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Research article

(L, M)-fuzzy weak hull groups

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Abstract: This study is carried out in the context of (L, M)-fuzzy weak hull operators. First, we introduce the concept of an (L, M)-fuzzy quasi-hull operator and prove that (L, M)-fuzzy quasi-hull operators are categorically isomorphic to (L, M)-fuzzy supratopologies. Second, by imposing a compatibility condition with group structures, we propose the notion of an (L, M)-fuzzy weak hull group and further investigate its relevant properties, such as those related to its subgroups and product groups.

Keywords: (L, M)-fuzzy weak hull operator; (L, M)-fuzzy supratopology; (L, M)-fuzzy quasi-hull operator; (L, M)-fuzzy weak hull group
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1. Introduction

Convexity theory [14] exists in many mathematical structures, such as lattices, topological spaces, and other related structures [4, 6, 7, 15, 24]. At the same time, with the development of fuzzy theory, convexity theory has also been extended to fuzzy settings.

In 1994, Rosa [11] first proposed the concept of fuzzy convex spaces and generalized relevant concepts from classical convex spaces [17], such as convex-preserving mappings, subspaces, quotient spaces, and product spaces, to the fuzzy context. In 2009, Maruyama [8] generalized fuzzy convex spaces by introducing the notion of L-fuzzy convex spaces under the condition that L is a completely distributive lattice, and extended five classical theorems about convex sets in Euclidean spaces to the fuzzy case.

In 2014, Shi and Xiu [12] characterized M-fuzzifying convex structures using M-fuzzifying closure operators, revealing a one-to-one correspondence between M-fuzzifying convex structures and M-fuzzifying closure operators. They further proposed concepts such as M-fuzzifying convex-preserving functions, substructures, product structures, and quotient structures, thereby investigating the related properties of M-fuzzifying convex structures. In 2017, under the assumption that L and M are completely distributive lattices, Shi and Xiu [13] defined (L, M)-fuzzy convex structures and studied their related properties, such as substructures, quotient structures, and product structures.

In 2019, Pang [9] introduced (L, M)-fuzzy hull operators and established the categorical relationship between (L, M)-fuzzy hull operators and (L, M)-fuzzy convex structures. Xiu et al. [19] proposed the concepts of (L, M)-fuzzy concave spaces, (L, M)-fuzzy interior spaces, (L, M)-fuzzy interior relations, and (L, M)-fuzzy hull relations, proving that the categories of (L, M)-fuzzy concave spaces, (L, M)-fuzzy interior spaces, (L, M)-fuzzy interior relation spaces, and (L, M)-fuzzy hull relation spaces are isomorphic. In addition, when L is a completely distributive lattice with an order-reversing involution, they verified that all these categories are isomorphic to the category of (L, M)-fuzzy convex spaces. Sayed et al. [16] proposed a new class of (L, M)-fuzzy hull operators (called Sayed (L, M)-fuzzy hull operators [20, 21]) and claimed that there is a one-to-one correspondence between this type of operators and (L, M)-fuzzy convex structures. In 2021, Zhao and Song [23] pointed out that the proofs of three results in [16] are incorrect and gave the corrected proofs. In 2023, Zhao and Hu [20] defined (L, M)-fuzzy weak hull operators. In this framework, they first introduced convex (L, M)-fuzzy hull operators and proved that (L, M)-fuzzy convex structures are categorically isomorphic to convex (L, M)-fuzzy hull operators; second, they gave a new characterization of the lattice structure of convex (L, M)-fuzzy hull operators and the product of (L, M)-fuzzy convex structures. Since (L, M)-fuzzy closure systems are dual to (L, M)-fuzzy supratopologies, a natural question is: Are (L, M)-fuzzy weak hull operators also categorically isomorphic to (L, M)-fuzzy supratopologies ((L, M)-fuzzy closure systems)? Motivated by the above, we will further investigate (L, M)-fuzzy weak hull operators in this paper.

The structure of this paper is arranged as follows: In Section 2, we review relevant concepts and properties of completely distributive lattices, (L, M)-fuzzy supratopologies, and (L, M)-fuzzy weak hull operators; in Section 3, we introduce the concept of (L, M)-fuzzy quasi-hull operators and study the categorical relationships between (L, M)-fuzzy quasi-hull operators and (L, M)-fuzzy supratopologies; and in Section 4, we define (L, M)-fuzzy weak hull groups and discuss their related structures, such as subgroups and product groups.

2. Preliminaries

In this paper, both L and M are completely distributive lattices with an order-reversing involution. Suppose that M (resp., L) is a complete lattice with the largest element 1_M (resp., 1_L) and the smallest element 0_M (resp., 0_L), respectively, and $M_{0_M} = M \setminus \{0_M\}$. An element $v \in M$ is called prime, if for each $p, q \in M$, $p \wedge q \leq v$ implies $p \leq v$ or $q \leq v$. $v \in M$ is called coprime if p' is a prime element. The set of all nonunit prime elements in M is denoted by $P(M)$. The set of all non-zero coprime elements of M is denoted by $\mathrm{J}(M)$. We say that p is wedge-below q in M, denoted by $p \lhd q$, if for any subset $R \subseteq M$ with $q \leq \bigvee R$ implies $p \leq v$ for some $v \in R$ (see [2]).

A complete lattice M is a completely distributive lattice if and only if $q = \bigvee\{p \in M \mid p \lhd q\}$ for each $q \in M$. $\beta(q) = \{p \in M \mid p \lhd q\}$ is the greatest minimal family of q and $\beta^*(q) = \beta(q) \cap \mathrm{J}(M)$ is a minimal family of q. $\alpha(q)$ denotes the greatest maximal family of q. When M is a completely distributive lattice, every element $q \in M$ has the greatest minimal family. In particular, for each $p \in \mathrm{J}(M)$, $p \in \beta^*(q)$ iff

$p \lhd q$, and $q = \bigvee \beta^*(q)$ for each $q \in M$ (see [18]).

In this paper, let X be a non-empty set. We denote by L^X the set of all L-subsets of X. The set L^X forms a completely distributive lattice equipped with an order-reversing involution under the pointwise order. The smallest and largest elements of L^X are denoted by 0_X and 1_X, respectively. For each $\mu \in L$, let $\underline{\mu}$ denote the constant L-fuzzy subset of X with the value μ.

Definition 2.1. *[10, 19] Let X and Y be nonempty sets and $f : X \to Y$. Define the forward image $f^{\to} : L^X \to L^Y$ and the preimage $f^{\leftarrow} : L^Y \to L^X$ as follows:*

1) For all $A \in L^X$ and $y \in Y$,

$$f^{\to}(A)(y) = \bigvee \{A(x) : x \in X,\ f(x) = y\}.$$

2) For all $B \in L^Y$ and $x \in X$,

$$f^{\leftarrow}(B)(x) = B(f(x)).$$

It can be shown that $(f^{\to}, f^{\leftarrow})$ forms a Galois connection between the completely distributive lattices $(L^X, \leq)$ and $(L^Y, \leq)$.

Definition 2.2. *[22] Let X be a group. For any $A, B \in L^X$, define A^{-1} and AB by*

$$A^{-1}(x) = A(x^{-1}), \quad \forall x \in X,$$

$$(AB)(z) = \bigvee_{xy=z} (A(x) \wedge B(y)), \quad \forall z \in X.$$

Definition 2.3. *[1, 3, 5] An (L, M)-fuzzy supratopology on a set X is a mapping $\mathcal{T} : L^X \to M$ such that:*

(T1) $\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1_M$;
(T2) For any family $\{B_j : j \in J\} \subseteq L^X$,

$$\mathcal{T}\Big(\bigvee_{j\in J} B_j\Big) \geq \bigwedge_{j\in J} \mathcal{T}(B_j).$$

The value $\mathcal{T}(B)$ is interpreted as the degree to which B is supraopen. The mapping $\mathcal{T}^\star(B) = \mathcal{T}(B')$ is called the degree to which B is supraclosed. The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy supratopological space.

A mapping $f : X \to Y$ between (L, M)-fuzzy supratopological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is called (L, M)-fuzzy supratopologically continuous (abbreviated as (L, M)-FSTL) if

$$\mathcal{T}_X(f^{\leftarrow}(B)) \geq \mathcal{T}_Y(B)$$

for all $B \in L^Y$.

*The category of all (L, M)-fuzzy supratopological spaces and their continuous mappings is denoted by (L, M)-**FSTP**.*

Definition 2.4. *[20] An (L, M)-fuzzy weak hull operator on a set X is a map $\mathcal{H} : L^X \times M_{0_M} \to L^X$ such that for all $A, B \in L^X$ and $r, s \in M_{0_M}$,*

(H1) $\mathcal{H}(0_X, r) = 0_X$;
(H2) $B \le \mathcal{H}(B, r)$;
(H3) *If* $r \le s$, *then* $\mathcal{H}(B, r) \le \mathcal{H}(B, s)$;
(H4) *If* $A \le B$, *then* $\mathcal{H}(A, r) \le \mathcal{H}(B, r)$;
(H5) $\mathcal{H}(\mathcal{H}(B, r), r) = \mathcal{H}(B, r)$.

If $\mathcal{H}$ *is an* (L, M)*-fuzzy weak hull operator on* X*, then the pair* $(X, \mathcal{H})$ *is called an* (L, M)*-fuzzy weak hull space. Let* $(X, \mathcal{H}_X)$ *and* $(Y, \mathcal{H}_Y)$ *be two* (L, M)*-fuzzy weak hull spaces. A mapping* $f : X \to Y$ *is called* (L, M)*-fuzzy weak hull-preserving (abbreviated as* (L, M)*-WHP) if*

$$\mathcal{H}_Y(f^{\to}(A), r) \ge f^{\to}(\mathcal{H}_X(A, r))$$

for all $(A, r) \in L^X \times M_{0_M}$.

Lemma 2.5. *[20] Let* X *be a set,* $\{(X_j, \mathcal{H}_j)\}_{j\in J}$ *a family of* (L, M)*-fuzzy weak hull spaces, and for each* $j \in J$*, let* $f_j : X \to X_j$ *be a mapping. Define* $\mathcal{H}^* : L^X \times M_{0_M} \to L^X$ *by*

$$\mathcal{H}^*(A, r) = \bigwedge_{j\in J} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\to}(A), r)), \quad \forall (A, r) \in L^X \times M_{0_M}.$$

Then:

1) $\mathcal{H}^*$ *is the finest* (L, M)*-fuzzy weak hull operator on* X *such that* $f_j : (X, \mathcal{H}^*) \to (X_j, \mathcal{H}_j)$ *is* (L, M)*-WHP for each* $j \in J$.
2) *Let* $(Y, \mathcal{H}_Y)$ *be an* (L, M)*-fuzzy weak hull space. Then* $g : (Y, \mathcal{H}_Y) \to (X, \mathcal{H}^*)$ *is* (L, M)*-WHP if and only if* $f_j \circ g : (Y, \mathcal{H}_Y) \to (X_j, \mathcal{H}_j)$ *is* (L, M)*-WHP for all* $j \in J$.

Corollary 2.6. *[20] Let* $\{(X_j, \mathcal{H}_j)\}_{j\in J}$ *be a family of* (L, M)*-fuzzy weak hull spaces, let* $X = \prod_{j\in J} X_j$*, and let* $p_j : X \to X_j$ *be the* j*-th projection. Let* $(X, \mathcal{H}^{\star})$ *be the product of* $\{(X_j, \mathcal{H}_j)\}_{j\in J}$*. Define* $\mathcal{H}^{\star} : L^X \times M_{0_M} \to L^X$ *by*

$$\mathcal{H}^{\star}(A, r) = \bigwedge_{j\in J} p_j^{\leftarrow}(\mathcal{H}_j(p_j^{\to}(A), r)), \quad \forall (A, r) \in L^X \times M_{0_M}.$$

Then, $\mathcal{H}^{\star}$ *is the coarsest* (L, M)*-fuzzy weak hull operator on* X *such that all projections* p_j *are* (L, M)*-WHP.*

3. (L, M)-fuzzy quasi-hull operators

In this section, based on the concept of (L, M)-fuzzy weak hull operators, we introduce the notion of (L, M)-fuzzy quasi-hull operators and investigate the categorical relationship between (L, M)-fuzzy quasi-hull operators and (L, M)-fuzzy supratopologies.

Definition 3.1. *Let* $\mathcal{H} : L^X \times M_{0_M} \to L^X$ *be an* (L, M)*-fuzzy weak hull operator on* X *and satisfy the following conditions: For all* $A \in L^X$ *and* $r, s \in M_{0_M}$,

(H6) *If* $r = \bigvee\{s \in M_{0_M} : A = \mathcal{H}(A, s)\}$*, then* $\mathcal{H}(A, r) = A$.

Then, $\mathcal{H}$ is called an (L, M)-fuzzy quasi-hull operator on X, and the pair $(X, \mathcal{H})$ is called an (L, M)-fuzzy quasi-hull space. The set of all (L, M)-fuzzy quasi-hull operators on X is denoted by $\mathbf{FQH}(X, L, M)$. *Define a relation $\leq$ on* $\mathbf{FQH}(X, L, M)$ *by*

$$\mathcal{H}_1 \leq \mathcal{H}_2 \Longleftrightarrow \mathcal{H}_1(A, r) \geq \mathcal{H}_2(A, r), \quad \forall (A, r) \in L^X \times M_{0_M}.$$

It is straightforward to verify that $(\mathbf{FQH}(X, L, M), \leq)$ *is a poset.*

Let $(X, \mathcal{H}_X)$ and $(Y, \mathcal{H}_Y)$ be two (L, M)-fuzzy quasi-hull spaces, and let $g : X \to Y$ be a mapping. If

$$g^{\to}(\mathcal{H}_X(A, a)) \leq \mathcal{H}_Y(g^{\to}(A), a)$$

for each $(A, a) \in L^X \times M_{0_M}$, we say g is (L, M)-fuzzy quasi-hull preserving mapping (abbreviated as (L, M)-QBP). It is easy to check that all (L, M)-fuzzy quasi-hull spaces as objects, and all corresponding (L, M)-QBP mappings as morphisms, form a category, denoted by (L, M)-$\mathbf{FQH}$.

Theorem 3.2. *Let $(X, \mathcal{T})$ be an (L, M)-fuzzy supratopology, and let $(A, r) \in L^X \times M_{0_M}$. Define a mapping $\mathcal{H}^{\mathcal{T}} : L^X \times M_{0_M} \to L^X$ by*

$$\mathcal{H}^{\mathcal{T}}(A, r) = \bigwedge \{B \in L^X : A \leq B,\ \mathcal{T}(B') \geq r\}.$$

Then, $\mathcal{H}^{\mathcal{T}}$ is an (L, M)-fuzzy quasi-hull operator.

Proof. It suffices to prove that $\mathcal{H}^{\mathcal{T}}$ satisfies conditions (H1)–(H6).

(H1) For any $r \in M_{0_M}$,

$$0_X \leq \mathcal{H}^{\mathcal{T}}(0_X, r) = \bigwedge \{B \in L^X : 0_X \leq B,\ \mathcal{T}(B') \geq r\} \leq 0_X.$$

Thus $\mathcal{H}^{\mathcal{T}}(0_X, r) = 0_X$.

(H2) This condition is easy to prove and is therefore omitted.

(H3) If $r \leq s$, then

$$\begin{aligned} \mathcal{H}^{\mathcal{T}}(A, r) &= \bigwedge \{B \in L^X : A \leq B,\ \mathcal{T}(B') \geq r\} \\ &\leq \bigwedge \{B \in L^X : A \leq B,\ \mathcal{T}(B') \geq s\} \\ &= \mathcal{H}^{\mathcal{T}}(A, s). \end{aligned}$$

(H4) If $A \leq B$, then

$$\begin{aligned} \mathcal{H}^{\mathcal{T}}(A, r) &= \bigwedge \{C \in L^X : A \leq C,\ \mathcal{T}(C') \geq r\} \\ &\leq \bigwedge \{C \in L^X : B \leq C,\ \mathcal{T}(C') \geq r\} \\ &= \mathcal{H}^{\mathcal{T}}(B, r). \end{aligned}$$

(H5) By (H2), we have $\mathcal{H}^{\mathcal{T}}(A, r) \leq \mathcal{H}^{\mathcal{T}}(\mathcal{H}^{\mathcal{T}}(A, r), r)$. Conversely, let

$$S = \{B \in L^X : A \leq B,\ \mathcal{T}(B') \geq r\}.$$

For every $B \in S$, we have $\mathcal{H}^{\mathcal{T}}(A, r) \leq B$ and $\mathcal{T}(B') \geq r$. Hence,

$$B \in \{D \in L^X : \mathcal{H}^{\mathcal{T}}(A, r) \leq D,\ \mathcal{T}(D') \geq r\},$$

which implies

$$S \subseteq \{D \in L^X : \mathcal{H}^{\mathcal{T}}(A, r) \leq D,\ \mathcal{T}(D') \geq r\}.$$

Therefore,

$$\mathcal{H}^{\mathcal{T}}(\mathcal{H}^{\mathcal{T}}(A, r), r) = \bigwedge\{D \in L^X : \mathcal{H}^{\mathcal{T}}(A, r) \leq D,\ \mathcal{T}(D') \geq r\} \leq \bigwedge S = \mathcal{H}^{\mathcal{T}}(A, r).$$

Thus $\mathcal{H}^{\mathcal{T}}(\mathcal{H}^{\mathcal{T}}(A, r), r) = \mathcal{H}^{\mathcal{T}}(A, r)$.

(H6) Let

$$r = \bigvee\{s \in M_{0_M} : A = \mathcal{H}^{\mathcal{T}}(A, s)\}.$$

Then $r \in M_{0_M}$, and for every $s \in \{s \in M_{0_M} : A = \mathcal{H}^{\mathcal{T}}(A, s)\}$, we have $A = \mathcal{H}^{\mathcal{T}}(A, s)$.

Hence,

$$\begin{array}{rcl} \mathcal{T}(A') & = & \mathcal{T}((\mathcal{H}^{\mathcal{T}}(A, s))') \\ & = & \mathcal{T}\left(\left(\bigwedge\{D \in L^X : A \leq D,\ \mathcal{T}(D') \geq s\}\right)'\right) \\ & = & \mathcal{T}\left(\bigvee\{D' \in L^X : A \leq D,\ \mathcal{T}(D') \geq s\}\right) \\ & \geq & \bigwedge \mathcal{T}(D') \geq s. \end{array}$$

This means

$$\mathcal{T}(A') \geq \bigvee\{s \in M_{0_M} : A = \mathcal{H}^{\mathcal{T}}(A, s)\} = r.$$

Thus,

$$A \leq \mathcal{H}^{\mathcal{T}}(A, r) = \bigwedge\{D \in L^X : A \leq D,\ \mathcal{T}(D') \geq r\} \leq A,$$

so $\mathcal{H}^{\mathcal{T}}(A, r) = A$.

Therefore, $\mathcal{H}^{\mathcal{T}}$ is an (L, M)-fuzzy quasi-hull operator on X. □

Theorem 3.3. *Let $\mathcal{H}$ be an (L, M)-fuzzy quasi-hull operator. Define a mapping $\mathcal{T}^{\mathcal{H}} : L^X \to M$ by*

$$\mathcal{T}^{\mathcal{H}}(A) = \bigvee\{r \in M_{0_M} : A' = \mathcal{H}(A', r)\}, \quad \forall A \in L^X.$$

Then $\mathcal{T}^{\mathcal{H}}$ is an (L, M)-fuzzy supratopology.

Proof. It suffices to prove that $\mathcal{T}^{\mathcal{H}}$ satisfies (T1) and (T2).

(T1) Since $1_M \in M_{0_M}$, by (H1) and (H2),

$$\mathcal{H}((1_X)', 1_M) = \mathcal{H}(0_X, 1_M) = 0_X, \quad \mathcal{H}((0_X)', 1_M) = \mathcal{H}(1_X, 1_M) = 1_X.$$

Hence,

$$\mathcal{T}^{\mathcal{H}}(0_X) = \bigvee\{r \in M_{0_M} : 0_X' = \mathcal{H}(0_X', r)\} = \bigvee\{r \in M_{0_M} : 1_X = \mathcal{H}(1_X, r)\} = 1_M,$$

and similarly $\mathcal{T}^{\mathcal{H}}(1_X) = 1_M$.

(T2) Let $\{A_j\}_{j\in J} \subseteq L^X$, and let

$$q \in \beta^*\left(\bigwedge_{j\in J} \mathcal{T}^{\mathcal{H}}(A_j)\right).$$

Then, $q \lhd \bigwedge_{j\in J} \mathcal{T}^{\mathcal{H}}(A_j)$ and $q \in \mathrm{J}(M)$, so $q \lhd \mathcal{T}^{\mathcal{H}}(A_j)$ for all $j \in J$. By the definition of $\mathcal{T}^{\mathcal{H}}$, there exists $r_j \in M_{0_M}$ such that

$$A'_j = \mathcal{H}(A'_j, r_j), \quad q \leq r_j.$$

Let $r_0 = \bigwedge_{j\in J} r_j$. Then $q \leq r_0$. By (H3) and (H4),

$$\mathcal{H}\left(\left(\bigvee_{j\in J} A_j\right)', r_0\right) \leq \bigwedge_{j\in J} \mathcal{H}(A'_j, r_0) \leq \bigwedge_{j\in J} \mathcal{H}(A'_j, r_j) = \bigwedge_{j\in J} A'_j = \left(\bigvee_{j\in J} A_j\right)'.$$

Thus,

$$\mathcal{H}\left(\left(\bigvee_{j\in J} A_j\right)', r_0\right) \leq \left(\bigvee_{j\in J} A_j\right)'.$$

On the other hand, by (H2)

$$\mathcal{H}\left(\left(\bigvee_{j\in J} A_j\right)', r_0\right) \geq \left(\bigvee_{j\in J} A_j\right)'.$$

Hence,

$$\mathcal{H}\left(\left(\bigvee_{j\in J} A_j\right)', r_0\right) = \left(\bigvee_{j\in J} A_j\right)',$$

which implies

$$\mathcal{T}^{\mathcal{H}}\left(\bigvee_{j\in J} A_j\right) \geq r_0 \geq q.$$

Therefore,

$$\mathcal{T}^{\mathcal{H}}\left(\bigvee_{j\in J} A_j\right) \geq \bigwedge_{j\in J} \mathcal{T}^{\mathcal{H}}(A_j).$$

□

Proposition 3.4. *If $g : (X, \mathcal{H}_X) \to (Y, \mathcal{H}_Y)$ is an (L, M)-QBP mapping, then*

$$g : (X, \mathcal{T}^{\mathcal{H}_X}) \to (Y, \mathcal{T}^{\mathcal{H}_Y})$$

is an (L, M)-FSTL mapping.

Proof. Let $B \in L^Y$ and $q \in \beta^*(\mathcal{T}^{\mathcal{H}_Y}(B))$. Then,

$$q \lhd \mathcal{T}^{\mathcal{H}_Y}(B) = \bigvee\{r \in M_{0_M} : B' = \mathcal{H}_Y(B', r)\},$$

so there exists $r_0 \in M_{0_M}$ such that

$$B' = \mathcal{H}_Y(B', r_0), \quad q \leq r_0.$$

Since $g : (X, \mathcal{H}_X) \to (Y, \mathcal{H}_Y)$ is (L, M)-QBP,

$$g^{\leftarrow}(B') = g^{\leftarrow}(\mathcal{H}_Y(B', r_0)) \geq g^{\leftarrow}(\mathcal{H}_Y(g^{\rightarrow}(g^{\leftarrow}(B')), r_0)) \geq g^{\leftarrow}(g^{\rightarrow}(\mathcal{H}_X(g^{\leftarrow}(B'), r_0))) \geq \mathcal{H}_X(g^{\leftarrow}(B'), r_0).$$

On the other hand, by (H2), $g^{\leftarrow}(B') \leq \mathcal{H}_X(g^{\leftarrow}(B'), r_0)$. Thus,

$$\mathcal{H}_X(g^{\leftarrow}(B'), r_0) = g^{\leftarrow}(B').$$

Furthermore,

$$\mathcal{T}^{\mathcal{H}_X}(g^{\leftarrow}(B)) = \bigvee\{t \in M_{0_M} : g^{\leftarrow}(B') = \mathcal{H}_X(g^{\leftarrow}(B'), t)\} \geq r_0 \geq q.$$

This yields

$$\mathcal{T}^{\mathcal{H}_X}(g^{\leftarrow}(B)) \geq \mathcal{T}^{\mathcal{H}_Y}(B)$$

for all $B \in L^Y$.

Therefore,

$$g : (X, \mathcal{T}^{\mathcal{H}_X}) \to (Y, \mathcal{T}^{\mathcal{H}_Y})$$

is an (L, M)-FSTL mapping. □

Proposition 3.5. *If* $g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ *is an* (L, M)*-FSTL mapping, then*

$$g : (X, \mathcal{H}^{\mathcal{T}_X}) \to (Y, \mathcal{H}^{\mathcal{T}_Y})$$

is an (L, M)*-QBP mapping.*

Proof. Since $g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is (L, M)-FSTL, we have

$$\mathcal{T}_X(g^{\leftarrow}(B)) \geq \mathcal{T}_Y(B), \quad \forall B \in L^Y.$$

Thus, for all $(A, r) \in L^X \times M_{0_M}$,

$$\begin{aligned} g^{\leftarrow}(\mathcal{H}^{\mathcal{T}_Y}(g^{\rightarrow}(A), r)) &= g^{\leftarrow}\left(\bigwedge\{B \in L^Y : g^{\rightarrow}(A) \leq B,\ \mathcal{T}_Y(B') \geq r\}\right) \\ &= \bigwedge\{g^{\leftarrow}(B) \in L^X : g^{\rightarrow}(A) \leq B,\ \mathcal{T}_Y(B') \geq r\} \\ &\geq \bigwedge\{g^{\leftarrow}(B) \in L^X : A \leq g^{\leftarrow}(B),\ \mathcal{T}_X(g^{\leftarrow}(B')) \geq r\} \\ &\geq \bigwedge\{C \in L^X : A \leq C,\ \mathcal{T}_X(C') \geq r\} \\ &= \mathcal{H}^{\mathcal{T}_X}(A, r). \end{aligned}$$

It follows that

$$\mathcal{H}^{\mathcal{T}_Y}(g^{\rightarrow}(A), r) \geq g^{\rightarrow}(g^{\leftarrow}(\mathcal{H}^{\mathcal{T}_Y}(g^{\rightarrow}(A), r))) \geq g^{\rightarrow}(\mathcal{H}^{\mathcal{T}_X}(A, r)).$$

Hence,

$$g : (X, \mathcal{H}^{\mathcal{T}_X}) \to (Y, \mathcal{H}^{\mathcal{T}_Y})$$

is an (L, M)-QBP mapping. □

By Theorems 3.2 and 3.3, and Propositions 3.4 and 3.5, we obtain the following two functors:

$$\mathbb{H}^{\mathbb{T}}:\begin{cases}(L,M)\text{-}\mathbf{FSTP}\longrightarrow(L,M)\text{-}\mathbf{FQH},\\ (X,\mathcal{T})\longmapsto(X,\mathcal{H}^{\mathcal{T}}),\\ g\longmapsto g,\end{cases}$$

$$\mathbb{T}^{\mathbb{H}}:\begin{cases}(L,M)\text{-}\mathbf{FQH}\longrightarrow(L,M)\text{-}\mathbf{FSTP},\\ (X,\mathcal{H})\longmapsto(X,\mathcal{T}^{\mathcal{H}}),\\ g\longmapsto g.\end{cases}$$

We now prove that $\mathbb{H}^{\mathbb{T}}$ and $\mathbb{T}^{\mathbb{H}}$ yield an isomorphism of categories.

Theorem 3.6. *The categories* (L,M)-**FSTP** *and* (L,M)-**FQH** *are isomorphic.*

Proof. It is enough to show that

$$\mathbb{T}^{\mathbb{H}}\circ\mathbb{H}^{\mathbb{T}}=\mathbb{I}_{(L,M)\text{-}\mathbf{FSTP}}\quad\text{and}\quad\mathbb{H}^{\mathbb{T}}\circ\mathbb{T}^{\mathbb{H}}=\mathbb{I}_{(L,M)\text{-}\mathbf{FQH}},$$

which is equivalent to:

1) $\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}=\mathcal{T}$;
2) $\mathcal{H}^{\mathcal{T}^{\mathcal{H}}}=\mathcal{H}$.

(1) If $\mathcal{T}(A)=0_M$, then $\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}(A)\geq 0_M=\mathcal{T}(A)$. If $\mathcal{T}(A)\neq 0_M$ (i.e., $\mathcal{T}(A)\in M_{0_M}$), by (H2),

$$A'\leq\mathcal{H}^{\mathcal{T}}(A',\mathcal{T}(A))=\bigwedge\{B\in L^X:A'\leq B,\ \mathcal{T}(B')\geq\mathcal{T}(A)\}\leq A'.$$

Hence $A'=\mathcal{H}^{\mathcal{T}}(A',\mathcal{T}(A))$. By the definition of $\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}$,

$$\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}(A)\geq\mathcal{T}(A).$$

Conversely, let $q\in\beta^*(\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}(A))$. Then,

$$q\lhd\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}(A)=\bigvee\{r\in M_{0_M}:A'=\mathcal{H}^{\mathcal{T}}(A',r)\},$$

so there exists $r_0\in M_{0_M}$ such that

$$A'=\mathcal{H}^{\mathcal{T}}(A',r_0),\quad q\leq r_0.$$

Then,

$$\mathcal{T}(A)=\mathcal{T}((\mathcal{H}^{\mathcal{T}}(A',r_0))')\geq r_0\geq q,$$

which implies $\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}(A)\leq\mathcal{T}(A)$. Thus $\mathcal{T}^{\mathcal{H}^{\mathcal{T}}}=\mathcal{T}$.

(2) For all $(A,r)\in L^X\times M_{0_M}$, by (H2) and (H5),

$$A\leq\mathcal{H}(A,r)=\mathcal{H}(\mathcal{H}(A,r),r).$$

Let $B_0=\mathcal{H}(A,r)$. By Theorem 3.3,

$$\mathcal{T}^{\mathcal{H}}(B_0')=\bigvee\{s\in M_{0_M}:B_0=\mathcal{H}(B_0,s)\},$$

so $\mathcal{T}^{\mathcal{H}}(B_0') \geq r$. Hence,

$$\mathcal{H}(A,r) \in \{B \in L^X : A \leq B,\ \mathcal{T}^{\mathcal{H}}(B') \geq r\},$$

and therefore

$$\mathcal{H}^{\mathcal{T}^{\mathcal{H}}}(A,r) = \bigwedge\{B \in L^X : A \leq B,\ \mathcal{T}^{\mathcal{H}}(B') \geq r\} \leq \mathcal{H}(A,r).$$

Conversely, if

$$B \in \{B \in L^X : A \leq B,\ \mathcal{T}^{\mathcal{H}}(B') \geq r\},$$

then $A \leq B$ and $\mathcal{T}^{\mathcal{H}}(B') \geq r$. By (H2) and (H6),

$$B \leq \mathcal{H}(B,r) \leq \mathcal{H}(B,\mathcal{T}^{\mathcal{H}}(B')) = \mathcal{H}\left(B, \bigvee\{s \in M_{0_M} : B = \mathcal{H}(B,s)\}\right) = B.$$

It follows that $B = \mathcal{H}(B,r) \geq \mathcal{H}(A,r)$. Thus,

$$\mathcal{H}^{\mathcal{T}^{\mathcal{H}}}(A,r) = \bigwedge\{B \in L^X : A \leq B,\ \mathcal{T}^{\mathcal{H}}(B') \geq r\} \geq \mathcal{H}(A,r).$$

Hence $\mathcal{H}^{\mathcal{T}^{\mathcal{H}}} = \mathcal{H}$. □

4. (L,M)-fuzzy weak hull groups

In this section, we investigate the compatibility between (L,M)-fuzzy weak hull operators and group structures. We first introduce the notion of (L,M)-fuzzy weak hull groups and then study their related properties. The category consisting of classical groups and surjective homomorphisms is denoted by **Grp** throughout this paper.

Definition 4.1. *Let $(X,\cdot)$ be a group, and $\mathcal{H}$ an (L,M)-fuzzy weak hull operator on X. The triple $(X,\cdot,\mathcal{H})$ is called an (L,M)-fuzzy weak hull group if the following mappings are (L,M)-WHP:*

(1)

$$f : (X,\cdot,\mathcal{H}) \times (X,\cdot,\mathcal{H}) \to (X,\cdot,\mathcal{H}),\ \ (x,y) \mapsto xy,$$

(2)

$$g : (X,\cdot,\mathcal{H}) \to (X,\cdot,\mathcal{H}),\ \ x \mapsto x^{-1},$$

*where $X \times X$ is equipped with the product (L,M)-fuzzy weak hull operator. The category of all (L,M)-fuzzy weak hull groups as objects and (L,M)-fuzzy weak hull-preserving mappings ((L,M)-WHP) as morphisms is denoted by (L,M)-***WHPG**.

Example 4.2. *Let $(X,\cdot)$ be a singleton group, $L = \{0_L, a, b, 1_L\}$ be the diamond lattice in Figure 1, and $M = \{0_M, c, d, m, n, 1_M\}$ be the six-element lattice in Figure 2. Then,*

$$L^X = \{0_X, \underline{a}, \underline{b}, 1_X\}, \quad L^{X\times X} = \{0_{X\times X}, \underline{a}, \underline{b}, 1_{X\times X}\}.$$

Define a mapping $\mathcal{H} : L^X \times M_{0_M} \to L^X$ as follows: For any $r \neq 0_M$,

$$\mathcal{H}(0_X, r) = 0_X, \quad \mathcal{H}(1_X, r) = 1_X,$$

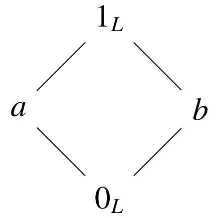

Figure 1. Diamond lattice $L = \{0_L, a, b, 1_L\}$.

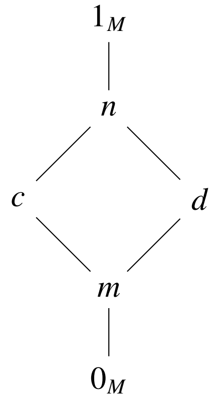

Figure 2. Six-element lattice $M = \{0_M, m, c, d, n, 1_M\}$.

$$\mathcal{H}(\underline{a}, r) = \begin{cases} \underline{a}, & \textit{if } r \in \{m, c, d\}, \\ 1_X, & \textit{otherwise}. \end{cases} \quad \mathcal{H}(\underline{b}, r) = \begin{cases} \underline{b}, & \textit{if } r \in \{m, c, d\}, \\ 1_X, & \textit{otherwise}. \end{cases}$$

It can be verified that $\mathcal{H} : L^X \times M_{0_M} \to L^X$ is an (L, M)-fuzzy weak hull operator on X. By the formula of product operator in Corollary 2.6, we compute $\mathcal{H}^\star = \mathcal{H} \times \mathcal{H} : L^{X\times X} \times M_{0_M} \to L^{X\times X}$ as follows:

$$\mathcal{H}^\star(A, r) = \bigwedge_{j=1}^{2} p_j^{\leftarrow}(\mathcal{H}(p_j^{\rightarrow}(A), r)), \quad \forall (A, r) \in L^{X\times X} \times M_{0_M}.$$

Thus for any $r \neq 0_M$, we have

$$\mathcal{H}^\star(0_{X\times X}, r) = 0_{X\times X}, \quad \mathcal{H}^\star(1_{X\times X}, r) = 1_{X\times X},$$

$$\mathcal{H}^\star(\underline{a}, r) = \begin{cases} \underline{a}, & \textit{if } r \in \{m, c, d\}, \\ 1_{X\times X}, & \textit{otherwise}. \end{cases} \quad \mathcal{H}^\star(\underline{b}, r) = \begin{cases} \underline{b}, & \textit{if } r \in \{m, c, d\}, \\ 1_{X\times X}, & \textit{otherwise}. \end{cases}$$

It is easy to prove that $(X, \cdot, \mathcal{H}^\star)$ is an (L, M)-fuzzy weak hull group.

Theorem 4.3. *Let $(X, \cdot, \mathcal{H})$ be an (L, M)-fuzzy weak hull group on X. Then the following hold:*

(1) For all $(A, r) \in L^X \times M_{0_M}$, $x \in X$,

$$\bigvee_{ab=x} \Big(\mathcal{H}(p_1^{\rightarrow}(A), r)(a) \wedge \mathcal{H}(p_2^{\rightarrow}(A), r)(b)\Big) \leq \mathcal{H}(f^{\rightarrow}(A), r)(x).$$

(2) For all $(A, r) \in L^X \times M_{0_M}$, $x \in X$,

$$\mathcal{H}(A, r)(x^{-1}) \leq \mathcal{H}(A^{-1}, r)(x).$$

Proof. (1) Since $(X,\cdot,\mathcal{H})$ is an (L,M)-fuzzy weak hull group on X, by Definition 4.1(1),

$$f:(X,\cdot,\mathcal{H})\times(X,\cdot,\mathcal{H})\to(X,\cdot,\mathcal{H}),\ \ (x,y)\mapsto xy$$

is an (L,M)-WHP. That is, for all $(A,r)\in L^X\times M_{0_M},\ x\in X$,

$$\begin{aligned}\bigvee_{ab=x}\Big(\mathcal{H}(p_1^{\to}(A),r)(a)\wedge\mathcal{H}(p_2^{\to}(A),r)(b)\Big) &= \bigvee_{ab=x}\Big(p_1^{\leftarrow}(\mathcal{H}(p_1^{\to}(A),r))\wedge p_2^{\leftarrow}(\mathcal{H}(p_2^{\to}(A),r))\Big)(a,b)\\ &= \bigvee_{f(a,b)=x}\Big(p_1^{\leftarrow}(\mathcal{H}(p_1^{\to}(A),r))\wedge p_2^{\leftarrow}(\mathcal{H}(p_2^{\to}(A),r))\Big)(a,b)\\ &= f^{\to}((\mathcal{H}\times\mathcal{H})(A,r))(x)\\ &\le \mathcal{H}(f^{\to}(A),r)(x).\end{aligned}$$

This shows that f is an (L,M)-WHP.

(2) By Definition 4.1(2),

$$g:(X,\cdot,\mathcal{H})\to(X,\cdot,\mathcal{H}),\ \ x\mapsto x^{-1}$$

is an (L,M)-WHP. For all $(A,r)\in L^X\times M_{0_M},\ x\in X$,

$$\mathcal{H}(A,r)(x^{-1})=\bigvee_{a^{-1}=x}\mathcal{H}(A,r)(a)=\bigvee_{g(a)=x}\mathcal{H}(A,r)(a)=g^{\to}(\mathcal{H}(A,r))(x)\le\mathcal{H}(g^{\to}(A),r)(x)=\mathcal{H}(A^{-1},r)(x).$$

Hence g is an (L,M)-WHP. By a similar argument, the converse is also true. So, $(X,\cdot,\mathcal{H})$ is an (L,M)-fuzzy weak hull group on X iff conditions (1) and (2) hold. □

Theorem 4.4. *Let $(X,\cdot,\mathcal{H})$ be an (L,M)-fuzzy weak hull group, and Y a subgroup of X. Define a mapping $\mathcal{H}|_Y:L^Y\times M_{0_M}\to L^Y$ by*

$$\mathcal{H}|_Y(A,r)=id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\to}(A),r)),\quad\forall(A,r)\in L^Y\times M_{0_M}.$$

Then, $\mathcal{H}|_Y$ is an (L,M)-fuzzy weak hull operator on Y, where $\mathrm{id}_Y:Y\to X$ *is the inclusion mapping, and we call $(Y,\cdot,\mathcal{H}|_Y)$ an (L,M)-fuzzy weak hull subgroup of $(X,\cdot,\mathcal{H})$.*

Proof. First, we verify that $\mathcal{H}|_Y$ is an (L,M)-fuzzy weak hull operator, i.e., it satisfies (H1)–(H5) in Definition 2.4. Let $A,B\in L^Y,\ r,s\in M_{0_M}$.

(H1)

$$\mathcal{H}|_Y(0_Y,r)=id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\to}(0_Y),r))=id_Y^{\leftarrow}(\mathcal{H}(0_X,r))=id_Y^{\leftarrow}(0_X)=0_Y.$$

(H2)

$$A\le id_Y^{\leftarrow}(id_Y^{\to}(A))\le id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\to}(A),r))=\mathcal{H}|_Y(A,r).$$

(H3) If $r\le s$, then

$$\mathcal{H}|_Y(A,r)=id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\to}(A),r))\le id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\to}(A),s))=\mathcal{H}|_Y(A,s).$$

(H4) If $A\le B$, then similarly, we can obtain

$$\mathcal{H}|_Y(A,r)\le\mathcal{H}|_Y(B,r).$$

(H5) By (H2), $\mathcal{H}|_Y(A,r) \leq \mathcal{H}|_Y(\mathcal{H}|_Y(A,r),r)$. Conversely,

$$\begin{aligned}\mathcal{H}|_Y(\mathcal{H}|_Y(A,r),r) &= id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(\mathcal{H}|_Y(A,r)),r))\\ &= id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(A),r))),r))\\ &\leq id_Y^{\leftarrow}(\mathcal{H}(\mathcal{H}(id_Y^{\rightarrow}(A),r),r))\\ &= id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(A),r))\\ &= \mathcal{H}|_Y(A,r).\end{aligned}$$

Hence,

$$\mathcal{H}|_Y(\mathcal{H}|_Y(A,r),r) = \mathcal{H}|_Y(A,r).$$

Next, we prove that the multiplication

$$f : (Y,\cdot,\mathcal{H}|_Y) \times (Y,\cdot,\mathcal{H}|_Y) \to (Y,\cdot,\mathcal{H}|_Y), \ \ (x,y) \mapsto xy$$

is (L,M)-WHP, i.e.,

$$f^{\rightarrow}(\mathcal{H}|_Y \times \mathcal{H}|_Y(A,r)) \leq \mathcal{H}|_Y(f^{\rightarrow}(A),r), \forall (A,r) \in L^{Y\times Y} \times M_{0_M}.$$

Since $(X,\cdot,\mathcal{H})$ is an (L,M)-fuzzy weak hull group,

$$f_X : (X,\cdot,\mathcal{H}) \times (X,\cdot,\mathcal{H}) \to (X,\cdot,\mathcal{H}), \ \ (x',y') \mapsto x'y'$$

is (L,M)-WHP, so

$$f_X^{\rightarrow}(\mathcal{H} \times \mathcal{H}(B,r)) \leq \mathcal{H}(f_X^{\rightarrow}(B),r), \forall (B,r) \in L^{X\times X} \times M_{0_M}.$$

Let $B = (id_Y \times id_Y)^{\rightarrow}(A)$. Then,

$$\begin{aligned}&f^{\rightarrow}(\mathcal{H}|_Y \times \mathcal{H}|_Y(A,r))\\ &= f^{\rightarrow}(p_1^{\leftarrow}(\mathcal{H}|_Y(p_1^{\rightarrow}(A),r)) \wedge p_2^{\leftarrow}(\mathcal{H}|_Y(p_2^{\rightarrow}(A),r)))\\ &= f^{\rightarrow}(p_1^{\leftarrow}(id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(p_1^{\rightarrow}(A)),r))) \wedge p_2^{\leftarrow}(id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(p_2^{\rightarrow}(A)),r))))\\ &\leq id_Y^{\leftarrow}(f_X^{\rightarrow}(\mathcal{H} \times \mathcal{H}((id_Y \times id_Y)^{\rightarrow}(A),r)))\\ &\leq id_Y^{\leftarrow}(\mathcal{H}(f_X^{\rightarrow}((id_Y \times id_Y)^{\rightarrow}(A)),r))\\ &= id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(f^{\rightarrow}(A)),r))\\ &= \mathcal{H}|_Y(f^{\rightarrow}(A),r).\end{aligned}$$

Finally, we prove that the inverse mapping

$$g : (Y,\cdot,\mathcal{H}|_Y) \to (Y,\cdot,\mathcal{H}|_Y), \ \ x \mapsto x^{-1}$$

is (L,M)-WHP, i.e.,

$$g^{\rightarrow}(\mathcal{H}|_Y(A,r)) \leq \mathcal{H}|_Y(g^{\rightarrow}(A),r), \forall (A,r) \in L^Y \times M_{0_M}.$$

Since $(X,\cdot,\mathcal{H})$ is an (L,M)-fuzzy weak hull group,

$$g_X : (X,\cdot,\mathcal{H}) \to (X,\cdot,\mathcal{H}), \ \ x' \mapsto x'^{-1}$$

is (L, M)-WHP, so

$$g_X^{\rightarrow}(\mathcal{H}(B, r)) \leq \mathcal{H}(g_X^{\rightarrow}(B), r), \forall (B, r) \in L^X \times M_{0_M}.$$

Let $B = id_Y^{\rightarrow}(A)$. Then,

$$\begin{aligned} g^{\rightarrow}(\mathcal{H}|_Y(A, r)) &= g^{\rightarrow}(id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(A), r))) \\ &= id_Y^{\leftarrow}(g_X^{\rightarrow}(\mathcal{H}(id_Y^{\rightarrow}(A), r))) \\ &\leq id_Y^{\leftarrow}(\mathcal{H}(g_X^{\rightarrow}(id_Y^{\rightarrow}(A)), r)) \\ &= id_Y^{\leftarrow}(\mathcal{H}(id_Y^{\rightarrow}(g^{\rightarrow}(A)), r)) \\ &= \mathcal{H}|_Y(g^{\rightarrow}(A), r). \end{aligned}$$

Therefore, $(Y, \cdot, \mathcal{H}|_Y)$ is an (L, M)-fuzzy weak hull group. □

Theorem 4.5. (L, M)-**WHPG** *is topological over* **Grp**.

Proof. Let $\{(X_j, \cdot, \mathcal{H}_j)\}_{j\in\Gamma}$ be a family of (L, M)-fuzzy weak hull groups, $(X, \cdot)$ a group, and $\{f_j : (X, \cdot) \to (X_j, \cdot, \mathcal{H}_j)\}$ a family of group homomorphisms. Define $\mathcal{H}^* : L^X \times M_{0_M} \to L^X$ by

$$\mathcal{H}^*(A, r) = \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(A), r)), \quad \forall (A, r) \in L^X \times M_{0_M}.$$

By Lemma 2.5(i), $\mathcal{H}^*$ is the coarsest (L, M)-fuzzy weak hull operator on X such that each f_j is (L, M)-WHP.

Next, we show that the multiplication

$$f : (X, \cdot, \mathcal{H}^*) \times (X, \cdot, \mathcal{H}^*) \to (X, \cdot, \mathcal{H}^*), \ \ (x, y) \mapsto xy$$

is (L, M)-WHP, i.e.,

$$f^{\rightarrow}((\mathcal{H}^* \times \mathcal{H}^*)(A, r)) \leq \mathcal{H}^*(f^{\rightarrow}(A), r), \forall\, A \in L^{X\times X}, \ r \in M_{0_M}.$$

For each $j \in \Gamma$,

$$f^j : (X_j, \cdot, \mathcal{H}_j) \times (X_j, \cdot, \mathcal{H}_j) \to (X_j, \cdot, \mathcal{H}_j), \ \ (x_j, y_j) \mapsto x_j y_j$$

is (L, M)-WHP, so

$$f^{j\rightarrow}((\mathcal{H}_j \times \mathcal{H}_j)(B, r)) \leq \mathcal{H}_j(f^{j\rightarrow}(B), r), \forall\, B \in L^{X_j\times X_j}, \ r \in M_{0_M}.$$

Let $B = (f_j \times f_j)^{\rightarrow}(A)$. Then,

$$\begin{aligned} & f^{\rightarrow}((\mathcal{H}^* \times \mathcal{H}^*)(A, r)) \\ &= f^{\rightarrow}(p_1^{\leftarrow}(\mathcal{H}^*(p_1^{\rightarrow}(A), r)) \wedge p_2^{\leftarrow}(\mathcal{H}^*(p_2^{\rightarrow}(A), r))) \\ &= f^{\rightarrow}\Big(p_1^{\leftarrow}\Big(\bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(p_1^{\rightarrow}(A)), r))\Big) \wedge p_2^{\leftarrow}\Big(\bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(p_2^{\rightarrow}(A)), r))\Big)\Big) \\ &\leq \bigwedge_{j\in\Gamma} f^{\rightarrow}(p_1^{\leftarrow}(f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(p_1^{\rightarrow}(A)), r))) \wedge p_2^{\leftarrow}(f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(p_2^{\rightarrow}(A)), r)))) \end{aligned}$$

$$
\begin{aligned}
&\leq \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(f^{j\rightarrow}((\mathcal{H}_j\times\mathcal{H}_j)((f_j\times f_j)^{\rightarrow}(A),r)))\\
&\leq \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f^{j\rightarrow}((f_j\times f_j)^{\rightarrow}(A)),r))\\
&= \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(f^{\rightarrow}(A)),r))\\
&= \mathcal{H}^*(f^{\rightarrow}(A),r).
\end{aligned}
$$

Similarly, the inverse mapping

$$g:(X,\cdot,\mathcal{H}^*)\to(X,\cdot,\mathcal{H}^*),\ \ x\mapsto x^{-1}$$

is (L,M)-WHP, i.e.,

$$g^{\rightarrow}(\mathcal{H}^*(A,r))\leq\mathcal{H}^*(g^{\rightarrow}(A),r),\quad \forall(A,r)\in L^X\times M_{0_M}.$$

For each $j\in\Gamma$,

$$g_j:(X_j,\cdot,\mathcal{H}_j)\to(X_j,\cdot,\mathcal{H}_j),\ \ x_j\mapsto x_j^{-1}$$

is (L,M)-WHP, so

$$g_j^{\rightarrow}(\mathcal{H}_j(B,r))\leq\mathcal{H}_j(g_j^{\rightarrow}(B),r),\quad \forall(B,r)\in L^{X_j}\times M_{0_M}.$$

Let $B=f_j^{\rightarrow}(A)$. Then,

$$
\begin{aligned}
g^{\rightarrow}(\mathcal{H}^*(A,r)) &= g^{\rightarrow}\Big(\bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(A),r))\Big)\\
&\leq \bigwedge_{j\in\Gamma} g^{\rightarrow}(f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(A),r)))\\
&= \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(g_j^{\rightarrow}(\mathcal{H}_j(f_j^{\rightarrow}(A),r)))\\
&\leq \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(g_j^{\rightarrow}(f_j^{\rightarrow}(A)),r))\\
&= \bigwedge_{j\in\Gamma} f_j^{\leftarrow}(\mathcal{H}_j(f_j^{\rightarrow}(g^{\rightarrow}(A)),r))\\
&= \mathcal{H}^*(g^{\rightarrow}(A),r).
\end{aligned}
$$

Hence, $(X,\cdot,\mathcal{H}^*)$ is an (L,M)-fuzzy weak hull group.

Finally, by Lemma 2.5(ii), if $(Y,\cdot,\mathcal{H}_Y)$ is an (L,M)-fuzzy weak hull group and $g:(Y,\cdot,\mathcal{H}_Y)\to(X,\cdot,\mathcal{H}^*)$ is a group homomorphism, then g is (L,M)-WHP if and only if $f_j\circ g$ is (L,M)-WHP for every $j\in\Gamma$.

Therefore, (L,M)-**WHPG** is topological over **Grp**. □

Corollary 4.6. *If* $\{(X_j,\cdot,\mathcal{H}_j)\}_{j\in\Gamma}$ *is a family of* (L,M)*-fuzzy weak hull groups, let* $X=\prod_{j\in\Gamma}X_j$. *Then the product*

$$\Big(X,\ \cdot,\ \prod_{j\in\Gamma}\mathcal{H}_j\Big)$$

is also an (L, M)*-fuzzy weak hull group. Moreover, for all* $(A, r) \in L^X \times M_{0_M}$*,*

$$\Big(\prod_{j\in\Gamma} \mathcal{H}_j\Big)(A, r) = \bigwedge_{j\in\Gamma} p_j^{\leftarrow}(\mathcal{H}_j(p_j^{\rightarrow}(A), r)).$$

5. Conclusions

Based on the fundamental concept of the fuzzy weak hull operator, this paper carries out a systematic and in-depth investigation into its relevant algebraic structures and categorical properties. Firstly, the notion of the (L, M)-fuzzy quasi-hull operator is introduced, and a rigorous proof is provided to demonstrate that there exists a categorical isomorphism between the category of (L, M)-fuzzy quasi-hull spaces and the category of (L, M)-fuzzy supratopological spaces. Secondly, the definition of the (L, M)-fuzzy weak hull group is established, and the structural characteristics of its subgroups and product groups are further analyzed and discussed in detail. Finally, it is strictly proved that the category formed by all (L, M)-fuzzy weak hull groups is precisely a topological category over the category of classical groups.

Author contributions

Runmei Shang: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review and editing; Hu Zhao: Conceptualization, Methodology, Validation, Writing-review and editing, Supervision, Funding Acquisition; Xiongwei Zhang: Writing-review and editing, Funding Acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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