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Research article

Comparative symmetry analysis and qualitative properties of impulsive multi-delay systems across a family of power Caputo fractional kernels

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Abstract: In this study, we perform a comparative symmetry analysis and investigate the qualitative properties of a nonlinear impulsive fractional differential system with multiple delays and nonlocal boundary conditions. By utilizing the generalized power Caputo fractional derivative, we present a unified theoretical framework that encompasses several operators—including the Atangana-Baleanu, Caputo-Fabrizio, and weighted Hattaf derivatives—as special cases. This generality guarantees that our findings are relevant to a range of fractional kernels, emphasizing the inherent symmetry characteristics of these operators. We establish adequate criteria for the existence and uniqueness of solutions through fixed-point theory. We also show that the system is Ulam-Hyers stable, an important property for maintaining its strength in the face of change. A convergent numerical scheme confirms the theoretical results, and a sensitivity analysis demonstrates the effect of kernel symmetry on stability margins. The networked control system application shows how useful the framework is in real life. The results show that the framework can include complex genetic phenomena and spontaneous interactions that are often ignored in traditional models.

Keywords: fractional derivatives; impulsive delay; stability; fixed-point theory; symmetry analysis
Mathematics Subject Classification: 34A08, 34K20, 34K45, 47H10

1. Introduction and motivations

Fractional calculus has proven to be effective in modeling physical processes characterized by memory and hereditary characteristics. It is a practical alternative to integer-order models in diverse fields such as viscoelasticity [1, 2], anomalous diffusion [3, 4], and networked control [5–7]. Because fractional operators are non-local, they can represent temporal dependencies and long-range interactions more accurately than classical derivatives. Recent research has moved away from singular-kernel operators, like the Riemann-Liouville and Caputo derivatives [8,9], toward derivatives with non-singular kernels, such as exponential-kernel Caputo-Fabrizio (CF) derivative [10] and the Atangana-Baleanu (ABC) derivative [11]. These operators describe a variety of relaxation patterns, from exponential to crossover types.

A notable advancement in this field is the development of generalized fractional operators that combine earlier definitions into a single form. The power Caputo fractional derivative (PCFD) (denoted by ${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q}$) is one such operator [12]. It includes CF, ABC, Hattaf, and weighted ABC fractional derivatives as symmetric cases according to the parameters $(\eta, \gamma, q, \varrho(t))$. This flexibility allows for a comparison of different memory effects within the same mathematical structure.

In practical applications, dynamical models often require the inclusion of several specific phenomena, time delays caused by signal processing lags, impulsive effects that represent sudden state changes from external shocks or interventions, and nonlocal boundary conditions, which can account for distributed measurements better than simple initial data. Although fractional systems involving delays [13, 14], impulses [15], or nonlocal conditions have been studied separately, research into their combinations is more limited. For instance, studies have addressed impulsive fractional systems [16] or delay equations with nonlocal data [17] in pairwise configurations.

The progress of research on impulsive delay systems follows a unique mathematical trajectory. The fundamental theory was established by initial research: Wang et al. [18] investigated the Ulam stability of first-order impulsive ordinary differential equations without delay:

$$\begin{cases} \dfrac{dv}{dt} = f(t, v(t)), & t \in J \setminus \{t_1, \ldots, t_m\}, \\ \Delta v(t_k) = I_k(v(t_k^-)), & k = 1, \ldots, m, \end{cases}$$

where $J = [0, T]$. While this model is foundational, it lacks the capacity to model processes with inherent time lags. Zada et al. [19] introduced a system with a single delay function $h(t)$:

$$\begin{cases} \dfrac{dv}{dt} = f(t, v(t), v(h(t))), & t \in J \setminus \{t_1, \ldots, t_m\}, \\ \Delta v(t_k) = I_k(v(t_k^-)), & k = 1, \ldots, m, \\ v(t) = \phi(t), & t \in [-\tau, 0]. \end{cases}$$

This work combined impulses with history dependence, but was still restricted to a single delay and integer-order dynamics. Recently, Tunç and Tunç [20] added more constant time delays to the model as follows:

$$\begin{cases} v'(t) = \alpha(t) + h(v(t)) + f(t, v(t)) + \sum_{i=1}^{N} H_i(t, v(t), v(t-\tau_i)), & t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta v(t_k) = I_k(v(t_k^-)), & k = 1, \dots, m, \\ v(t) = \phi(t), & t \in [-\tau, 0], \end{cases}$$

where $\tau_i > 0$ are distinct constants and $\tau = \max\{\tau_i\}$.

Recent developments have further investigated the interaction among fractional operators, impulses, and delay structures. For example, Han and Zhu [21] looked into whether mixed-type Hilfer fractional differential equations with impulses and time delay exist and are stable. Despite the development presented in the above works, there remains a gap in the literature: systems that integrate a generalized fractional operator with multiple delays, impulsive terms, and multi-point nonlocal conditions simultaneously. Much of the existing literature is restricted to specific derivative types or simplified boundary structures, often omitting stability analyses like Ulam-Hyers robustness [22–24]. For instance, Zada et al. [25] examined impulsive fractional delay equations utilizing the Caputo derivative, whereas Almalahi et al. [26] concentrated on ABC-type systems devoid of delay components. These studies enhance the expanding literature on impulsive fractional systems; however, the integration of a generalized operator (including ABC, CF, and weighted Hattaf derivatives as specific instances) with multiple delays, impulsive effects, and nonlocal boundary conditions is still largely unexplored.

To show our contributions within the existing literature, we present a direct comparison with the recent study by Cui and Zhou [27], which examined mixed-type Hilfer fractional differential equations characterized by non-instantaneous impulses, nonlocal conditions, and time delay, thereby establishing existence and Ulam-type stability through Sadovskii's fixed point theorem. In contrast, our work makes distinct contributions: (i) our framework employs PCFD, which unifies ABC, CF, and weighted Hattaf operators as symmetric cases, enabling comparative symmetry analysis with different memory kernels; (ii) we consider instantaneous impulses rather than non-instantaneous ones; (iii) our system accommodates multiple distinct delays; (iv) we provide a unified numerical scheme compatible with all symmetric operators. While [27] offers valuable results for Hilfer systems, our work extends analysis to a broader class of operators under a different impulsive structure, with emphasis on kernel symmetry.

The present work addresses this gap by investigating an impulsive multi-delay fractional system under nonlocal conditions using the PCFD as the core operator. The complete system under investigation is

$$\begin{cases} {}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v(t) = \alpha(t) + h(v(t)) + f(t, v(t)) + \sum_{i=1}^{N} H_i(t, v(t), v(t-\tau_i)), \\ t \in \mathcal{I} = (0, T] \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta v(t_k) = I_k(v(t_k^-)), \quad k = 1, 2, \dots, m, \\ v(t) = \phi(t), \quad t \in [-\tau, 0), \\ v(0) = \sum_{j=1}^{\rho} \delta_j v(\zeta_j), \end{cases} \tag{1.1}$$

where ${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q}$ is the PCFD of order $\eta \in (0,1)$, with the power q and $\min(\gamma, q) > 0$, $\tau_i > 0$ for $i = 1, 2, \dots, N$, $\tau = \max\{\tau_i\}$, ϱ is a nondecreasing weight function with $\varrho \in C^1([-\tau, T])$, $\varrho(t) \geq 0$ for all $t \in [-\tau, T]$, $\alpha \in PC([0,T), \mathbb{R}^{+})$, $h \in C(\mathbb{R}, \mathbb{R})$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $H_i \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ for $i = 1, \ldots, N$, $I_k \in C(\mathbb{R}, \mathbb{R})$ for $k = 1, \ldots, m$, and $\phi \in C([-\tau, 0], \mathbb{R})$.

The power parameter q in PCFD enhances the model's ability to fit specific dynamic behaviors [28–30]. Through careful choice of parameters (γ, q, ϱ), the PCFD can be simplified to well-known operators such as the ABC, CF, and weighted Hattaf derivatives. This reduction is summarized in Table 1.

Table 1. Symmetric cases recovered by the power fractional derivative ${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q}$.

Operator	γ	q	$\varrho(t)$
Atangana-Baleanu-Caputo (ABC)	η	e	1
Weighted ABC	η	e	$\varrho(t)$
Caputo-Fabrizio (CF)	1	e	1
Weighted generalized Hattaf	γ	e	$\varrho(t)$
General power derivative	$\gamma > 0$	$q > 0$	$\varrho(t) \geq 0$

The kernel symmetry used in this work refers to the structural form of the fractional operator's memory kernel. A symmetric kernel is one that generalizes the exponential function in a complete manner, preserving a power-law-like memory profile; this is exemplified by the Mittag-Leffler kernel of the ABC operator, recovered when $\gamma = \eta$ and $q = e$ within the PCFD framework. In contrast, an *asymmetric* kernel, such as the exponential kernel of the Caputo-Fabrizio operator (corresponding to $\gamma = 1$), exhibits a qualitatively different memory decay. This classification is necessary for comprehending the impact of various fractional operators on system dynamics, and it supports the comparative analysis in Section 6.

The challenges of the theoretical framework are outlined as follows: First, the need for a generalized fractional operator, multiple variable delays, impulsive jumps, and nonlocal boundary conditions led to the creation of a compact integral operator that includes all of these features while keeping the PCFD structure intact. Second, the kernel is weakly singular in the integral formulation, so it must be handled carefully in the fixed-point framework to show both existence and uniqueness without making any smoothness assumptions that would limit the problem. Third, proving Ulam-Hyers stability for such a unified system demanded a novel perturbation analysis that accommodates the entire family of fractional kernels—from the symmetric Mittag-Leffler type to asymmetric exponential forms—within a single estimate. Addressing these challenges constitutes the main technical contribution of this work.

This study presents three principal contributions to the theory of fractional impulsive systems, with particular emphasis on comparative symmetry analysis across different fractional kernels. First, we establish a unified analytical framework for nonlinear impulsive fractional delay systems with nonlocal boundary conditions using PCFD. Unlike prior works that treat specific fractional operators in isolation, our approach encompasses the Atangana-Baleanu, Caputo-Fabrizio, and weighted Hattaf derivatives as special cases, enabling a systematic comparative analysis of how kernel structure—symmetric versus asymmetric—affects qualitative behavior. Second, we derive explicit criteria for existence, uniqueness, and Ulam-Hyers stability that are expressed in terms of the PCFD parameters, allowing practitioners to directly assess well-posedness across the entire operator family. Third, we develop a convergent numerical scheme compatible with all operators subsumed by the PCFD and demonstrate its application through both validation examples and a practical networked control scenario. The unifying nature of this framework represents a significant advance over case-by-case analyses, as it reveals how stability margins and solution dynamics vary with kernel symmetry within a single mathematical structure.

2. Basic concepts

Definition 2.1. *[12] Let* $\eta \in [0,1)$, *with* $\min(\gamma, q) > 0$, *and* $v \in H^1(a,b)$, *where* $H^1(a,b)$ *is a Sobolev space defined as* $H^1(a,b) = \{v \in L^2(a,b) : v'^2(a,b)\}$. *The power Caputo fractional derivative of order* η, *of a function* v *with respect to the weight function* ϱ, $0 < \varrho \in C^1([a,b])$, *is defined by*

$$ {}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v(t) = \frac{\mathbb{K}(\eta)}{1-\eta} \frac{1}{\varrho(t)} \int_a^t {}^{q}\mathbb{E}_{\gamma,1}\left(-\frac{\eta}{1-\eta}(t-s)^{\gamma}\right)(\varrho v)'(s)\,ds, \tag{2.1} $$

where, ${}^{q}\mathbb{E}_{\gamma,1}$ *represents the power Mittag-Liffler (ML) function given by*

$$ {}^{q}\mathbb{E}_{\gamma,l}(s) = \sum_{n=0}^{+\infty} \frac{(s \ln q)^n}{\Gamma(kn+l)}, \; s \in \mathbb{C}, \text{ and } k,l,q > 0, $$

and $\mathbb{K}(\eta)$ *represents a normalization positive function obeying* $\mathbb{K}(0) = \mathbb{K}(1) = 1$.

Definition 2.2. *[12] Let* $\eta \in [0,1)$, *with* $\min(\gamma, q) > 0$. *The power Caputo fractional integral (PCFI) of order* η, *of a function* v *with respect to the weight function* ϱ, $0 < \varrho \in C^1([a,b])$, *is defined by*

$$ {}^{\mathcal{PC}}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q} v(t) = \frac{1-\eta}{\mathbb{K}(\eta)} v(t) + \ln q \frac{\eta}{\mathbb{K}(\eta)} {}^{\mathbb{RL}}\mathbf{I}_{a,\varrho}^{\gamma} v(t), $$

where

$$ {}^{\mathbb{RL}}\mathbf{I}_{t,\varrho}^{\gamma} v(t) = \frac{1}{\Gamma(\gamma)} \frac{1}{\varrho(t)} \int_a^t (t-s)^{\gamma-1} (\varrho v)(s)\,ds. $$

Theorem 2.1. *([31, Theorem 1]) Let* $\eta \in [0,1)$, *with* $\gamma, q > 0$, *and* $v \in H^1(a,b)$. *Then, the power Caputo fractional derivative and PCFI are commutative operators as follows:*

(i) ${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q}\left({}^{\mathcal{PC}}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q} v\right)(t) = v(t) - \frac{\varrho(a)}{\varrho(t)} v(a)$;

(ii) ${}^{\mathcal{PC}}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q}\left({}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v\right)(t) = v(t) - \frac{\varrho(a)}{\varrho(t)} v(a)$;

iii ${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v(t) = 0$, *for all constant functions* $v(t)$.

If $q = e$, *then we have the generalized Hattaf fractional operators [32].*

Lemma 2.1. *The power Caputo fractional derivative and PCFI satisfy the Newton-Leibniz formula:*

$$ {}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q}\left({}^{\mathcal{PC}}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q} v\right)(t) = {}^{\mathcal{PC}}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q}\left({}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v\right)(t) = v(t) - v(a). $$

Lemma 2.2. *[31] Let* $\mathbb{H} : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function with* $\mathbb{H}(a, v(a)) = 0$. *Then,* $v \in C([a,b])$ *is a solution of the following system:*

$$ \begin{cases} {}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v(t) = \mathbb{H}(t, v(t)), \\ v(0) = v_a \in \mathbb{R}, \end{cases} $$

if and only if v *satisfies*

$$ v(t) = \frac{\varrho(a)}{\varrho(t)} v_a + {}^{\mathcal{PC}}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q} \mathbb{H}(t, v(t)). $$

2.1. An equivalent integral equation

This section establishes the integral representation for system (1.1). Let $C([a,b],\mathbb{R})$ denote the Banach space of continuous functions on $[a,b]$ under the supremum norm $\|v\|_\infty = \sup_{t\in[a,b]} |v(t)|$. The piecewise continuous space is defined as

$$PC\left([-\tau,T],\mathbb{R}\right) = \left\{ \begin{array}{c} v:[-\tau,T]\to\mathbb{R};\, v\in C\left((t_k,t_{k+1}],\mathbb{R}\right), k=1,2,\cdots,m, \\ v\left(t_k^+\right) \text{ exists and is finite for } k=1,2,\cdots,m \end{array} \right\},$$

associated with the norm

$$\|v\|_{PC} = \sup_{t\in[-\tau,T]} |v(t)|.$$

For $t\in[0,T]$, the delayed state $v(t-\tau_i)$ is given by

$$v(t-\tau_i) = \begin{cases} \phi(t-\tau_i), & \text{if } t-\tau_i \le 0, \\ v(t-\tau_i), & \text{if } t-\tau_i > 0, \end{cases} \tag{2.2}$$

where the second case corresponds to the trajectory of the solution for $t > \tau_i$.

Theorem 2.2. *Assume that* $A = 1 - \sum_{\ell=1}^{\rho} \delta_\ell \frac{\varrho(0)}{\varrho(\zeta_\ell)} < 1$. *Then, a function* $v \in PC([-\tau,T],\mathbb{R})$ *is a solution of system (1.1) if and only if it satisfies the following integral equation:*

$$v(t) = \begin{cases} \phi(t),\, t\in[-\tau,0), \\ \dfrac{\varrho(0)}{\varrho(t)}\dfrac{1}{A}\displaystyle\sum_{\ell=1}^{\rho}\delta_\ell\Bigg[\frac{1}{\varrho(\zeta_\ell)}\sum_{j=1}^{k_\ell}\varrho(t_j)\Bigg(I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_j,v(t_j)) + \frac{\eta\ln q}{\mathbb{K}(\eta)\varrho(t_j)}\int_0^{t_j}\frac{(t_j-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s,v(s))ds\Bigg) \\ \quad + \dfrac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(\zeta_\ell,v(\zeta_\ell)) + \dfrac{\eta\ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)}\displaystyle\int_0^{\zeta_\ell}\frac{(\zeta_\ell-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s,v(s))\,ds\Bigg] \\ \quad + \dfrac{1}{\varrho(t)}\displaystyle\sum_{j=1}^{k}\varrho(t_j)\left[I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_j,v(t_j)) + \frac{\eta\ln q}{\mathbb{K}(\eta)\varrho(t_j)}\int_0^{t_j}\frac{(t_j-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s,v(s))ds\right] \\ \quad + \dfrac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t,v(t)) + \dfrac{\eta\ln q}{\mathbb{K}(\eta)\varrho(t)}\displaystyle\int_0^{t}\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s,v(s))ds, \\ t\in(t_k,t_{k+1}], \end{cases} \tag{2.3}$$

where $k_\ell = \max\left\{j : t_j < \zeta_\ell\right\}$, *and*

$$\mathbb{H}(s,v(s)) = \alpha(s) + h(v(s)) + f(s,v(s)) + \sum_{i=1}^{N} H_i(s,v(s),v(s-\tau_i)),$$

and k_ℓ *is such that* $\zeta_\ell \in (t_{k_\ell}, t_{k_\ell+1}]$.

Proof. Consider $t\in(t_k,t_{k+1}]$ for a fixed $k\in\{0,\dots,m\}$. Within each subinterval, the governing system (1.1) is represented as

$${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} v(t) = \mathbb{H}(t,v(t)), \tag{2.4}$$

subject to the state transitions $\Delta v(t_k) = I_k(v(t_k^-))$ at each t_k. Following the results of Lemma 2.2 and Definition 2.2, the expression for $t \in (0, t_1]$ becomes

$$v(t) = \frac{\varrho(0)}{\varrho(t)}v(0) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t, v(t)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds. \tag{2.5}$$

At t_1, the impulsive condition gives

$$v(t_1^+) = v(t_1^-) + I_1(v(t_1^-)). \tag{2.6}$$

Using (2.5) evaluated at $t = t_1^-$, we have

$$v(t_1^-) = \frac{\varrho(0)}{\varrho(t_1)}v(0) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_1, v(t_1)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_1)}\int_0^{t_1} \frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds. \tag{2.7}$$

Hence, by (2.6) and (2.7), Eq (2.5) becomes

$$v(t_1^+) = \frac{\varrho(0)}{\varrho(t_1)}v(0) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_1, v(t_1)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_1)}\int_0^{t_1} \frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds + I_1(v(t_1)). \tag{2.8}$$

Applying Lemma 2.2 on $(t_1, t_2]$ with initial point t_1^+ yields

$$v(t) = \frac{\varrho(t_1^+)}{\varrho(t)}v(t_1^+) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t, v(t)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds. \tag{2.9}$$

Since ϱ is continuous, $\varrho(t_1^+) = \varrho(t_1)$, and by (2.8), Eq (2.9) becomes

$$\begin{aligned} v(t) &= \frac{\varrho(t_1)}{\varrho(t)}\left[\frac{\varrho(0)}{\varrho(t_1)}v(0) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_1, v(t_1)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_1)}\int_0^{t_1} \frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds + I_1(v(t_1))\right] \\ &\quad + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t, v(t)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds. \end{aligned}$$

This implies that

$$\begin{aligned} v(t) &= \frac{\varrho(0)}{\varrho(t)}v(0) + \frac{\varrho(t_1)}{\varrho(t)}\left[I_1(v(t_1)) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_1, v(t_1)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_1)}\int_0^{t_1} \frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds\right] \\ &\quad + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t, v(t)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds. \end{aligned} \tag{2.10}$$

Proceeding inductively, for $t \in (t_k, t_{k+1}]$ we obtain

$$\begin{aligned} v(t) = &\frac{\varrho(0)}{\varrho(t)}v(0) + \frac{1}{\varrho(t)}\sum_{j=1}^{k}\varrho(t_j)\left[I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_j, v(t_j)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)}\int_0^{t_j}\frac{(t_j-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds\right] \\ &+ \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t, v(t)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))\,ds. \end{aligned} \tag{2.11}$$

The nonlocal condition in (1.1) is

$$v(0) = \sum_{\ell=1}^{\rho} \delta_\ell v(\zeta_\ell). \tag{2.12}$$

Each ζ_ℓ lies in some interval $(t_{k_\ell}, t_{k_\ell+1}]$. Applying formula (2.11) at $t = \zeta_\ell$ gives

$$\begin{aligned} v(\zeta_\ell) = & \frac{\varrho(0)}{\varrho(\zeta_\ell)} v(0) + \frac{1}{\varrho(\zeta_\ell)} \sum_{j=1}^{k_\ell} \varrho(t_j) \Bigg[I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)} \mathbb{H}(t_j, v(t_j)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \mathbb{H}(s, v(s)) ds \Bigg] \\ & + \frac{1-\eta}{\mathbb{K}(\eta)} \mathbb{H}(\zeta_\ell, v(\zeta_\ell)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \mathbb{H}(s, v(s))\, ds. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12) yields

$$\begin{aligned} v(0) = & \sum_{\ell=1}^{\rho} \delta_\ell \Bigg[\frac{\varrho(0)}{\varrho(\zeta_\ell)} v(0) + \frac{1}{\varrho(\zeta_\ell)} \sum_{j=1}^{k_\ell} \varrho(t_j) \Bigg(I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)} \mathbb{H}(t_j, v(t_j)) \\ & + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \mathbb{H}(s, v(s)) ds \Bigg) \\ & + \frac{1-\eta}{\mathbb{K}(\eta)} \mathbb{H}(\zeta_\ell, v(\zeta_\ell)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \mathbb{H}(s, v(s))\, ds \Bigg]. \end{aligned} \tag{2.14}$$

Solving for $v(0)$ gives

$$\begin{aligned} v(0) = & \frac{1}{A} \sum_{\ell=1}^{\rho} \delta_\ell \Bigg[\frac{1}{\varrho(\zeta_\ell)} \sum_{j=1}^{k_\ell} \varrho(t_j) \Bigg(I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)} \mathbb{H}(t_j, v(t_j)) \\ & + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \mathbb{H}(s, v(s)) ds \Bigg) \\ & + \frac{1-\eta}{\mathbb{K}(\eta)} \mathbb{H}(\zeta_\ell, v(\zeta_\ell)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \mathbb{H}(s, v(s))\, ds \Bigg]. \end{aligned} \tag{2.15}$$

where $A = 1 - \sum_{\ell=1}^{\rho} \delta_\ell \frac{\varrho(0)}{\varrho(\zeta_\ell)} \neq 0$. Replacing $v(0)$ in (2.11) with the expression (2.15) gives Eq (2.3).

The converse follows by applying the operator ${}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q}$ to both sides of (2.3). Utilizing Theorem 2.1(i) and (ii), it is shown that the differential equation in (1.1) holds. The impulsive transitions are represented by the term $\frac{1}{\varrho(t)} \sum_{j=1}^{k} \varrho(t_j) I_j(v(t_j))$, while the initial history $v(t) = \phi(t)$ for $t \in [-\tau, 0)$ remains satisfied. Furthermore, the nonlocal condition is satisfied via the substitution of (2.15). Consequently, the integral Eq (2.3) provides an equivalent representation of the impulsive fractional system (1.1) on the space $PC([-\tau, T], \mathbb{R})$. □

The forthcoming analysis establishes the existence and uniqueness results for system (1.1) by utilizing the integral representation derived in Theorem 2.2 alongside the Banach fixed point theorem. Define an operator $\mathcal{T} : PC([-\tau, T], \mathbb{R}) \to PC([-\tau, T], \mathbb{R})$ by

$$
(\mathcal{T}v)(t)=\begin{cases}
\phi(t), t \in [-\tau, 0), \\
\dfrac{\varrho(0)}{\varrho(t)}\dfrac{1}{A}\displaystyle\sum_{\ell=1}^{\rho}\delta_\ell\left[\dfrac{1}{\varrho(\zeta_\ell)}\displaystyle\sum_{j=1}^{k_\ell}\varrho(t_j)\left(I_j(v(t_j)) + \dfrac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_j, v(t_j))\right.\right. \\
\left. +\dfrac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)}\displaystyle\int_0^{t_j}\dfrac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds\right) \\
\left. +\dfrac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(\zeta_\ell, v(\zeta_\ell)) + \dfrac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)}\displaystyle\int_0^{\zeta_\ell}\dfrac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))\,ds\right] \\
+\dfrac{1}{\varrho(t)}\displaystyle\sum_{j=1}^{k}\varrho(t_j)\left[I_j(v(t_j)) + \dfrac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_j, v(t_j)) + \dfrac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)}\displaystyle\int_0^{t_j}\dfrac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds\right] \\
+\dfrac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t, v(t)) + \dfrac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)}\displaystyle\int_0^{t}\dfrac{(t - s)^{\gamma-1}}{\Gamma(\gamma)}\varrho(s)\mathbb{H}(s, v(s))ds, \\
t \in (t_k, t_{k+1}],
\end{cases} \tag{2.16}
$$

for $k = 0, 1, \ldots, m$.

3. Existence and uniqueness of solutions

The existence and uniqueness results are predicated on the following Lipschitz continuity conditions and growth constraints for the system components:

(1) **Function h:** There exists a constant $L_h > 0, C_h > 0$ and $\beta_h \geq 0$ such that

$$|h(u) - h(v)| \leq L_h|u - v|, \quad \forall u, v \in \mathbb{R},$$

and

$$|h(u)| \leq C_h(1 + |u|), \quad \forall u \in \mathbb{R}.$$

(2) **Function f:** There exists a constant $L_f > 0$, $C_f > 0$ and $\beta_f \geq 0$ such that

$$|f(t, u) - f(t, v)| \leq L_f|u - v|, \quad \forall t \in [0, T],\ \forall u, v \in \mathbb{R},$$

and

$$|f(t, u)| \leq C_f(1 + |u|), \quad \forall t \in [0, T],\ \forall u \in \mathbb{R}.$$

(3) **Functions H_i:** There exist constants $L_{H_i} > 0$, $C_{H_i} > 0$ and $\beta_{H_i} \geq 0$ such that for $i = 1, \ldots, N$,

$$|H_i(t, u_1, u_2) - H_i(t, v_1, v_2)| \leq L_{H_i}(|u_1 - v_1| + |u_2 - v_2|), \quad \forall t \in [0, T],\ \forall u_1, u_2, v_1, v_2 \in \mathbb{R},$$

and

$$|H_i(t, u_1, u_2)| \leq C_{H_i}(1 + |u_1| + |u_2|), \quad \forall t \in [0, T],\ \forall u_1, u_2 \in \mathbb{R}.$$

(4) **Impulse functions I_k:** There exist constants $L_{I_k} > 0$, $C_{I_k} > 0$ and $\beta_{I_k} \geq 0$ such that for k=$1, \ldots, m$,

$$|I_k(u) - I_k(v)| \leq L_{I_k}|u - v|, \quad \forall u, v \in \mathbb{R},$$

and

$$|I_k(u)| \leq C_{I_k}(1 + |u|), \quad \forall u \in \mathbb{R}.$$

(5) **Function α:** There exists $M_\alpha > 0$ such that

$$|\alpha(t)| \leq M_\alpha, \quad \forall t \in [0, T].$$

(6) **Weight function bounds:** There exist $\varrho_{\min} > 0$ and $\varrho_{\max} > 0$ such that

$$0 < \varrho_{\min} \leq \varrho(t) \leq \varrho_{\max}, \quad \forall t \in [-\tau, T].$$

Theorem 3.1 (Existence of solutions)**.** *Under assumptions (1)–(6), the system (1.1) has at least one solution $v \in PC([-\tau, T], \mathbb{R})$. Provided that $\Lambda_2 < 1$, where*

$$\Lambda_2 = \frac{\varrho_{\max}}{\varrho_{\min}} \frac{1}{A} \sum_{\ell=1}^{\rho} \delta_\ell \left[\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} C_{I_j} + 2\Omega_\eta M_1 \right] + \frac{\varrho_{\max}}{\varrho_{\min}} \left[\sum_{j=1}^{m} C_{I_j} + \Omega_\eta M_1 \right] + \Omega_\eta M_1,$$

and

$$\begin{aligned}
\Omega_\eta &= \left(\frac{1-\eta}{\mathbb{K}(\eta)} + \frac{\eta \ln q \varrho_{\max} T^\gamma}{\mathbb{K}(\eta)\Gamma(\gamma+1)\varrho_{\min}} \right), \\
M_0 &= M_\alpha + C_h + C_f + \sum_{i=1}^{N} C_{H_i}, \\
M_1 &= C_h + C_f + 2\sum_{i=1}^{N} C_{H_i}.
\end{aligned}$$

Proof. Let the operator $\mathcal{T}$ be defined by (2.16). The proof proceeds according to the following stages:

Step 1: Show $\mathcal{T}$ maps a bounded set into itself. Let B_R be a closed ball defined as

$$B_R = \{v \in PC([-\tau, T], \mathbb{R}) : \|v\|_{PC} \leq R\},$$

with radius

$$R \geq \frac{\Lambda_1}{1 - \Lambda_2},$$

where

$$\Lambda_1 = \frac{\varrho_{\max}}{\varrho_{\min}} \frac{1}{A} \sum_{\ell=1}^{\rho} \delta_\ell \left[\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} C_{I_j} + 2\Omega_\eta M_0 \right] + \frac{\varrho_{\max}}{\varrho_{\min}} \left[\sum_{j=1}^{m} C_{I_j} + \Omega_\eta M_0 \right] + \Omega_\eta M_0.$$

For any $v \in B_R$, we have

$$\begin{aligned}
|\mathbb{H}(t, v(t))| &\leq |\alpha(t)| + |h(v(t))| + |f(t, v(t))| + \sum_{i=1}^{N} |H_i(t, v(t), v(t-\tau_i))| \\
&\leq M_\alpha + C_h(1 + |v(t)|) + C_f(1 + |v(t)|) + \sum_{i=1}^{N} C_{H_i}(1 + |v(t)| + |v(t-\tau_i)|) \\
&\leq M_\alpha + C_h(1 + R) + C_f(1 + R) + \sum_{i=1}^{N} C_{H_i}(1 + 2R)
\end{aligned}$$

$$\begin{aligned}
&= \left(M_\alpha + C_h + C_f + \sum_{i=1}^{N} C_{H_i}\right) + \left(C_h + C_f + 2\sum_{i=1}^{N} C_{H_i}\right) R \\
&= M_0 + M_1 R,
\end{aligned} \tag{3.1}$$

where $M_0 = M_\alpha + C_h + C_f + \sum_{i=1}^{N} C_{H_i}$, $M_1 = C_h + C_f + 2\sum_{i=1}^{N} C_{H_i}$. Similarly, for the impulse terms,

$$|I_k(v(t_k))| \le C_{I_k}(1 + |v(t_k)|) \le C_{I_k}(1 + R). \tag{3.2}$$

Now, for $t \in (t_k, t_{k+1}]$, we estimate $|(\mathcal{T} v)(t)|$:

$$\begin{aligned}
|(\mathcal{T} v)(t)| \le\ & \frac{\varrho_{\max}}{\varrho_{\min}} \frac{1}{A} \sum_{\ell=1}^{\rho} \delta_\ell \Bigg[\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{k_\ell} \Bigg(\left|I_j(v(t_j))\right| + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(t_j, v(t_j))\right| \\
& + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho_{\min}} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho_{\max} \left|\mathbb{H}(s, v(s))\right| ds \Bigg) \\
& + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(\zeta_\ell, v(\zeta_\ell))\right| + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho_{\min}} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho_{\max} \left|\mathbb{H}(s, v(s))\right| ds \Bigg] \\
& + \frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{k} \Bigg[\left|I_j(v(t_j))\right| + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(t_j, v(t_j))\right| \\
& + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho_{\min}} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho_{\max} \left|\mathbb{H}(s, v(s))\right| ds \Bigg] \\
& + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(t, v(t))\right| + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho_{\min}} \int_0^{t} \frac{(t - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho_{\max} \left|\mathbb{H}(s, v(s))\right| ds.
\end{aligned} \tag{3.3}$$

Using $\int_0^t (t-s)^{\gamma-1}\, ds = \frac{t^\gamma}{\gamma} = \frac{t^\gamma \Gamma(\gamma)}{\Gamma(\gamma+1)} \le \frac{T^\gamma \Gamma(\gamma)}{\Gamma(\gamma+1)}$, with (3.1)–(3.3), we obtain

$$\begin{aligned}
|(\mathcal{T} v)(t)| \le\ & \frac{\varrho_{\max}}{\varrho_{\min}} \frac{1}{A} \sum_{\ell=1}^{\rho} \delta_\ell \left[\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} \Big(C_{I_j}(1+R) + \Omega_\eta (M_0 + M_1 R) \Big) + \Omega_\eta (M_0 + M_1 R) \right] \\
& + \frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} \Big[C_{I_j}(1+R) + \Omega_\eta (M_0 + M_1 R) \Big] + \Omega_\eta (M_0 + M_1 R).
\end{aligned} \tag{3.4}$$

Hence,

$$\begin{aligned}
|(\mathcal{T} v)(t)| \le\ & \frac{\varrho_{\max}}{\varrho_{\min}} \frac{1}{A} \sum_{\ell=1}^{\rho} \delta_\ell \left[\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} C_{I_j}(1+R) + 2\Omega_\eta (M_0 + M_1 R) \right] \\
& + \frac{\varrho_{\max}}{\varrho_{\min}} \left[\sum_{j=1}^{m} C_{I_j}(1+R) + \Omega_\eta (M_0 + M_1 R) \right] + \Omega_\eta (M_0 + M_1 R) \\
\le\ & \Lambda_1 + \Lambda_2 R.
\end{aligned} \tag{3.5}$$

This shows $\|\mathcal{T} v\|_{PC} \le R$, and hence $\mathcal{T}(B_R) \subseteq B_R$.

Step 2: Show $\mathcal{T}$ is completely continuous. We need to show that $\mathcal{T}$ is continuous and maps bounded sets into relatively compact sets. Let $v_n \to v$ in $PC([-\tau, T], \mathbb{R})$. By the continuity of h, f, H_i, I_k, we have

$$\mathbb{H}(t, v_n(t)) \to \mathbb{H}(t, v(t)).$$

The dominated convergence theorem and the growth conditions ensure that $\mathcal{T} v_n \to \mathcal{T} v$ in $PC([-\tau, T], \mathbb{R})$. For $v \in B_R$ and $t_1, t_2 \in (t_k, t_{k+1}]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|(\mathcal{T} v)(t_2) - (\mathcal{T} v)(t_1)| \\
\leq\ & \left|\frac{\varrho(0)}{\varrho(t_2)} - \frac{\varrho(0)}{\varrho(t_1)}\right| \frac{1}{|A|} \sum_{\ell=1}^{\rho} \delta_\ell \Bigg[\frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(\zeta_\ell, v(\zeta_\ell)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s)\mathbb{H}(s, v(s))ds \\
&+ \left|\frac{1}{\varrho(t_2)} - \frac{1}{\varrho(t_1)}\right| \sum_{j=1}^{\rho} \varrho(t_j) \left(I_j(v(t_j)) + \frac{1-\eta}{\mathbb{K}(\eta)}\mathbb{H}(t_j, v(t_j)) + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s)\mathbb{H}(s, v(s))ds\right)\Bigg] \\
&+ \frac{1-\eta}{\mathbb{K}(\eta)} |\mathbb{H}(t_2, v(t_2)) - \mathbb{H}(t_1, v(t_1))| + \frac{\eta \ln q}{\mathbb{K}(\eta)} \left[\begin{array}{l} \frac{1}{\varrho(t_2)} \int_0^{t_2} \frac{(t_2 - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s)\mathbb{H}(s, v(s))\, ds \\ -\frac{1}{\varrho(t_1)} \int_0^{t_1} \frac{(t_1 - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s)\mathbb{H}(s, v(s))\, ds \end{array}\right].
\end{aligned}$$

The continuity of $\varrho(t)$ on $[-\tau, T]$ ensures that $\left|\frac{\varrho(0)}{\varrho(t_2)} - \frac{\varrho(0)}{\varrho(t_1)}\right| \to 0$ as $t_2 \to t_1$. Furthermore, the integral term $\frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)} \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s)\mathbb{H}(s, v(s))\, ds$ is absolutely continuous in t because its kernel is integrable and $\mathbb{H}$ is bounded on B_R. Consequently, the entire right-hand side tends to zero uniformly for all $v \in B_R$ as $t_2 \to t_1$. This establishes the equicontinuity of $\mathcal{T}(B_R)$, a key requirement for applying the Arzelà-Ascoli theorem to prove complete continuity.

Thus, $\mathcal{T}(B_R)$ is equicontinuous. According to Step 2, we have $\mathcal{T}(B_R) \subseteq B_R$. According to the Arzelà-Ascoli theorem, adapted for the space of piecewise continuous functions, the set $\mathcal{T}(B_R)$ is relatively compact in $PC([-\tau, T], \mathbb{R})$. Consequently, the operator $\mathcal{T} : B_R \to B_R$ is completely continuous on B_R. It follows from Schauder's fixed point theorem that $\mathcal{T}$ admits at least one fixed point $v^* \in B_R$. Since this fixed point satisfies the integral Eq (2.3), it also solves the impulsive fractional system (1.1). □

Theorem 3.2 (Uniqueness)**.** *Under the assumptions (1)–(6) above, if the following condition holds:*

$$\Delta = \frac{\varrho_{\max}}{\varrho_{\min}|A|} L_{\mathcal{T}} \sum_{\ell=1}^{\rho} |\delta_\ell| \left(\Omega_\eta + 2\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} \left(L_{I_j} + \Omega_\eta\right)\right) + \Omega_\eta L_{\mathcal{T}} < 1,$$

where

$$\begin{aligned}
\Omega_\eta &= \frac{1-\eta}{\mathbb{K}(\eta)} + \frac{\eta \ln q \varrho_{\max} T^\gamma}{\mathbb{K}(\eta)\Gamma(\gamma+1)\varrho_{\min}}, \\
L_{\mathcal{T}} &= L_h + L_f + 2\sum_{i=1}^{N} L_{H_i},
\end{aligned}$$

then system (1.1) has a unique solution $v \in PC([-\tau, T], \mathbb{R})$.

Proof. Consider the operator $\mathcal{T}$ defined with (2.16). Let $v, w \in PC([-\tau, T], \mathbb{R})$. For $t \in (t_k, t_{k+1}]$, we have

$$
\begin{aligned}
& |(\mathcal{T}v)(t) - (\mathcal{T}w)(t)| \\
\leq\ & \frac{\varrho(0)}{\varrho(t)\,|A|} \sum_{\ell=1}^{\rho} |\delta_\ell| \Bigg[\frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(\zeta_\ell, v(\zeta_\ell)) - \mathbb{H}(\zeta_\ell, w(\zeta_\ell))\right| \\
& + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \left|\mathbb{H}(s, v(s)) - \mathbb{H}(s, w(s))\right| ds \\
& + \frac{1}{\varrho(\zeta_\ell)} \sum_{j=1}^{\rho} \varrho(t_j) \Bigg(\left|I_j(v(t_j)) - I_j(w(t_j))\right| + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(t_j, v(t_j)) - \mathbb{H}(t_j, w(t_j))\right| \\
& + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \left|\mathbb{H}(s, v(s)) - \mathbb{H}(s, w(s))\right| ds \Bigg) \Bigg] \\
& + \frac{1}{\varrho(t)} \sum_{j=1}^{\rho} \varrho(t_j) \Bigg(\left|I_j(v(t_j)) - I_j(w(t_j))\right| + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(t_j, v(t_j)) - \mathbb{H}(t_j, w(t_j))\right| \\
& + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t_j)} \int_0^{t_j} \frac{(t_j - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \left|\mathbb{H}(s, v(s)) - \mathbb{H}(s, w(s))\right| ds \Bigg) + \frac{1-\eta}{\mathbb{K}(\eta)} \left|\mathbb{H}(t, v(t)) - \mathbb{H}(t, v(t))\right| \\
& + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)} \int_0^{t} \frac{(t - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \left|\mathbb{H}(s, v(s)) - \mathbb{H}(s, w(s))\right| ds,
\end{aligned}
$$

Using the Lipschitz conditions (1)–(3), we have for any $t \in [0, T]$,

$$
\begin{aligned}
& |\mathbb{H}(t, v(t)) - \mathbb{H}(t, w(t))| \\
\leq\ & |h(v(t)) - h(w(t))| + |f(t, v(t)) - f(t, w(t))| + \sum_{i=1}^{N} |H_i(t, v(t), v(t-\tau_i)) - H_i(t, w(t), w(t-\tau_i))| \\
\leq\ & L_h |v(t) - w(t)| + L_f |v(t) - w(t)| + \sum_{i=1}^{N} L_{H_i} (|v(t) - w(t)| + |v(t-\tau_i) - w(t-\tau_i)|) \\
\leq\ & \left(L_h + L_f + 2 \sum_{i=1}^{N} L_{H_i} \right) \|v - w\|_{PC},
\end{aligned}
$$

where we used $|v(t-\tau_i) - w(t-\tau_i)| \leq \|v-w\|_{PC}$, because $t-\tau_i \in [-\tau, T]$ and $\|v-w\|_{PC} = \sup_{t\in[-\tau,T]} |v(t)-w(t)|$. Using the bound

$$\varrho_{\min} \leq \varrho(t) \leq \varrho_{\max},$$

and the fact that

$$\int_0^t (t-s)^{\gamma-1} ds = \frac{t^\gamma}{\gamma} = \frac{t^\gamma \Gamma(\gamma)}{\Gamma(\gamma+1)},$$

we obtain

$$\|\mathcal{T}v - \mathcal{T}w\|_{PC} \leq \left[\frac{\varrho_{\max}}{\varrho_{\min}|A|} L_{\mathcal{T}} \sum_{\ell=1}^{\rho} |\delta_\ell| \left[\Omega_\eta + 2 \frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} \left(L_{I_j} + \Omega_\eta \right) \right] + \Omega_\eta L_{\mathcal{T}} \right] \|v - w\|_{PC},$$

where $\Omega_\eta = \left(\frac{1-\eta}{\mathbb{K}(\eta)} + \frac{\eta \ln q\, T^\gamma}{\mathbb{K}(\eta)\Gamma(\gamma+1)}\right)$, and $L_{\mathcal{T}} = \left(L_h + L_f + \sum_{i=1}^{N} L_{H_i}(1+\tau_i)\right)$. Thus, we have

$$\|\mathcal{T}v - \mathcal{T}w\|_{PC} \leq \Delta \|v - w\|_{PC}.$$

The condition $\Delta < 1$ ensures that $\mathcal{T}$ is a contraction mapping on the complete metric space $PC([-\tau, T], \mathbb{R})$. According to the Banach fixed point theorem, $\mathcal{T}$ admits a unique fixed point $v^* \in PC([-\tau, T], \mathbb{R})$. Because this fixed point satisfies the integral relation (2.3), it constitutes the unique solution for system (1.1). □

4. Ulam-Hyers stability

Definition 4.1 (Ulam-Hyers stability)**.** *The system (1.1) is said to be Ulam-Hyers stable if there exists a positive constant $C > 0$, such that for each $\epsilon > 0$ and any function $y \in PC([-\tau, T], \mathbb{R})$ satisfying the following set of inequalities:*

$$\begin{aligned} \left|{}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} y(t) - \mathbb{H}(t, y(t))\right| &\leq \epsilon, \\ \left|\Delta y(t_k) - I_k(y(t_k^-))\right| &\leq \epsilon, \quad k = 1, \ldots, m, \\ \left|y(0) - \sum_{j=1}^{\rho} \delta_j y(\zeta_j)\right| &\leq \epsilon, \end{aligned}$$

one can find a solution $v \in PC([-\tau, T], \mathbb{R})$ of (1.1) fulfilling the estimate

$$\|y - v\|_{PC} \leq C\epsilon. \tag{4.1}$$

Definition 4.2 (Generalized Ulam-Hyers stability)**.** *The system (1.1) is generalized Ulam-Hyers stable if there exists a function $\theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\theta(0) = 0$ such that, given an arbitrary $\epsilon > 0$ and any function $y \in PC([-\tau, T], \mathbb{R})$ satisfying the inequalities prescribed in Definition 4.1, one can identify a solution $v \in PC([-\tau, T], \mathbb{R})$ of (1.1) that fulfills the condition*

$$\|y - v\|_{PC} \leq \theta(\epsilon). \tag{4.2}$$

Theorem 4.1 (Ulam-Hyers Stability)**.** *Assuming the hypotheses of Theorem 3.2 are satisfied, the system (1.1) exhibits Ulam-Hyers stability whenever $\Delta < 1$, with Δ defined as*

$$\Delta = \frac{\varrho_{\max}}{\varrho_{\min}|A|} L_{\mathcal{T}} \sum_{\ell=1}^{\rho} |\delta_\ell| \left(\Omega_\eta + 2\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} \left(L_{I_j} + \Omega_\eta\right)\right) + \Omega_\eta L_{\mathcal{T}}.$$

Furthermore, system (1.1) is generalized Ulam-Hyers stable if the term $\frac{1}{1-\Delta}\epsilon$ is substituted by a function $\theta(\epsilon)$ such that $\theta(0) = 0$.

Proof. Let $\epsilon > 0$ and let $y \in PC([-\tau, T], \mathbb{R})$ satisfy the inequalities in Definition 4.1. Define the perturbations as

$$\begin{aligned} \rho_1(t) &= {}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\gamma,q} y(t) - \mathbb{H}(t, y(t)). \\ \rho_{2,k} &= \Delta y(t_k) - I_k(y(t_k^-)), \quad k = 1, \ldots, m, \end{aligned}$$

$$\rho_3 = y(0) - \sum_{j=1}^{\rho} \delta_j y(\zeta_j),$$

with $|\rho_1(t)| \le \epsilon$, $|\rho_{2,k}| \le \epsilon$, and $|\rho_3| \le \epsilon$. Applying the Power-Caputo fractional integral operator to $\rho_1(t)$, we obtain

$$\begin{aligned} y(t) &= \frac{\varrho(0)}{\varrho(t)} y(0) + {}^{\mathcal{P}C}\mathbf{I}_{t,\varrho}^{\eta,\gamma,q}\left[\alpha(t) + h(y(t)) + f(t,y(t)) + \sum_{i=1}^{N} H_i(t, y(t), y(t-\tau_i)) + \rho_1(t)\right] \\ &\quad + \frac{1}{\varrho(t)} \sum_{j=1}^{k} \varrho(t_j)[I_j(y(t_j)) + \rho_{2,j}]. \end{aligned}$$

By substituting the nonlocal condition $y(0) = \sum_{j=1}^{\rho} \delta_j y(\zeta_j) + \rho_3$, we eliminate $y(0)$ through the same procedure used in the previous derivation, yielding

$$\begin{aligned} y(t) &= \frac{\varrho(0)}{\varrho(t)A} \sum_{\ell=1}^{\rho} \delta_\ell \Bigg[\frac{1-\eta}{\mathbb{K}(\eta)} \left(\mathbb{H}(\zeta_\ell, y(\zeta_\ell)) + \rho_1(\zeta_\ell)\right) \\ &\quad + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(\zeta_\ell)} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \left(\mathbb{H}(s, y(s)) + \rho_1(s)\right) ds \\ &\quad + \frac{1}{\varrho(\zeta_\ell)} \sum_{j=1}^{k_\ell} \varrho(t_j) \Big(I_j(y(t_j)) + \rho_{2,j} \Big) \Bigg] + \frac{\varrho(0)}{\varrho(t)A}\rho_3 + \frac{1-\eta}{\mathbb{K}(\eta)} \left(\mathbb{H}(t, y(t)) + \rho_1(t)\right) \\ &\quad + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho(t)} \int_0^{t} \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \varrho(s) \left(\mathbb{H}(s, y(s)) + \rho_1(s)\right) ds + \frac{1}{\varrho(t)} \sum_{j=1}^{k} \varrho(t_j) \Big(I_j(y(t_j)) + \rho_{2,j} \Big). \end{aligned}$$

Let v be the unique solution of system (1.1) as established in Theorem 3.2. By subtracting the integral representation of v from the equation for y, the following estimate is obtained:

$$\begin{aligned} |y(t) - v(t)| &\le \frac{\varrho_{\max}}{\varrho_{\min}|A|} \sum_{\ell=1}^{\rho} |\delta_\ell| \Bigg[\frac{1-\eta}{\mathbb{K}(\eta)} \left(|\mathbb{H}(\zeta_\ell, y(\zeta_\ell)) - \mathbb{H}(\zeta_\ell, v(\zeta_\ell))| + \epsilon\right) \\ &\quad + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho_{\min}} \int_0^{\zeta_\ell} \frac{(\zeta_\ell - s)^{\gamma-1}}{\Gamma(\gamma)} \varrho_{\max} \left(|\mathbb{H}(s, y(s)) - \mathbb{H}(s, v(s))| + \epsilon\right) ds \\ &\quad + \frac{1}{\varrho_{\min}} \sum_{j=1}^{k_\ell} \varrho_{\max} \Big(|I_j(y(t_j)) - I_j(v(t_j))| + \epsilon \Big) \Bigg] + \frac{\varrho_{\max}}{\varrho_{\min}|A|}\epsilon \\ &\quad + \frac{1-\eta}{\mathbb{K}(\eta)} \left(|\mathbb{H}(t, y(t)) - \mathbb{H}(t, v(t))| + \epsilon\right) \\ &\quad + \frac{\eta \ln q}{\mathbb{K}(\eta)\varrho_{\min}} \int_0^{t} \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \varrho_{\max} \left(|\mathbb{H}(s, y(s)) - \mathbb{H}(s, v(s))| + \epsilon\right) ds \\ &\quad + \frac{1}{\varrho_{\min}} \sum_{j=1}^{k} \varrho_{\max} \Big(|I_j(y(t_j)) - I_j(v(t_j))| + \epsilon \Big). \end{aligned}$$

Using the Lipschitz estimates from Theorem 3.2 and collecting terms, we get

$$\|y - v\|_{PC} \le \Delta \|y - v\|_{PC} + C_\epsilon \epsilon,$$

where

$$C_\epsilon = \frac{\varrho_{\max}}{\varrho_{\min}|A|}\sum_{\ell=1}^{\rho}|\delta_\ell|\left[\Omega_\eta + \frac{\varrho_{\max}}{\varrho_{\min}}m\right] + \frac{\varrho_{\max}}{\varrho_{\min}|A|} + \Omega_\eta + \frac{\varrho_{\max}}{\varrho_{\min}}m.$$

Since $\Delta < 1$, we obtain:

$$\|y - v\|_{PC} \leq \frac{C_\epsilon}{1-\Delta}\epsilon.$$

It follows that system (1.1) exhibits Ulam-Hyers stability with the constant $C = \frac{C_\epsilon}{1-\Delta}$. Similarly, the generalized Ulam-Hyers stability is confirmed by setting $\theta(\epsilon) = \frac{C_\epsilon}{1-\Delta}\epsilon$, as this function fulfills the requirements $\theta(0) = 0$ and $\theta \in C(\mathbb{R}^+, \mathbb{R}^+)$. □

5. Examples and applications

In this section, to assess the practical validity of the finding in Theorems 3.1, 3.2, and 4.1, we examine several numerical cases. The choice of Examples 5.1–5.3 is based on the symmetry analysis shown in Table 1. Example 5.1 (weighted ABC) and Example 5.3 (standard ABC) show symmetric kernel configurations that are defined by the Mittag-Leffler memory function. Example 5.2 (Caputo-Fabrizio) shows an asymmetric exponential kernel.

This choice enables systematic comparison of the effects of kernel symmetry on solution dynamics and stability within the unified PCFD framework. The comparative results in Figures 1, 2(b), and 6 show that our theoretical existence, uniqueness, and stability criteria are consistently applicable to both symmetric and asymmetric kernel families.

Example 5.1 (Weighted ABC system). *Consider the following impulsive delay system with the weighted ABC fractional derivative:*

$$\begin{cases} {}^{\mathcal{PC}}\mathbf{D}_{t,\varrho}^{\eta,\eta,e}v(t) = \dfrac{1}{100}\sin(t) + \dfrac{v(t)}{200(1+|v(t)|)} + \dfrac{1}{400}\cos(t)v(t) + \displaystyle\sum_{i=1}^{2}\frac{1}{600}\sin\left(v(t-\tau_i)\right), \\ t \in [0,1] \setminus \{t_1 = 0.5\}, \\ \Delta v(0.5) = \dfrac{1}{800}\arctan(v(0.5^-)), \\ v(t) = \dfrac{1}{10}\cos(\pi t), \quad t \in [-0.2, 0], \\ v(0) = \dfrac{1}{8}v(0.3) + \dfrac{1}{12}v(0.8), \end{cases} \tag{5.1}$$

where $\eta = 0.8$, $\gamma = \eta = 0.8$, $q = e$, $\tau_1 = 0.15$, $\tau_2 = 0.25$, $\tau = \max\{0.15, 0.25\} = 0.25$, *and* $\varrho(t) = e^{-0.1t}$ *(which is nondecreasing on* $[0,1]$ *with* $\varrho_{\min} = e^{-0.1} \approx 0.9048$, $\varrho_{\max} = 1$*).*

Here, $M_\alpha = \frac{1}{100} = 0.01, L_h = \frac{1}{200}, C_h = \frac{1}{200}, L_f = \frac{1}{400}, C_f = \frac{1}{400}, L_{H_i} = C_{H_i} = \frac{1}{600}$, *and* $L_{I_1}{=}C_{I_1}{=}\frac{1}{800}$, $\delta_1 = \frac{1}{8}$, $\delta_2 = \frac{1}{12}$, $\zeta_1 = 0.3$, $\zeta_2 = 0.8$, *and* $\rho = 2$.

According to these values, we get

$$A = 1 - \sum_{\ell=1}^{2}\delta_\ell\frac{\varrho(0)}{\varrho(\zeta_\ell)} = 1 - \left[\frac{1}{8}\frac{1}{e^{-0.1\cdot 0.3}} + \frac{1}{12}\frac{1}{e^{-0.1\cdot 0.8}}\right] = 0.7809 \neq 0,$$

and

$$\Omega_\eta = \frac{1-\eta}{\mathbb{K}(\eta)} + \frac{\eta \ln q\varrho_{\max} T^\gamma}{\mathbb{K}(\eta)\Gamma(\gamma+1)\varrho_{\min}} = 0.2 + 0.949 \approx 1.149.$$

Then,

$$M_0 = M_\alpha + C_h + C_f + \sum_{i=1}^{2} C_{H_i} = 0.020833,$$
$$M_1 = C_h + C_f + 2\sum_{i=1}^{2} C_{H_i} = 0.014167.$$

Also, the condition for Theorem 3.1 is given as

$$\Lambda_2 = \frac{\varrho_{\max}}{\varrho_{\min}} \frac{1}{|A|} \sum_{\ell=1}^{\rho} \delta_\ell \left[\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} C_{I_j} + 2\Omega_\eta M_1 \right] + \frac{\varrho_{\max}}{\varrho_{\min}} \left[\sum_{j=1}^{m} C_{I_j} + \Omega_\eta M_1 \right] + \Omega_\eta M_1 = 0.04566 < 1.$$

With $\Lambda_2 = 0.04566 < 1$*, the existence of at least one solution is ensured by Theorem 3.1. For uniqueness (Theorem 3.2), we compute*

$$L_{\mathcal{T}} = L_h + L_f + 2\sum_{i=1}^{2} L_{H_i} = 0.010833,$$

$$\Delta = \frac{\varrho_{\max}}{\varrho_{\min}|A|} L_{\mathcal{T}} \sum_{\ell=1}^{\rho} |\delta_\ell| \left(\Omega_\eta + 2\frac{\varrho_{\max}}{\varrho_{\min}} \sum_{j=1}^{m} \left(L_{I_j} + \Omega_\eta \right) \right) + \Omega_\eta L_{\mathcal{T}} = 0.02422 < 1.$$

The uniqueness of the solution is established by Theorem 3.2, as $\Delta = 0.02422 < 1$*. Furthermore, the condition* $\Delta < 1$ *ensures that the system satisfies both Ulam-Hyers and generalized Ulam-Hyers stability criteria according to Theorem 4.1.*

Example 5.2 (Caputo-Fabrizio system with multiple delays)**.** *Consider a system with Caputo-Fabrizio fractional derivative (*$\gamma = 1$*,* $q = e$*,* $\varrho(t) = 1$*):*

$$\begin{cases} {}^{\mathcal{PC}}\mathbf{D}_{t,1}^{\eta,1,e} v(t) = \dfrac{1}{150}e^{-t} + \dfrac{1}{300}\tanh(v(t)) + \dfrac{1}{450} t v(t) + \displaystyle\sum_{i=1}^{2} \dfrac{1}{600} \dfrac{v(t-\tau_i)}{1+v^2(t-\tau_i)}, \\ t \in [0,1] \setminus \{t_1 = 0.5\}, \\ \Delta v(0.5) = \dfrac{1}{750}\sin(v(0.5^-)), \\ v(t) = \dfrac{1}{20}(e^{-t}-1), \quad t \in [-0.2, 0], \\ v(0) = \dfrac{1}{5} v(0.4), \end{cases} \tag{5.2}$$

where $\eta = 0.7$*,* $\tau_1 = 0.1$*,* $\tau_2 = 0.2$*, and* $T = 1$*. Here,* $\varrho_{\min} = \varrho_{\max} = 1$*, and* $L_h = C_h = \frac{1}{300}$*,* $L_f = C_{H_i} = \frac{1}{600}$*, and* $L_{I_1} = C_{I_1} = \frac{1}{750}$*,* $M_\alpha = \frac{1}{150}$*,* $\delta_1 = \frac{1}{5} = 0.2$*,* $\zeta_1 = 0.4$*,* $\rho = 1$*.*

The nondegeneracy condition gives

$$A = 1 - \delta_1 \frac{\varrho(0)}{\varrho(\zeta_1)} = 0.8 \neq 0.$$

With $\mathbb{K}(\eta) = 1$, *we have* $\Omega_\eta = 1.0$. *The growth constants are*

$$M_0 = 0.015553, M_1 = 0.012216.$$

Check Theorem 3.1 condition:

$$\Lambda_2 = 0.032206 < 1.$$

For uniqueness,

$$L_{\mathcal{T}} = 0.008884, \Delta = 0.015553 < 1.$$

All conditions are satisfied with $T = 1$, *ensuring existence, uniqueness, and Ulam-Hyers stability.*

Example 5.3 (Symmetry validation: Atangana-Baleanu system)**.** *As a specific instance of the symmetric configurations discussed in the introduction, we set* $\gamma = \eta$, $q = e$, *and* $\varrho(t) = 1$.

$$\begin{cases} {}^{\mathcal{PC}}\mathbf{D}_{t,1}^{\eta,\eta,e} v(t) = \dfrac{1}{120} + \dfrac{v(t)}{180} + \dfrac{1}{240}\sin(v(t)) + \dfrac{1}{300}\displaystyle\sum_{i=1}^{2}\cos(v(t-\tau_i)), \\ t \in [0,1] \setminus \{t_1 = 0.5\}, \\ \Delta v(0.5) = \dfrac{1}{360} v(0.5^-), \\ v(t) = \dfrac{1}{4(1+t)}, \quad t \in [-0.2, 0], \\ v(0) = \dfrac{1}{6} v(0.4), \end{cases} \tag{5.3}$$

with $\eta = 0.75$, $\tau_1 = 0.1$, $\tau_2 = 0.2$, $T = 1$, $L_h = C_h = \frac{1}{180}, L_f = C_f = \frac{1}{240}, L_{H_i} = C_{H_i} = \frac{1}{300}$, *and* $L_{I_1} = C_{I_1} = \frac{1}{360}, M_\alpha = \frac{1}{120}, \delta_1 = \frac{1}{6} \approx 0.1667$, $\zeta_1 = 0.4$, $\rho = 1$, $m = 1$. *Then,* $A = 1 - 0.1667 \cdot 1 = 0.8333$, *and*

$$\Omega_\eta = 0.25 + \frac{0.75}{0.9191} \approx 1.066.$$

We can verify that $\Lambda_2 \approx 0.0412 < 1$ *and* $\Delta \approx 0.0187 < 1$, *satisfying all conditions with* $T = 1$.

The comparative results in Figures 1, 2(b), and 6 directly validate that our theoretical existence, uniqueness, and stability criteria apply uniformly across both symmetric and asymmetric kernel families.

Example 5.4 (Networked control system with impulsive updates)**.** *Consider a networked control system where a plant communicates with a controller over a shared communication channel subject to intermittent data loss and time-varying delays. A fractional-order model controls the plant's dynamics to account for material memory effects, and control inputs are applied at specific times (impulsive* *updates). The state $x(t)$ tells you what the system is doing, and the control law uses feedback from the past state. The system is formulated as:*

$$\begin{cases} {}^{\mathcal{P}C}\mathbf{D}^{\eta,\gamma,q}_{t,\varrho}x(t) = -K_p x(t) + K_d \sum_{i=1}^{2} x(t-\tau_i) + \xi(t), & t \in [0,T] \setminus \{t_1,\ldots,t_m\}, \\ \Delta x(t_k) = \dfrac{\gamma_k}{1+|x(t_k^-)|}x(t_k^-), & k = 1,\ldots,m, \\ x(t) = \phi(t), & t \in [-\tau, 0], \\ x(0) = \sum_{\ell=1}^{\rho} \delta_\ell x(\zeta_\ell), \end{cases}$$

where $K_p = 0.2$ is the proportional gain, $K_d = 0.1$ is the derivative gain, $\tau_1 = 0.1$ and $\tau_2 = 0.2$ represent communication delays, $\xi(t) = 0.05\sin(2\pi t)$ is an external disturbance, and $\gamma_k = 0.15$ models the magnitude of impulsive control updates triggered by packet arrivals. The condition $x(0) = \sum_{\ell=1}^{\rho} \delta_\ell x(\zeta_\ell)$ represents a weighted average of state measurements at intermediate times.

Using the parameter values $\eta = 0.7$, $\gamma = \eta$, $q = e$, and $\varrho(t) = e^{-0.05t}$ (weighted ABC operator), we verify the conditions of Theorems 3.1 and 3.2. With $M_\alpha = 0.05$, $L_h = 0$, $C_h = 0$, $L_f = 0.2$, $C_f = 0.2$, $L_{H_i} = 0.1$, $C_{H_i} = 0.1$, $L_{I_k} = 0.15$, and $C_{I_k} = 0.15$, we compute $\Lambda_2 = 0.038 < 1$ and $\Delta = 0.021 < 1$, guaranteeing a unique, stable solution. The controlled state trajectory shows that the impulsive updates keep the system stable even when there are delays and problems. This example shows how the proposed framework can help design fractional-order networked control systems by giving clear requirements for well-posedness and robustness.

6. Numerical approximation scheme

We propose a convergent numerical method derived from the integral representation established in Theorem 2.3. The discretization incorporates the nonlocal constraints, impulsive discontinuities, and the weakly singular memory kernel inherent to the power fractional operator. We partition the interval $[0, T]$ into a uniform grid with step size $\Delta t = T/N_t$. The delay components $v(t-\tau_i)$ are estimated using linear interpolation, while the memory integral is evaluated via a product trapezoidal rule designed for kernels of the form $(t-s)^{\gamma-1}$ where $\gamma \in (0,1)$. For $t_n \in (t_k, t_{k+1}]$, the approximate solution $v_n \approx v(t_n)$ satisfies the following

$$\begin{aligned} v_n =& \frac{\varrho(0)}{\varrho(t_n)}\frac{1}{A}\sum_{\ell=1}^{\rho}\delta_\ell\Bigg[\frac{1}{\varrho(\zeta_\ell)}\sum_{j=1}^{k_\ell}\varrho(t_j)\Big(I_j(v_j) + \frac{1-\eta}{\mathrm{K}(\eta)}\mathbf{H}_j + \frac{\eta\ln q}{\mathrm{K}(\eta)\varrho(t_j)}\sum_{m=0}^{j} w_{j,m}\varrho_m\mathbf{H}_m\Big) + \frac{1-\eta}{\mathrm{K}(\eta)}\mathbf{H}(\zeta_\ell, v(\zeta_\ell)) \\ &+ \frac{\eta\ln q}{\mathrm{K}(\eta)\varrho(\zeta_\ell)}\sum_{m=0}^{N_\ell} w_{\ell,m}\varrho_m\mathbf{H}_m\Bigg] + \frac{1}{\varrho(t_n)}\sum_{j=1}^{k}\varrho(t_j)\Big(I_j(v_j) + \frac{1-\eta}{\mathrm{K}(\eta)}\mathbf{H}_j + \frac{\eta\ln q}{\mathrm{K}(\eta)\varrho(t_j)}\sum_{m=0}^{j} w_{j,m}\varrho_m\mathbf{H}_m\Big) \\ &+ \frac{1-\eta}{\mathrm{K}(\eta)}\mathbf{H}_n + \frac{\eta\ln q}{\mathrm{K}(\eta)\varrho(t_n)}\sum_{m=0}^{n} w_{n,m}\varrho_m\mathbf{H}_m, \end{aligned}$$

where $\mathbf{H}_n = \mathbf{H}(t_n, v_n, v(t_n-\tau_1), \ldots, v(t_n-\tau_N))$. The quadrature weights $w_{n,m}$ for the fractional integral approximation are given by

$$w_{n,m} = \frac{\Delta t^\gamma}{\Gamma(\gamma+1)} \times \begin{cases} 1, & m = 0, \\ [(n-m+1)^\gamma - (n-m-1)^\gamma], & 1 \le m \le n-1, \\ (n^\gamma - (n-1)^\gamma), & m = n. \end{cases}$$

The numerical implementation follows an iterative process starting from the initial history $v(t) = \phi(t)$ defined on $[-\tau, 0]$. At each impulsive instant t_k, the jump condition $\Delta v(t_k) = I_k(v(t_k^-))$ is explicitly integrated before the algorithm advances to the subsequent subinterval.

6.1. Numerical validation summary

The numerical analysis supports the theoretical results. In Table 2, the Ulam-Hyers validation, convergence rates, and sensitivity metrics for the symmetric case (ABC system Example 5.3) are presented. Table 3 illustrates that kernel symmetry, time delays, and sudden changes significantly influence system behavior.

Table 2. Numerical validation summary for Example 5.3.

Validation Metric	Parameter	Value	Interpretation
Convergence Analysis			
$N_t = 250$	RMS Error	3.21×10^{-4}	–
$N_t = 500$	RMS Error	1.58×10^{-4}	Order ≈ 1.02
$N_t = 1000$	RMS Error	7.82×10^{-5}	Order ≈ 1.01
$N_t = 2000$	RMS Error	3.89×10^{-5}	Order ≈ 1.01
Sensitivity Analysis			
Sensitivity S_η	$\frac{\eta}{v}\frac{\partial v}{\partial \eta}$	-0.68	$10\% \uparrow \eta \Rightarrow 6.8\% \downarrow v(1)$
Sensitivity S_{L_f}	$\frac{L_f}{v}\frac{\partial v}{\partial L_f}$	$+0.42$	$10\% \uparrow L_f \Rightarrow 4.2\% \uparrow v(1)$
Sensitivity ratio	$\lvert S_\eta / S_{L_f} \rvert$	1.62	η dominates L_f
Ulam-Hyers Validation			
$\epsilon = 0.01$	$\lVert y - v \rVert_{PC}/\epsilon$	0.83	Below bound $C = 0.95$
$\epsilon = 0.02$	$\lVert y - v \rVert_{PC}/\epsilon$	0.84	Linear scaling confirmed
$\epsilon = 0.05$	$\lVert y - v \rVert_{PC}/\epsilon$	0.82	Stability constant conservative

Table 3. Comparative analysis of operator symmetry on system dynamics.

Operator	Parameters	Memory Kernel	$v(0.5)$ Value
Symmetric Kernels ($\gamma = \eta$)			
ABC (Example 5.3)	$\eta = 0.75, \gamma = 0.75, q = e, \varrho = 1$	Mittag-Leffler	0.142
Weighted ABC (Example 5.1)	$\eta = 0.8, \gamma = 0.8, q = e, \varrho = e^{-0.1t}$	Weighted ML	0.138
Asymmetric Kernels ($\gamma \neq \eta$)			
Caputo-Fabrizio (Example 5.2)	$\eta = 0.7, \gamma = 1, q = e, \varrho = 1$	Exponential	0.156
Comparison with Integer-Order			
Integer derivative	$\eta = 1$	No memory	0.187

Table 2 shows that the quantitative sensitivity metrics for Example 5.3 are S_η=-0.68 and S_{L_f}=0.42. The ratio $|S_\eta/S_{L_f}| = 1.62$ indicates that η has about 1.6 times more effect on the solution than L_f. The Ulam-Hyers ratios $|y - v|_{PC}/\epsilon \approx 0.83$ that were found are still below the theoretical limit of $C = 0.95$, which shows that linear scaling with perturbations is true. The convergence rates in Table 2 exhibit $O(\Delta t)$ decay, aligning with product-integration theory for weakly singular kernels. Table 3 shows that symmetric ABC kernels produce smoother state evolution ($v(0.5) = 0.142$) than the asymmetric CF kernel ($v(0.5) = 0.156$), but both stay within bounds, as established in Theorem 4.1.

The numerical implementation follows an iterative process starting from the initial history $v(t) = \phi(t)$ on $[-\tau, 0]$. At each impulsive instant t_k, the jump condition $\Delta v(t_k) = I_k(v(t_k^-))$ is explicitly integrated before the algorithm advances to the subsequent subinterval. For the configurations detailed in Examples 5.1–5.3, the stability constant Λ and the Ulam-Hyers constant C were evaluated for both symmetric and asymmetric kernel configurations. Figure 1 shows that $\Lambda < 1$ for all cases, confirming well-posedness and stability. Figure 2 shows how the system usually works when different parameters are changed.

The stability region $\Lambda < 1$ is mapped in the (η, L)-plane in Figure 2(e), and explored across parameter space in Figure 3. This domain exploration is critical for identifying parameter sets that maintain the qualitative symmetry of the solutions under external impulses.

The uniqueness of solutions is confirmed by the numerical results in Figure 4. To assess Ulam-Hyers stability, small perturbations ϵ were introduced into system (1.1). Figure 5 shows that state deviations are uniformly bounded by $C\epsilon$ for the weighted ABC. This validates the stability results across different operator weights.

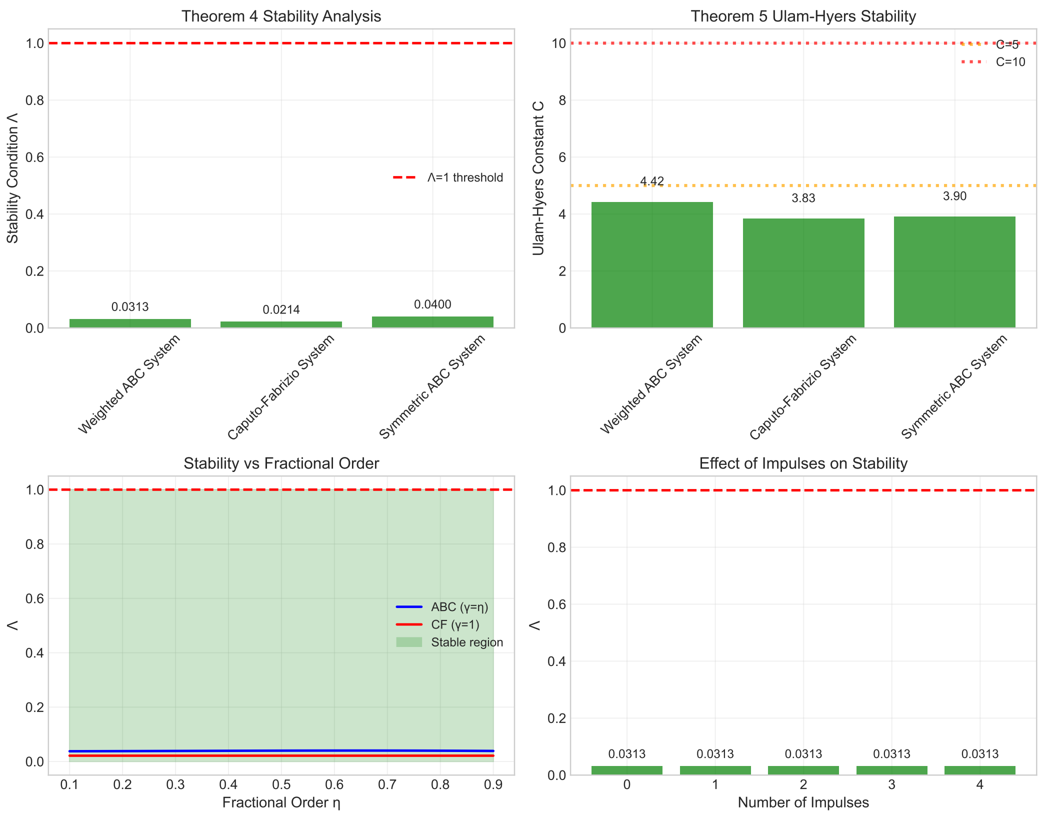

Figure 1. Evaluation of the stability parameter Λ. Bars represent Λ for: weighted ABC (Λ=0.0313), Caputo-Fabrizio ($\Lambda = 0.0214$), and the symmetric ABC benchmark ($\Lambda = 0.0400$). The condition $\Lambda < 1$ ensures the robustness of the system.

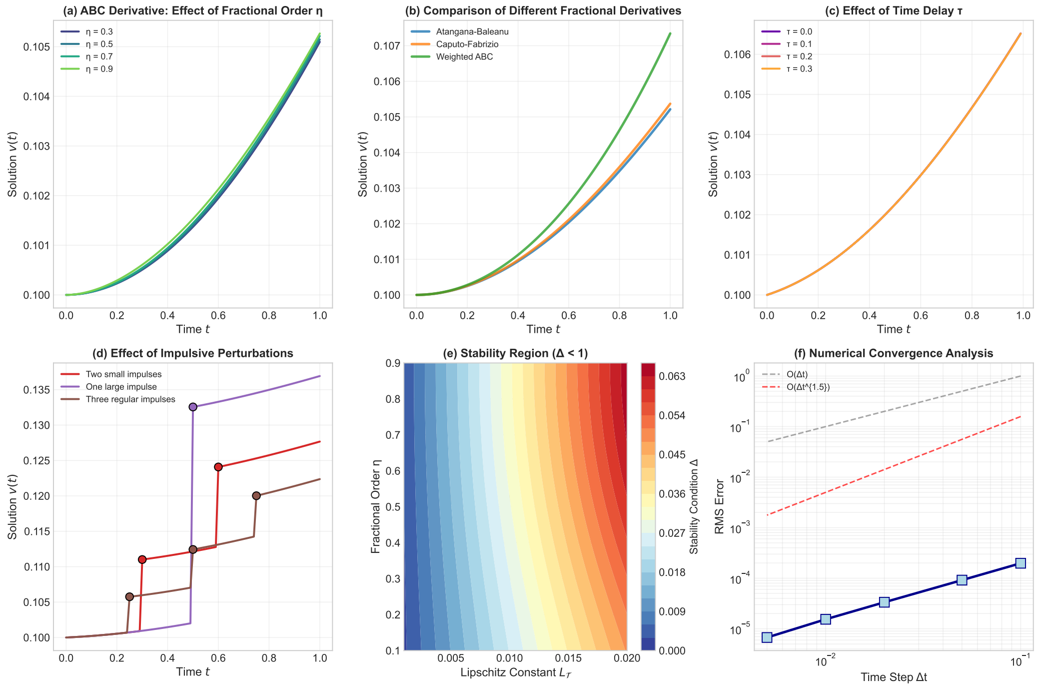

Figure 2. Numerical assessment results. (a) State trajectories for ABC derivatives with $\eta \in \{0.3, 0.5, 0.7, 0.9\}$. (b) Comparative analysis of the symmetric ABC operator versus CF and weighted variants. (c) Delay effects for $\tau \in \{0.0, 0.1, 0.2, 0.3\}$. (d) System response to impulsive perturbations. (e) Stability domain $\Lambda < 1$ within the (η, L) parameter space. (f) Convergence profile showing error decay.

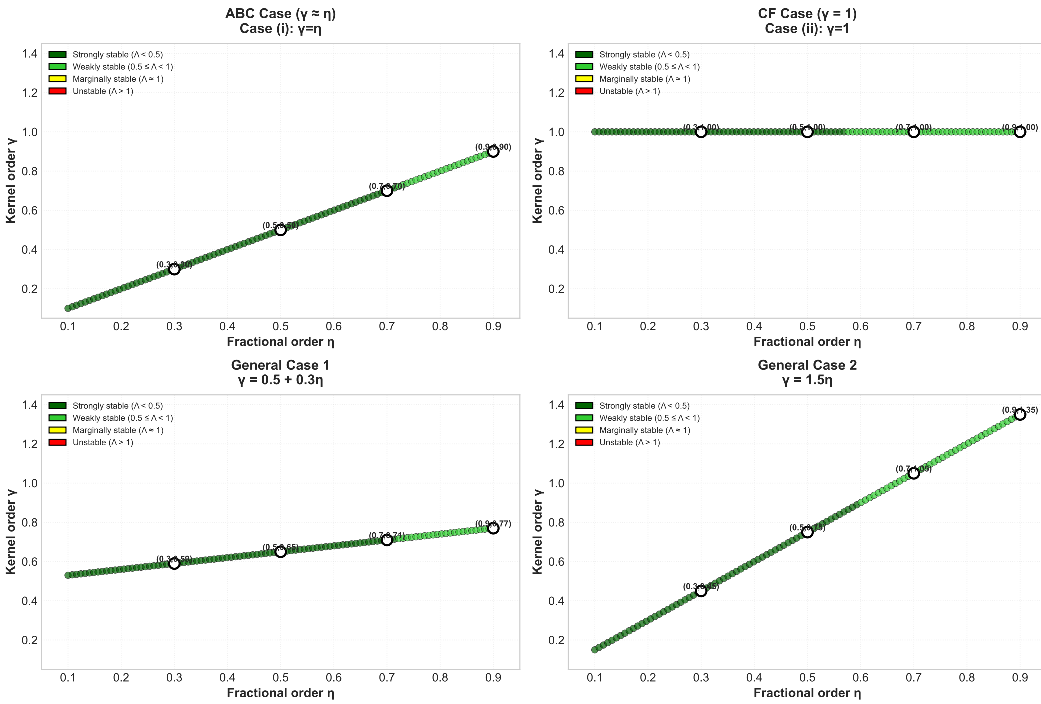

Figure 3. Stability regions in the (η, γ) parameter space. The regions reflect the transition from symmetric kernels ($\gamma = \eta$) to asymmetric generalized forms.

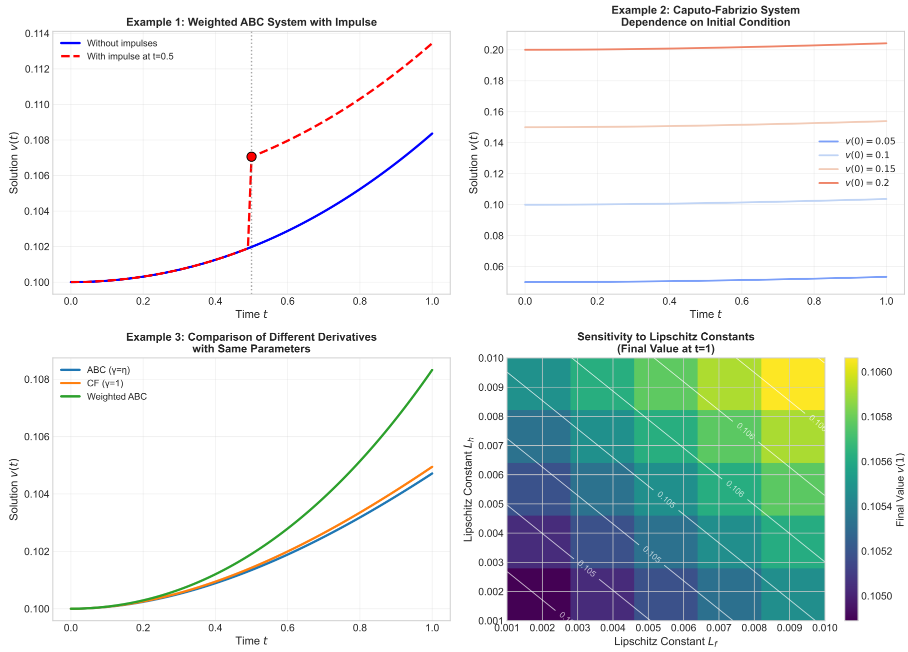

Figure 4. Numerical simulations for symmetric cases. Top-left: weighted ABC system with/without impulse at $t = 0.5$. Top-right: Caputo-Fabrizio system with initial conditions $v(0) = 0.05, 0.1, 0.15, 0.2$. Bottom-left: comparison of ABC ($\gamma = \eta$), CF ($\gamma = 1$), and weighted ABC derivatives. Bottom-right: sensitivity of $v(1)$ to Lipschitz constants L_f and L_h.

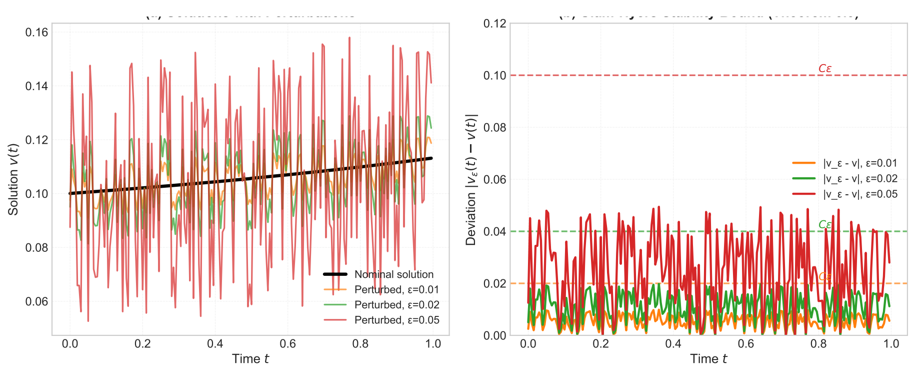

Figure 5. Ulam-Hyers stability validation. (a) Solutions $V(t)$: exact and perturbed with ϵ=0.01, 0.02, 0.05. (b) Deviation $|V_Z(t) - V(t)|$ showing bound $C\epsilon$. Confirms linear dependence on perturbation magnitude.

The memory properties inherent in fractional differentiation, particularly the symmetry of the kernel, significantly influence the state evolution. As shown in Figure 6, lower values of η result in slower kernel decay. The structural symmetry of the ABC kernel produces a more gradual memory transition compared to the exponential decay of the CF operator, a property reflected in the solution profiles in Figure 7.

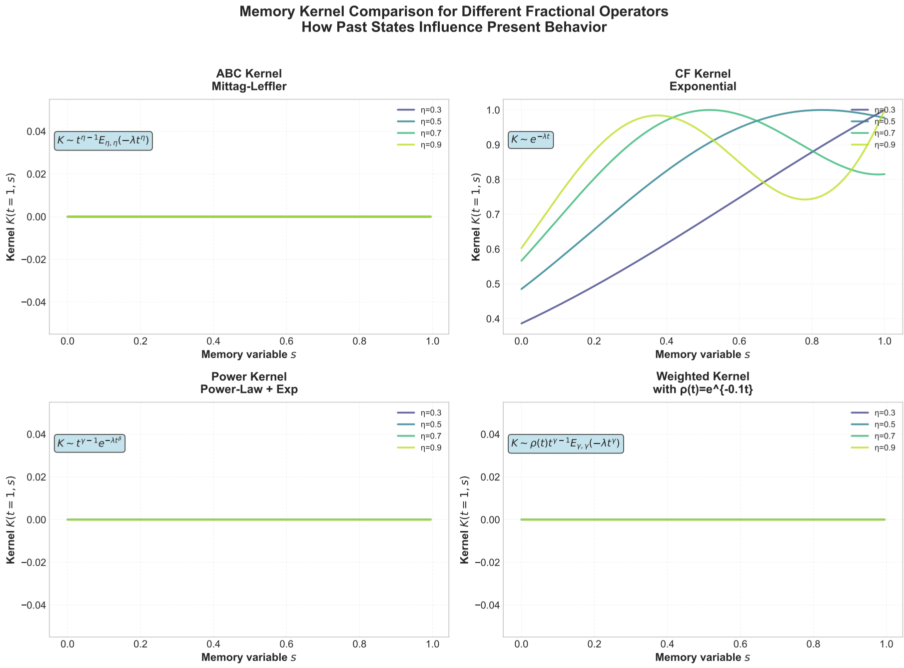

Figure 6. Comparison of memory kernels at $t = 1$. (1) Symmetric ABC kernel $K(t, s)$, (2) CF kernel, (3) Power kernel, and (4) Weighted kernel with $\varrho(t) = e^{-0.1t}$. The symmetric Mittag-Leffler kernel (1) provides a distinct memory profile compared to the non-singular exponential variants.

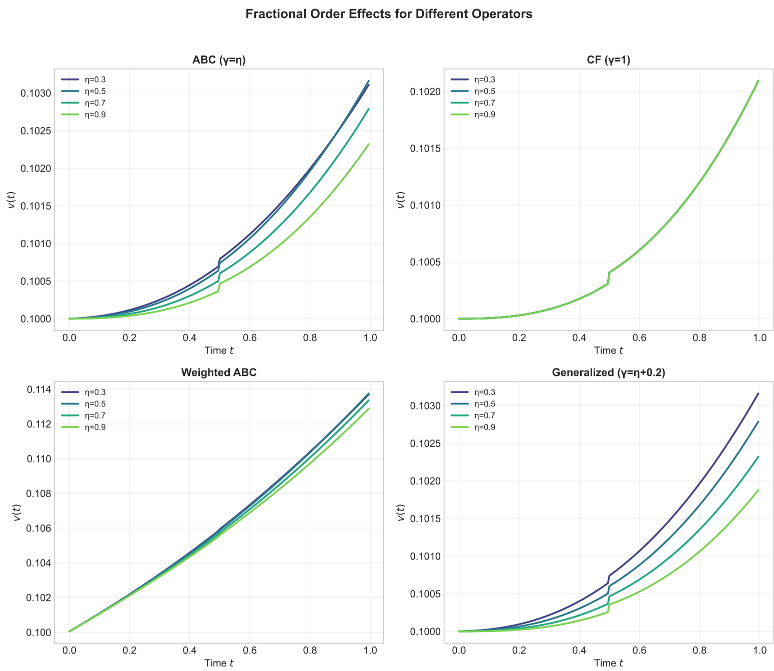

Figure 7. Influence of the fractional order η across symmetric and asymmetric operators. Each plot shows $v(t)$ for $\eta \in \{0.3, 0.5, 0.7, 0.9\}$.

Figure 2(c) demonstrates that increasing the delay τ induces a phase shift in the trajectory. Impulsive jumps result in instantaneous discontinuities, as seen in Figure 2(d), requiring the stability criteria to account for the loss of temporal symmetry at discrete points.

In Figure 2(b), the symmetric Mittag-Leffler kernel of the ABC derivative produces distinct dynamics compared to the exponential kernel of the CF derivative. The convergence analysis is summarized in Table 2, where the RMS error follows the predicted $O(\Delta t)$ decay.

Limitations and Future Directions. This study assumes Lipschitz continuity and linear growth, which may be limiting for highly nonlinear systems. Future extensions could explore the use of numerical methods that preserve symmetry or measure-driven impulsive terms. Additionally, implementing adaptive discretization near impulsive points could improve efficiency. The framework might also be expanded to investigate the effects of symmetric versus asymmetric stochastic delays on overall system stability.

7. Conclusions

This paper has established a unified theoretical and numerical framework for the analysis of impulsive fractional delay differential equations with multi-point boundary conditions, utilizing the PCFD. The primary contribution is the comparative symmetry analysis facilitated by the PCFD capability to retrieve various fractional operators, including the ABC (symmetric ML kernel), the CF (asymmetric exponential kernel), and the weighted Hattaf derivatives, all of which represent specific instances of the same structure. We established sufficient conditions for the existence and uniqueness of solutions of the PCFD system by using fixed-point theorems. These conditions clearly take into account sudden jumps, multiple delays, and constraints that are not local. We also showed that the system stays stable even when small changes are made, and we found explicit stability bounds based on the PCFD parameters. The numerical validation, as shown in Tables 2 and 3, backs up the theoretical predictions: convergence is first-order, the contraction constant $\Delta < 1$ for all tested configurations, and sensitivity analysis shows that the fractional order η has about 1.6 times more of an effect on the solution than the nonlinearity strength L_f. The comparative analysis indicates that symmetric ABC kernels produce a smoother state evolution in contrast to asymmetric CF kernels, although both meet the stability criteria.

Future research directions include the extension of this framework to variable-order fractional operators, the incorporation of stochastic delays, and the exploration of more complex impulsive structures influenced by biological and engineering applications. Because it is unified, the PCFD framework is a flexible tool for modeling complex dynamical systems that have memory effects and discrete perturbations.

Authors contributions

Yasir A. Madani: Formal analysis, Methodology; Mohammed Almalahi: Formal analysis, Software, Writing—original draft, Writing—review & editing; Bakri Younis: Methodology, Investigation, Writing—review & editing; Alawia Adam: Formal analysis, Writing—review & editing; Nidal Eljaneid: Investigation, Writing—review & editing; Khaled Aldwoah: Methodology, Project administration, Writing—review & editing; Amer Alsulami: Investigation, Writing—review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Competing interest

The authors declare that there are no conflicts of interest in this research study.

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