10, 12\Big\}.$$

Note that $(\varphi(\omega), \tau_{\mathbb{B}_\omega})$ has disconnected and nondiscrete properties.

Table 1. Extremes in Figure 5(b), where $k = 1, \cdots, 5, 8, 10, 13, \cdots, 17, 19, 21$, and $m = 6, 7, 9, 11, 12, 18, 20$.

x	$\eta_\omega(x)$	$\theta_\omega(x)$	$\pi_\omega(x)$	π^x_ω
k	$\{k, k'\}$	$\emptyset$	$\{k, k'\}$	$\{k, k'\}$
m	$\{m\}$	$\emptyset$	$\{m\}$	$\{m\}$
k'	$\{k'\}$	$\{k\}$	$\{k, k'\}$	$\{k, k'\}$
19	$\{19, 19', 19''\}$	$\emptyset$	$\{19, 19', 19''\}$	$\{19, 19''\}$
$19''$	$\{19''\}$	$\{19, 19''\}$	$\{19, 19''\}$	$\{19, 19''\}$
n_1	$\{n_1\}$	$\emptyset$	$\{n_1\}$	$\{n_1\}$
n_2	$\{n_2\}$	$\emptyset$	$\{n_2\}$	$\{n_2\}$
n_3	$\{n_3\}$	$\emptyset$	$\{n_3\}$	$\{n_3\}$
s_1	$\{s_1\}$	$\emptyset$	$\{s_1\}$	$\{s_1\}$

Table 2. Extremes in Figure 6(b), where $k = 1, \cdots, 5, 8, 9, 11, 13, \cdots, 17, 19, 21, n_1$, and $m = 6, 7, 10, 12$.

x	$\eta_\omega(x)$	$\theta_\omega(x)$	$\pi_\omega(x)$	π^x_ω
k	$\{k, k'\}$	$\emptyset$	$\{k, k'\}$	$\{k, k'\}$
m	$\{m\}$	$\emptyset$	$\{m\}$	$\{m\}$
k'	$\{k'\}$	$\{k\}$	$\{k, k'\}$	$\{k, k'\}$
11	$\{11, 11', 11''\}$	$\emptyset$	$\{11, 11', 11''\}$	$\{11, 11''\}$
$11''$	$\{11''\}$	$\{11, 11''\}$	$\{11, 11''\}$	$\{11, 11''\}$
n_2	$\{n_2\}$	$\emptyset$	$\{n_2\}$	$\{n_2\}$
0_1	$\{o_1\}$	$\{9\}$	$\{o_1, 9\}$	$\{o_1, 9\}$

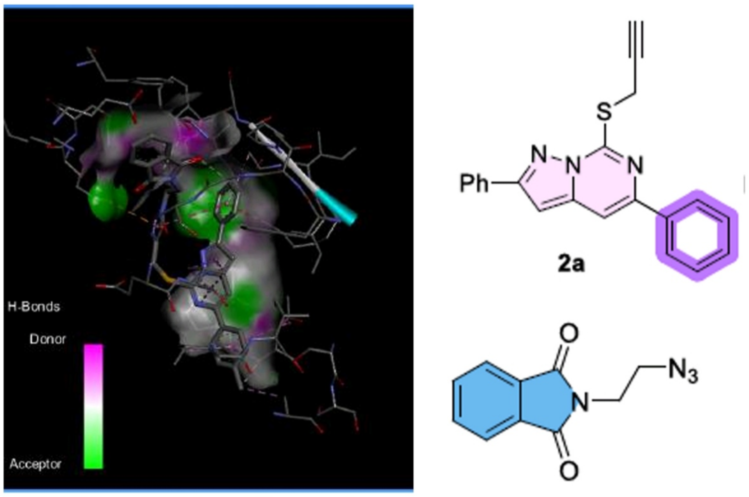
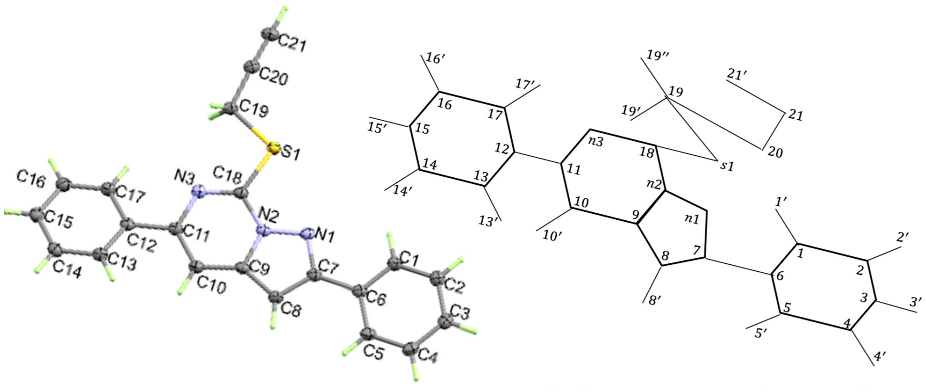
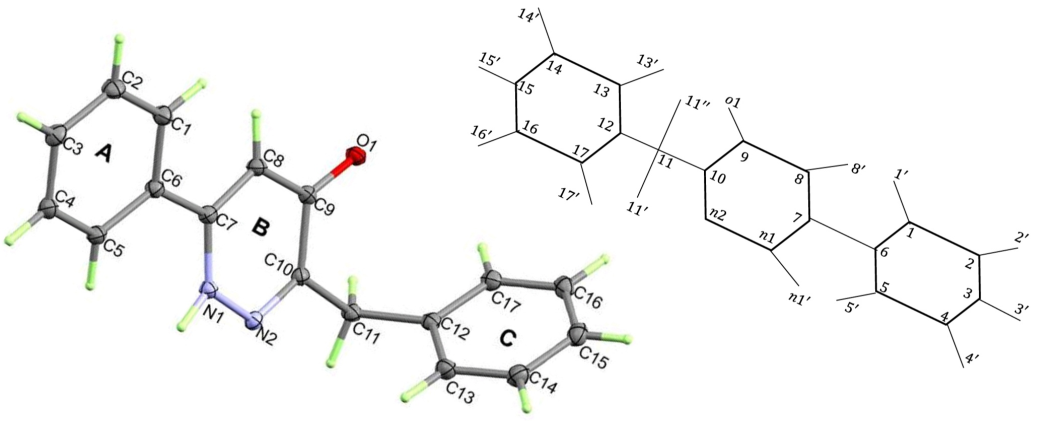


Figure 4. Cyclin D kinase-6.



(a) X-ray structure of compound 2a (b) X-ray representation of compound 2a

Figure 5. Slide clip for the optical system of human eyes.



(a) X-ray structure of compound 4

(b) X-ray representation of compound 4

Figure 6. Representation of optical system in slide clip I.

5. Conclusions

The concept of connectedness, whether considered from a graphical or topological perspective, reveals a notable correspondence between these two frameworks. Our analysis of the X-ray crystallographic representations of compounds 2a and 4 indicates that these structures lack connectedness and instead exhibit discrete characteristics. These observations highlight the need for further theoretical development in this area. In particular, we propose the future exploration of extreme graphical topological spaces within the context of connected geodesic directed graphs, which may provide new insights into the structural and mathematical properties of such systems.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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