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*Research article*

# Solution continuous dependence of novel Caputo–Hadamard type fuzzy fractional partial differential coupled systems with applications

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**Abstract:** Our purpose of this paper was to investigate a class of novel Caputo–Hadamard type fuzzy fractional partial differential coupled systems with generalized Hukuhara difference and integral boundary conditions. We proposed properties of the solution, including its existence and uniqueness, continuous dependence on the initial conditions, and chaotic behavior in specific cases. Using Banach fixed-point theorem, we established the existence and uniqueness theorems of solutions for the partial differential coupled systems, and subsequently discussed continuous dependence of the solutions on initial conditions. Furthermore, a numerical example is presented to validate the major conclusions. The local solution ehibited chaotic behavior, which was accompanied by a corresponding circuit implementation. Finally, the existence and uniqueness of the solution for a novel fuzzy projection neural network system were established.

**Keywords:** existence and uniqueness theorem; solution continuous dependence; chaotic behavior of the solution; novel Caputo–Hadamard generalized Hukuhara type fuzzy fractional partial differential coupled system; Banach fixed-point theorem; projection neural network
**Mathematics Subject Classification:** 34A12, 35R11, 35R13, 47H10

## Abbreviations

H-difference: Hukuhara difference; gH: generalized Hukuhara; FDE: fractional-order differential equation; PDE: partial differential equation; FPDE: fractional partial differential equation; FFPDE: fuzzy fractional partial differential equation; CH-gH: Caputo–Hadamard gH

## 1. Introduction

Optimization problems are often encountered in the engineering, management science, and finance industries (see [1]). Constrained optimization problems are equivalent to projection neural network systems, and further equivalent to solving systems of equations. Constrained optimization problems are essentially solving equations. Abdulkadirov et al. [2] proposed modifications to the first, second, and information geometry order optimization algorithms related to Fisher-Rao and Bregman metrics. These optimizers have had a significant impact on the development of neural networks through geometric and probabilistic tools. Summers and Dinneen [3] established an experimental protocol to understand the impact of optimizing uncertainty on model diversity, enabling the isolation of the effects of non deterministic sources. Eshaghnezhad et al. [4] discussed the nonlinear projection equation:

$$y = P_{\overline{\Omega}}[y - \beta F(y)],$$

here $\overline{\Omega}$ is a closed convex set with $\overline{\Omega} = \{y \in \mathbb{R}^n | \overline{d}_i \le y_i \le \widetilde{d}_i, \forall i \in \bar{I} \subseteq \mathbb{N}\} \subseteq \mathbb{R}^n$, $\mathbb{N} = \{1, 2, \ldots, n\}$, $\alpha > 0$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function. Moreover, $P_{\overline{\Omega}}(\cdot)$ is

$$P_{\Omega}(y) = \arg\min_{x \in \Omega} \| y - x \| .$$

In recent years, increasing attention has been devoted to projection operators under fuzzy environments. In particular, researchers from India introduced a 2–fuzzy 2–metric projection operator $P_{\mathfrak{L}} : \mathfrak{I}(X) \to \mathfrak{L}$ (see [5]):

$$P_{\mathfrak{L}}(f) = \{g \in \mathfrak{L} : M(f, g, h, t) = \inf_{g \in \mathfrak{L}} \{\inf\{t : N(f - g, h, t) \ge \alpha, \alpha \in (0, 1)\}\}\}.$$

On the other hand, Wu et al. [6] incorporated the following projection operator: $P_K : R^n \to K$:

$$P_K[x] = \{\xi \in K : \|x - \xi\| = \min_{v \in K} \|x - v\|\}$$

into the fuzzy fractional differential inclusion system. Moreover, Dawood and Jabur [7] investigated fuzzy soft projection operators and fuzzy soft orthogonal projection operators. Additionally, Kavari et al. [8] employed $r$-cuts to project vectors onto fuzzy sets, which effectively solves constrained optimization problems within the $r$-cuts domain. These researchers collectively demonstrate the significant importance of developing fuzzy projection operators for solving optimization problems.

Coupling serves as an effective tool for describing species competition processes in natural ecosystems. As noted by Ding et al. [9], coupling mechanisms are capable of characterizing the interactions between two competing species in ecological systems. A representative example of such a coupled structure is the following elliptic system (see [10]):

$$\begin{cases} \Delta u_k = \varphi_{k_1}(x, u_k, v_k) & \text{in } \Omega, \\ \Delta v_k = \varphi_{k_2}(x, u_k, v_k) & \text{in } \Omega, \\ u_k = v_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\varphi_{k_1}, \varphi_{k_2} \in C(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, $\Omega \in \mathbb{R}^n (n > 3)$ is a smooth bounded domain.

Wu et al. [6] proposed that $P_K$ establishes the equivalence between variational inequalities and fixed-point problems. Consequently, solving the equilibrium point of a projection neural network is equivalent to solving a variational inequality system [11]. Moreover, Zhang et al. [11] proposed a class of fuzzy fractional-order partial differential coupled projection neural network systems

$$\begin{cases} {}_{gH}^{C}\mathcal{D}_k^{\alpha} u(x,y) = -u(x,y) + P_{F(D)}(m(x,y)v(x,y)), \\ {}_{gH}^{C}\mathcal{D}_k^{\beta} v(x,y) = \hat{g}(x,y,u(x,y)), \\ u(x,0) = \xi_1(x), v(x,0) = \eta_1(x), \\ u(0,y) = \xi_2(y), v(0,y) = \eta_2(y). \end{cases} \tag{1.2}$$

Here, ${}_{gH}^{C}\mathcal{D}_k^{\alpha}$ denotes the Caputo generalized Hukuhara (gH-type) derivative operator, $m(x)$, $u(x)$, and $v(x)$ are continuous fuzzy functions defined on $J = [0,a] \times [0,b]$, where $\alpha \in (0,1]$ and $\beta \in (0,1]$ represent the orders of the derivatives; and the initial value functions $\xi_1(x)$, $\eta_1(x) \in [0,a]$, and $\xi_2(y)$, $\eta_2(x) \in [0,b]$ are all continuous. It is noteworthy that (1.2) is derived from

$$\begin{cases} {}_{gH}^{C}D_k^{\alpha} u(x,y) = \widehat{f}(x,y,u(x,y),v(x,y)), \\ {}_{gH}^{C}D_k^{\beta} v(x,y) = \widehat{g}(x,y,u(x,y)), \\ u(x,0) = \xi_1(x),\ v(x,0) = \eta_1(x), \\ u(0,y) = \xi_2(y),\ v(0,y) = \eta_2(y), \end{cases} \tag{1.3}$$

which was proposed by Zhang et al. [11]. (1.3) modifies the form of the derivative on the left-hand side of (1.1), incorporates a fuzzy environment, and alters the initial conditions. Its formulation provides a concrete and feasible mathematical model for addressing fuzzy optimal control problems. In essence, fuzzy fractional-order partial differential coupled projection neural network systems (1.2) can be regarded as an extension of the biological competition coupled model (1.1), thereby providing a concrete and feasible mathematical framework for addressing fuzzy optimal control problems.

By introducing a fuzzy projection operator into the second equation on the right-hand side of (1.2), modifying $\xi_1(x)$, $\eta_1(x)$, $\xi_2(y)$, and $\eta_2(x)$, as integral boundary conditions, and changing the derivative type to ${}_{gH}^{CH}D_k^{\theta}$, ${}_{gH}^{CH}D_k^{\theta}$, we obtain

$$\begin{cases} {}_{gH}^{CH}D_k^{\theta} u(x,y) = P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)v(x,y)), \\ {}_{gH}^{CH}D_k^{\vartheta} v(x,y) = P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u(x,y)), \\ u(x,1) = \lambda \int_1^x \int_1^e u(s,t)\frac{ds}{s}\frac{dt}{t} + \xi_1(x), \\ u(1,y) = \lambda \int_1^e \int_1^y u(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_2(y), \\ v(x,1) = \eta_1(x), v(1,y) = \eta_2(y), \end{cases} \tag{1.4}$$

here $\mathbb{F}_\iota(\mathbb{D}_\iota) \subset \mathbb{F}_\iota$ denotes the set of all fuzzy numbers on $\mathbb{D}_\iota \subset \mathbb{R}$ that are Holder continuous, $P_{\mathbb{F}_\iota(\mathbb{D}_\iota)}$ : $\mathbb{F}_\iota \to \mathbb{F}_\iota(\mathbb{D}_\iota)$ is the fuzzy projection operator, and $m_1$, $m_2$ are continuous real-valued functions defined on $\mathcal{K} = [1,p] \times [1,q]$, $\iota \in \{1,2\}$. For $k \in \{1,2\}$, ${}_{gH}^{CH}D_k^{\theta}$ is a Caputo–Hadamard generalized Hukuhara (CH-gH) type differentiable operator (see similarly Definition 7 in [34]) and the order $\theta = (\theta_1, \theta_2) \in (0,1] \times (0,1]$, $\vartheta = (\vartheta_1, \vartheta_2) \in (0,1] \times (0,1]$, $u \in \mathbb{C}_{gH}^{2,2}(\mathcal{K}, \mathbb{F}_1) \triangleq \{\mathfrak{u} : \mathcal{K} \to \mathbb{F}_1 \mid \mathfrak{u}$ is gH-type partially differentiable}, and $\eta_k$ and $\xi_k$ are fuzzy-valued continuous functions. Motivated by (1.2), we pose the following question (i.e., Question 1):

**Question 1.** *Zhang et al.* [11] *investigated existence and uniqueness of solutions for* (1.2). *An interesting and worthy research question is whether the proposed system* (1.4) *also admits an existence and uniqueness result?*

To better address Question 1, we replace the right-hand side of (1.4) with more general functions $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1) \triangleq \{\mathfrak{f} : \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_2) \to \mathbb{F}_1 \mid \mathfrak{f} \text{ is jointly continuous}\}$ and $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$, $C(\mathcal{K}, \mathbb{F}_2)$ denotes the space of continuous functions from $\mathcal{K}$ to $\mathbb{F}_2$. Consequently, (1.4) is extended to the following fuzzy fractional partial differential equation (FFPDE):

$$\begin{cases} {}^{CH}_{gH}D^{\theta}_{k}u(x,y) = f(x,y,v(x,y)), \\ {}^{CH}_{gH}D^{\vartheta}_{k}v(x,y) = g(x,y,u(x,y)), \\ u(x,1) = \lambda \displaystyle\int_1^x \int_1^e u(s,t)\frac{ds}{s}\frac{dt}{t} + \xi_1(x), \\ u(1,y) = \lambda \displaystyle\int_1^e \int_1^y u(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_2(y), \\ v(x,1) = \eta_1(x), v(1,y) = \eta_2(y) \end{cases} \tag{1.5}$$

with integral boundary conditions and CH-gH type differentiable operator ${}^{CH}_{gH}D^{\theta}_{k}$.

**Remark 1.** (i) *Given that CH-type derivative is more suitable for modeling slow diffusion phenomena* [12]*, we incorporate it into the framework established by Zhang et al.* [13]. *This approach not only accounts for the competitive relationships among species within ecosystems, but also acknowledges that species do not always experience rapid growth. Due to natural constraints, species may undergo slower developmental phases, for which CH-type derivative provides an adequate description.*

(ii) *Similarly, in the dynamic behavior of ecosystems, species are influenced not only by the current environment but also by historical environmental conditions. Integral boundary conditions are effective in describing the population dynamics of ecosystems* [14]. *Therefore, building on the work of Zhang et al.* [13]*, we further integrate integral boundary conditions into FFPDEs to enhance the ability of FFPDEs to accurately represent the fundamental laws of nature.*

(iii) *For complex neural systems* (*e.g., those involving dynamic memory and time-varying propagation mechanisms* )*, the CH-type derivative, with its logarithmic kernel that inherently models multiplicative processes with geometric scaling, offers significant advantages over other fractional derivatives* [15].

(iv) *When* $\lambda = 0$ *and the derivative is of Caputo type in* (1.5)*, the system* (1.5) *degenerates to the initial value problem due to Zhang et al.* [13].

Thus, the relationships among (1.1)–(1.5) can be illustrated more intuitively in Figure 1.

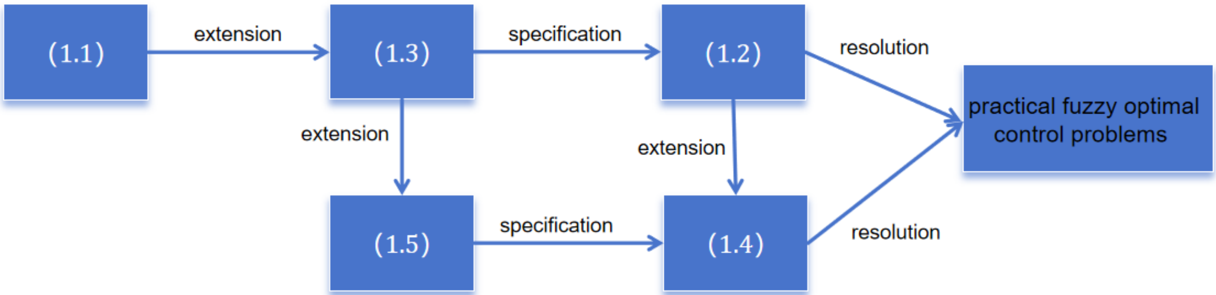


**Figure 1.** A diagram of the relationships among (1.1)–(1.5).

In fact, in the description of complex real-world dynamic systems, traditional integer-order derivatives are limited to capturing local variations. In contrast, fractional-order differential equations (FDEs) offer a more powerful tool to capture complex real-world dynamics, owing to their inherent memory, hereditary characteristics and long-range correlations. Sun et al. [16] presented a series of concise summaries authored by prominent scholars in the field of fractional calculus, and highlighted the powerful functionality of FDEs in real-world applications. Hattaf [17] and Jannelli [18] studied the qualitative properties of solutions to FDEs, which include stability, asymptotic stability, and Mittag–Leffler stability, and applied these results to a nonlinear biological system arising from epidemiology as well as to problems in engineering sciences. Using the Riemann–Liouville derivative, Alzaidy [19] derived analytical solutions for the space-time fractional modified Benjamin–Bona–Mahony equation and the space-time fractional Zakharov–Kuznetsov–Benjamin–Bona–Mahony equation. As an effective approach for numerically solving FDEs, the finite difference method has attracted widespread attention. Chen et al. [20] investigated numerical methods for two-dimensional integro-differential equations with two fractional Riemann-Liouville kernels. In the following year, Liu et al. [21] proposed a new compact Crank–Nicolson alternating-direction implicit (ADI) difference scheme, which is applicable to three-dimensional nonlinear partial integro-differential equations with multiple weakly singular kernels. Similarly, in the field of chemical dynamics, Farman et al. [22] developed a fractional-order model that holds irreplaceable significance for systems such as brine pool cascades and regenerative brine pool cascades. Given the strong memory characteristics inherent in real financial data, fractional-order derivative models are particularly well-suited for such applications. Najafi et al. [23] considered a fractional Liouville geometric model and calibrated its parameters to forecast the future value of market assets. It is evident that FDEs have permeated fields such as physical engineering [16, 18, 19], epidemiological biology [17, 18], chemical dynamics [22], and finance [23], and have offered extensive application potential and significant practical value.

For solution systems of FDEs, research has extended beyond existence and uniqueness. Researchers have investigated qualitative properties from multiple perspectives: Choi and Koo [24] studied the monotonicity and stability of solutions to FDEs. Khan and Ahmad [25] examined the boundedness of FDE solutions. Cong et al. [26] analyzed the asymptotic behavior of solutions to fractional differential equations. Based on the above considerations of the qualitative properties of solutions to FDEs, this, in turn, gives rise to another question (i.e., Question 2) as follows:

**Question 2.** *Beyond the existence and uniqueness of solutions, it is natural to ask whether the proposed FFPDE* (1.5) *possesses other qualitative properties, such as continuous dependence on* *initial conditions. Moreover, under certain conditions, can the solutions of* (1.5) *exhibit chaotic behavior, and if so, is it possible to realize via an appropriate circuit implementation?*

It is worth noting that capturing real-world phenomena solely from a single-variable perspective often fails to provide a comprehensive representation. Practical problems like fluid dynamics, electromagnetism, gravity, and quantum mechanics often involve complex interactions between variables. Accordingly, partial differential equations (PDEs), as a natural extension of classical ordinary differential equations, serve as a more appropriate framework for describing such complex interactions in real-world applications [27]. Fractional partial differential equations (FPDEs), which integrate the strengths of PDEs and FDEs, afford a powerful modeling framework that combines multi-variable perspectives with inherent properties like heredity, memory and long-range correlations. This unique blend greatly enhances their ability to simulate complex, real-world systems and anomalous phenomena in nature. FPDEs have been proven to be particularly sharp tools for modeling anomalous diffusion processes, such as ultra-slow (subdiffusion) and ultra-fast (superdiffusion) behaviors [28]. In exploring the solvability of FPDEs, many researchers have approached from two perspectives: Obtaining exact analytical solutions and deriving approximate numerical solutions. Zheng and Wen [29] employed the fractional version of the known $(G'/G)$ method to obtain exact solutions of the following FDEs denoted by

$$D_{\xi}^{2\alpha}G(\xi) + \lambda D_{\xi}^{\alpha}G(\xi) + \mu G(\xi) = 0,$$

where $D_{\xi}^{\alpha}G(\xi)$ represents the Riemann–Liouville fractional derivative of order $\alpha$ for $G(\xi)$ with respect to $\xi$ and $\lambda, \mu \in \mathbb{R}$. Additionally, Jafari et al. [30] utilized the iterative Laplace transform method to derive exact solutions for the proposed system. However, it is regrettable that, in practical applications, FPDEs for which exact solutions are available remain limited. Thus, we note that the CH-type fractional derivative, which integrates the features of the Caputo fractional derivative and the Hadamard type, is particularly suitable for describing scenarios on the semi-axis with slow diffusion [31]. This derivative captures the non-local behavior of systems, and is it especially useful for modeling systems with memory effects. In certain natural phenomena, the dynamic characteristics of the system are more appropriately described using the CH-type fractional derivative, as its introduction enhances the accuracy and generality of the model [12]. Systems incorporating the CH-type derivative may also exhibit chaotic effects. Investigating the chaotic properties of CH-type derivatives holds significant importance. Leveraging the chaotic phenomena of CH-type derivatives in image encryption offers a notable advantage, as it enables the selection of a greater number of parameters as keys [32]. This indicates substantial practical value in the field of communications [33]. For further details, please refer to [34, 35] and references therein. However, research on the circuit implementation of systems with CH-type derivatives remains relatively scarce in the literature.

On the other hand, in real-world scenarios, numerous uncertain phenomena cannot be accurately described by precise numerical values. To address this ambiguity and represent the inherent uncertainty, Zadeh [36] introduced the theory of fuzzy sets in 1965. This theory uses membership functions to transform fuzzy uncertainty into a deterministic quantitative expression, which is a process known as defuzzification. The introduction of fuzzy sets has significantly advanced fields such as control theory, artificial intelligence, and intelligent decision-making [37]. Incorporating a fuzzy environment into traditional FPDEs typically involves using the gH- difference [38], which overcomes the limitation of the cancellation law in Minkowski addition (i.e., $A, B, C \in k_c^n$, and $k_c^n$

represents a non-empty, convex, and compact set. $A + B = B + C \nRightarrow A = C$) and addresses the deficiencies of the standard Hukuhara (H-) difference [39, 40]. This improves the ability of FPDEs to model practical problems and provids a better representation of uncertainty [40]. Consequently, FFPDEs have emerged. The differential transform method and fuzzy variational iteration method [41] are effective tools for obtaining the corresponding numerical solutions of FFPDEs. In [42], Viet et al. examined the fuzzy hyperbolic Darboux problem under the Caputo gH-derivative. Zhang et al. [13] extended the work of Viet et al. [42]. Zhang et al. [13] investigated the solvability, continuous dependence on initial conditions, and approximation properties of the subsequent FFPDE coupled systems. Moreover, Lin et al. [34] first integrated the CH-type derivative with gH-difference and introduced CH-gH type (mixed) partial differential operator to enhance its applicability to uncertainty modeling.

In the dynamic evolution of ecosystems, species are influenced not only by the current environmental conditions but also by the cumulative effects of past environmental factors. It is appropriate to characterize population dynamics using integral boundary conditions [14]. Additionally, integral boundary conditions are applicable in various fields such as chemical engineering [43], groundwater flow [44], and thermoplastic processes [45]. Ardjouni and Djoudi [35] explored the existence of positive solutions for the following systems with integral boundary conditions involving the CH-type derivative:

$$\begin{cases} D_1^{\alpha} u(x) = f(x, u(x)), 1 < x \le e, \\ u(1) = \lambda \displaystyle\int_1^e u(s)ds + d, \end{cases} \tag{1.6}$$

where $D_1^{\alpha}$ is a CH-type fractional derivative of order $0 < \alpha < 1$, and constants $\lambda \ge 0$ and $d > 0$. $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1) \triangleq \{\mathfrak{f} : \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_2) \to \mathbb{F}_1 \mid \mathfrak{f} \text{ is jointly continuous}\}$, and $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$, $C(\mathcal{K}, \mathbb{F}_2)$ denotes the space of continuous functions from $\mathcal{K}$ to $\mathbb{F}_2$.

Based on studies of FFPDE systems [11], along with the proposed CH-gH type differentiability functions and the CH-type vector form of the Gronwall inequality, our major contributions are as follows:

(i) We propose a novel CH-type FFPDE coupled system with integral boundary conditions. The inclusion of CH-type derivatives and integral boundary conditions enables (1.5) to not only describe the mutual competition among species, but also capture the influence of historical conditions and environmental carrying capacity. This significantly enhances its capability to represent natural phenomena.

(ii) We establish the existence and uniqueness theorems for (1.5) under different conditions by applying the Banach fixed point theorem. For different values of $\lambda$, we establish the existence and uniqueness Theorems 1, 2, 4, and 5 for various cases. In addition, based on the fuzzy projection of fuzzy functions, we present the application of the fuzzy projection neural network (1.4). Furthermore, under varying parameter configurations, we establish existence and uniqueness theorems for †-type and ‡-type solutions of (1.4). Moreover, Theorems 9, 10, 7, and 8, derived from Theorems 1, 2, 4, and 5, effectively address Question 1.

(iii) Besides the existence and uniqueness of the solution, we also investigate other properties of the solution. We derive the Gronwall inequality in the vector form involving Hadamard type integrals. Moreover, we establish the continuous dependence of the solution to (1.5) on the initial data (i.e., Theorem 3) by employing the generalized Gronwall inequality involving the H-integral, as developed in Lemma 5. Similarly, we also establish the continuous dependence of the solutions of the projection neural network system (1.4) on the initial conditions, as stated in Theorem 6. In the section on numerical examples, we conduct a Chaos analysis for the local solution systems and implement them using electrical circuits. This provides a satisfactory answer to a portion of the questions raised in Question 2.

The overall structure of this paper is organized as follows: In Section 2, we review fundamental concepts and properties and extend certain notions relevant to the CH-type derivative. In Section 3, we provide a detailed discussion on the existence and uniqueness of solutions for (1.5), as well as their continuous dependence on initial values. In Section 4, we present a numerical example, within whose solution system we identify iterative schemes exhibiting chaotic behavior. Based on these iterative schemes, we design and implement circuit configurations. Finally, the work conducted in this study is summarized in Section 5, along with potential avenues for future research.

## 2. Preliminaries

To effectively handle (1.5), in this section, we review key results related to fuzzy functions and differentiability, and introduce extended concepts rooted in CH-type derivative.

We begin by defining $\mathbb{F}$, $\mathbb{F}_1$ and $\mathbb{F}_2$ as fuzzy number spaces composed of mappings that fulfill the following conditions for all $\iota \in \mathbb{F}$ and any $m_1, m_2 \in \mathbb{R}$:

(i) $\iota(m_1) = 1$, in other words, $\iota$ is normal.

(ii) $\iota(\mathbf{r}m_1 + (1-\mathbf{r})m_2) \geq \min\{\iota(m_1), \iota(m_2)\}$ for all $\mathbf{r} \in [0,1]$, that is to say, $\iota$ is fuzzy convex.

(iii) $\iota$ is also upper semi-continuous.

(iv) $\text{supp}\iota \triangleq cl\{m_1 \in \mathbb{R} \mid \iota(m_1) > 0\}$ is compactly supported, here $cl$ denotes the closure of the sets.

**Definition 1.** *$d_\infty$ is the supremum measure defined on $\mathbb{F} \times \mathbb{F}$, which serves as an extension of the classical Hausdorff measure:*

$$d_\infty(\iota, \varrho) = \sup_{\gamma\in[0,1]} d_H([\iota]^\gamma, [\varrho]^\gamma) = \sup_{\gamma\in[0,1]} \max\left\{\left|\underline{\iota}_\gamma - \underline{\varrho}_\gamma\right|, \left|\overline{\iota}_\gamma - \overline{\varrho}_\gamma\right|\right\}, \tag{2.1}$$

*where $[\iota]^\gamma = [\underline{\iota}_\gamma, \overline{\iota}_\gamma]$ means the $\gamma$-level set of $\iota \in \mathbb{F}$ as*

$$[\iota]^\gamma = \begin{cases} \{\lambda \in \mathbb{R} | \iota(\lambda) \geq \gamma\}, & \text{for} \quad 0 < \gamma \leq 1, \\ cl(\text{supp}\ \iota), & \text{for} \quad \gamma = 0. \end{cases}$$

**Definition 2.** ([46]) *H-difference can be defined using the following relation for any $\varrho_1, \varrho_2 \in \mathbb{F}$:*

$$\varrho_1 \ominus \varrho_2 = \nu \Leftrightarrow \varrho_1 = \varrho_2 + \nu.$$

*Furthermore, the $\gamma$-level sets for the addition $\varrho_1 + \varrho$, scalar multiplication $a\rho$ and $\rho \ominus \varrho$ satisfy*

(i) $[\varrho_1 + \varrho_2]^\gamma = [\varrho_1]^\gamma + [\varrho_2]^\gamma = \{c + d | c \in [\varrho_1]^\gamma, d \in [\varrho_2]^\gamma\}$;

(ii) $[a\varrho_1]^\gamma = a[\varrho_1]^\gamma = \{ac | c \in [\varrho_1]^\gamma, a \in \mathbb{R}\}$;

(iii) $[\varrho_1 \ominus \varrho_2]^\gamma = [\underline{\varrho}_{1\gamma} - \underline{\varrho}_{2\gamma}, \overline{\varrho}_{1\gamma} - \overline{\varrho}_{2\gamma}]$.

**Lemma 1.** ([46]) *When $l_1, l_2, l_3, l_4 \in \mathbb{F}$, the following relationship is valid*
($a_1$) $d_\infty(l_1 + l_2, l_3 + l_4) \le d_\infty(l_1, l_3) + d_\infty(l_2, l_4)$;
($a_2$) *if* $l_1 \ominus (l_2 + l_3)$ *and* $l_1 \ominus l_2 \ominus l_3$ *exist, then* $l_1 \ominus (l_2 + l_3) = l_1 \ominus l_2 \ominus l_3$;
($a_3$) $d_\infty(l_1 \ominus l_2, l_3 \ominus l_4) \le d_\infty(l_1, l_3) + d_\infty(l_2, l_4)$ *when* $l_1 \ominus l_2, l_3 \ominus l_4$; *exist*
($a_4$) *if* $l_1 \ominus l_2$ *and* $(-1)l_1 \ominus (-1)l_2$ *both exist, then* $(-1)(l_1 \ominus l_2) = (-1)l_1 \ominus (-1)l_2$;
($a_5$) $l_1 \ominus (l_2 \ominus l_3) = l_1 \ominus l_2 + l_3$ *is valid if* $l_1 \ominus (l_2 \ominus l_3)$ *and* $l_1 \ominus l_2$ *both exist.*

**Definition 3.** ([46]) *For all $\varpi, \omega, \nu \in \mathbb{F}$, gH-difference is defined as follows*

$$\varpi \ominus_{gH} \omega = \nu \Longleftrightarrow \begin{cases} (\dagger)\ \varpi = \omega + \nu, \\ \text{or } (\ddagger)\ \omega = \varpi + (-1)\nu. \end{cases} \tag{2.2}$$

**Remark 2.** *If H-difference $\varpi \ominus \omega$ and gH-difference $\varpi \ominus_{gH} \omega$ exist simultaneously, and $\varpi \ominus \omega = \varpi \ominus_{gH} \omega$, while the deduction rules ($\dagger$) and ($\ddagger$) are satisfied, then $\psi$ is defined as a crisp quantity with $\varpi \ominus \omega = \varpi \ominus_{gH} \omega = 0$.*

**Definition 4.** *If, for $\hbar \in C(\mathcal{K}, \mathbb{F})$ (i.e., $\hbar : \mathcal{K} \to \mathbb{F}$ is a continuous function) and a point $(x_0, y_0) \in \mathcal{K}$, then $\hbar$ can be defined as gH-type partially differentiable at $x$ and is expressed by the following equation:*

$$\lim_{\epsilon \to 0} \frac{\hbar(x_0 + \epsilon, y_0) \ominus_{gH} \hbar(x_0, y_0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\hbar(x_0, y_0) \ominus_{gH} \hbar(x_0 - \epsilon, y_0)}{\epsilon} = \frac{\partial \hbar(x_0, y_0)}{\partial x},$$

*here, $\epsilon$ is a sufficiently small quantity, and it holds that $\frac{\partial \hbar(x_0, y_0)}{\partial x} \in \mathbb{F}$, $\hbar(x_0+\epsilon, y_0) \ominus_{gH} \hbar(x_0, y_0)$, $\hbar(x_0, y_0) \ominus_{gH} \hbar(x_0 - \epsilon, y_0) \in \mathcal{K}$.*

**Remark 3.** *Similar to Definition 4, we can also define gH-type partial differentiability with respect to y, as well as higher-order gH-type partial differentiability with respect to x, and y, respectively. However, these definitions will not be elaborated here. Furthermore, in the following definitions and properties, the existence of gH-type differences is assumed as a prerequisite.*

**Lemma 2.** *( [47] Banach fixed-point theorem) Let $\mathbb{S}$ be a complete metric space, which means that every Cauchy sequence in $\mathbb{S}$ converges to a point in $\mathbb{S}$. If $\Phi : \mathbb{S} \to \mathbb{S}$ is contraction mapping, then it admits a unique fixed point $x^* \in \mathbb{S}$ such that $\Phi(x^*) = x^*$.*

*A contraction mapping is defined as a function, satisfying*

$$d(\Phi(x), \hbar(y)) \le \varsigma \cdot d(x, y)$$

*for some constant $0 \le \varsigma < 1$ and all $x, y \in \mathbb{S}$, where $d$ is the metric on $\mathbb{S}$.*

**Remark 4.** *If $\Phi$ in Lemma 2 satisfies $d\left(\Phi^n(x), \Phi^n(y)\right) \le c_n d(x, y)$, where $\lim_{n\to\infty} c_n = 0$ and all other conditions remain unchanged as in Lemma 2, then we conclude that there exists a unique fixed point $x^* \in \mathbb{S}$ such that $\Phi(x^*) = x^*$. This represents an alternative formulation of Banach fixed-point theorem.*

**Definition 5.** *Let $\mathcal{K} = [1, p] \times [1, q]$, $u \in C(\mathcal{K}, \mathbb{F}_1) \cap \mathbb{L}(\mathcal{K}, \mathbb{F}_1)$, here $C(\mathcal{K}, \mathbb{F}_1)$ is the same as in Definition 4 and $\mathbb{L}(\mathcal{K}, \mathbb{F}_1) \triangleq \{u |\ u : \mathcal{K} \to \mathbb{F}_1)$ is Lebesgue integrable$\}$, and $v \in C(\mathcal{K}, \mathbb{F}_2) \cap \mathbb{L}(\mathcal{K}, \mathbb{F}_2)$. Then Hadamard fractional fuzzy integral operators of u and v can be severally defined by*

$${}^{H}_{F}\mathcal{I}^{\theta}_{1} u(x, y) = \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{\theta_1 - 1} \left(\ln \frac{y}{t}\right)^{\theta_2 - 1} u(s, t) \frac{dt}{t} \frac{ds}{s},$$

$$ {}^{H}_{F}\mathcal{I}^{\vartheta}_{1}v(x,y) = \frac{1}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{\vartheta_1 - 1} \left(\ln\frac{y}{t}\right)^{\vartheta_2 - 1} v(s,t)\frac{dt}{t}\frac{ds}{s}, $$

*where* $\theta = (\theta_1, \theta_2), \vartheta = (\vartheta_1, \vartheta_2) \in [0,1) \times [0,1)$ *are the orders. Moreover, for each* $(x,y) \in \mathcal{K}$, $\gamma_1$*-level set of* $u(x,y)$ *and* $\gamma_2$*-level set for* $v(x,y)$ *are denoted as* $[u(x,y)]^{\gamma_1} = [\underline{u}(x,y;\gamma_1), \overline{u}(x,y;\gamma_1)]$, $[v(x,y)]^{\gamma_2} = [\underline{v}(x,y;\gamma_2), \overline{v}(x,y;\gamma_2)]$*, and one gets*

$$ \left[{}^{H}_{F}\mathcal{I}^{\theta}_{1}u(x,y)\right]^{\gamma_1} = \left[{}^{H}_{F}\mathcal{I}^{\theta}_{1}\underline{u}(x,y;\gamma_1), {}^{H}_{F}\mathcal{I}^{\theta}_{1}\overline{u}(x,y;\gamma_1)\right] $$

*for* $\gamma_1$*-level set of the Hadamard fractional fuzzy integral operator* ${}^{H}_{F}\mathcal{I}^{\theta}_{1}u(x,y)$, ${}^{H}_{F}\mathcal{I}^{\theta}_{1}\underline{u}(x,y;\gamma_1)$ *is referred to as the left endpoint function, and* ${}^{H}_{F}\mathcal{I}^{\theta}_{1}\overline{u}(x,y;\gamma_1)$ *as the right endpoint function. Similarly, there are analogous results for* $\left[{}^{H}_{F}\mathcal{I}^{\vartheta}_{1}v(x,y)\right]^{\gamma_2} = \left[{}^{H}_{F}\mathcal{I}^{\vartheta}_{1}\underline{v}(x,y;\gamma_2), {}^{H}_{F}\mathcal{I}^{\vartheta}_{1}\overline{v}(x,y;\gamma_2)\right]$.

**Definition 6.** *If, for any* $\varepsilon > 0$*, there exist* $\delta_1$ *and* $\delta_2$ *such that for each* $(x,y,u) \in \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_1)$ *and* $(x,y,v) \in \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_2)$, $|x - x_0| + |y - y_0| + d_\infty(u,\psi) < \delta_1$ *and* $|x - x_0| + |y - y_0| + d_\infty(v,\varphi) < \delta_2$ *hold, then we define* $f : \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_2) \to \mathbb{F}_1$ *and* $g : \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_1) \to \mathbb{F}_2$ *to be jointly continuous at point* $(x_0, y_0, \varphi)$ *and* $(x_0, y_0, \psi)$ *if* $f$ *and* $g$ *satisfy the following conditions:*

$$ d_\infty(f(x,y,v), f(x_0,y_0,\varphi)) < \varepsilon, \quad d_\infty(g(x,y,u), g(x_0,y_0,\psi)) < \varepsilon, $$

*where* $\varphi(x,y)$ *and* $\psi(x,y)$ *are determined by*

$$ \psi(x,y) = \lambda \int_1^x \int_1^e u(s,t)\frac{ds}{s}\frac{dt}{t} + \lambda \int_1^e \int_1^y u(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_2(y) + \xi_1(x) \ominus \xi_1(0), \tag{2.3} $$

$$ \varphi(x,y) = \eta_2(y) + \eta_1(x) \ominus \eta_1(0). \tag{2.4} $$

Similarly, based on Remark 3, we know that $\xi_1(y) \ominus \xi_1(0)$ and $\eta_1(x) \ominus \eta_1(0)$ exist. In the following discussion, unless otherwise specified, $\psi(x,y)$ and $\varphi(x,y)$ are defined by (2.3) and (2.4), respectively. Furthermore, $\xi_1(x)$, $\int_1^x \int_1^e u(x,y)\frac{dx}{x}\frac{dy}{y} \in C([1,p], \mathbb{F}_1)$; $\xi_2(y)$, $\lambda \int_1^e \int_1^y u(x,y)\frac{dy}{y}\frac{dx}{x} \in C([1,q], \mathbb{F}_1)$; $\eta_1(x) \in C([1,p], \mathbb{F}_2)$; $\eta_2(y) \in C([1,q], \mathbb{F}_2)$. Subsequently, we denote

$$ \widetilde{\mathbb{C}}^{f}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_2) \triangleq \{\mathrm{v}(x,y) \in C(\mathcal{K}, \mathbb{F}_2) \,|\, \psi(x,y) \ominus (-1){}^{H}_{F}\mathcal{I}^{\theta}_{1}f(x,y,\mathrm{v}(x,y)) \text{ exists}, \forall (x,y) \in \mathcal{K}\}, \tag{2.5} $$

$$ \widetilde{\mathbb{C}}^{g}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_1) \triangleq \{\mathrm{u}(x,y) \in C(\mathcal{K}, \mathbb{F}_1) \,|\, \varphi(x,y) \ominus (-1){}^{H}_{F}\mathcal{I}^{\theta}_{1}g(x,y,\mathrm{u}(x,y)) \text{ exists}, \forall (x,y) \in \mathcal{K}\}. \tag{2.6} $$

Based on (2.1), we state the ultra-high measure $\rho$ as

$$ \rho(u(x,y), v(x,y)) = \sup_{(x,y)\in\mathcal{K}} d_\infty(u(x,y), v(x,y)). \tag{2.7} $$

Additionally, we clarify the weighted metric involving $w = (w_1, w_2) \in [0,1] \times [0,1]$ by

$$ d_w(u,v) = \sup_{(x,y)\in\mathcal{K}} \{(\ln x)^{w_1}(\ln y)^{w_2} d_\infty(u(x,y), v(x,y))\}, \tag{2.8} $$

and $C_K(\mathcal{K}, \mathbb{F}_{k_1}, \mathbb{F}_{k_2}) = \{f : \mathcal{K} \times C(\mathcal{K}, \mathbb{F}_{k_1}) \to \mathbb{F}_{k_2} \,|\, f \text{ is jointly continuous}\}$, $k_1, k_2 \in \{1,2\}$.

**Definition 7.** *Let* $\ell : \mathbb{R}^2 \to \mathbb{F}_i (i \in \{1,2\})$ *be a gH-type partially differentiable operator with the highest order of* $x$ *being* $m \in \{0,1,2\}$ *and the highest order of* $y$ *being* $n \in \{0,1,2\}$*. The set of all* *such $\ell$ is denoted by $\mathbb{C}^{m,n}_{gH}(\mathbb{R}^2, \mathbb{F}_i)$. Under $\theta = (\theta_1, \theta_2) \in (0, 1]$, $\vartheta = (\vartheta_1, \vartheta_2) \in (0, 1]$, $u \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$, $v \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$, for $k = 1, 2$ we separately define CH-gH type differentiability of u and v as*

$$
{}^{CH}_{gH}D^{\theta}_{k}u(x,y) = \frac{1}{\Gamma(1-\theta_1)\Gamma(1-\theta_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{-\theta_1} \left(\ln \frac{y}{t}\right)^{-\theta_2} \frac{\partial^2(u(s,t))}{\partial s \partial t} dtds,
$$

$$
{}^{CH}_{gH}D^{\vartheta}_{k}v(x,y) = \frac{1}{\Gamma(1-\vartheta_1)\Gamma(1-\vartheta_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{-\vartheta_1} \left(\ln \frac{y}{t}\right)^{-\vartheta_2} \frac{\partial^2(v(s,t))}{\partial s \partial t} dtds.
$$

(i) *We denote ${}^{CH}_{gH}D^{\theta}_{1}u(x,y)$ as the †-type $\theta$ order CH-gH differentiability with respect to $(x, y)$ (†-type CH-gH differentiable), when $k = 1$ and at the point $(s, t)$, $\frac{\partial^2(u(s,t))}{\partial s \partial t}$ fulfills the conditions of †-type gH-differentiability as in Definition* 3.

(ii) *If $k = 2$ and $\frac{\partial^2(u(s,t))}{\partial s \partial t}$ at the point $(s, t)$ satisfies the conditions of ‡-type gH-differentiability as defined in Definition* 3*, we denote ${}^{CH}_{gH}D^{\theta}_{2}u(x,y)$ as the ‡-type $\theta$ order CH-gH differentiability with respect to $(x, y)$ (‡-type CH-gH differentiable).*

**Remark 5.** *Due to the development of fractional calculus, various types of fractional derivatives have been introduced; for instance, the Riemann-Liouville type derivative, ${}^{RL}_{a}D^{\alpha}_{t}(\cdot)$, has been employed by Rametse et al. [48]. Murillo-Arcila et al. [49] employed the Caputo-type fractional derivative, denoted as ${}^{C}_{0}D^{\alpha}_{t}(\cdot)$. Zhang et al. [13] extended the classical Caputo fractional derivative and proposed the Caputo gH-type derivative operators ${}^{C}_{gH}\mathcal{D}^{\alpha}(\cdot)$. In recent years, attention has been devoted to the study of composite fractional derivatives. For instance, positive analysis was performed by Ardjouni [35] et al. based on the Caputo-Hadamard-type fractional derivative $\mathfrak{D}^{\alpha}_{1}(\cdot)$, while identification problems were examined by Kahouli et al. [50] using the Caputo-Katugampola derivative ${}^{c}D^{\alpha,\rho}_{a}$. Lin et al. [34] extended the classical Caputo-Hadamard-type fractional derivative and proposed the Caputo-Hadamard gH-type mixed partial differential operator ${}^{\theta}_{CH}D^{i,l}_{xy}(\cdot)$. Similarly, Jiang et al. [51] extended the classical Caputo-Katugampola derivative and introduced the Caputo-Katugampola gH-type derivative operator ${}^{CK}_{gH}\mathcal{D}^{\alpha,p}_{k}(\cdot)$.*

*Unlike the one-dimensional Riemann-Liouville derivative ${}^{RL}_{a}D^{\alpha}_{t}(\cdot)$, the Caputo derivative ${}^{C}_{0}D^{\alpha}_{t}(\cdot)$, the Caputo-Hadamard type fractional derivative $\mathfrak{D}^{\alpha}_{1}(\cdot)$, and the Caputo-Katugampola derivative ${}^{c}D^{\alpha,\rho}_{a}$, the CH-gH type differentiability derivative proposed in Definition 7 was extended to two variable functions, incorporating a fuzzy environment and modifying the form of the integral kernel.*

*For definitions involving two-variable functions, particularly those incorporating a fuzzy environment, unlike the ${}^{C}_{gH}\mathcal{D}^{\alpha}(\cdot)$ operator defined by Zhang et al. [13] and the ${}^{CK}_{gH}\mathcal{D}^{\alpha,p}_{k}(\cdot)$ operator defined by Jiang et al. [51], the integral kernel in Definition 7 was modified to adopt a logarithmic form. Differing from the ${}^{\theta}_{CH}D^{i,l}_{xy}(\cdot)$ derivative defined by Lin [34], here we consider only the case where $i = l = 1, 2$, that is, $k = 1, 2$ and $n = 1$. Moreover, in Definition 7, $\theta$ is a vector.*

**Lemma 3.** *Assume that we take $h_i \in C(\mathcal{K}, \mathbb{F}_i)$ $(i = 1, 2)$ as a continuous function, and the fuzzy functions*

$$
\overline{\mathscr{H}}_1(x,y) = \psi(x,y) +^{H}_{F} \mathcal{I}^{\vartheta}_{1} h_1(x,y),
$$
$$
\overline{\mathscr{H}}_2(x,y) = \varphi(x,y) \ominus^{H}_{F} \mathcal{I}^{\vartheta}_{1} h_2(x,y)
$$

*are †-type CH-gH differentiable and ‡-type CH-gH differentiable, respectively. Then it indicates that*

$$
{}^{CH}_{gH}D^{\vartheta}_{1}\overline{\mathscr{H}}_1(x,y) = h_1(x,y), \tag{2.9}
$$

$$ {}^{CH}_{gH}D_2^{\vartheta}\overline{\mathscr{H}}_2(x,y) = -h_2(x,y). \tag{2.10}$$

*Proof.* We provide a detailed explanation of (2.9). The expression for (2.10) can be obtained through a similar process, as detailed in Definition 2.1 of [52]. In fact, applying operator ${}^{CH}_{gH}D_1^{\vartheta}$ to $\overline{\mathscr{H}}_1(x,y)$, the following result yields:

$$\left[{}^{CH}_{gH}D_1^{\vartheta}\overline{\mathscr{H}}_1(x,y)\right]^{\gamma} = \left[{}^{CH}_{gH}D_1^{\vartheta}\left(\underline{\psi}(x,y;\gamma) + {}^{H}_{F}\mathcal{I}_1^{\vartheta}\underline{h_1}(x,y;\gamma)\right), {}^{CH}_{gH}D_1^{\vartheta}\left(\overline{\psi}(x,y;\gamma) + {}^{H}_{F}\mathcal{I}_1^{\vartheta}\overline{h_1}(x,y;\gamma)\right)\right],$$

from the fundamental theorem of fractional calculus given by Gambo et al. [53]. From Lemma 2 in [13], it follows that

$$\begin{aligned}&\left[{}^{CH}_{gH}D_1^{\vartheta}\left(\underline{\psi}(x,y;\gamma) + {}^{H}_{F}\mathcal{I}_1^{\vartheta}\underline{h_1}(x,y;\gamma)\right), {}^{CH}_{gH}D_1^{\vartheta}\left(\overline{\psi}(x,y;\gamma) + {}^{H}_{F}\mathcal{I}_1^{\vartheta}\overline{h_1}(x,y;\gamma)\right)\right] = \left[\underline{h_1}(x,y;\gamma), \overline{h_1}(x,y;\gamma)\right]\\ &= [h_1(x,y)]^{\gamma}.\end{aligned}$$

Thus,

$$ {}^{CH}_{gH}D_1^{\vartheta}\overline{\mathscr{H}}_1(x,y) = h_1(x,y).$$

Hence, the proof is completed. □

The symbol □ above indicates the completion of the proof. Similar symbols will also appear after subsequent proofs. Similar to [54], the following Lemma 4 provides an integral system that is equivalent to (1.5), and the subsequent theorem demonstrates that this integral system indeed represents the solution system of the original (1.5).

**Lemma 4.** *Let $f : \mathcal{K} \times \mathbb{F}_2 \to \mathbb{F}_1$ and be a continuous function, $g \in: \mathcal{K} \times \mathbb{F}_1 \to \mathbb{F}_2$, and $u$, $v$ be fuzzy functions. Then system* (1.5) *can be transformed into the following equivalent Volterra integral coupled system:*

(i) *When $k = 1$,*

$$\begin{cases} u(x,y) = \psi(x,y) +{}^{H}_{F}\mathcal{I}_1^{\theta} f(x,y,v(x,y)) \\ v(x,y) = \varphi(x,y) +{}^{H}_{F}\mathcal{I}_1^{\vartheta} g(x,y,u(x,y)); \end{cases} \tag{2.11}$$

(ii) *When $k = 2$,*

$$\begin{cases} u(x,y) = \psi(x,y) \ominus (-1){}^{H}_{F}\mathcal{I}_1^{\theta} f(x,y,v(x,y)) \\ v(x,y) = \varphi(x,y) \ominus (-1){}^{H}_{F}\mathcal{I}_1^{\vartheta} g(x,y,u(x,y)). \end{cases} \tag{2.12}$$

*Proof.* "⇒": Set $\omega(x,y) = {}^{CH}_{gH}D_k^{\theta}u(x,y)$, then

$$ {}^{H}_{F}\mathcal{I}_1^{\theta}[\omega](x,y) = \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{\theta_1-1}\left(\ln\frac{y}{t}\right)^{\theta_2-1}\omega(s,t)\frac{dt}{t}\frac{ds}{s}.$$

Furthermore, we have

$$\omega(s,t) = \frac{1}{\Gamma(1-\theta_1)\Gamma(1-\theta_2)}\int_1^s\int_1^t \left(\ln\frac{s}{\xi}\right)^{-\theta_1}\left(\ln\frac{t}{\eta}\right)^{-\theta_2}\frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta}d\eta d\xi.$$

Thus, we obtain

$$ {}^{F}_{H}\mathcal{I}_1^{\theta}[\omega](x,y) = \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(1-\theta_1)\Gamma(1-\theta_2)}\int_1^x\int_1^y \left(\ln\frac{x}{s}\right)^{\theta_1-1}\left(\ln\frac{y}{t}\right)^{\theta_2-1}$$

$$\times\left[\int_1^s\int_1^t\left(\ln\frac{s}{\xi}\right)^{-\theta_1}\left(\ln\frac{t}{\eta}\right)^{-\theta_2}\frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta}d\eta\,d\xi\right]\frac{dt\,ds}{t\,s}.$$

By applying Fubini theorem, we obtain

$$\begin{aligned}{}^F_H\mathcal{I}^\theta_1[\omega](x,y) &= \frac{1}{\Gamma(\theta_1)\Gamma(1-\theta_1)\Gamma(\theta_2)\Gamma(1-\theta_2)}\int_1^x\int_1^y\frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta}\\ &\times\left[\int_\xi^x\left(\ln\frac{x}{s}\right)^{\theta_1-1}\left(\ln\frac{s}{\xi}\right)^{-\theta_1}\frac{ds}{s}\right]\left[\int_\eta^y\left(\ln\frac{y}{t}\right)^{\theta_2-1}\left(\ln\frac{t}{\eta}\right)^{-\theta_2}\frac{dt}{t}\right]d\eta\,d\xi.\end{aligned}$$

Consequently, invoking the Beta function yields:

$${}^F_H\mathcal{I}^\theta_1\,{}^{CH}_{gH}D^\theta_k u(x,y)=\int_1^x\int_1^y\frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta}d\eta\,d\xi.$$

Case I: $k=1$, we have

$$\int_1^x\left(\int_1^y\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right)d\eta\right)d\xi=\int_1^x\left(\frac{\partial u}{\partial\xi}(\xi,\eta)\ominus\frac{\partial u}{\partial\xi}(\xi,1)\right)d\xi.$$

Thus, we have

$$\int_1^x\left[\frac{\partial u}{\partial s}(s,\eta)\ominus\frac{\partial u}{\partial s}(s,1)\right]ds={}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y))$$

or

$$\int_1^x\frac{\partial u}{\partial s}(s,\eta)ds=\int_1^x\frac{\partial u}{\partial s}(s,1)ds+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y)).$$

If $u(x,y)$ is †-type gH-differentiable as mentioned in reference [42], then we have

$$u(x,y)\ominus u(1,y)=u(x,1)\ominus u(1,1)+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y))$$

or

$$u(x,y)=u(1,y)+[u(x,1)\ominus u(1,1)]+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y)).$$

Finally, one can get

$$u(x,y)=\psi(x,y)+^H_F\mathcal{I}^\theta_1 f(x,y,v(x,y)).$$

On the other hand, if $u(x,y)$ is ‡-type gH-differentiable as mentioned in reference [42], then we have

$$\begin{aligned}&u(1,1)\ominus u(x,1)=u(1,y)\ominus u(x,y)+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y))\\ \Longleftrightarrow\ &u(1,1)=u(x,1)+[u(1,y)\ominus u(x,y)]+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y))\\ \Longleftrightarrow\ &u(1,y)\ominus u(x,y)=u(1,1)\ominus[u(x,1)+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y))]\\ \Longleftrightarrow\ &u(1,y)=u(x,y)+u(1,1)\ominus[u(x,1)+{}^F_H\mathcal{I}^\theta_1 f(x,y,v(x,y))].\end{aligned}$$

Thus, we obtain

$$\begin{aligned}u(x,y)&=u(1,y)\ominus[u(1,1)\ominus(u(x,1)+{}^F_H\mathcal{I}^\theta_1 f(x,y,u(x,y)))]\\ &=\psi(x,y)+{}^F_H\mathcal{I}^\theta_1 f(x,y,u(x,y)).\end{aligned}$$

Case II: $k = 2$, Similarly, we first consider the case where $u$ ‡-type gH-differentiable as mentioned in reference [42]; then we have

$$(-1)u(0,y) \ominus (-1)u(x,y) = (-1)u(0,0) \ominus (-1)u(x,0) + {}_{H}^{F}\mathcal{I}_{1}^{\theta} f(x,y,u(x,y))$$

$$\Longleftrightarrow \; u(0,y) \ominus u(x,y) = u(0,0) \ominus u(x,0) + {}_{H}^{F}\mathcal{I}_{1}^{\theta} f(x,y,u(x,y))$$

$$\Longleftrightarrow \; u(0,y) = u(x,y) + u(0,0) \ominus u(x,0) + {}_{H}^{F}\mathcal{I}_{1}^{\theta} f(x,y,u(x,y))$$

$$\Longleftrightarrow \; u(x,y) = u(0,y) \ominus \Big(u(0,0) \ominus u(x,0) + {}_{H}^{F}\mathcal{I}_{1}^{\theta} f(x,y,u(x,y))\Big)$$

$$= u(x,y) = \psi(x,y) \ominus (-1){}_{F}^{H}\mathcal{I}_{1}^{\theta} f(x,y,v(x,y)).$$

For the case where $u(x,y)$ is ‡-type gH-differentiable as mentioned in reference [42], the discussion is analogous to that of Case I and thus will not be elaborated in detail here. Similarly, for $k = 1, 2$, the situation for $v(x,y)$ is almost identical to that of $u(x,y)$, and therefore will also be omitted.

"⇐": Case I: $k = 1$, apply the operator ${}_{gH}^{CH}D_{1}^{\theta}$ to both sides of (2.11). One can get

$${}_{gH}^{CH}D_{1}^{\theta} u(x,y) = z(x,y)$$

which intends

$${}_{gH}^{CH}D_{1}^{\theta} u(x,y) = f(x,y,v(x,y)).$$

Case II: $k = 2$. Then, similarly, apply the operator ${}_{gH}^{CH}D_{2}^{\theta}$ to both sides of (2.12)

$${}_{gH}^{CH}D_{2}^{\theta} u(x,y) = z(x,y)$$

which intends

$${}_{gH}^{CH}D_{1}^{\theta} u(x,y) = f(x,y,v(x,y)).$$

Similar to the proof of sufficiency, the case for $v(x,y)$ is nearly identical to that of $u(x,y)$ and will not be repeated here. □

**Lemma 5.** *For $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1)$ and $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$ with the Lipschitz constants $\mathcal{L}_1, \mathcal{L}_2 \in (0,1)$, and $l_1, l_2 \in C(\mathcal{K}, \mathbb{F}_1)$, $\delta_1, \delta_2 \in C(\mathcal{K}, \mathbb{F}_2)$, the following Lipschitz condition is satisfied:*

$$\begin{cases} d_\infty(f(x,y,l_1), f(x,y,l_2)) \le \mathcal{L}_1 d_\infty(l_1,l_2), \\ d_\infty(g(x,y,\delta_1), g(x,y,\delta_2)) \le \mathcal{L}_2 d_\infty(\delta_1,\delta_2). \end{cases} \tag{2.13}$$

*To denote Riemann–Liouville integral operator $\lambda \int_1^x \int_1^e \cdot \frac{ds}{s}\frac{dt}{t} + \lambda \int_1^e \int_1^y \cdot \frac{dt}{t}\frac{ds}{s}$, we use the notation ${}_{R}^{L}\mathcal{I}(\cdot)$. We extend the results of Zhang et al.* [13] *and establish the following vector form of Gronwall's inequality involving Hadamard fractional integrals ${}_{F}^{H}\mathcal{I}_{1}^{\theta}$ and ${}_{F}^{H}\mathcal{I}_{1}^{\vartheta}$:*

$$P(x,y) \le QP(x,y) + O,$$

*where* $P(x,y) = \begin{pmatrix} u_1(x,y) \\ v_1(x,y) \end{pmatrix}$, $O(x,y) = \begin{pmatrix} o_1(x,y) \\ o_2(x,y) \end{pmatrix} \in C(\mathcal{K},\mathbb{F}_1) \times C(\mathcal{K},\mathbb{F}_2)$, $Q = \begin{pmatrix} 0 & \mathcal{L}_1 \cdot_F^H \mathcal{I}_1^{\vartheta} \\ \mathcal{L}_2 \cdot_F^H \mathcal{I}_1^{\theta} + \mathcal{L}_2 \cdot_R^L \mathcal{I} & 0 \end{pmatrix}$. *Assume that the following conditions are satisfied:*

($\mathbf{C_1}$) *constants* $p, q \in (1, e)$ *and* $\theta, \vartheta \in (0, 1)$*;*

($\mathbf{C_2}$) $\max\{\mathcal{L}_1, \mathcal{L}_2\} < \frac{1}{A}$, *here*

$$A = \max\left\{\frac{(\ln p)^{\theta_1}(\ln q)^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}, \lambda\ln(pq) + \frac{(\ln p)^{\vartheta_1}(\ln q)^{\vartheta_2}}{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}\right\}.$$

*Then* $P(x,y) \leq \sum_{i=0}^{\infty} Q^i O$.

*Proof.* We define the operator $\mathcal{B} : C(\mathcal{K},\mathbb{F}_1) \times C(\mathcal{K},\mathbb{F}_2) \to C(\mathcal{K},\mathbb{F}_1) \times C(\mathcal{K},\mathbb{F}_2)$ by

$$(\mathcal{B}P)(x,y) = QP(x,y) + O.$$

Letting $\kappa_1 = \begin{pmatrix} l_1 \\ r_1 \end{pmatrix}$, $\kappa_2 = \begin{pmatrix} l_2 \\ r_2 \end{pmatrix}$, and assume $\kappa_1 \leq \kappa_1$, where $l_1$, $l_2 \in C(\mathcal{K},\mathbb{F}_1)$ and $r_1$, $r_2 \in C(\mathcal{K},\mathbb{F}_2)$. It can be deduced that

$$\mathcal{B}(\kappa_2) - \mathcal{B}(\kappa_1) = \begin{pmatrix} \mathcal{L}_1 \cdot_F^H \mathcal{I}_1^{\vartheta}(l_2 - l_1) \\ \left(\mathcal{L}_2 \cdot_F^H \mathcal{I}_1^{\theta} + \mathcal{L}_2 \cdot_R^L \mathcal{I}\right)(r_2 - r_1) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $\mathcal{B}$ is a monotone operator. Moreover, when $\tau \in [0,1]$, based on Remark 5 proposed by Zhang et al. [13], we can conclude that

$$\|\kappa_1\| = \max\left\{\sup_{(x,y)\in\mathcal{K}} \max\{|\underline{l_1}_{\tau}|, |\overline{l_1}_{\tau}|\}, \sup_{(x,y)\in\mathcal{K}} \max\{|\underline{r_1}_{\tau}|, |\overline{r_1}_{\tau}|\}\right\} = 1,$$

it follows that

$$\begin{aligned} \|Q\| &= \sup_{\|\kappa_1\|=1} \|Q\kappa_1\| \\ &\leq \max\{\mathcal{L}_1, \mathcal{L}_2\} \times \sup_{(x,y)\in\mathcal{K}} \max\left\{\frac{(\ln p)^{\theta_1}(\ln q)^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}, \lambda\ln(pq) + \frac{(\ln p)^{\vartheta_1}(\ln b)^{\vartheta_2}}{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}\right\} \\ &\leq 1. \end{aligned}$$

Additionally, it can be concluded that $\mathcal{B}$ has a unique fixed point $\widehat{P}$ and

$$\lim_{n\to\infty}(\mathcal{B}^n P)(x,y) = \widehat{P}(x,y).$$

Analogous to Lemma 4 of [13], it can be derived that $P(x,y) \leq \sum_{i=0}^{\infty} Q^i O$. □

## 3. Continuous dependence of the solution

In this section, we discuss the cases where $\lambda = 0$ and $\lambda \neq 0$, respectively, and analyze solvability of (1.5). Due to the presence of the gH-difference, as expressed in (2.2), there exist two corresponding types of solutions. Here, we refer to them as †-type solution and ‡-type solution.

**Case I**: $\lambda \neq 0$

We present some preliminary conditions as follows:

($\mathbf{H_1}$) For any $u \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$, and each $v \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$, there always exists ${}^{H}_{F}\mathcal{I}^{\vartheta}_{1} g\left(x, y, \lambda \int_1^x \int_1^e u(s,t)\frac{ds}{s}\frac{dt}{t} + \lambda \int_1^e \int_1^y u(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_2(y) + \xi_1(x) \ominus \xi_1(0)\right)$ $\in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$. In addition, we have $1 \leq x \leq p \leq e$, $1 \leq y \leq q \leq e$.

($\mathbf{H_2}$) When $\mathcal{L}_1, \mathcal{L}_2 \in (0,1)$, and $(\theta_1, \theta_2), (\vartheta_1, \vartheta_2) \in (0,1] \times (0,1]$, the following relationships hold: for each $p, q$ that are the same as in ($\mathbf{H}_1$),

$$\frac{\lambda \ln p \ln q \left[(\ln q)^{-\vartheta_2} + (\ln p)^{-\vartheta_2}\right]}{\vartheta_1 \vartheta_2} + \frac{\mathcal{L}_1 \mathcal{L}_2 (\ln p)^{\theta_1 + \vartheta_1} (\ln q)^{\theta_2 + \vartheta_2}}{\Gamma(2\vartheta_1 + \theta_1)\Gamma(2\vartheta_2 + \theta_2)} < 1.$$

($\mathbf{H_3}$) Letting $\mathcal{L}_i$, $\theta_i$, $\vartheta_i$ $(i = 1, 2)$, and $p, q$ be identical to those in ($\mathbf{H}_2$), then it follows that

$$\frac{\lambda \mathcal{L}_2 \Gamma(\theta_1 + 1)(\ln p)^{\theta_1 + 1} \ln q}{\theta_1 \theta_2 \vartheta_2 \Gamma(\vartheta_2)\Gamma(\theta_1 + \vartheta_1 + 1)} + \frac{\lambda \mathcal{L}_2 \Gamma(\theta_2 + 1) \ln p (\ln q)^{\theta_2 + 1}}{\theta_1 \theta_2 \vartheta_1 \Gamma(\vartheta_1)\Gamma(\theta_2 + \vartheta_2 + 1)} + \frac{\mathcal{L}_1 \mathcal{L}_2 \Gamma(\theta_1)\Gamma(\theta_2)(\ln p)^{2\theta_1}(\ln q)^{2\theta_2}}{\Gamma(2\theta_1 + \vartheta_1)\Gamma(2\theta_2 + \vartheta_2)} < 1.$$

**Remark 6.** *For hypotheses* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$)*, we specifically set the parameters as* $\lambda = 0.5$*,* $\mathcal{L}_1 = \mathcal{L}_2 = 0.2$*,* $\theta_1 = \theta_2 = \vartheta_1 = 0.1$*,* $\vartheta_2 = 0.4$*,* $p = q = 1.2$*. Under these settings,* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *are satisfied, which further supports the validity and rationality of these hypotheses* ($\mathbf{H_1}$)*–*($\mathbf{H_3}$)*.*

**Theorem 1.** *Suppose that* $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1)$*,* $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$ *fulfill the Lipschitz condition* (2.13) *with constants* $\mathcal{L}_1, \mathcal{L}_2 \in (0,1)$*, and conditions* ($\mathbf{H_1}$)*–*($\mathbf{H_3}$) *are satisfied. With these prerequisites,* (1.5) *admits a unique †-type solution on* $\mathcal{K}$*.*

*Proof.* Based on Lemma 4, we define the following two operators:

$$\begin{aligned}
\widehat{T_1}(u(x,y)) =& \lambda \int_1^x \int_1^e u(s,t)\frac{ds}{s}\frac{dt}{t} + \lambda \int_1^e \int_1^y u(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_1(x) \\
&+ \xi_2(y) \ominus \xi_1(0) +^{H}_{F} \mathcal{I}^{\theta}_{1} f\left(x, y, \varphi(x,y) +^{H}_{F} \mathcal{I}^{\vartheta}_{1} g(x, y, u(x,y))\right), \\
\widehat{G_1}(v(x,y)) =& {}^{H}_{F}\mathcal{I}^{\vartheta}_{1} g\Big(x, y, \lambda \int_1^x \int_1^e u(s,t)\frac{ds}{s}\frac{dt}{t} + \lambda \int_1^e \int_1^y u(s,t)\frac{dt}{t}\frac{ds}{s} \\
&+ \xi_2(y) + \xi_1(x) \ominus \xi_1(0) +^{H}_{F} \mathcal{I}^{\theta}_{1} f(x, y, v(x,y))\Big) + \varphi(x,y).
\end{aligned}$$

According to ($\mathbf{H_2}$), one knows that $\widehat{T_1} : \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1) \to \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$ and $\widehat{G_1} : \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2) \to \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$. For $u_1 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$ and $v_1 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$, let $u_2 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$ and $v_2 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$ be selected to satisfy

$$d_{1-\vartheta}(u_1, u_2) \leq d_{1-\vartheta}(v_1, v_2).$$

Next, by Lemma 1 and (2.13), we derive the following:

$$d_\infty(\widehat{T_1}(u_1(x,y)), \widehat{T_1}(u_2(x,y)))$$

$$
\begin{aligned}
&\leq d_\infty\left(\lambda\int_1^x\int_1^e u_1(s,t)\frac{ds}{s}\frac{dt}{t},\lambda\int_1^x\int_1^e u_2(s,t)\frac{ds}{s}\frac{dt}{t}\right)\\
&\quad+d_\infty\left(\lambda\int_1^e\int_1^y u_1(s,t)\frac{dt}{t}\frac{ds}{s},\lambda\int_1^e\int_1^y u_2(s,t)\frac{dt}{t}\frac{ds}{s}\right)\\
&\quad+d_\infty\left({}^H_F\mathcal{I}^\theta_1 f\left(x,y,\varphi(x,y)+^H_F\mathcal{I}^\vartheta_1 g(x,y,u_1(x,y))\right),{}^H_F\mathcal{I}^\theta_1 f\left(x,y,\varphi(x,y)+^H_F\mathcal{I}^\vartheta_1 g(x,y,u_2(x,y))\right)\right)\\
&\leq\lambda\int_1^x\int_1^e d_\infty(u_1(s,t),u_2(s,t))\frac{ds}{s}\frac{dt}{t}+\lambda\int_1^e\int_1^y d_\infty(u_1(s,t),u_2(s,t))\frac{dt}{t}\frac{ds}{s}\\
&\quad+\frac{\mathcal{L}_1}{\Gamma(\theta_1)\Gamma(\theta_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\theta_1-1}\left(\ln\frac{y}{t}\right)^{\theta_2-1}d_\infty\left({}^H_F\mathcal{I}^\vartheta_1 g(s,t,u_1(s,t)),{}^H_F\mathcal{I}^\vartheta_1 g(s,t,u_2(s,t))\right)\frac{ds}{s}\frac{dt}{t},
\end{aligned}
$$

and by (2.8), one can derive

$$
\begin{aligned}
&d_\infty(\widehat{T}_1(u_1(x,y)),\widehat{T}_1(u_2(x,y)))\\
&\leq\left[\lambda\frac{(\ln x)^{\vartheta_1}}{\vartheta_1\vartheta_2}+\lambda\frac{(\ln y)^{\vartheta_2}}{\vartheta_1\vartheta_2}+\frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)(\ln x)^{2\vartheta_1+\theta_1-1}(\ln y)^{2\vartheta_2+\theta_2-1}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\right]d_{1-\vartheta}(u_1(x,y),u_2(x,y)).
\end{aligned}
$$

Then, one gets

$$
\begin{aligned}
&d_{1-\vartheta}(\widehat{T}_1(u_1(x,y)),\widehat{T}_1(u_2(x,y)))\\
&\leq\left[\frac{\left(\lambda(\ln p)(\ln q)^{1-\vartheta_2}+\lambda(\ln q)(\ln p)^{1-\vartheta_2}\right)\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}{\vartheta_1\vartheta_2\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\right.\\
&\qquad\left.+\frac{\mathcal{L}_1\mathcal{L}_2\vartheta_1\vartheta_2(\ln p)^{\theta_1+\vartheta_1}(\ln q)^{\theta_2+\vartheta_2}}{\vartheta_1\vartheta_2\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\right]d_{1-\vartheta}(u_1(x,y),u_2(x,y))\\
&=\left[\frac{\lambda\ln p\ln q\left[(\ln q)^{-\vartheta_2}+(\ln p)^{-\vartheta_2}\right]}{\vartheta_1\vartheta_2}+\frac{\mathcal{L}_1\mathcal{L}_2(\ln p)^{\theta_1+\vartheta_1}(\ln q)^{\theta_2+\vartheta_2}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\right]d_{1-\vartheta}(u_1(x,y),u_2(x,y)). \qquad (3.1)
\end{aligned}
$$

From ($\mathbf{H_2}$), (3.1) and Lemma 2, it can be concluded that there exists a unique $u(x,y)\in\mathbb{C}^{2,2}_{gH}(\mathcal{K},\mathbb{F}_1)$ that satisfies (1.5).

Now we will prove that $v(x,y)$ satisfying (1.5) is unique:

$$
\begin{aligned}
d_\infty(\widehat{G}_1(v_1(x,y)),\widehat{G}_1(v_2(x,y)))&=d_\infty(\varphi(x,y)+^H_F\mathcal{I}^\vartheta_1 g(x,y,u_1(x,y)),\varphi(x,y)+^H_F\mathcal{I}^\vartheta_1 g(x,y,u_2(x,y)))\\
&\leq d_\infty\Bigg({}^H_F\mathcal{I}^\vartheta_1 g\Bigg(x,y,\lambda\int_1^x\int_1^e u_1(s,t)\frac{ds}{s}\frac{dt}{t}+\lambda\int_1^e\int_1^y u_1(s,t)\frac{dt}{t}\frac{ds}{s}\\
&\qquad+\xi_2(y)+\xi_1(x)\ominus\xi_1(0)+^H_F\mathcal{I}^\theta_1 f(x,y,v_1(x,y))\Bigg),\\
&\qquad{}^H_F\mathcal{I}^\vartheta_1 g\Bigg(x,y,\lambda\int_1^x\int_1^e u_2(s,t)\frac{ds}{s}\frac{dt}{t}+\lambda\int_1^e\int_1^y u_2(s,t)\frac{dt}{t}\frac{ds}{s}\\
&\qquad+\xi_2(y)+\xi_1(x)\ominus\xi_1(0)+^H_F\mathcal{I}^\theta_1 f(x,y,v_2(x,y))\Bigg)\Bigg).
\end{aligned}
$$

From (2.13), ($\mathbf{H_1}$) and Lemma 1, it can be concluded that

$$
d_\infty(\widehat{G}_1(v_1(x,y)),\widehat{G}_1(v_2(x,y)))
$$

$$
\begin{aligned}
&\leq \frac{\mathcal{L}_2}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\vartheta_1-1}\left(\ln\frac{y}{t}\right)^{\vartheta_2-1}\left[\lambda\int_1^s\int_1^e d_\infty(v_1(s,t),v_2(s,t))\frac{ds}{s}\frac{dt}{t}\right.\\
&\quad\left.+\lambda\int_1^e\int_1^t d_\infty(v_1(s,t),v_2(s,t))\frac{dt}{t}\frac{ds}{s}+d_\infty({}_F^H\mathcal{I}_1^\theta f(s,t,v_1(s,t)),{}_F^H\mathcal{I}_1^\theta f(s,t,v_2(s,t)))\right]\frac{dt}{t}\frac{ds}{s}\\
&\leq\left[\frac{\mathcal{L}_2\lambda\Gamma(\theta_1+1)(\ln x)^{\theta_1+\vartheta_1}(\ln y)^{\vartheta_2}}{\theta_1\theta_2\vartheta_2\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)}+\frac{\mathcal{L}_2\lambda\Gamma(\theta_2+1)(\ln x)^{\vartheta_1}(\ln y)^{\theta_2+\vartheta_2}}{\theta_1\theta_2\vartheta_2\Gamma(\vartheta_1)\Gamma(\theta_2+\vartheta_2+1)}\right.\\
&\quad\left.+\frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\theta_1)\Gamma(\theta_2)(\ln x)^{2\theta_1+\vartheta_1-1}(\ln y)^{2\theta_2+\vartheta_2-1}}{\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\right]d_{1-\vartheta}(v_1,v_2),
\end{aligned}
$$

and from (2.8), it can be derived that

$$
\begin{aligned}
&d_{1-\vartheta}(\widehat{G}_1(v_1(x,y)),\widehat{G}_1(v_2(x,y)))\\
&\quad\leq\left[\frac{\lambda\mathcal{L}_2\vartheta_1\Gamma(\vartheta_1)\Gamma(\theta_1+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)\Gamma(\theta_2+\vartheta_2+1)(\ln p)^{\theta_1+1}(\ln q)}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\right.\\
&\qquad+\frac{\lambda\mathcal{L}_2\vartheta_2\Gamma(\vartheta_2)\Gamma(\theta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)(\ln p)(\ln q)^{\theta_2+1}}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\\
&\qquad+\frac{\mathcal{L}_1\mathcal{L}_2\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\\
&\qquad\left.\cdot\frac{\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)(\ln p)^{2\theta_1}(\ln q)^{2\theta_2}}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\right]d_{1-\vartheta}(v_1,v_2)\\
&\quad=\left[\frac{\lambda\mathcal{L}_2\Gamma(\theta_1+1)(\ln p)^{\theta_1+1}\ln q}{\theta_1\theta_2\vartheta_2\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)}+\frac{\lambda\mathcal{L}_2\Gamma(\theta_2+1)\ln p(\ln q)^{\theta_2+1}}{\theta_1\theta_2\vartheta_1\Gamma(\vartheta_1)\Gamma(\theta_2+\vartheta_2+1)}\right.\\
&\qquad\left.+\frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\theta_1)\Gamma(\theta_2)(\ln p)^{2\theta_1}(\ln q)^{2\theta_2}}{\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\right]d_{1-\vartheta}(v_1,v_2). \qquad (3.2)
\end{aligned}
$$

From ($\mathbf{H_3}$) and (3.2), it follows that $\widehat{G}_1$ is a contraction mapping. Thus, we have demonstrated the existence and uniqueness of $u\in\mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_1)$ and $v\in\mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_2)$, satisfying (1.5) through Lemma 2, which establishes existence and uniqueness of the †-type solution for (1.5). □

**Remark 7.** *It is precisely due to assumptions* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *that the coefficients preceding in* (3.1) *and* (3.2) *remain "$<1$". This further ensures that $\widehat{T}_1$, $\widehat{G}_1$ are contraction operators, thereby guaranteeing the existence and uniqueness of the solution.* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *play a similar role in the following Theorem 2. To more intuitively illustrate the roles of* ($H_2$) *and* ($H_3$)*, the flowchart of the proof of Theorem 1 is presented as Figure 2.*

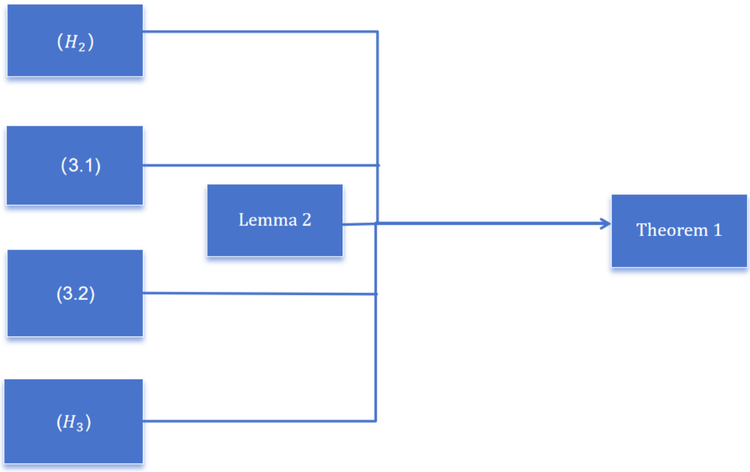


**Figure 2.** Flowchart of the proof of Theorem 1.

Subsequently, we introduce the following additional assumptions based on (2.5) and (2.6) to further ensure the existence and uniqueness of the ‡-type solution:

($\mathbf{H_4}$) $\widetilde{\mathbb{C}}^{f}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_2) \neq \emptyset$ and $\widetilde{\mathbb{C}}^{g}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_1) \neq \emptyset$.

($\mathbf{H_5}$) For $\forall (x, y) \in \mathcal{K}$, given $U(x, y)$, $V(x, y)$ as

$$U(x, y) = \psi(x, y) \ominus (-1)^{H}_{F}\mathcal{I}^{\theta}_{1} f(x, y, v(x, y)),$$

$$V(x, y) = \varphi(x, y) \ominus (-1)^{H}_{F}\mathcal{I}^{\theta}_{1} g(x, y, u(x, y)),$$

if $u(x, y) \in \widetilde{\mathbb{C}}^{f}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_2)$ and $v(x, y) \in \widetilde{\mathbb{C}}^{g}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_1)$, then $U(x, y) \in \widetilde{\mathbb{C}}^{f}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_2)$ and $V(x, y) \in \widetilde{\mathbb{C}}^{g}_{\mathbb{F}}(\mathcal{K}, \mathbb{F}_1)$.

**Theorem 2.** *Assume that* $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1)$ *and* $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$ *meet the Lipschitz condition* (2.13) *with constants* $\mathcal{L}_1, \mathcal{L}_2 \in (0, 1)$*, and conditions* ($\mathbf{H_1}$)–($\mathbf{H_5}$) *satisfied. With these prerequisites, then* (1.5) *admits a unique* ‡*-type solution on* $\mathcal{K}$*.*

*Proof.* We denote the operators

$$\begin{aligned}\widehat{T}_2(u(x, y)) =& \lambda \int_1^x \int_1^e u(s, t) \frac{ds}{s}\frac{dt}{t} + \lambda \int_1^e \int_1^y u(s, t) \frac{dt}{t}\frac{ds}{s} + \xi_2(y) \\ &+ \xi_1(x) \ominus \xi_1(0) \ominus (-1)^{H}_{F}\mathcal{I}^{\theta}_{1} f\left(x, y, \varphi(x, y) +^{H}_{F} \mathcal{I}^{\vartheta}_{1} g(x, y, u(x, y))\right),\end{aligned}$$

and

$$\begin{aligned}\widehat{G}_2(v(x, y)) =& \varphi(x, y) \ominus (-1)^{H}_{F}\mathcal{I}^{\vartheta}_{1} g\Big(x, y, \lambda \int_1^x \int_1^e u(s, t) \frac{ds}{s}\frac{dt}{t} + \xi_2(y) \\ &+ \xi_1(x) \ominus \xi_1(0) + \lambda \int_1^e \int_1^y u(s, t) \frac{dt}{t}\frac{ds}{s} +^{H}_{F} \mathcal{I}^{\theta}_{1} f\,(x, y, v(x, y))\Big).\end{aligned}$$

Moreover, as indicated from ($\mathbf{H_4}$), one can know that the H-difference contained in $\widehat{T}_2$ and $\widehat{G}_2$ exist. Furthermore, from ($\mathbf{H_1}$) and ($\mathbf{H_5}$), it can be deduced that $\widehat{T}_2 : \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1) \to \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$ and $\widehat{G}_2 : \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2) \to \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$. For $u_1 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$ and $v_1 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$, suppose that $u_2 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$

and $v_2 \in \mathbb{C}^{2,2}_{gH}(\mathcal{K},\mathbb{F}_2)$ are selected to satisfy inequality $d_{1-\vartheta}(u_1,u_2) \le d_{1-\vartheta}(v_1,v_2)$. Building on the $(a_3)$ of Lemma 1 and employing a similar approach as used in the proof of Theorem 1, we can deduce that

$$\begin{aligned}
& d_\infty(\widehat{T}_2(u_1(x,y)),\widehat{T}_2(u_2(x,y))) \\
&\le \Bigg[\frac{\left(\lambda(\ln p)(\ln q)^{1-\vartheta_2}+\lambda(\ln q)(\ln p)^{1-\vartheta_2}\right)\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}{\vartheta_1\vartheta_2\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)} \\
&\quad + \frac{\mathcal{L}_1\mathcal{L}_2\vartheta_1\vartheta_2(\ln p)^{\theta_1+\vartheta_1}(\ln q)^{\theta_2+\vartheta_2}}{\vartheta_1\vartheta_2\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\Bigg] d_{1-\vartheta}(u_1(x,y),u_2(x,y)) \\
&= \left[\frac{\lambda\ln p\ln q\left[(\ln q)^{-\vartheta_2}+(\ln p)^{-\vartheta_2}\right]}{\vartheta_1\vartheta_2} + \frac{\mathcal{L}_1\mathcal{L}_2(\ln p)^{\theta_1+\vartheta_1}(\ln q)^{\theta_2+\vartheta_2}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\right] d_{1-\vartheta}(u_1(x,y),u_2(x,y)),
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
& d_{1-\vartheta}(\widehat{G}_1(v_1(x,y)),\widehat{G}_1(v_2(x,y))) \\
&\le \Bigg[\frac{\lambda\mathcal{L}_2\vartheta_1\Gamma(\vartheta_1)\Gamma(\theta_1+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)\Gamma(\theta_2+\vartheta_2+1)(\ln p)^{\theta_1+1}(\ln q)}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)} \\
&\quad + \frac{\lambda\mathcal{L}_2\vartheta_2\Gamma(\vartheta_2)\Gamma(\theta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)(\ln p)(\ln q)^{\theta_2+1}}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)} \\
&\quad + \frac{\mathcal{L}_1\mathcal{L}_2\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)} \\
&\quad \cdot \frac{\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)(\ln p)^{2\theta_1}(\ln q)^{2\theta_2}}{\theta_1\theta_2\vartheta_1\vartheta_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)\Gamma(\theta_2+\vartheta_2+1)\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\Bigg] d_{1-\vartheta}(v_1,v_2) \\
&= \Bigg[\frac{\lambda\mathcal{L}_2\Gamma(\theta_1+1)(\ln p)^{\theta_1+1}\ln q}{\theta_1\theta_2\vartheta_2\Gamma(\vartheta_2)\Gamma(\theta_1+\vartheta_1+1)} + \frac{\lambda\mathcal{L}_2\Gamma(\theta_2+1)\ln p(\ln q)^{\theta_2+1}}{\theta_1\theta_2\vartheta_1\Gamma(\vartheta_1)\Gamma(\theta_2+\vartheta_2+1)} \\
&\quad + \frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\theta_1)\Gamma(\theta_2)(\ln p)^{2\theta_1}(\ln q)^{2\theta_2}}{\Gamma(2\theta_1+\vartheta_1)\Gamma(2\theta_2+\vartheta_2)}\Bigg] d_{1-\vartheta}(v_1,v_2).
\end{aligned} \tag{3.4}$$

Relying on $(\mathbf{H_1})$–$(\mathbf{H_5})$ and Lemma 2, it follows that $(u(x,y),v(x,y)) \in \mathbb{C}^{2,2}_{gH}(\mathcal{K},\mathbb{F}_1)\times\mathbb{C}^{2,2}_{gH}(\mathcal{K},\mathbb{F}_2)$ is the unique ‡-type solution of (1.5). □

In the following, we further investigate continuous dependence of the solutions on the integral boundary conditions. By modifying the initial conditions of (1.5), we derive the following novel coupled system of CH-type FFPDE with integral boundary conditions:

$$\begin{cases}
{}^{CH}_{gH}D^{\theta}_{k}u_1(x,y) = f(x,y,v_1(x,y)), \\
{}^{CH}_{gH}D^{\vartheta}_{k}v_1(x,y) = g(x,y,u_1(x,y)), \\
u_1(x,1) = \lambda\displaystyle\int_1^x\int_1^e u_1(s,t)\frac{ds}{s}\frac{dt}{t} + \xi_{11}(x), \\
u_1(1,y) = \lambda\displaystyle\int_1^e\int_1^y u_1(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_{21}(y), \\
v_1(x,1) = \eta_{11}(x), v_1(1,y) = \eta_{21}(y),
\end{cases} \tag{3.5}$$

and

$$\begin{cases} {}^{CH}_{gH}D^{\theta}_{k}u_2(x,y) = f(x,y,v_2(x,y)),\\ {}^{CH}_{gH}D^{\vartheta}_{k}v_2(x,y) = g(x,y,u_2(x,y)),\\ u_2(x,1) = \lambda \int_1^x \int_1^e u_2(s,t)\dfrac{ds}{s}\dfrac{dt}{t} + \xi_{12}(x),\\ u_2(1,y) = \lambda \int_1^e \int_1^y u_2(s,t)\dfrac{dt}{t}\dfrac{ds}{s} + \xi_{22}(y),\\ v_2(x,1) = \eta_{12}(x), v_2(1,y) = \eta_{22}(y). \end{cases} \tag{3.6}$$

Thus, for (3.5) and (3.6), the †-type solutions are

$$\begin{cases} u_1(x,y) = \psi_1(x,y) +^H_F \mathcal{I}^{\theta}_{1} f(x,y,v_1(x,y)),\\ v_1(x,y) = \varphi_1(x,y) +^H_F \mathcal{I}^{\vartheta}_{1} g(x,y,u_1(x,y)),\\ u_2(x,y) = \psi_2(x,y) +^H_F \mathcal{I}^{\theta}_{2} f(x,y,v_2(x,y)),\\ v_2(x,y) = \varphi_2(x,y) +^H_F \mathcal{I}^{\vartheta}_{2} g(x,y,u_2(x,y)), \end{cases} \tag{3.7}$$

while the ‡-type solutions are

$$\begin{cases} u_1(x,y) = \psi_1(x,y) \ominus (-1)^H_F \mathcal{I}^{\theta}_{1} f(x,y,v_1(x,y)),\\ v_1(x,y) = \varphi_1(x,y) \ominus (-1)^H_F \mathcal{I}^{\vartheta}_{1} g(x,y,u_1(x,y)),\\ u_2(x,y) = \psi_2(x,y) \ominus (-1)^H_F \mathcal{I}^{\theta}_{2} f(x,y,v_2(x,y)),\\ v_2(x,y) = \varphi_2(x,y) \ominus (-1)^H_F \mathcal{I}^{\vartheta}_{2} g(x,y,u_2(x,y)), \end{cases} \tag{3.8}$$

and

$$\begin{cases} \psi_1(x,y) = \lambda \int_1^x \int_1^e u_1(x,y)\dfrac{dx}{x}\dfrac{dy}{y} + \lambda \int_1^e \int_1^y u_1(x,y)\dfrac{dy}{y}\dfrac{dx}{x}\\ \qquad + \xi_{21}(y) + \xi_{11}(x) \ominus \xi_{11}(0),\\ \psi_2(x,y) = \lambda \int_1^x \int_1^e u_2(x,y)\dfrac{dx}{x}\dfrac{dy}{y} + \lambda \int_1^e \int_1^y u_2(x,y)\dfrac{dy}{y}\dfrac{dx}{x}\\ \qquad + \xi_{22}(y) + \xi_{12}(x) \ominus \xi_{12}(0),\\ \varphi_1(x,y) = \eta_{21}(y) + \eta_{11}(x) \ominus \eta_{11}(0),\\ \varphi_2(x,y) = \eta_{22}(y) + \eta_{12}(x) \ominus \eta_{12}(0). \end{cases} \tag{3.9}$$

The following theorem provides a comprehensive answer to the first part of Question 2.

**Theorem 3.** *If* $(\mathbf{C_1})$ *and* $(\mathbf{C_2})$ *in Lemma* 5 *hold,* $\lambda < [\ln(pq)]^{-1}$, $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1)$ *and* $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$ *satisfy the Lipschitz condition* (2.13) *with constants* $\mathcal{L}_1, \mathcal{L}_2 \in (0,1)$*, then for different initial conditions* $\psi_1(x,y)$*,* $\psi_2(x,y)$*,* $\varphi_1(x,y)$ *and* $\varphi_2(x,y)$ *decided by* (3.9)*, the inequality*

$$\begin{aligned} &\begin{pmatrix} \rho(u_1(x,y), u_2(x,y)) \\ \rho(v_1(x,y), v_2(x,y)) \end{pmatrix} \\ &\leq \begin{pmatrix} \frac{l(\ln p)^{\theta_1}(\ln q)^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)} & 0 \\ 0 & \lambda \ln(pq) + \frac{(\ln p)^{\vartheta_1}(\ln q)^{\vartheta_2}}{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)} \end{pmatrix} \times \sum_{i=0}^{\infty} Q^i \begin{pmatrix} \mathcal{L}_1 \rho(\varphi_1(x,y), \varphi_2(x,y)) \\ \mathcal{L}_2 \rho(\chi_1(x,y), \chi_2(x,y)) \end{pmatrix} \end{aligned}$$

$$+\begin{pmatrix} l\rho(\chi_1(x,y),\chi_2(x,y)) \\ \rho(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix} \tag{3.10}$$

*holds, where $Q$ is the same as in Lemma 5, $l = [1-\lambda\ln(pq)]^{-1}$, the functions $u_1$, $u_2$, $v_1$, $v_2$ are separately defined as in (3.7) and (3.8), and $\chi_1(x,y)$, $\chi_2(x,y)$, $\varphi_1(x,y)$, $\varphi_2(x,y)$ represent different initial conditions with $\chi_1(x,y) = \xi_{21}(y) + \xi_{11}(x) \ominus \xi_{11}(0)$ and $\chi_2(x,y) = \xi_{22}(y) + \xi_{12}(x) \ominus \xi_{12}(0)$.*

*Proof.* Letting $l = [1-\lambda\ln(pq)]^{-1}$, then, according to (3.7) and (3.8), one obtains

$$\begin{aligned} d_\infty(u_1(x,y),u_2(x,y)) \le & \lambda \int_1^x \int_1^e d_\infty(u_1(s,t),u_2(s,t)) \frac{ds}{s}\frac{dt}{t} \\ & + \lambda \int_1^e \int_1^y d_\infty(u_1(s,t),u_2(s,t)) \frac{dt}{t}\frac{ds}{s} \\ & + {}^H_F\mathcal{I}_1^\theta \left[ d_\infty(f(x,y,v_1(x,y)), f(x,y,v_2(x,y))) \right] \\ & + d_\infty(\chi_1(x,y),\chi_2(x,y)) \\ \le & l {}^H_F\mathcal{I}_1^\theta \left[ d_\infty(f(x,y,v_1(x,y)), f(x,y,v_2(x,y))) \right] \\ & + l d_\infty(\chi_1(x,y),\chi_2(x,y)), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} d_\infty(v_1(x,y),v_2(x,y)) \le & d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ & + {}^H_F\mathcal{I}_1^\theta d_\infty \left[ g(x,y,u_1(x,y)), g(x,y,u_2(x,y)) \right]. \end{aligned} \tag{3.12}$$

Next, for each $(x,y) \in \mathcal{K} = [1,p]\times[1,q]$, let $s_1 = {}^{CH}_{gH} D_k^\theta u_1(x,y)$, $s_2 = {}^{CH}_{gH} D_k^\theta u_2(x,y)$, $t_1 = {}^{CH}_{gH} D_k^\vartheta v_1(x,y)$, $t_2 = {}^{CH}_{gH} D_k^\vartheta v_2(x,y)$. It is evident that

$$\begin{cases} s_1(x,y) = f(x,y,\varphi_1(x,y) + {}^H_F\mathcal{I}_1^\vartheta t_1(x,y))), \\ t_1(x,y) = g\left(x,y,\psi_1(x,y) + {}^H_F\mathcal{I}_1^\theta s_1(x,y)\right), \end{cases}$$

and

$$\begin{cases} s_2(x,y) = f(x,y,\varphi_2(x,y) + {}^H_F\mathcal{I}_1^\vartheta t_2(x,y))), \\ t_2(x,y) = g\left(x,y,\psi_2(x,y) + {}^H_F\mathcal{I}_1^\theta s_2(x,y)\right). \end{cases}$$

Furthermore, we obtain

$$\begin{aligned} d_\infty(s_1,s_2) \le & \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) + \mathcal{L}_1 \cdot {}^H_F\mathcal{I}_1^\vartheta d_\infty(t_1(x,y),t_2(x,y)), \\ d_\infty(t_1,t_2) \le & \mathcal{L}_2\lambda \int_1^x \int_1^e d_\infty(s_1,s_2)\frac{dx}{x}\frac{dy}{y} + \mathcal{L}_2\lambda \int_1^e \int_1^y d_\infty(s_1,s_2)\frac{dy}{y}\frac{dx}{x} \\ & + \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) + \mathcal{L}_2 \cdot {}^H_F\mathcal{I}_1^\theta d_\infty(s_1(x,y),s_2(x,y)) \end{aligned}$$

and

$$\begin{pmatrix} d_\infty(s_1,s_2) \\ d_\infty(t_1,t_2) \end{pmatrix} \le \begin{pmatrix} 0 & \mathcal{L}_1 \cdot {}^H_F\mathcal{I}_1^\vartheta \\ \mathcal{L}_2 \cdot {}^H_F\mathcal{I}_1^\theta + \mathcal{L}_2 \cdot {}^L_R\mathcal{I} & 0 \end{pmatrix} \begin{pmatrix} d_\infty(s_1,s_2) \\ d_\infty(t_1,t_2) \end{pmatrix} + \begin{pmatrix} \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix}.$$

From Lemma 5, it is follows that

$$\begin{pmatrix} d_\infty(s_1,s_2) \\ d_\infty(t_1,t_2) \end{pmatrix} \le \sum_{i=0}^{\infty} \begin{pmatrix} 0 & \mathcal{L}_1 \cdot {}_F^H \mathcal{I}_1^\vartheta \\ \mathcal{L}_2 \cdot {}_F^H \mathcal{I}_1^\theta + \mathcal{L}_2 \cdot {}_R^L \mathcal{I} & 0 \end{pmatrix}^i \times \begin{pmatrix} \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix}. \tag{3.13}$$

By (3.11)–(3.13), we know that

$$\begin{aligned} \begin{pmatrix} d_\infty(u_1,u_2) \\ d_\infty(v_1,v_2) \end{pmatrix} &\le \begin{pmatrix} l d_\infty(\chi_1(x,y),\chi_2(x,y)) \\ d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix} + \begin{pmatrix} l_F^H \mathcal{I}_1^\theta & 0 \\ 0 & {}_F^H \mathcal{I}_1^\vartheta \end{pmatrix} \begin{pmatrix} d_\infty(s_1,s_2) \\ d_\infty(t_1,t_2) \end{pmatrix} \\ &= \begin{pmatrix} l d_\infty(\chi_1(x,y),\chi_2(x,y)) \\ d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix} + \begin{pmatrix} l_F^H \mathcal{I}_1^\theta & 0 \\ 0 & {}_F^H \mathcal{I}_1^\vartheta \end{pmatrix} \sum_{i=0}^{\infty} Q^i \begin{pmatrix} \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix}, \end{aligned}$$

and further derived from (2.7) that

$$\begin{aligned} \begin{pmatrix} \rho(u_1(x,y),u_2(x,y)) \\ \rho(v_1(x,y),v_2(x,y)) \end{pmatrix} \le & \begin{pmatrix} \frac{l(\ln p)^{\theta_1}(\ln q)^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)} & 0 \\ 0 & \lambda \ln(pq) + \frac{(\ln p)^{\vartheta_1}(\ln q)^{\vartheta_2}}{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)} \end{pmatrix} \\ & \times \sum_{i=0}^{\infty} Q^i \begin{pmatrix} \mathcal{L}_1 \rho(\varphi_1(x,y),\varphi_2(x,y)) \\ \mathcal{L}_2 \rho(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix} + \begin{pmatrix} l\rho(\chi_1(x,y),\chi_2(x,y)) \\ \rho(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix}. \end{aligned}$$

This proves (4.11) of Theorem 3. For (3.8), a similar conclusion holds, and the proof follows a similar process, which will not be repeated here for brevity. □

**Case II**: $\lambda = 0$.

In this case, we demonstrate that Theorem 1.5 is no longer dependent on the existence and uniqueness results under assumptions ($\mathbf{H_1}$)–($\mathbf{H_3}$), and we also illustrate the continuous dependence of the solution on the initial conditions.

**Theorem 4.** *If $f \in C_K(\mathcal{K},\mathbb{F}_2,\mathbb{F}_1)$, $g \in C_K(\mathcal{K},\mathbb{F}_1,\mathbb{F}_2)$ fulfill the Lipschitz condition* (2.13) *with constants $\mathcal{L}_1, \mathcal{L}_2 \in (0,1)$, then* (1.5) *admits a unique †-type solution on $\mathcal{K}$.*

*Proof.* Similarly, we respectively define two operators $\widetilde{T}_1 : \mathbb{C}_{gH}^{2,2}(\mathcal{K}^2,\mathbb{F}_1) \to \mathbb{C}_{gH}^{2,2}(\mathcal{K}^2,\mathbb{F}_1)$ and $\widetilde{G}_1 : \mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_2) \to \mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_2)$ as

$$\widetilde{T}_1(u(x,y)) = \xi_1(x) + \xi_2(y) \ominus \xi_1(0) +_F^H \mathcal{I}_1^\theta f\left(x,y,\varphi(x,y) +_F^H \mathcal{I}_1^\vartheta g(x,y,u(x,y))\right)$$

and

$$\widetilde{G}_1(v(x,y)) = \varphi(x,y) +_F^H \mathcal{I}_1^\vartheta g\left(x,y,\xi_2(y) + \xi_1(x) \ominus \xi_1(0) +_F^H \mathcal{I}_1^\theta f(x,y,v(x,y))\right).$$

By (2.8), one can obtain

$$\begin{aligned} & d_\infty\left({}_F^H\mathcal{I}_1^\vartheta g(s,t,u_1(s,t)), {}_F^H \mathcal{I}_1^\vartheta g(s,t,u_2(s,t))\right) \\ \le & \frac{\mathcal{L}_2}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \int_1^s \int_1^t \left(\ln \frac{s}{\zeta}\right)^{\vartheta_1-1} \left(\ln \frac{t}{\sigma}\right)^{\vartheta_2-1} d_\infty(u_1(\zeta,\sigma),u_2(\zeta,\sigma)) \frac{d\zeta}{\zeta}\frac{d\sigma}{\sigma} \\ = & \frac{\mathcal{L}_2 (\ln s)^{2\vartheta_1-1} (\ln t)^{2\vartheta_2-1} \Gamma^2(\vartheta_1)\Gamma^2(\vartheta_2)}{\Gamma(2\vartheta_1)\Gamma(2\vartheta_2)} d_{1-\vartheta}(u_1,u_2), \end{aligned}$$

which means from Lemma 1 and (2.13) that

$$\begin{aligned}
&d_\infty(\widetilde{T}_1(u_1(x,y)),\widetilde{T}_1(u_2(x,y)))\\
&\leq d_\infty\big({}_F^H\mathcal{I}_1^\theta f\left(x,y,\varphi(x,y)+{}_F^H\mathcal{I}_1^\vartheta g(x,y,u_1(x,y))\right),{}_F^H\mathcal{I}_1^\theta f\left(x,y,\varphi(x,y)+{}_F^H\mathcal{I}_1^\vartheta g(x,y,u_2(x,y))\right)\big)\\
&\leq \mathcal{L}_1\cdot{}_F^H\mathcal{I}_1^\theta d_\infty\left({}_F^H\mathcal{I}_1^\vartheta g(s,t,u_1(s,t)),{}_F^H\mathcal{I}_1^\vartheta g(s,t,u_2(s,t))\right)\\
&\leq \frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)(\ln x)^{2\vartheta_1+\theta_1-1}(\ln y)^{2\vartheta_2+\theta_2-1}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\cdot d_{1-\vartheta}(u_1,u_2).
\end{aligned}$$

Thereafter, we use the method of mathematical induction to prove the following inequality:

$$d_{1-\vartheta}(\widetilde{T}_1^n(u_1(x,y)),\widetilde{T}_2^n(u_2(x,y)))\leq\frac{\mathcal{L}_1^n\mathcal{L}_2^n(\ln x)^{n\vartheta_1+n\theta_1}(\ln y)^{n\vartheta_2+n\theta_2}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((n+1)\vartheta_1+n\theta_1)\Gamma((n+1)\vartheta_2+n\theta_2)}\cdot d_{1-\vartheta}(u_1,u_2),\tag{3.14}$$

which further supports Theorem 4. Indeed, we assume that

$$\begin{aligned}
&d_\infty(\widetilde{T}_1^k(u_1(x,y)),\widetilde{T}_1^k(u_2(x,y)))\\
&\leq\frac{\mathcal{L}_1^k\mathcal{L}_2^k(\ln x)^{(k+1)\vartheta_1+k\theta_1-1}(\ln y)^{(k+1)\vartheta_2+k\theta_2-1}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((k+1)\vartheta_1+k\theta_1)\Gamma((k+1)\vartheta_2+k\theta_2)}\cdot d_{1-\vartheta}(u_1,u_2)
\end{aligned}\tag{3.15}$$

holds for $n=k$. Then, for $n=k+1$, we have

$$\begin{aligned}
&d_\infty(\widetilde{T}_1^{k+1}(u_1(x,y)),\widetilde{T}_1^{k+1}(u_2(x,y)))\\
&\leq d_\infty\Big({}_F^H\mathcal{I}_1^\theta f\left(x,y,\varphi(x,y)+{}_F^H\mathcal{I}_1^\vartheta g(x,y,\widetilde{T}_1^k(u_1(x,y)))\right),{}_F^H\mathcal{I}_1^\theta f\left(x,y,\varphi(x,y)+{}_F^H\mathcal{I}_1^\vartheta g(x,y,\widetilde{T}_1^k(u_2(x,y)))\right)\Big)\\
&\leq \mathcal{L}_1\cdot{}_F^H\mathcal{I}_1^\theta d_\infty({}_F^H\mathcal{I}_1^\vartheta g(s,t,\widetilde{T}_1^k(u_1(s,t)),{}_F^H\mathcal{I}_1^\vartheta g(s,t,\widetilde{T}_1^k(u_2(s,t))).
\end{aligned}$$

From (3.15), it can be concluded that

$$\begin{aligned}
&d_\infty\left({}_F^H\mathcal{I}_1^\vartheta g(s,t,\widetilde{T}_1^k(u_1(s,t)),{}_F^H\mathcal{I}_1^\vartheta g(s,t,\widetilde{T}_1^k(u_2(s,t))\right)\\
&\leq\frac{\mathcal{L}_1^k\mathcal{L}_2^{k+1}(\ln s)^{(k+2)\vartheta_1+k\theta_1-1}(\ln t)^{(k+2)\vartheta_2+k\theta_2-1}}{\Gamma((k+2)\vartheta_1+k\theta_1)\Gamma((k+2)\vartheta_2+k\theta_2)}\cdot d_{1-\vartheta}(u_1,u_2)
\end{aligned}$$

owns, and one can derive

$$\begin{aligned}
&d_{1-\vartheta}\left(\widetilde{T}_1^{k+1}(u_1(x,y)),\widetilde{T}_1^{k+1}(u_2(x,y))\right)\\
&\leq\frac{\mathcal{L}_1^{k+1}\mathcal{L}_2^{k+1}(\ln x)^{(k+2)\vartheta_1+(k+1)\theta_1-1}(\ln y)^{(k+2)\vartheta_2+(k+1)\theta_2-1}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((k+2)\vartheta_1+(k+1)\theta_1)\Gamma((k+2)\vartheta_2+(k+1)\theta_2)}d_{1-\vartheta}(u_1,u_2).
\end{aligned}$$

We have successfully proved (3.14) using mathematical induction, and get the limit equation as follows:

$$\lim_{n\to 0}\frac{\mathcal{L}_1^n\mathcal{L}_2^n(\ln x)^{(n+1)\vartheta_1+n\theta_1-1}(\ln y)^{(n+1)\vartheta_2+n\theta_2-1}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((n+1)\vartheta_1+n\theta_1)\Gamma((n+1)\vartheta_2+n\theta_2)}=0\tag{3.16}$$

which implies that $\widetilde{T}_1^n$ is a contraction operator. This means from Remark 4 that there exists a unique $u(x,y)\in\mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_1)$, satisfying (1.5). Similarly, it can be shown that $\widetilde{G}_1^n$ is also a contraction operator, which guarantees the existence of a unique $v(x,y)\in\mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_2)$, filling (1.5). □

In the following, we use ($\mathbf{H_4}$) and ($\mathbf{H_5}$) to establish an existence and uniqueness result for the ‡-type solution of (1.5).

**Theorem 5.** *Suppose that $f \in C_K(\mathcal{K}, \mathbb{F}_2, \mathbb{F}_1)$ and $g \in C_K(\mathcal{K}, \mathbb{F}_1, \mathbb{F}_2)$ fulfill the Lipschitz condition* (2.13) *with constants $\mathcal{L}_1, \mathcal{L}_2 \in (0, 1)$, and conditions* ($\mathbf{H_4}$) *and* ($\mathbf{H_5}$) *are satisfied. Then,* (1.5) *has a unique ‡-type solution on $\mathcal{K}$.*

*Proof.* In a similar manner, the operators $\widehat{T}_2 : \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1) \to \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_1)$ and $\widehat{G}_2 : \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2) \to \mathbb{C}^{2,2}_{gH}(\mathcal{K}, \mathbb{F}_2)$ are separately represented as

$$\widehat{T}_2(u(x,y)) = \xi_1(x) + \xi_2(y) \ominus \xi_1(0) \ominus (-1)^H_F \mathcal{I}^\vartheta_1 f\left(x, y, \varphi(x,y) +^H_F \mathcal{I}^\vartheta_1 g(x, y, u(x,y))\right)$$

and

$$\widehat{G}_2(v(x,y)) = \varphi(x,y) \ominus (-1)^H_F \mathcal{I}^\vartheta_1 g\left(x, y, \xi_2(y) + \xi_1(x) \ominus \xi_1(0) +^H_F \mathcal{I}^\vartheta_1 f(x, y, v(x,y))\right).$$

By Lemma 1, it can be concluded that

$$d_{1-\vartheta}\left(\widehat{T}_2(u_1(x,y)), \widehat{T}_2(u_2(x,y))\right) \leq \frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)(\ln x)^{\vartheta_1+\theta_1}(\ln y)^{\vartheta_2+\theta_2}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)} \cdot d_{1-\vartheta}(u_1, u_2).$$

Following a similar approach to the proof of Theorem 4, we obtain

$$d_{1-\vartheta}\left(\widehat{T}^n_2(u_1(x,y)), \widehat{T}^n_2(u_2(x,y))\right) \leq \frac{\mathcal{L}^n_1\mathcal{L}^n_2(\ln x)^{n\vartheta_1+n\theta_1}(\ln y)^{n\vartheta_2+n\theta_2}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((n+1)\vartheta_1+n\theta_1)\Gamma((n+1)\vartheta_2+n\theta_2)} \cdot d_{1-\vartheta}(u_1, u_2), \tag{3.17}$$

We can deduce from (3.16) and (4.15) that $\widehat{T}^n_2$ is a contraction operator. Similarly, it can be shown that $\widehat{G}^n_2$ is also a contraction operator. Therefore, under the assumption that conditions ($\mathbf{H_4}$) and ($\mathbf{H_5}$) hold, we have proven that there exists a unique ‡-type solution, satisfying (1.5). □

**Remark 8.** *Since Theorem* 3 *is independent of conditions* ($\mathbf{H_1}$)–($\mathbf{H_5}$)*, the continuous dependence on the initial values discussed here represents a special case of Theorem* 3 *with $\lambda = 0$ and so it is omitted.*

## 4. Numerical examples and applications

In this section, we illustrate the existence and uniqueness results through a numerical example. As mentioned, the CH-type fractional derivative is more suitable for describing slow diffusion phenomena. Hence, the examples in this section better align with the objective laws of ecosystems (i.e., species reproduction rates are constrained by the environmental carrying capacity). Furthermore, in the solution system of the example, we identify an iterative map exhibiting chaotic behavior, which is then implemented using a circuit. Furthermore, in this section, we present the applications of Theorems 1,2, 4, and 5 in the context of projection neural network systems.

### 4.1. Numerical examples and circuit design of the solution

To enhance the ability of the system (1.1) to capture the intrinsic dynamics of ecological systems, the slow diffusion behavior, and the effects of historical uncertainty, we introduce the CH-gH type differentiable operator ${}^{CH}_{gH}D^{(\frac{1}{2},\frac{1}{2})}_k$ on the left hand side of (1.1), specify the right hand side more concretely, and impose fuzzy initial conditions. Thus, in this section, we present the following example for ecological systems, which does not rely on hypotheses ($\mathbf{H_1}$)–($\mathbf{H_3}$):

**Example 1.** *We consider the following coupled system of CH-type FFPDE involving triangular fuzzy numbers $M = (3, 4, 5)$:*

$$\begin{cases} {}^{CH}_{gH}D_k^{(\frac{1}{2},\frac{1}{2})}u(x,y) = \dfrac{-4\,(\ln x)^{\frac{1}{2}}\,(\ln y)^{\frac{1}{2}}}{\left[\Gamma(\frac{1}{2})\right]^2}v(x,y) + \dfrac{4M\left[(\ln x)(\ln y) - (\ln x)^2(\ln y)^2\right]}{\left[\Gamma(\frac{1}{2})\right]^2}, \\ {}^{CH}_{gH}D_k^{(\frac{1}{2},\frac{1}{2})}v(x,y) = \dfrac{9\pi}{32}u(x,y) - \dfrac{9\pi}{16}M, \\ u(x,1) = u(1,y) = u(1,1) = 2M, \\ v(x,1) = v(1,y) = v(1,1) = -M. \end{cases} \tag{4.1}$$

It is easy to say that for $(x, y) \in [1, 2] \times [1, 2]$ and $k = 1, 2$, $f(x, y, v(x, y)) = \dfrac{-4\,(\ln x)^{\frac{1}{2}}\,(\ln y)^{\frac{1}{2}}}{\left[\Gamma(\frac{1}{2})\right]^2} v(x,y) + \dfrac{4M\left[(\ln x)(\ln y) - (\ln x)^2(\ln y)^2\right]}{\left[\Gamma(\frac{1}{2})\right]^2}$ and $g(x, y, u(x, y)) = \dfrac{9\pi}{32}u(x,y) - \dfrac{9\pi}{16}M$ satisfy the Lipschitz condition (2.13), with constants $\mathcal{L}_1 = \dfrac{4(\ln 2)}{[\Gamma(\frac{1}{2})]^2}$, $\mathcal{L}_2 = \dfrac{9\pi}{32}$. Furthermore, $\psi(x, y) = 2M$ and $\varphi(x, y) = -M$ can be derived from (2.3) and (2.4). In the following, we focus on discussing the existence and uniqueness of ‡-type solution for (4.1). As the existence of †-type solution does not depend on ($\mathbf{H_4}$)–($\mathbf{H_5}$), we do not elaborate on it in detail here.

By incorporating the BF method proposed by Viet Long et al [42], commonly referred to as the defuzzification process, we obtain the †-type solution of (4.1) as

$$\begin{cases} u(x,y) = 2M - 2M\ln(x)\ln(y), \\ v(x,y) = -M + M(\ln x)^{\frac{1}{2}}(\ln y)^{\frac{1}{2}} - M(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}}, \end{cases}$$

and the ‡-type solution by

$$\begin{cases} u(x,y) = 2M \ominus 2M\ln(x)\ln(y), \\ v(x,y) = -M \ominus (-1)M(\ln x)^{\frac{1}{2}}(\ln y)^{\frac{1}{2}} \ominus M(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}}. \end{cases}$$

For the triangular fuzzy number $M = (3, 4, 5)$, its corresponding $\gamma_1$-level set and $\gamma_2$-level set can be determined as

$$[M]^{\gamma_1} = [3 + \gamma_1, 5 - \gamma_1], \quad [M]^{\gamma_2} = [3 + \gamma_2, 5 - \gamma_2].$$

Similarly, it is easy to see

$$\begin{aligned} [\psi(x,y)]^{\gamma_1} &= [2M]^{\gamma_1} = 2[3 + \gamma_1, 5 - \gamma_1], \\ [\varphi(x,y)]^{\gamma_2} &= [-M]^{\gamma_2} = -[3 + \gamma_2, 5 - \gamma_2]. \end{aligned}$$

Subsequently, one has

$$[u(x,y)]^{\gamma_1} = [2(3 + \gamma_1) - 2(3 + \gamma_1)\ln x \ln y, 2(5 - \gamma_1) - 2(5 - \gamma_1)\ln x \ln y], \tag{4.2}$$

$$[v(x,y)]^{\gamma_2} = \Big[ -(3+\gamma_2) + (3+\gamma_2)(\ln x)^{\frac{1}{2}}(\ln y)^{\frac{1}{2}} - (3+\gamma_2)(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}}, \\ -(5-\gamma_2) + (5-\gamma_2)(\ln x)^{\frac{1}{2}}(\ln y)^{\frac{1}{2}} - (5-\gamma_2)(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}} \Big]. \quad (4.3)$$

In Figure 3, we present the $\gamma_1$-level set and $\gamma_2$-level set of the fuzzy ‡-type solution $u(x,y)$ and $v(x,y)$, respectively. We fix $y$ at five constant values: $y = 1.18, 1.20, 1.22, 1.24$, and $1.26$, such that $[u(x,y)]^{\gamma_1}/[v(x,y)]^{\gamma_2}$ varies with $x$ and $\gamma_1/\gamma_2$, respectively, to obtain the left and right images. In both images, the upper part represents $[u(x,y)]^{\gamma_1}$, while the lower part represents $[v(x,y)]^{\gamma_2}$. It can be observed that each plane in the left image represents $[u(x,y)]^{\gamma_1}/[v(x,y)]^{\gamma_2}$ with a fixed $y$. Similarly, each plane in the right image represents $[u(x,y)]^{\gamma_1}/[v(x,y)]^{\gamma_2}$ with a fixed $x$, where $x = 1.18, 1.20, 1.22$, and $1.24$.

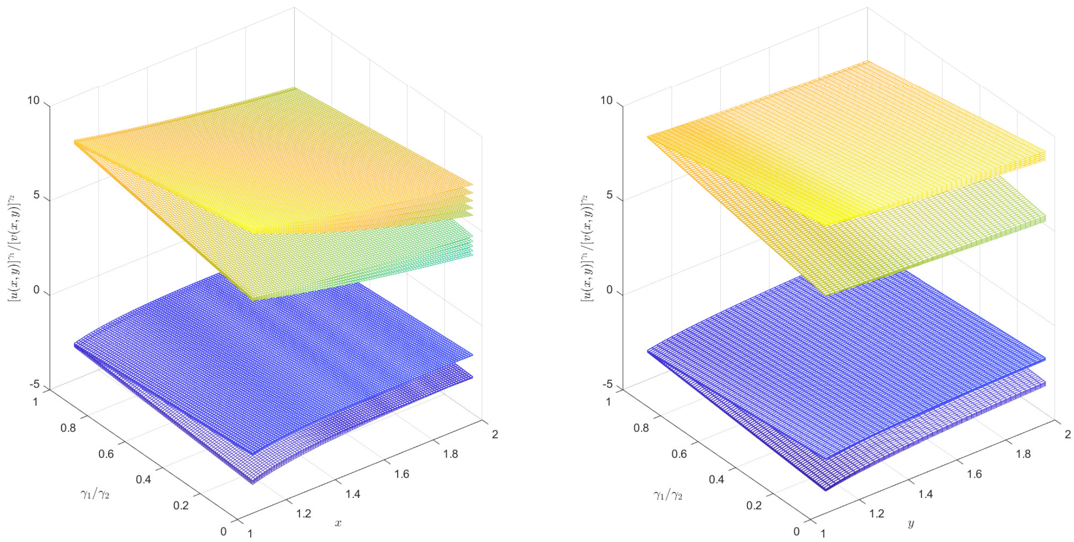


**Figure 3.** The level set of the ‡-type solution in (4.1).

For the ‡-type solution, it is evident that $(\mathbf{H_4})$ is satisfied. The following will provide an explanation for the satisfaction of $(\mathbf{H_5})$. By combining (4.2) and (4.3),

$$len[u(x,y)]^{\gamma_1} = (4-4\gamma_1)(1-\ln x \ln y),$$
$$len[v(x,y)]^{\gamma_2} = (2-2\gamma_2)\left[-1 + (\ln x)^{\frac{1}{2}}(\ln y)^{\frac{1}{2}} - (\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}}\right],$$

and

$$\left[(-1)^H_F \mathcal{I}^{\frac{1}{2}}_1 g(x,y,u(x,y))\right]^{\gamma_2} = \left(-\frac{27}{4}(\ln x)^{\frac{1}{2}}(\ln y)^{\frac{1}{2}} + 9(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}} - \frac{135}{64}(\ln x)(\ln y)\left[\Gamma\left(\frac{1}{2}\right)\right]^4\right)[M]^{\gamma_2}$$

can be further obtained. Then, we get

$$len\left[(-1)^H_F \mathcal{I}^{\frac{1}{2}}_1 g(x,y,u(x,y))\right]^{\gamma_2} \geq 2\gamma_2 - 2,$$

and

$$len\left[(-1)_F^H\mathcal{I}_1^{\frac{1}{2}}g(x,y,u(x,y))\right]^{\gamma_2} \geq len[\varphi(x,y)]^{\gamma_2}. \tag{4.4}$$

Combining (4.4) with Properties 21 in [55], we can conclude that the following H-difference exists:

$$\varphi(x,y) \ominus (-1)_F^H\mathcal{I}_1^{\frac{1}{2}}g(x,y,u(x,y)).$$

Setting

$$\widetilde{U}(x,y) = \psi(x,y) \ominus (-1)_F^H\mathcal{I}_1^{\frac{1}{2}}f(x,y,v(x,y)),$$

then one obtains

$$\left[(-1)_F^H\mathcal{I}_1^{\theta}f(x,y,v(x,y))\right]^{\gamma_1} = \left(-\frac{55}{9}(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}} + \frac{55}{9}(\ln x)^{\frac{5}{2}}(\ln y)^{\frac{5}{2}} - (\ln x)(\ln y)\right)[M]^{\gamma_1}, \tag{4.5}$$

and

$$len[(-1)_F^H\mathcal{I}_1^{\theta}g(x,y,\widetilde{U}(x,y))]^{\gamma_2} = \Big[-\frac{(144-9\pi^2)}{64}\ln x\ln y - \left(\frac{42624+729\pi^2}{9216}\right)(\ln x)^2(\ln y)^2 - \frac{9}{8}(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}} + \frac{495}{72}(\ln x)^3(\ln y)^3\Big]len[M]^{\gamma_2}.$$

Since $len[(-1)_F^H\mathcal{I}_1^{\theta}g(x,y,\widetilde{U}(x,y))]^{\gamma_2} \geq len[\varphi(x,y)]^{\gamma_2}$, and based on Properties 21 in [55], we can conclude that $\varphi(x,y) \ominus (-1)_F^H\mathcal{I}_1^{\frac{1}{2}}g(x,y,\widetilde{U}(x,y))$ exists. Similarly, letting $\widetilde{V}(x,y) = \varphi(x,y) \ominus (-1)_F^H\mathcal{I}_1^{\frac{1}{2}}g(x,y,u(x,y))$, we proceed to show that $\psi(x,y) \ominus (-1)f(x,y,\widetilde{V}(x,y))$ exists. By using (4.5),

$$len\left[(-1)_F^H\mathcal{I}_1^{\theta}f(x,y,v(x,y))\right]^{\gamma_1} = (2-2\gamma_1)\left(-\frac{55}{9}(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}} + \frac{55}{9}(\ln x)^{\frac{5}{2}}(\ln y)^{\frac{5}{2}} - (\ln x)(\ln y)\right)$$

can be derived. Since $len\left[(-1)_F^H\mathcal{I}_1^{\theta}f(x,y,v(x,y))\right]^{\gamma_1} \geq len\left[\psi(x,y)\right]^{\gamma_1}$, this suggests that $\psi(x,y) \ominus (-1)_F^H\mathcal{I}_1^{\theta}f(x,y,v(x,y))$ exists. Subsequently, one has

$$len\left[(-1)_F^H\mathcal{I}_1^{\theta}f\left(x,y,\widetilde{V}(x,y)\right)\right]^{\gamma_1} = \Big[2(\ln x)(\ln y) - \left(\frac{243+512\pi^2}{36\pi^2}\right)(\ln x)^{\frac{3}{2}}(\ln y)^{\frac{3}{2}} - \frac{135\pi^2}{64}(\ln x)^2(\ln y)^2 + \left(\frac{128}{9\pi^2}-9\right)(\ln x)^{\frac{5}{2}}(\ln y)^{\frac{5}{2}} + \frac{2375\pi^2(\ln x)^3(\ln y)^3}{1024}\Big]len[M]^{\gamma_1}.$$

On the other hand, $len\left[(-1)_F^H\mathcal{I}_1^{\theta}f\left(x,y,\widetilde{V}(x,y)\right)\right]^{\gamma_1} \geq len\left[\psi(x,y)\right]^{\gamma_1}$ is obtained, which naturally implies the existence of $\psi(x,y) \ominus (-1)_F^H\mathcal{I}_1^{\theta}f\left(x,y,\widetilde{V}(x,y)\right)$. The above process demonstrates that the H-difference in ‡-type solution of (4.1) exist, which further satisfies ($\mathbf{H_5}$).

As Lemma 4 indicates, (1.5) is equivalent to the Volterra integral system (2.11) and (2.12). Furthermore, by Theorems 1, 2, 4, and 5, we can conclude that the solution of (1.5) is also equivalent to the Volterra integral system (2.11) and (2.12). This demonstrates the equivalence between (1.5) and its solution system. Hence, investigating the chaotic behavior of the solution system is of practical significance. For instance, Chang and Hong [56] investigated the necessary and sufficient conditions for chaos in the solution semigroup of the Lasota equation. Moreover, Criens [57] explored how chaotic properties propagate in the mild solutions of semilinear stochastic partial differential equations with weak interactions. In this section, we further investigate the chaotic phenomena in the solution system of (4.1).

In order to deal with Question 2, we use the Lyapunov exponent [58] to investigate the local stability (i.e., $2(3+\gamma_1) - 2(3+\gamma_1)\ln x \ln y$) of the ‡-type solution system and simulate the resulting chaotic behavior through circuit simulation. To investigate the stability of the local ‡-type solution system, we propose the following iterative scheme

$$x_{n+1} = 2(3+\gamma_1) - 2(3+\gamma_1)\ln|x_n|\,\ln y. \tag{4.6}$$

By fixing different values of $y$ in (4.6), we generated bifurcation diagrams corresponding to each $y$ using (4.6), as shown in Figure 4.

In Figure 4, it can be observed that as the value of $\gamma_1$ increases, the system described by (4.6) gradually transitions from a stable state to a chaotic state. Moreover, the onset of chaos occurs earlier as the fixed value of $y$ increases.

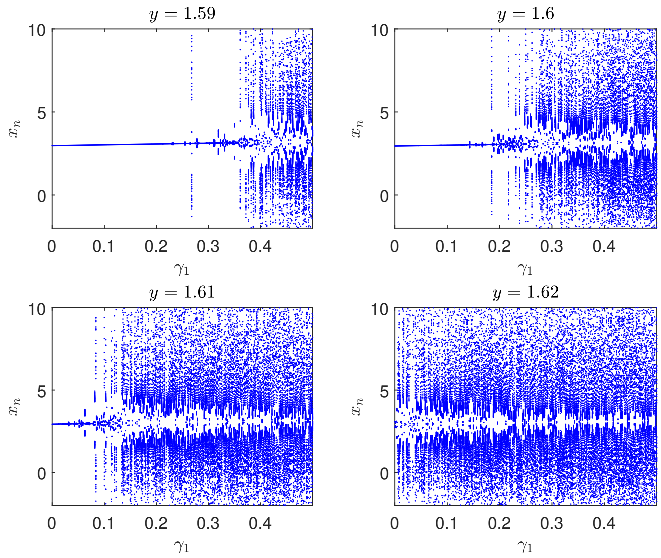


**Figure 4.** bifurcation diagrams.

On the other hand, we also present the Lyapunov exponent (Le) plots for different values of $y$, as presented in Figure 5.

The Le index plots (see Figure 5) for each fixed value of $y$ exhibit similar trends as the bifurcation diagrams (see Figure 5). Specifically, as the fixed value of $y$ increases, chaotic behavior emerges at an earlier stage. In the following, we fix parameter $\gamma_1$ in (4.6) to 0.4 and $y$ to 1.6, and further implement the (4.6) using a circuit. We reduce (4.6) into a non-iterative form $f(x) = 6.8 - 6.8\ln(1.6)\ln|x|$, which is realized by the circuit shown in Figure 6.

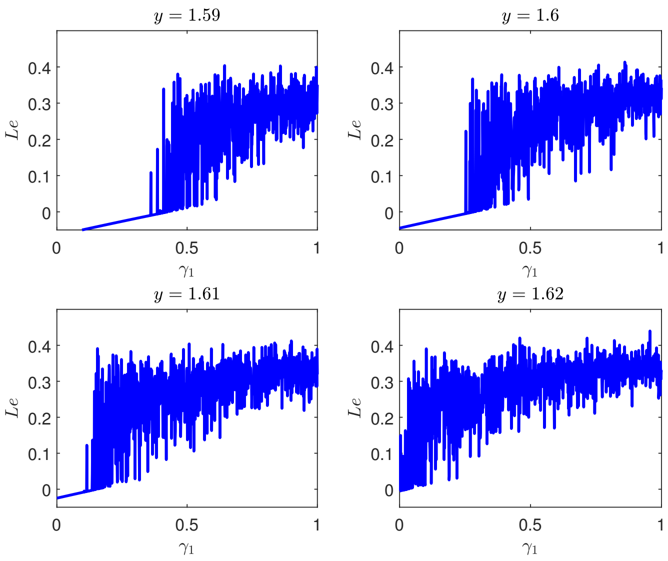


**Figure 5.** Lyapunov exponent (Le) plots.

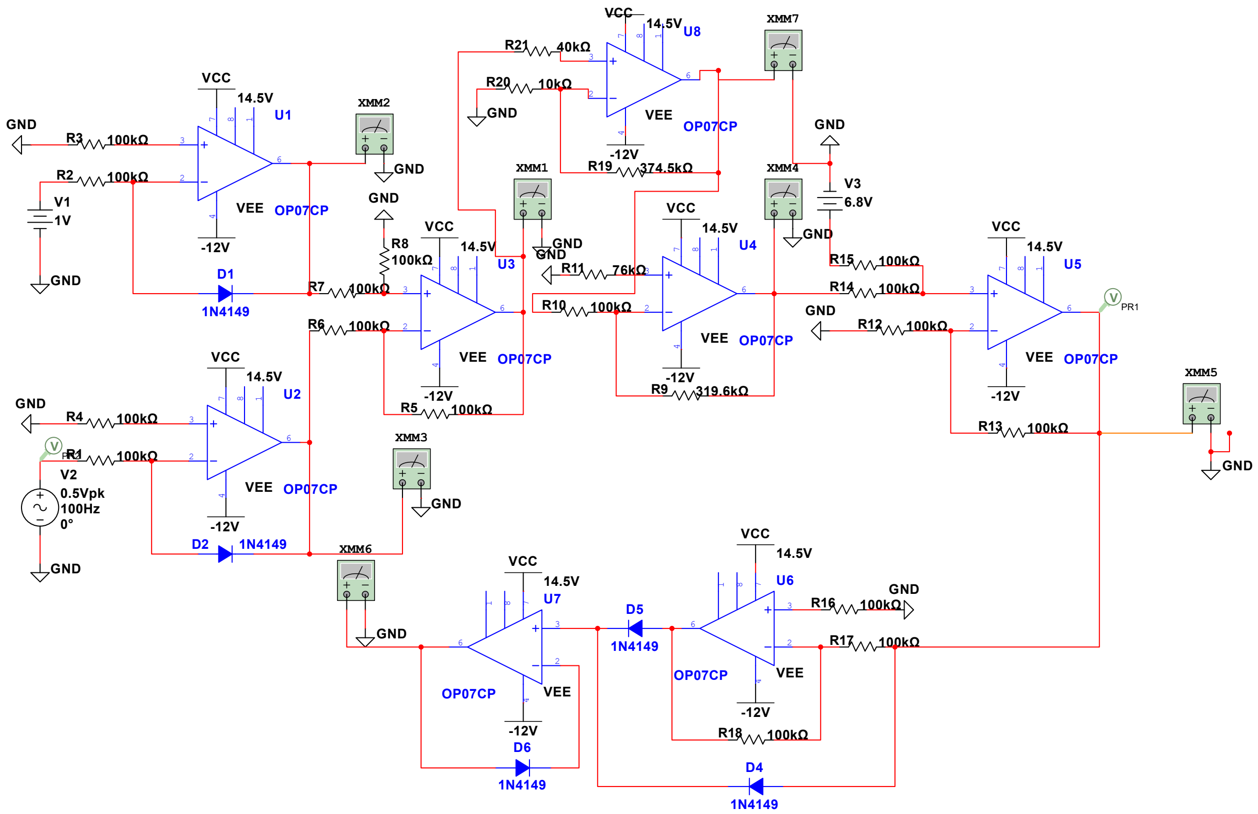

**Figure 6.** The circuit implementation of the degenerate form $f(x) = 6.8 - 6.8\ln(1.6)\ln|x|$.

To validate the circuit, a sinusoidal wave $x = 0.4\sin(200\pi t) + 0.6$ is applied as the input, and the actual output (see Figure 7) is compared with the theoretical values (see Figure 8).

Comparing the actual and theoretical values, we can observe that the results are in good agreement, which demonstrates that the circuit design shown in Figure 6 is reasonable. Based on the above results, we implement the circuit realization of (4.6), as shown in Figure 9.

The meanings of the components in Figures 6 and 9 are listed in Table 1.

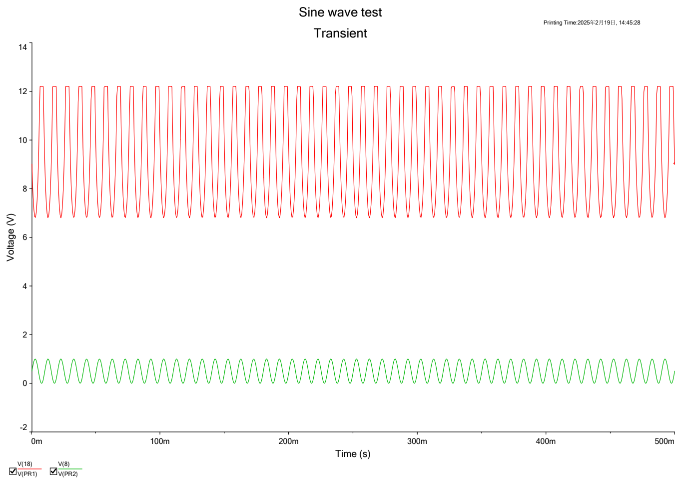


**Figure 7.** The experimental values under sinusoidal wave input.

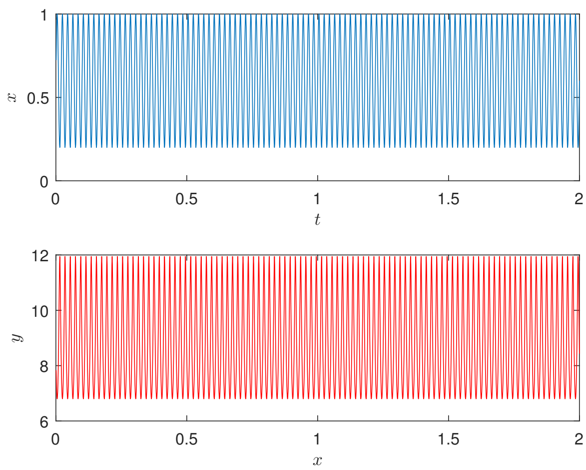


**Figure 8.** The theoretical values under sinusoidal wave input.

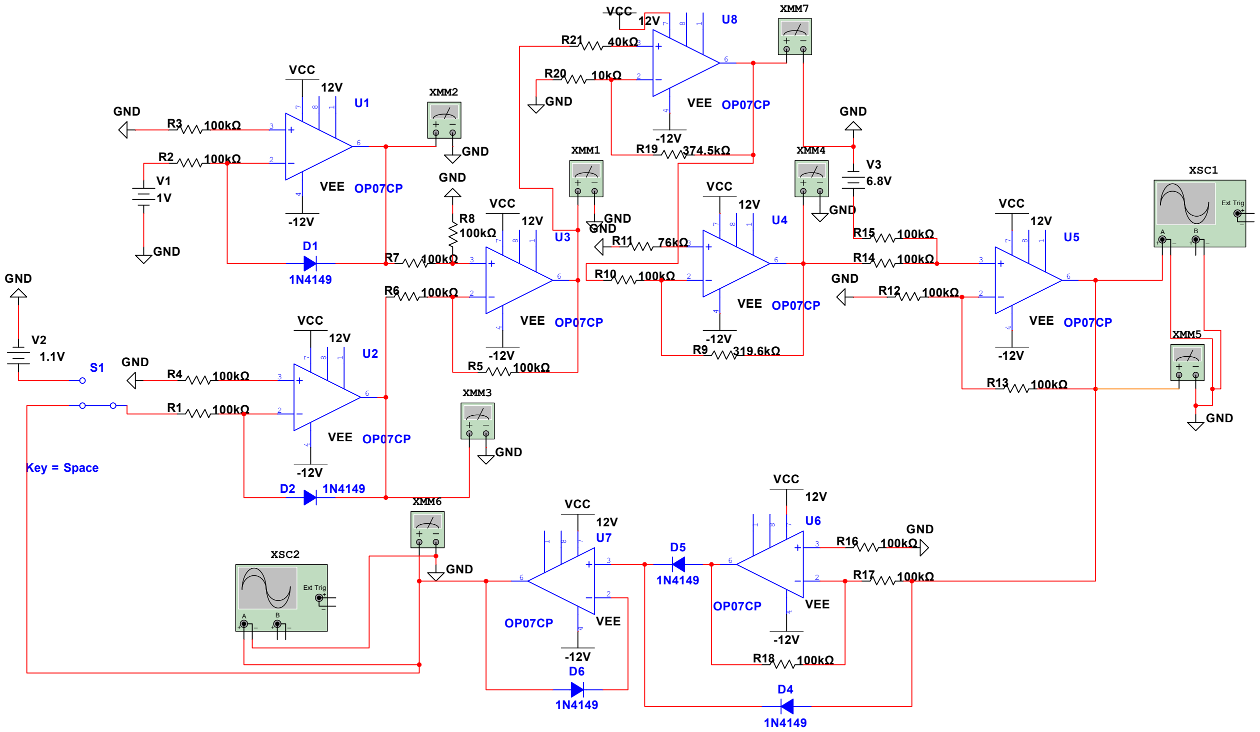


**Figure 9.** The circuit implementation of (4.6).

**Table 1.** Names of electronic components.

| component | name |
|---|---|
| $R_i$ $(i = 1, 2, ..., 21)$ | Resistor |
| $V_i$ $(i = 1, 2, 3)$ | Voltage |
| $S_1$ | Switch |
| GND | Ground |
| VCC | Positive supply voltage |
| VEE | Negative supply voltag |
| OP07CP $(U_i$ $i = 1, 2, ..., 8)$ | Precision operational amplifier |
| 1N4149 $(D_i$ $i = 1, 2, ..., 6)$ | Small Signal Diode |
| XSC$_i$ $(i = 1, 2)$ | Virtual Oscilloscope |
| XMM$_i$ $(i = 1, 2, ..., 7)$ | Virtual Voltmeter |

We select a portion of the numerical results from the iterative process of the circuit, and using MATLAB, we obtain the circuit simulation results, as shown in Figure 10. From Figure 10, it can be observed that in the early stages, the simulated values agree well with the true values; however, as the number of iterations increases, the discrepancy between the simulated and true values gradually becomes more pronounced. This behavior is attributable to the errors generated by the simulation components in Multisim, which tend to accumulate with increasing iteration count.

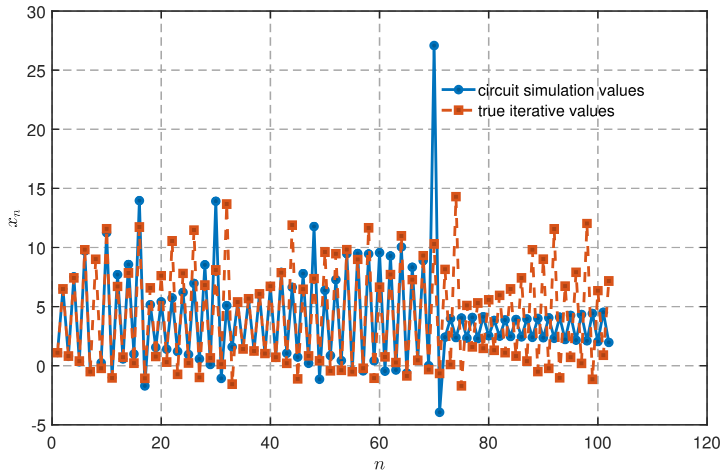


**Figure 10.** Circuit simulation results of (4.6).

*4.2. Applications to fuzzy projection neural networks*

In this subsection, we establish relevant theorems for the fuzzy fractional-order partial differential coupled projection neural network system (1.4), which is derived from (1.5), by reformulating the Lipschitz condition (2.13) and incorporating assumptions ($\mathbf{H_4}$) and ($\mathbf{H_5}$) based on the concept of fuzzy projection. Thus, the theorems in this subsection can be readily derived from Theorems 1–5. Hence, Theorems 6–8 are selected for detailed proof.

We introduce the fundamental concepts of fuzzy projection as follows:

**Definition 8.** [59] *For any fuzzy function* $\Lambda(x,y) \in \mathbb{F}_\iota$ *and* $\iota \in \{1,2\}$, *its projection* $P_{\mathbb{F}_\iota(\mathbb{D}_\iota)} : \mathbb{F}_\iota \to \mathbb{F}_\iota(\mathbb{D}_\iota)$ *on* $\mathbb{F}_\iota(\mathbb{D}_\iota) \subset \mathbb{F}_\iota$ *is defined as*

$$P_{\mathbb{F}_\iota(\mathbb{D}_\iota)}(\Lambda(x,y)) = \underset{\overline{\Lambda}\in\mathbb{F}_\iota(D_\iota)}{\arg\min}\rho(\Lambda,\overline{\Lambda}),$$

*where* $\mathbb{F}_\iota(\mathbb{D}_\iota)$ *denotes the set of all fuzzy numbers defined on* $\mathbb{D}_\iota \subset \mathbb{R}_\iota$ *with holder continuity, and* $\rho$ *is defined by* (2.7).

It is readily observed that $P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y), v(x,y))$ and $P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y), u(x,y))$ in (1.4) are special cases of $f(x,y,v(x,y))$ and $g(x,y,u(x\ ,y))$ in (1.5). By modifying the Lipschitz condition into the form of

$$\begin{cases} d_\infty(P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)l_1(x,y)), P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)l_2(x,y))) \le \overline{\mathcal{L}}_1 d_\infty(l_1,l_2), \\ d_\infty(P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\delta_1(x,y)), P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\delta_2(x,y))) \le \overline{\mathcal{L}}_2 d_\infty(\delta_1,\delta_2), \end{cases} \tag{4.7}$$

with constants $\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2 \in (0,1)$, where $l_1, l_2 \in C(\mathcal{K},\mathbb{F}_1)$, $\delta_1,\delta_2 \in C(\mathcal{K},\mathbb{F}_2)$, we can obtain the existence and uniqueness of the †-type and ‡-type solutions of the fuzzy projection neural network (1.4), respectively.

We consider the continuous dependence of the solution on the initial value. Similarly, we modify the initial conditions of (1.4) as follows:

$$\begin{cases} {}^{CH}_{gH}D^{\theta}_{k}\widehat{u}_1(x,y) = P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v}_1(x,y)), \\ {}^{CH}_{gH}D^{\vartheta}_{k}\widehat{v}_1(x,y) = P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u}_1(x,y)), \end{cases} \tag{4.8}$$

$$\begin{cases}\widehat{u}_1(x,1) = \lambda \int_1^x \int_1^e \widehat{u}_1(s,t)\frac{ds}{s}\frac{dt}{t} + \xi_{11}(x),\\ \widehat{u}_1(1,y) = \lambda \int_1^e \int_1^y \widehat{u}_1(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_{21}(y),\\ \widehat{v}_1(x,1) = \eta_{11}(x), \widehat{v}_1(1,y) = \eta_{21}(y),\end{cases}$$

and

$$\begin{cases}{}^{CH}_{gH}D^{\theta}_{k}\widehat{u}_2(x,y) = P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v}_2(x,y)),\\ {}^{CH}_{gH}D^{\vartheta}_{k}\widehat{v}_2(x,y) = P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u}_2(x,y)),\\ \widehat{u}_2(x,1) = \lambda \int_1^x \int_1^e \widehat{u}_2(s,t)\frac{ds}{s}\frac{dt}{t} + \xi_{12}(x),\\ \widehat{u}_2(1,y) = \lambda \int_1^e \int_1^y \widehat{u}_2(s,t)\frac{dt}{t}\frac{ds}{s} + \xi_{22}(y),\\ \widehat{v}_2(x,1) = \eta_{12}(x), \widehat{v}_2(1,y) = \eta_{22}(y).\end{cases} \tag{4.9}$$

Thus, the †-type and ‡-type solutions of (4.8), (4.9) are severally denoted by

$$\begin{cases}\widehat{u}_1(x,y) = \widehat{\psi}_1(x,y) +^H_F \mathcal{I}^{\theta}_1 P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v}_1(x,y)),\\ \widehat{v}_1(x,y) = \widehat{\varphi}_1(x,y) +^H_F \mathcal{I}^{\vartheta}_1 P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u}_1(x,y)),\\ \widehat{u}_2(x,y) = \widehat{\psi}_2(x,y) +^H_F \mathcal{I}^{\theta}_2 P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v}_2(x,y)),\\ \widehat{v}_2(x,y) = \widehat{\varphi}_2(x,y) +^H_F \mathcal{I}^{\vartheta}_2 P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u}_2(x,y)),\end{cases}$$

and

$$\begin{cases}\widehat{u}_1(x,y) = \widehat{\psi}_1(x,y) \ominus (-1)^H_F\mathcal{I}^{\theta}_1 P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v}_1(x,y)),\\ \widehat{v}_1(x,y) = \widehat{\varphi}_1(x,y) \ominus (-1)^H_F\mathcal{I}^{\vartheta}_1 P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u}_1(x,y)),\\ \widehat{u}_2(x,y) = \widehat{\psi}_2(x,y) \ominus (-1)^H_F\mathcal{I}^{\theta}_2 P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v}_2(x,y)),\\ \widehat{v}_2(x,y) = \widehat{\varphi}_2(x,y) \ominus (-1)^H_F\mathcal{I}^{\vartheta}_2 P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u}_2(x,y)).\end{cases}$$

Here, $\psi_1(x,y)$, $\varphi_1(x,y)$, $\psi_2(x,y)$, and $\varphi_2(x,y)$ are defined in the similar manner as in (4.10):

$$\begin{cases}\widehat{\psi}_1(x,y) = \lambda \int_1^x \int_1^e \widehat{u}_1(x,y)\frac{dx}{x}\frac{dy}{y} + \lambda \int_1^e \int_1^y \widehat{u}_1(x,y)\frac{dy}{y}\frac{dx}{x} + \xi_{21}(y) + \xi_{11}(x) \ominus \xi_{11}(0),\\ \widehat{\psi}_2(x,y) = \lambda \int_1^x \int_1^e \widehat{u}_2(x,y)\frac{dx}{x}\frac{dy}{y} + \lambda \int_1^e \int_1^y \widehat{u}_2(x,y)\frac{dy}{y}\frac{dx}{x} + \xi_{22}(y) + \xi_{12}(x) \ominus \xi_{12}(0),\end{cases} \tag{4.10}$$

$$\begin{cases}\varphi_1(x,y) = \eta_{21}(y) + \eta_{11}(x) \ominus \eta_{11}(0),\\ \varphi_2(x,y) = \eta_{22}(y) + \eta_{12}(x) \ominus \eta_{12}(0).\end{cases}$$

We now establish the continuous dependence of the solutions of (4.8) and (4.9) on the initial conditions.

**Theorem 6.** *If* $(\mathbf{C_1})$ *and* $(\mathbf{C_2})$ *in Lemma* 5 *hold,* $\lambda < [\ln(pq)]^{-1}$*,* $P_{\mathbb{F}_1(\mathbb{D}_1)} \in C_K(\mathbb{F}_1, \mathbb{F}_1(\mathbb{D}_1))$ *and* $P_{\mathbb{F}_2(\mathbb{D}_2)} \in C_K(\mathbb{F}_2, \mathbb{F}_2(\mathbb{D}_2))$ *satisfy the Lipschitz condition* (4.7) *with constants* $\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2 \in (0,1)$*, then for different initial conditions* $\widehat{\psi}_1(x,y)$*,* $\widehat{\psi}_2(x,y)$*,* $\varphi_1(x,y)$ *and* $\varphi_2(x,y)$ *decided by* (4.10)*, the inequality*

$$\begin{pmatrix}\rho(\widehat{u}_1(x,y), \widehat{u}_2(x,y))\\ \rho(\widehat{v}_1(x,y), \widehat{v}_2(x,y))\end{pmatrix} \leq \begin{pmatrix}\frac{l(\ln p)^{\theta_1}(\ln q)^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)} & 0\\ 0 & \lambda\ln(pq) + \frac{(\ln p)^{\vartheta_1}(\ln q)^{\vartheta_2}}{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}\end{pmatrix}$$

$$\times \sum_{i=0}^{\infty} Q^i \begin{pmatrix} \overline{\mathcal{L}}_1\rho(\varphi_1(x,y),\varphi_2(x,y)) \\ \overline{\mathcal{L}}_2\rho(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix} + \begin{pmatrix} l\rho(\chi_1(x,y),\chi_2(x,y)) \\ \rho(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix} \quad (4.11)$$

*holds, where* $Q$ *is the same as in Lemma* 5*,* $l = [1 - \lambda \ln(pq)]^{-1}$*, the functions* $\widehat{u_1}$*,* $\widehat{u_2}$*,* $\widehat{v_1}$*, and* $\widehat{v_2}$ *are separately defined as in* (3.7) *and* (3.8)*, and* $\chi_1(x,y)$*,* $\chi_2(x,y)$*,* $\varphi_1(x,y)$*, and* $\varphi_2(x,y)$ *represent different initial conditions with* $\chi_1(x,y) = \xi_{21}(y) + \xi_{11}(x) \ominus \xi_{11}(0)$ *and* $\chi_2(x,y) = \xi_{22}(y) + \xi_{12}(x) \ominus \xi_{12}(0)$*.*

*Proof.* Letting $l = [1 - \lambda \ln(pq)]^{-1}$, then, according to (3.7) and (3.8), one obtains

$$\begin{aligned} d_\infty(\widehat{u_1}(x,y),\widehat{u_2}(x,y)) \leq & \lambda \int_1^x \int_1^e d_\infty(\widehat{u_1}(s,t),\widehat{u_2}(s,t)) \frac{ds}{s}\frac{dt}{t} \\ & + \lambda \int_1^e \int_1^y d_\infty(\widehat{u_1}(s,t),\widehat{u_2}(s,t)) \frac{dt}{t}\frac{ds}{s} \\ & +_F^H \mathcal{I}_1^\theta \left[d_\infty(P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v_1}(x,y)), P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v_2}(x,y)))\right] \\ & + d_\infty(\chi_1(x,y),\chi_2(x,y)) \\ \leq & \; l_F^H \mathcal{I}_1^\theta \left[d_\infty(P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v_1}(x,y)), P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)\widehat{v_2}(x,y)))\right] \\ & + l d_\infty(\chi_1(x,y),\chi_2(x,y)), \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} d_\infty(\widehat{v_1}(x,y),\widehat{v_2}(x,y)) \leq & d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ & +_F^H \mathcal{I}_1^\theta d_\infty \left[P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u_1}(x,y)), P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widehat{u_1}(x,y)))\right]. \end{aligned} \quad (4.13)$$

Next, for each $(x,y) \in \mathcal{K}$, let $s_1' = {}^{CH}_{gH}D_k^\theta \widehat{u_1}(x,y)$, $s_2' = {}^{CH}_{gH}D_k^\theta \widehat{u_2}(x,y)$, $t_1' = {}^{CH}_{gH}D_k^\vartheta \widehat{v_1}(x,y)$, $t_2' = {}^{CH}_{gH}D_k^\vartheta \widehat{v_2}(x,y)$. It is evident that

$$\begin{cases} s_1'(x,y) = P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)[\varphi_1(x,y) +_F^H \mathcal{I}_1^\vartheta t_1'(x,y)]\right), \\ t_1'(x,y) = P_{\mathbb{F}_2(\mathbb{D}_2)}\left(m_2(x,y)[\widehat{\psi_1}(x,y) +_F^H \mathcal{I}_1^\vartheta s_1'(x,y)]\right), \end{cases}$$

and

$$\begin{cases} s_2'(x,y) = P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)[\varphi_2(x,y) +_F^H \mathcal{I}_2^\vartheta t_2'(x,y)]\right), \\ t_2'(x,y) = P_{\mathbb{F}_2(\mathbb{D}_2)}\left(m_2(x,y)[\widehat{\psi_2}(x,y) +_F^H \mathcal{I}_2^\vartheta s_2'(x,y)]\right). \end{cases}$$

Furthermore, we obtain

$$\begin{aligned} d_\infty(s_1', s_2') \leq & \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) + \mathcal{L}_1 \cdot_F^H \mathcal{I}_1^\vartheta d_\infty(t_1'(x,y), t_2'(x,y)), \\ d_\infty(t_1', t_2') \leq & \mathcal{L}_2 \lambda \int_1^x \int_1^e d_\infty(s_1', s_2') \frac{dx}{x}\frac{dy}{y} + \mathcal{L}_2 \lambda \int_1^e \int_1^y d_\infty(s_1', s_2') \frac{dy}{y}\frac{dx}{x} \\ & + \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) + \mathcal{L}_2 \cdot_F^H \mathcal{I}_1^\theta d_\infty(s_1'(x,y), s_2'(x,y)) \end{aligned}$$

and

$$\begin{pmatrix} d_\infty(s_1', s_2') \\ d_\infty(t_1', t_2') \end{pmatrix} \leq \begin{pmatrix} 0 & \mathcal{L}_1 \cdot_F^H \mathcal{I}_1^\vartheta \\ \mathcal{L}_2 \cdot_F^H \mathcal{I}_1^\theta + \mathcal{L}_2 \cdot_R^L \mathcal{I} & 0 \end{pmatrix} \begin{pmatrix} d_\infty(s_1', s_2') \\ d_\infty(t_1', t_2') \end{pmatrix} + \begin{pmatrix} \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_2(x,y)) \\ \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix}.$$

By (4.12) and (4.13) and Lemma 5, now one can get

$$\begin{pmatrix} d_\infty(\widehat{u_1},\widehat{u_2}) \\ d_\infty(\widehat{v_1},\widehat{v_2}) \end{pmatrix} \leq \begin{pmatrix} ld_\infty(\chi_1(x,y),\chi_2(x,y)) \\ d_\infty(\varphi_1(x,y),\varphi_1(x,y)) \end{pmatrix} + \begin{pmatrix} l{}_F^H\mathcal{I}_1^\theta & 0 \\ 0 & {}_F^H\mathcal{I}_1^\vartheta \end{pmatrix}\begin{pmatrix} d_\infty(s_1',s_2') \\ d_\infty(t_1',t_2') \end{pmatrix}$$
$$= \begin{pmatrix} ld_\infty(\chi_1(x,y),\chi_2(x,y)) \\ d_\infty(\varphi_1(x,y),\varphi_1(x,y)) \end{pmatrix} + \begin{pmatrix} l{}_F^H\mathcal{I}_1^\theta & 0 \\ 0 & {}_F^H\mathcal{I}_1^\vartheta \end{pmatrix}\sum_{i=0}^{\infty} Q^i \begin{pmatrix} \mathcal{L}_1 d_\infty(\varphi_1(x,y),\varphi_1(x,y)) \\ \mathcal{L}_2 d_\infty(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix},$$

and further derive from (2.7) that

$$\begin{pmatrix} \rho(\widehat{u_1}(x,y),\widehat{u_2}(x,y)) \\ \rho(\widehat{v_1}(x,y),\widehat{v_2}(x,y)) \end{pmatrix} \leq \begin{pmatrix} \frac{l(\ln p)^{\theta_1}(\ln q)^{\theta_2}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)} & 0 \\ 0 & \lambda\ln(pq) + \frac{(\ln p)^{\vartheta_1}(\ln q)^{\vartheta_2}}{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)} \end{pmatrix}$$
$$\times \sum_{i=0}^{\infty} Q^i \begin{pmatrix} \mathcal{L}_1\rho(\varphi_1(x,y),\varphi_2(x,y)) \\ \mathcal{L}_2\rho(\chi_1(x,y),\chi_2(x,y)) \end{pmatrix} + \begin{pmatrix} l\rho(\chi_1(x,y),\chi_2(x,y)) \\ \rho(\varphi_1(x,y),\varphi_2(x,y)) \end{pmatrix}$$

□

Let us discuss the case where $\lambda = 0$.

**Situation I**: $\lambda = 0$

**Theorem 7.** *If* $P_{\mathbb{F}_1(\mathbb{D}_1)} \in C_K(\mathbb{F}_1,\mathbb{F}_1(\mathbb{D}_1))$ *and* $P_{\mathbb{F}_2(\mathbb{D}_2)} \in C_K(\mathbb{F}_2,\mathbb{F}_2(\mathbb{D}_2))$ *fulfill the Lipschitz condition* (4.7) *with constants* $\overline{\mathcal{L}_1}, \overline{\mathcal{L}_2} \in (0,1)$*, then* (1.4) *admits a unique* †*-type solution on* $\mathcal{K}$*.*

*Proof.* Similar to the previous approach, we first define the operators $\widetilde{\mathbb{T}}_1 : \mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_1(\mathbb{D}_1)) \to \mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_1(\mathbb{D}_1))$ and $\widetilde{\mathbb{G}}_1 : \mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_2(\mathbb{D}_2)) \to \mathbb{C}_{gH}^{2,2}(\mathcal{K},\mathbb{F}_2(\mathbb{D}_2))$ as

$$\widetilde{\mathbb{T}}_1(u(x,y)) = \xi_1(x) + \xi_2(y) \ominus \xi_1(0) +_F^H \mathcal{I}_1^\theta P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)\left[\varphi(x,y) +_F^H \mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u(x,y))\right]\right)$$

and

$$\widetilde{\mathbb{G}}_1(v(x,y)) = \varphi(x,y) +_F^H \mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}\left(m_2(x,y)\left[\xi_2(y) + \xi_1(x) \ominus \xi_1(0) +_F^H \mathcal{I}_1^\theta P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)v(x,y))\right]\right).$$

Using the Lipschitz condition (4.7), we obtain

$$d_\infty\left({}_F^H\mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(s,t)u_1(s,t)),{}_F^H \mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(s,t)u_2(s,t))\right)$$
$$\leq \frac{\overline{\mathcal{L}_2}}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}\int_1^s\int_1^t\left(\ln\frac{s}{\zeta}\right)^{\vartheta_1-1}\left(\ln\frac{t}{\sigma}\right)^{\vartheta_2-1} d_\infty(u_1(\zeta,\sigma),u_2(\zeta,\sigma))\frac{d\zeta}{\zeta}\frac{d\sigma}{\sigma}$$
$$= \frac{\overline{\mathcal{L}_2}\,(\ln s)^{2\vartheta_1-1}(\ln t)^{2\vartheta_2-1}\,\Gamma^2(\vartheta_1)\Gamma^2(\vartheta_2)}{\Gamma(2\vartheta_1)\Gamma(2\vartheta_2)} d_{1-\vartheta}(u_1,u_2).$$

Furthermore, one can get

$$d_\infty\left(\widetilde{\mathbb{T}}_1(u_1(x,y)),\widetilde{\mathbb{T}}_1(u_2(x,y))\right)$$
$$\leq d_\infty\Big({}_F^H\mathcal{I}_1^\theta P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)\left[\varphi(x,y) +_F^H \mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u_1(x,y))\right]\right),$$
$${}_F^H\mathcal{I}_1^\theta P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)\left[\varphi(x,y) +_F^H \mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u_2(x,y))\right]\right)\Big)$$

$$\leq \overline{\mathcal{L}}_1 \cdot {}_F^H \mathcal{I}_1^{\theta} d_\infty \left( {}_F^H\mathcal{I}_1^{\vartheta} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u_1(x,y)), {}_F^H \mathcal{I}_1^{\vartheta} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u_2(x,y))(s,t)) \right)$$
$$\leq \frac{\overline{\mathcal{L}}_1\overline{\mathcal{L}}_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)(\ln x)^{2\vartheta_1+\theta_1-1}(\ln y)^{2\vartheta_2+\theta_2-1}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)} \cdot d_{1-\vartheta}(u_1,u_2).$$

Next, assume that the following inequality holds for $n = k$:

$$d_\infty(\widetilde{\mathbb{T}}_1^k(u_1(x,y)), \widetilde{\mathbb{T}}_1^k(u_2(x,y)))$$
$$\leq \frac{\overline{\mathcal{L}}_1^k\overline{\mathcal{L}}_2^k(\ln x)^{(k+1)\vartheta_1+k\theta_1-1}(\ln y)^{(k+1)\vartheta_2+k\theta_2-1}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((k+1)\vartheta_1+k\theta_1)\Gamma((k+1)\vartheta_2+k\theta_2)} \cdot d_{1-\vartheta}(u_1,u_2). \tag{4.14}$$

Then, for $n = k+1$, one can get

$$d_\infty(\widetilde{\mathbb{T}}_1^{k+1}(u_1(x,y)), \widetilde{\mathbb{T}}_1^{k+1}(u_2(x,y)))$$
$$\leq d_\infty\Big( {}_F^H\mathcal{I}_1^{\theta} P_{\mathbb{F}_1(\mathbb{D}_1)} \left( m_1(x,y) \left[ \varphi(x,y) + {}_F^H \mathcal{I}_1^{\vartheta} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widetilde{\mathbb{T}}_1^k(u_1(x,y))) \right] \right),$$
$${}_F^H\mathcal{I}_1^{\theta} P_{\mathbb{F}_1(\mathbb{D}_1)} \left( m_1(x,y) \left[ \varphi(x,y) + {}_F^H \mathcal{I}_1^{\vartheta} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widetilde{\mathbb{T}}_1^k(u_2(x,y))) \right] \right) \Big)$$
$$\leq \overline{\mathcal{L}}_1 \cdot {}_F^H \mathcal{I}_1^{\theta} d_\infty \left( {}_F^H\mathcal{I}_1^{\vartheta} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widetilde{\mathbb{T}}_1^k(u_1(x,y))), {}_F^H \mathcal{I} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widetilde{\mathbb{T}}_1^k(u_2(x,y))) \right).$$

From (4.14), it can be concluded that

$$d_\infty \left( {}_F^H\mathcal{I}_1^{\vartheta} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widetilde{\mathbb{T}}_1^k(u_1(x,y))), {}_F^H \mathcal{I} P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)\widetilde{\mathbb{T}}_1^k(u_2(x,y))) \right)$$
$$\leq \frac{\overline{\mathcal{L}}_1^k\overline{\mathcal{L}}_2^{k+1}(\ln s)^{(k+2)\vartheta_1+k\theta_1-1}(\ln t)^{(k+2)\vartheta_2+k\theta_2-1}}{\Gamma((k+2)\vartheta_1+k\theta_1)\Gamma((k+2)\vartheta_2+k\theta_2)} \cdot d_{1-\vartheta}(u_1,u_2)$$

owns, and ultimately, one can derive

$$d_{1-\vartheta}\left(\widetilde{\mathbb{T}}_1^{k+1}(u_1(x,y)), \widetilde{\mathbb{T}}_1^{k+1}(u_2(x,y))\right)$$
$$\leq \frac{\overline{\mathcal{L}}_1^{k+1}\overline{\mathcal{L}}_2^{k+1}(\ln x)^{(k+2)\vartheta_1+(k+1)\theta_1-1}(\ln y)^{(k+2)\vartheta_2+(k+1)\theta_2-1}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((k+2)\vartheta_1+(k+1)\theta_1)\Gamma((k+2)\vartheta_2+(k+1)\theta_2)} \cdot d_{1-\vartheta}(u_1,u_2).$$

Then, by mathematical induction, one knows that (4.14) holds, and so it follows that

$$\lim_{n\to 0} \frac{\overline{\mathcal{L}}_1^n\overline{\mathcal{L}}_2^n(\ln x)^{(n+1)\vartheta_1+n\theta_1-1}(\ln y)^{(n+1)\vartheta_2+n\theta_2-1}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((n+1)\vartheta_1+n\theta_1)\Gamma((n+1)\vartheta_2+n\theta_2)} = 0.$$

Hence, $\widetilde{\mathbb{T}}_1^n(u(x,y))$ is a contraction operator. Furthermore, $u(x,y)$ is the unique solution of (1.4). The situation for the operator $\widetilde{\mathbb{G}}_1(v(x,y))$ is similar to that of $\widetilde{\mathbb{T}}_1(u(x,y))$ and will not be repeated here. □

For the sake of convenience in demonstrating the existence and uniqueness of ‡-type solution, we rewrite (**H$_4$**) and (**H$_5$**) as ($\mathbb{H}_6$) and ($\mathbb{H}_7$), respectively, and further obtain the following two assumptions ($\mathbb{H}_6$)-($\mathbb{H}_7$) to establish the existence and uniqueness of ‡-type solution:

($\mathbb{H}_6$) $\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_1(\mathbb{D}_1)}}(\mathcal{K},\mathbb{F}_2) \neq \emptyset$ and $\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_2(\mathbb{D}_2)}}(\mathcal{K},\mathbb{F}_1) \neq \emptyset$.

($\mathbb{H}_7$) For $\forall(x,y) \in \mathcal{K}$, given $U(x,y)$, $V(x,y)$ as

$$U(x,y) = \psi(x,y) \ominus (-1){}_F^H\mathcal{I}_1^{\theta}\left(P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)v(x,y))\right),$$

$$V(x,y)=\varphi(x,y)\ominus(-1)_F^H\mathcal{I}_1^\theta\left(P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u(x,y))\right),$$

if $u(x,y)\in\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_1(\mathbb{D}_1)}}(\mathcal{K},\mathbb{F}_2)$ and $v(x,y)\in\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_2(\mathbb{D}_2)}}(\mathcal{K},\mathbb{F}_1)$, then $U(x,y)\in\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_1(\mathbb{D}_1)}}(\mathcal{K},\mathbb{F}_2)$ and $V(x,y)\in\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_2(\mathbb{D}_2)}}(\mathcal{K},\mathbb{F}_1)$. Moreover, here $\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_1(\mathbb{D}_1)}}(\mathcal{K},\mathbb{F}_2)$ and $\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_2(\mathbb{D}_2)}}(\mathcal{K},\mathbb{F}_1)$ denote

$$\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_1(\mathbb{D}_1)}}(\mathcal{K},\mathbb{F}_2)\triangleq\{\mathrm{v}(x,y)\in C(\mathcal{K},\mathbb{F}_2)\,|\,\psi(x,y)\ominus(-1)_F^H\mathcal{I}_1^\theta\left(P_{\mathbb{F}_1(\mathbb{D}_1)}(m_1(x,y)v(x,y))\right)\ \text{exists},\forall(x,y)\in\mathcal{K}\}$$

and

$$\widetilde{\mathbb{C}}_{\mathbb{F}}^{P_{\mathbb{F}_2(\mathbb{D}_2)}}(\mathcal{K},\mathbb{F}_1)\triangleq\{\mathrm{u}(x,y)\in C(\mathcal{K},\mathbb{F}_1)\,|\,\varphi(x,y)\ominus(-1)_F^H\mathcal{I}_1^\theta\left(P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u(x,y))\right)\ \text{exists},\forall(x,y)\in\mathcal{K}\},$$

respectively, where $\psi$ and $\varphi$ are defined by (2.3) and (2.4), respectively.

**Theorem 8.** *Suppose that $P_{\mathbb{F}_1(\mathbb{D}_1)}\in C_K(\mathbb{F}_1,\mathbb{F}_1(\mathbb{D}_1))$ and $P_{\mathbb{F}_2(\mathbb{D}_2)}\in C_K(\mathbb{F}_2,\mathbb{F}_2(\mathbb{D}_2))$ fulfill the Lipschitz condition* (4.7) *with constants $\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2\in(0,1)$, and the conditions* ($\mathbb{H}_6$) *and* ($\mathbb{H}_7$) *are satisfied. Then,* (1.4) *has a unique ‡-type solution on $\mathcal{K}$.*

*Proof.* Similar to Theorem 7 mentioned above, we define the operators $\widehat{\mathbb{T}}_1(u(x,y))$ and $\widehat{\mathbb{G}}_1(v(x,y))$ as follows.

$$\widehat{\mathbb{T}}_2(u(x,y))=\xi_1(x)+\xi_2(y)\ominus\xi_1(0)\ominus(-1)_F^H\mathcal{I}_1^\theta P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)\left[\varphi(x,y)+_F^H\mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}(m_2(x,y)u(x,y))\right]\right)$$

and

$$\widehat{\mathbb{G}}_2(v(x,y))=\varphi(x,y)\ominus(-1)_F^H\mathcal{I}_1^\vartheta P_{\mathbb{F}_2(\mathbb{D}_2)}\left(m_2(x,y)\left[\xi_2(y)+\xi_1(x)\ominus\xi_1(0)+_F^H\mathcal{I}_1^\theta P_{\mathbb{F}_1(\mathbb{D}_1)}\left(m_1(x,y)v(x,y)\right)\right]\right).$$

By Lemma 1, we can obtain

$$d_{1-\vartheta}\left(\widehat{\mathbb{T}}_2(u_1(x,y)),\widehat{\mathbb{T}}_2(u_2(x,y))\right)\le\frac{\mathcal{L}_1\mathcal{L}_2\Gamma(\vartheta_1)\Gamma(\vartheta_2)(\ln x)^{\vartheta_1+\theta_1}(\ln y)^{\vartheta_2+\theta_2}}{\Gamma(2\vartheta_1+\theta_1)\Gamma(2\vartheta_2+\theta_2)}\cdot d_{1-\vartheta}(u_1,u_2).$$

Following a similar approach to the proof of Theorem 4, one has

$$d_{1-\vartheta}\left(\widehat{\mathbb{T}}_2^n(u_1(x,y)),\widehat{\mathbb{T}}_2^n(u_2(x,y))\right)\le\frac{\mathcal{L}_1^n\mathcal{L}_2^n(\ln x)^{n\vartheta_1+n\theta_1}(\ln y)^{n\vartheta_2+n\theta_2}\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma((n+1)\vartheta_1+n\theta_1)\Gamma((n+1)\vartheta_2+n\theta_2)}\cdot d_{1-\vartheta}(u_1,u_2),\tag{4.15}$$

Similarly to Theorem 7, it can be shown that $\widehat{\mathbb{T}}_2$ and $\widehat{\mathbb{G}}_2$ are also contraction operators, thereby establishing the existence and uniqueness of the solution, satisfying (1.4). □

We now proceed to discuss the case when $\lambda\neq0$.

**Situation II**: $\lambda\neq0$

The remaining theorems in this subsection (i.e., Theorems 9 and 10) can be obtained by similar arguments as presented above and are thus omitted for brevity.

**Theorem 9.** *Under the assumptions that the conditions* (**H₁**)–(**H₃**) *meet, and $P_{\mathbb{F}_1(\mathbb{D}_1)}\in C_K(\mathbb{F}_1,\mathbb{F}_1(\mathbb{D}_1))$ and $P_{\mathbb{F}_2(\mathbb{D}_2)}\in C_K(\mathbb{F}_2,\mathbb{F}_2(\mathbb{D}_2))$ also satisfy the Lipschitz condition* (4.7)*, there exists a unique †-type solution to* (1.4).

**Theorem 10.** *Assume that $P_{\mathbb{F}_1(\mathbb{D}_1)}\in C_K(\mathbb{F}_1,\mathbb{F}_1(\mathbb{D}_1))$ and $P_{\mathbb{F}_2(\mathbb{D}_2)}\in C_K(\mathbb{F}_2,\mathbb{F}_2(\mathbb{D}_2))$ meet the Lipschitz condition* (4.7) *with constants $\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2\in(0,1)$, and conditions* (**H₁**)–(**H₃**)*,* ($\mathbb{H}_6$) *and* ($\mathbb{H}_7$) *are satisfied. With these prerequisites, then* (1.4) *admits a unique ‡-type solution on $\mathcal{K}$.*

## 5. Concluding remarks

In this paper, we focused on a coupled system of FFPDE with integral boundary conditions and CH-type derivatives as (1.5). We mostly obtained the following results:

(i) We proposed a novel CH-type FFPDE coupled system with integral boundary conditions (1.5). This significantly enhances its capability to represent natural phenomena.

(ii) We established the existence and uniqueness theorems for (1.5) under different conditions by applying the Banach fixed point theorem. In addition, we present the application of the fuzzy projection neural network (1.4).

(iii) We established the continuous dependence of the solution to (1.5) on the initial data by employing the generalized Gronwall inequality involving the H-integral. Similarly, we also established the continuous dependence of the solutions of the projection neural network system (1.4) on the initial conditions. In the section on numerical examples, we conducted a Chaos analysis for the local solution systems and implemented them using electrical circuits.

Furthermore, exploring the chaotic behavior of FFPDE systems with CH-type derivatives in future research holds significant practical importance and value, particularly for applications in communication encryption and related fields. Therefore, we pose the following question:

**Question 3.** *What is the relationship between the chaos exhibited by the proposed FFPDEs solution system and the chaos in the original system* (1.5)*? Is the intensity of chaos in the solution system consistent with that of the original system FFPDEs? Furthermore, under certain conditions, do both systems maintain the same stability?*

Moreover, we noted that Yang et al. [60] proposed a novel nonlinear finite volume scheme, which preserves the discrete maximum principle (DMP) for two-dimensional sub-diffusion equations on distorted meshes, ensuring the absence of spurious oscillations in the numerical solutions and maintaining the physical bounds of various quantities. In the same year, Yang et al. [61] also investigated the orthogonal Gauss collocation method (OGCM) with an arbitrary polynomial degree for the numerical solution of a two-dimensional (2D) fourth-order subdiffusion model and demonstrated that the numerical solution maintains robustness. These approaches further prompted our consideration of the following issues:

**Question 4.** *How can the numerical solution of* (1.5) *be obtained by employing the method proposed by Yang et al. [60, 61]? Which essential physical properties and structures, such as temperature and density, can be preserved? What necessary conditions need to be established?*

Additionally, considering Question 3 and Question 4 for other type fractional-order derivatives or relaxing the assumptions ($\mathbf{H_1}$)–($\mathbf{H_5}$) made in this paper for the case $\lambda \neq 0$ using alternative methods may serve as a key direction for future work.

## Author contributions

**Si-yuan Lin**: Writing - original draft, Software, Methodology, Conceptualization, Writing - review and editing, Funding acquisition, Validation, Visualization. **Heng-you Lan**: Writing - review and editing, Conceptualization, Funding acquisition, Methodology, Validation, Project administration. **Ji-hong Li**: Writing - review and editing, Visualization, Validation, Software, Funding acquisition.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflict of interest.

## References

1. Q. Han, L. Z. Liao, H. Qi, L. Qi, Stability analysis of gradient-based neural networks for optimization problems, *J. Global Optim.*, **19** (2001), 363–381. https://doi.org/10.1023/A:1011245911067
2. R. Abdulkadirov, P. Lyakhov, N. Nagornov, Survey of optimization algorithms in modern neural networks, *Mathematics*, **11** (2023), 2466. https://doi.org/10.3390/math11112466
3. C. Summers, M. J. Dinneen, Nondeterminism and instability in neural network optimization, 2021, arXiv: 2103.04514. https://doi.org/10.48550/arXiv.2103.04514
4. M. Eshaghnezhad, S. Effati, A. Mansoori, A neurodynamic model to solve nonlinear pseudo-monotone projection equation and its applications, *IEEE T. Cybernetics*, **47** (2017), 3050–3062. https://doi.org/10.1109/TCYB.2016.2611529
5. T. Beaula, R. Abirami, Some aspects of 2-fuzzy 2-metric projection operator of 2-fuzzy 2-Banach spaces, *Malaya J. Mat.*, **8** (2020), 1012–1021. https://doi.org/10.26637/MJM0803/0048
6. Z. B. Wu, M. Chao, N. J. Huang, On a system of fuzzy fractional differential inclusions with projection operators, *Fuzzy Sets Syst.*, **347** (2018), 70–88. https://doi.org/10.1016/j.fss.2018.01.005
7. S. Dawood, A. Q. Jabur, On fuzzy soft projection operators in Hilbert space, *Al-Qadisiyah J. Pure Sci.*, **26** (2021), 112–123.
8. F. kavari, M. R. Baloochshahryari, O. Sedaghatfar, Best approximation and projection operator onto a fuzzy set and properties, *J. Intell. Fuzzy Syst.*, **49** (2021), 1393–1413. https://doi.org/10.1177/18758967251345258
9. M. H. Ding, H. Liu, G. H. Zheng, On inverse problems for several coupled PDE systems arising in mathematical biology, *J. Math. Biol.*, **87** (2023), 86. https://doi.org/10.1007/s00285-023-02021-4
10. Z. Zhang, X. Cheng, Existence of positive solutions for a semilinear elliptic system, *Topol. Method. Nonl. An.*, **37** (2011), 103–116.

11. F. Zhang, H. Y. Lan, H. Y. Xu, Generalized Hukuhara weak solutions for a class of coupled systems of fuzzy fractional order partial differential equations without Lipschitz conditions, *Mathematics*, **10** (2022), 4033. https://doi.org/10.3390/math10214033
12. M. Cai, E. K. George, C.P. Li, Fractional SEIR model and data-driven predictions of COVID-19 dynamics of Omicron variant, *Chaos*, **32** (2022), 071101. https://doi.org/10.1063/5.0099450
13. F. Zhang, H. Y. Xu, H. Y. Lan, Initial value problems of fuzzy fractional coupled partial differential equations with Caputo gH-type derivatives, *Fractal Fract.*, **6** (2022), 132. https://doi.org/10.3390/fractalfract6030132
14. P. Shi, Weak solution to an evolution problem with a nonlocal constraint, *SIAM J. Math. Anal.*, **24** (1993), 46–58. https://doi.org/10.1137/0524004
15. S. Chen, H. Ding, High-order numerical method for Caputo-Hadamard fractional reaction-diffusion equations using nonuniform temporal mesh, *Math. Comput. Simulat.*, **241** (2026), 634–658. https://doi.org/10.1016/j.matcom.2025.09.022
16. H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Q. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019
17. K. Hattaf, On the stability and numerical scheme of fractional differential equations with application to biology, *Computation*, **10** (2022), 97. https://doi.org/10.3390/computation10060097
18. A. Jannelli, Numerical solutions of fractional differential equations arising in engineering sciences, *Mathematics*, **8** (2020), 215. https://doi.org/10.3390/math8020215
19. J. F. Alzaidy, Fractional sub-equation method and its applications to the space-time fractional differential equations in mathematical physics, *J. Adv. Math. Comput. Sci.*, **3** (2013), 153–163. https://doi.org/10.9734/BJMCS/2013/2908
20. Z. Chen, H. Zhang, H. Chen, ADI compact difference scheme for the two-dimensional integro-differential equation with two fractional Riemann–Liouville integral kernels, *Fractal Fract.*, **8** (2024), 707. https://doi.org/10.3390/fractalfract8120707
21. T. Liu, H. Zhang, S. Wang, A new high-order compact CN-ADI scheme on graded meshes for three-dimensional nonlinear PIDEs with multiple weakly singular kernels, *Appl. Math. Lett.*, **171** (2025), 109697. https://doi.org/10.1016/j.aml.2025.109697
22. M. Farman, M. Aslam, A. Akgul, F. Jarad, On solutions of the stiff differential equations in chemistry kinetics with fractal-fractional derivatives, *J. Comput. Nonlinear Dynam.*, **17** (2022), 071007. https://doi.org/10.1115/1.4054347
23. A. Najafi, R. Taleghani, Fractional Liu uncertain differential equation and its application to finance, *Chaos Soliton. Fract.*, **165** (2022), 112875. https://doi.org/10.1016/j.chaos.2022.112875
24. S. K. Choi, N. Koo, The monotonic property and stability of solutions of fractional differential equations, *Nonlinear Anal.-Theor.*, **74** (2011), 6530–6536. https://doi.org/10.1016/j.na.2011.06.037
25. Z. A. Khan, H. Ahmad, Qualitative properties of solutions of fractional differential and difference equations arising in physical models, *Fractals*, **29** (2021), 2140024. https://doi.org/10.1142/S0218348X21400247

26. N. D. Cong, H. T. Tuan, H. Trinh, On asymptotic properties of solutions to fractional differential equations, *J. Math. Anal. Appl.*, **484** (2020), 123759. https://doi.org/10.1016/j.jmaa.2019.123759

27. J. Ockendon, S. Howison, A. Lacey, A. Movchan, *Applied partial differential equation*, Oxford University Press, 2003.

28. A. A. Kilbas, Partial fractional differential equations and some of their applications, *Analysis*, **30** (2010), 35–66. https://doi.org/10.1524/anly.2010.0934

29. B. Zheng, C. B. Wen, Exact solutions for fractional partial differential equations by a new fractional sub-equation method, *Adv. Differ. Equ.*, **2013** (2013) 199. https://doi.org/10.1186/1687-1847-2013-199

30. H. Jafari, M. Nazari, D. Baleanu, C. M. Khalique, A new approach for solving a system of fractional partial differential equations, *Comput. Math. Appl.*, **66** (2013), 838–843. https://doi.org/10.1016/j.camwa.2012.11.014

31. D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **204** (1996), 609–625. https://doi.org/10.1006/jmaa.1996.0456

32. Z. Liu, J. Chen, W. Zhou, Caputo-Hadamard fractional chaotic maps, *Phys. Scr.*, **98** (2023), 074002. https://doi.org/10.1088/1402-4896/acd885

33. C. Yin, Chaos detection of the Chen system with Caputo-Hadamard fractional derivative, *Int. J. Bifurcat. Chaos*, **31** (2021), 2150016. https://doi.org/10.1142/S0218127421500164

34. S. Y. Lin, H. Y. Lan, J. H. Li, Solvability of fuzzy partially differentiable models for Caputo-Hadamard-type Goursat problems involving generalized Hukuhara difference, *Fractal Fract.*, **9** (2025), 395. https://doi.org/10.3390/fractalfract9060395

35. A. Ardjouni, A. Djoudi, Positive solutions for nonlinear Caputo-Hadamard fractional differential equations with integral boundary conditions, *Open J. Math. Anal.*, **3** (2019), 62–69. https://doi.org/10.30538/psrp-oma2019.0033

36. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

37. H. J. Zimmermann, *Fuzzy set theory–and its applications*, Beijing : World Publishing Corporation, 2011.

38. L. Stefanini, A generalization of Hukuhara difference, In: *Soft methods for handling variability and imprecision*, Berlin, Heidelberg: Springer, **48** (2008), 203–210. https://doi.org/10.1007/978-3-540-85027-4_25

39. L. T. Gomes, L. C. Barros, A note on the generalized difference and the generalized differentiability, *Fuzzy Set. Syst.*, **280** (2015), 142–145. https://doi.org/10.1016/j.fss.2015.02.015

40. L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Anal.-Theor.*, **71** (2009), 1311–1328. https://doi.org/10.1016/j.na.2008.12.005

41. M. Osman, A. Almahi, O. A. Omer, A. M. Mustafa, S. A. Altaie, Approximation solution for fuzzy fractional-order partial differential equations, *Fractal Fract.*, **6** (2022), 646. https://doi.org/10.3390/fractalfract6110646

42. H. Viet Long, N. Thi Kim Son, H. Thi Thanh Tam, The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability, *Fuzzy Set. Syst.*, **309** (2017), 35–63. https://doi.org/10.1016/j.fss.2016.06.018

43. Y. S. Choi, K. Y. Chan, A parabolic equation with nonlocal boundary conditions arising from electrochemistry, *Nonlinear Anal.-Theor.* **18** (1992), 317–331. https://doi.org/10.1016/0362-546X(92)90148-8

44. R. E. Ewing, T. Lin, A class of parameter estimation techniques for fluid flow in porous media, *Adv. Water Resour.*, **14** (1991), 89–97. https://doi.org/10.1016/0309-1708(91)90055-S

45. P. Shi, M. Shillor, *On design of contact patterns in one-dimensional thermoelasticity*, Theoretical Aspects of Industrial Design, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

46. P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Set. Syst.*, **35** (1990), 241–249. https://doi.org/10.1016/0165-0114(90)90197-E

47. K. Diethelm, N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.*, **265** (2002), 229–248. https://doi.org/10.1006/jmaa.2000.7194

48. S. M. Rametse, R. S. Herbst, Numerical discretization of Riemann-Liouville fractional derivatives with strictly positive eigenvalues, *AppliedMath*, **5** (2025), 130. https://doi.org/10.3390/appliedmath5040130

49. M. Murillo-Arcila, A. Peris, A. Vargas-Moreno, Dynamics of the Caputo fractional derivative, *Fract. Calc. Appl. Anal.*, **28** (2025), 1717–1731. https://doi.org/10.1007/s13540-025-00430-4

50. O. Kahouli, A. Jmal, O. Naifar, A. M. Nagy, A. Ben Makhlouf, New result for the analysis of Katugampola fractional-order systems-application to identification problems, *Mathematics*, **10** (2022), 1814. https://doi.org/10.3390/math10111814

51. L. C. Jiang, H. Y. Lan, Y. X. Yang, Ulam-Hyers stability of Caputo-Katugampola generalized Hukuhara type partial differential symmetry coupled systems, *Symmetry*, **17** (2025), 1707. https://doi.org/10.3390/sym17101707

52. H. V. Long, On random fuzzy fractional partial integro-differential equations under Caputo generalized Hukuhara differentiability, *Comput. Appl. Math.*, **37** (2018), 2738–2765. https://doi.org/10.1007/s40314-017-0478-1

53. Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2014** (2014), 10. https://doi.org/10.1186/1687-1847-2014-10

54. R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.-Theor.*, **72** (2010), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

55. L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, *Fuzzy Set. Syst.*, **161** (2010), 1564–1584. https://doi.org/10.1016/j.fss.2009.06.009

56. Y. H. Chang, C. H. Hong, The chaos of the solution semigroup for the quasi-linear lasota equation, *Taiwanese J. Math.*, **16** (2012), 1707–1717. https://doi.org/10.11650/twjm/1500406791

57. D. Criens, Propagation of chaos for weakly interacting mild solutions to stochastic partial differential equations, *J. Stat. Phys.*, **190** (2023), 114. https://doi.org/10.1007/s10955-023-03121-7

58. J. B. Dingwell, Lyapunov exponents, In: *Wiley encyclopedia of biomedical engineering*, 2006. https://doi.org/10.1002/9780471740360.ebs0702

59. M. Pakdaman, S. Effati, On fuzzy linear projection equation and applications, *Fuzzy Optim. Decis. Making*, **15** (2016), 219–236. https://doi.org/10.1007/s10700-015-9222-8

60. X. Yang, Z. Zhang, Analysis of a new NFV scheme preserving DMP for two-dimensional sub-diffusion equation on distorted meshes, *J. Sci. Comput.*, **99** (2024), 80. https://doi.org/10.1007/s10915-024-02511-7

61. X. Yang, Z. Zhang, Superconvergence analysis of a robust orthogonal Gauss collocation method for 2D fourth-order subdiffusion equations, *J. Sci. Comput.*, **100** (2024), 62. https://doi.org/10.1007/s10915-024-02616-z