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Research article

Past extropy for linear consecutive r-out-of-n: F systems and its properties, bounds, and estimation

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Abstract: This paper develops a rigorous information-theoretic framework for uncertainty quantification in lifetime analysis through the concept of past extropy. Particular attention is paid to linear consecutive r-out-of-n:F systems with independent and identically distributed component lifetimes. Explicit analytical expressions for the past extropy of system lifetimes are derived, and several new theoretical properties are established, including monotonicity results, probabilistic bounds, and characterization theorems, which reveal structural relationships between system reliability configurations and information-based uncertainty measures. The investigation is further extended to conditional past extropy, providing additional insight into uncertainty assessment under partial system information. To support practical implementation, a nonparametric kernel-based estimator of past extropy is proposed. Its performance is evaluated through Monte Carlo simulation experiments that illustrate stable finite-sample behavior and reliable estimation accuracy across different parameter settings. The presented results demonstrate that past extropy offers a flexible and informative framework for studying uncertainty in lifetime distributions and structured reliability systems, combining rigorous theoretical developments with practical data-driven estimation methodology.

Keywords: past extropy; reliability modeling; stochastic orders; characterization; nonparametric estimation
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1. Introduction

Quantifying uncertainty is a central objective in the modeling and analysis of complex engineering and stochastic systems. In reliability engineering, the lifetime of a system is influenced by the random behavior of its components and, more generally, by the structural mechanisms governing their joint operation. As a result, system performance is inherently uncertain. Information-theoretic measures, particularly those originating from Shannon's entropy, have emerged as powerful tools for describing and comparing uncertainties in such settings. Entropy characterizes the expected unpredictability of a random variable, and its numerous extensions have become fundamental across a wide range of scientific and engineering applications.

Recently, the study of uncertainty in system lifetimes has received considerable attention, especially through the use of information-based measures that provide quantitative insight into stochastic behavior. In parallel, an extensive body of research has focused on consecutive r-out-of-n systems, which arise naturally in a variety of technological and physical applications. These systems may be arranged linearly or circularly and, depending on their operational definition, can be categorized as failure (F) or good (G) systems. A linear consecutive r-out-of-n: F system consists of n independent and identically distributed components arranged in a sequence, and system failure occurs when at least r consecutive components fail. Classical system structures arise as special cases, including the series system when $r = 1$ and the parallel system when $r = n$. As an illustrative example, consider an oil pipeline equipped with n pumping stations arranged sequentially along the pipeline. If four or more consecutive stations fail, the flow of oil is disrupted, and the system fails. This configuration corresponds to a linear consecutive 4-out-of-n: F system.

The study of consecutive-type systems dates back to the seminal work of Kontoleon [1]; this topic has since become an active area of research in reliability theory. Numerous analytical and approximate methods have been developed to evaluate such systems, to characterize their performance, and to explore the effects of component-level dependence. Important contributions include those by Yin et al. [2], Cui et al. [3], Zhao et al. [4], Jung and Kim [5], Eryılmaz [6–9], Kuo and Zuo [10], and Salehi et al. [11]. A noteworthy observation by Eryılmaz [6] is that the lifetime distribution of a linear consecutive $r$out-of-n: F system becomes significantly simpler when $2r \geq n$. Motivated by this simplification, the present study focuses on systems satisfying this condition in order to derive new analytical results and to gain a deeper insight into their uncertainty characteristics. To this end, we assume that the lifetime of each component in these systems is represented by $T_1, T_2, \dots, T_{\mathrm{n}}$, with the corresponding order statistics denoted $T_{1:n}, T_{2:n}, \dots, T_{n:n}$. It is assumed that the lifetimes of the components correspond to a probability density function (pdf) $h(x)$ and a cumulative distribution function (cdf) $H(x) = \mathrm{P}(T \leq x)$. The system's lifetime is represented by $T_{r|n:F}$. Thus, when $2r \geq n$, the cdf of $T_{r|n:F}$ is given by (see Lemma 2.1 of [12])

$$H_{r|n:F}(x) = (n-r+1)H^r(x) - (n-r)H^{r+1}(x),\ x > 0, \tag{1}$$

and it then follows that

$$h_{r|n:F}(x) = h(x)[r(n-r+1)H^{r-1}(x) - (r+1)(n-r)H^{\mathrm{r}}(x)],\ x > 0. \tag{2}$$

It is worth noting that the representation in (1) can be interpreted as a distortion representation of the system lifetime distribution. Consequently, several structural properties of the system can also be derived using general results on distortion systems. In particular, preservation results for aging classes such as the decreasing reversed failure rate (DRFR) property may alternatively be obtained by applying Theorem 4.1 in Navarro [13]. Furthermore, the representation in (1) may also be viewed as a negative mixture representation, which provides additional insights into the probabilistic structure of the system lifetime.

Ebrahimi et al. [14] made a significant contribution to the understanding of the relationship between information theory and reliability, particularly through their exploration of the information properties of order statistics. Shannon's differential entropy, introduced in the seminal work of Shannon [15], remains a fundamental measure of uncertainty in probability theory. If T denotes a nonnegative random variable with pdf h, then the Shannon differential entropy of T is given by $\mathcal{H}(T) = -\mathbb{E}[\log h(T)]$, provided that the expectation exists. Recently, Lad et al. [16] introduced extropy, a dual measure of uncertainty to entropy, defined as

$$J(T) = -\frac{1}{2}\int_0^\infty h^2(x)dx = -\frac{1}{2}\mathbb{E}[h(H^{-1}(U))], \tag{3}$$

where $H^{-1}(u) = \inf\{x; H(x) \geq u\}$, for $u \in [0,1]$, denotes the quantile function.

As $J(T)$ increases, $h(x)$ approaches a uniform distribution, indicating that extropy measures the uniformity of the distribution. This phenomenon results in a decrease in probability concentration, thereby increasing the difficulty of predicting outcomes. In contrast to entropy, which can assume both negative and positive values, extropy is consistently negative. Engineers frequently consider and quantify uncertainty in the lifetime of systems to evaluate reliability. Let T denote the lifetime of a system and $J(T)$ represent its uncertainty. When information regarding the current age *t* of the system is available, the remaining lifetime $T^t = [T - t|T > t]$ of the system becomes pertinent. To measure the uncertainty of the residual lifetime, residual extropy is then defined as (Toomaj et al. [17])

$$J(T;t) = -\frac{1}{2}\int_0^\infty \left(\frac{h(x+t)}{H(t)}\right)^2 dx = -\frac{1}{2}\int_t^\infty \left(\frac{h(x)}{H(t)}\right)^2 dx, \qquad \text{for all } t > 0.$$

Uncertainty pervades real systems and influences both future and past events. To address past uncertainty, researchers have introduced the concept of past extropy, as exemplified in the work of Toomaj et al. [17] and Kamari and Buono [18]. For a random variable T representing a system's lifetime with pdf $h(x)$ and cdf $H(x)$, the past lifetime $[t - T|T < t]$ for $t > 0$ has pdf $h(t - x)/H(t)$ and cdf $H(t-x)/H(t)$, for $0 \leq x \leq t$. The past extropy at time t is then defined as

$$\bar{J}(T;t) = -\frac{1}{2}\int_0^t \left(\frac{h(x)}{H(t)}\right)^2 dx = -\frac{1}{2}\int_0^{H(t)} \frac{h(H^{-1}(u))}{H^2(t)} du, \tag{4}$$

where $H^{-1}(u) = \inf\{x; H(x) \geq u\}$ denotes the quantile function of $H(t)$. Recalling (4), $\bar{J}(T;t)$ corresponds to the extropy of $T_t = [T|T < t]$. Let $h_t(x) = h(x)/H(t)$ and $H_t(x) = H(x)/H(t)$, for $0 \leq x \leq t$, represent the pdf and cdf of T_t. Using the transformation $u = H_t(x)$, an alternative representation for the past extropy can be obtained as

$$\bar{J}(T;t) = -\frac{1}{2}\int_0^1 h_t(H_t^{-1}(u))\mathrm{d}u, \tag{5}$$

where $H_t^{-1}(u) = \inf\{x; H_t(x) \geq u\}$ for $0 < u < 1$. It is important to note that $\bar{J}(T;t)$ is always nonpositive whenever the defining integral exists, and therefore its range is contained in $(-\infty, 0)$. In system failure analysis, it quantifies the uncertainty of a system's past lifetime, especially when comparing different lifetime distributions.

Extensive research has been carried out on the properties and applications of extropy and residual extropy; see, for example, Lad et al. [16], Qiu [19], Qiu and Jia [20,21], Qiu et al. [22], and Shrahili and Kayid [23]. These studies have examined extropy for order statistics, record values, mixed systems, and related models, and have established various characterization results, monotonicity properties, and bounds. Here, we aim to explore the variability characteristics of system lifetimes through the lens of past extropy, emphasizing how it reflects the uncertainty inherent in the system's elapsed or historical lifetime. The analysis provides a unified framework for assessing the information content of a system's reliability function, whose simplicity and adaptability make it suitable for a wide range of lifetime models. Understanding the behavior of past extropy is crucial for quantifying the uncertainty associated with a system's aging process and for comparing different lifetime distributions under various dependence structures. This work therefore extends the role of information measures beyond residual uncertainty to encompass the system's historical uncertainty, offering new insights into reliability modeling and information-based assessment.

To clearly position this work relative to the existing literature, the main contributions of the present study are as follows:

1) We extend the concept of past extropy to the lifetime analysis of linear consecutive r-out-of-n:F systems with independent and identically distributed components, a system structure that has not previously been investigated within the framework of past extropy.
2) Using probabilistic transformations and order-statistic representations, we derive explicit analytical expressions for the past extropy associated with these systems.
3) We establish several structural properties of past extropy, including monotonicity results and preservation properties under aging classes such as the decreasing reversed failure rate (DRFR) class.
4) We obtain characterization results showing that the underlying component lifetime distribution can be uniquely determined from the past extropy of consecutive systems under suitable conditions.
5) From a statistical perspective, we develop a nonparametric kernel-based estimator for the past extropy of consecutive systems and investigate its finite-sample performance through extensive Monte Carlo simulation studies.

These results extend the existing literature on past extropy, which has mainly focused on individual lifetimes, order statistics, and record values, to the context of structured reliability systems, thereby providing new theoretical insights and practical estimation tools for uncertainty quantification in reliability analysis.

The remainder of the paper is organized as follows: Section 2 derives analytical expressions and structural properties of the past extropy for linear consecutive r-out-of-n: F systems. In addition, that section examines the preservation of stochastic ordering properties and develops several informative bounds that characterize the behavior of past extropy for consecutive system configurations. Section 3 investigates the conditional past extropy associated with these systems and establishes several related properties. Section 4 introduces the proposed nonparametric estimation procedure and presents the results of a Monte Carlo simulation study. Finally, Section 5 summarizes the principal findings and contributions of the study and outlines potential directions for future research.

Throughout this paper, the terms "increasing" and "decreasing" are used in a non-strict sense. Moreover, we also use the following concepts: the usual stochastic order denoted by $T \leq_{st} Z$, the hazard rate order denoted by $T \leq_{hr} Z$, the likelihood ratio order denoted by $T \leq_{lr} Z$, and the dispersive order denoted by $T \leq_{d} Z$. For formal definitions and properties of these concepts, interested readers may refer to the book by Shaked and Shanthikumar [24]. For clarity, we summarize the main notation used throughout the paper. In particular, $T_{r|n:F}$ denotes the lifetime of a linear consecutive r-out-of-$n:F$ system, while $T_{n:n}$ represents the lifetime of a parallel system. Conditional system lifetimes are denoted by $T^{r}_{n,F}(t)$, as introduced in Section 3.

2. Past extropy of consecutive systems

We begin by examining the past extropy of $T_{r|n:F}$, a measure quantifying the uncertainty in the density of $[t - T_{r|n:F} | T_{r|n:F} \leq t]$. This metric sheds light on the predictability of past lifetime of consecutive r-out-of-n: F systems. In particular, the lifetime of a parallel system $T_{n:n}$ is equivalent in distribution to $T_{n|n:F}$ under the assumption of interchangeable components. The past extropy of the consecutive r-out-of-n: F system, denoted by $\bar{J}(T_{r|n:F}; t)$, can be expressed as the extropy of $T_{r|n:F} = H^{-1}(U_{r|n:F})$. It is known that the transformed component lifetimes $U_i = H(T_i)$, for $i = 1, \cdots, n$, are iid uniform random variables on the interval $[0,1]$. Consequently, when $2r \geq n$, the pdf of $U_{r|n:F}$ can be formulated as

$$g_{r|n:F}(u) = r(n-r+1)u^{r-1} - (r+1)(n-r)u^{r}, \tag{6}$$

where $0 < u < 1$. The pdf in (6) is derived by considering the Jacobian transformation from $T_{r|n:F} = H^{-1}(U_{r|n:F})$, which is given by $1/h(H^{-1}(u))$. Thus, we see that $g_{r|n:F}(\mathrm{u}) = h_{r|n:F}(H^{-1}(u))/h(H^{-1}(u))$, $0 < u < 1$. Alternatively, the cdf of $U_{r|n:F}$ can be determined as

$$G_{r|n:F}(u) = (n-r+1)u^r - (n-r)u^{r+1}, \tag{7}$$

for all $0 < u < 1$. Throughout the rest of this paper, we employ the notation $Y \sim \mathcal{B}(t; a, b)$ to denote a random variable Y following a truncated beta distribution. This distribution is characterized by the pdf

$$h_Y(y) = \frac{1}{\mathcal{B}(t;a,b)} y^{a-1}(1-y)^{b-1},\ 0 < y < t,\ \text{for all}\ a, b > 0, \tag{8}$$

where

$$\mathcal{B}(t; a, b) = \int_0^t x^{a-1}(1-x)^{b-1},\ 0 < t < 1, \tag{9}$$

represents the lower incomplete beta function.

The following lemma plays a pivotal role in our subsequent analyses.

Lemma 2.1. Suppose $U_{r|n:F}$ denotes the lifetime of the consecutive r-out-of-n: F system having common iid component lifetimes uniformly distributed in $[0,1]$. Then, for $n/2 \le r < n$, we have

$$\bar{J}(U_{r|n:F}; t) = -\frac{(r(n-r+1))^{2r+1}\mathcal{B}\left(\frac{(r+1)(n-r)}{r(n-r+1)}t; 2r-1,3\right)}{2((r+1)(n-r))^{2r-1}G^2_{r|n:F}(t)}, \qquad \text{for all } 0 < \mathrm{t} < 1.$$

Proof. Setting $A = r(n-r+1)$ and $B = (r+1)(n-r)$, from Eqs (4)–(7), we have

$$\bar{J}(U_{r|n:F}; t) = -\frac{1}{2}\int_0^t \left(\frac{g_{r|n:F}(u)}{G_{r|n:F}(t)}\right)^2 du = -\frac{1}{2G^2_{r|n:F}(t)}\int_0^t (Au^{r-1} - Bu^r)^2 du$$

$$= -\frac{A^2}{2G^2_{r|n:F}(t)}\int_0^t z^{2r-2}\left(1 - \frac{B}{A}z\right)^2 dz = -\frac{A^{2r+1}\mathcal{B}\left(\frac{B}{A}t; 2r-1,3\right)}{2B^{2r-1}G^2_{r|n:F}(t)}, \tag{10}$$

which completes the proof of the lemma.

It is important to recognize that Lemma 2.1 is valid for the case $n/2 \le r < n$. However, for $r = n$ (which corresponds to a parallel system), we can present the following lemma, whose proof is straightforward.

Lemma 2.2. If $U_{n|n:F}$ represents the lifetime of the parallel system with common iid component lifetimes uniformly distributed in $[0,1]$, then, for all $n \ge 1$, we have

$$\bar{J}(U_{n|n:F}; t) = -\frac{n^2\mathcal{B}(t; 2n-1,1)}{2t^{2n}}, \qquad \text{for all } 0 < t < 1.$$

Using the identity $\mathcal{B}(t; 2n-1,1) = \frac{t^{2n-1}}{2n-1}$, the above expression simplifies to

$$\bar{J}(U_{n|n:F}; t) = -\frac{n^2}{2(2n-1)t}.$$

Remark 2.1. We can now derive an explicit expression for $\bar{J}(U_{r|n:F};t)$. From (10), we obtain

$$\bar{J}(U_{r|n:F};t) = -\frac{1}{2G^2_{r|n:F}(t)}\int_0^t (Az^{r-1} - Bz^r)^2 dz$$

$$= -\frac{1}{2G^2_{r|n:F}(t)}\left(\frac{A^2}{2r-1}t^{2r-1} + \frac{B^2}{2r+1}t^{2r+1} - \frac{AB}{r}t^{2r}\right)$$

$$= -\frac{t^{2r-1}\left(\frac{(r(n-r+1))^2}{2r-1} + \frac{((r+1)(n-r))^2}{2r+1}t^2 - (r+1)(n-r)(n-r+1)t\right)}{2t^{2r}((n-r+1)-(n-r)t)^2}.$$

Recalling identity (7), $G_{r|n:F}(t) = t^r((\mathrm{n}-\mathrm{r}+1)-(\mathrm{n}-\mathrm{r})\mathrm{t})$, and hence $G^2_{r|n:F}(t) = t^{2r}((\mathrm{n}-\mathrm{r}+1)-(\mathrm{n}-\mathrm{r})\mathrm{t})^2$, then

$$\bar{J}(U_{r|n:F};t) = \frac{(r+1)(n-r)(n-r+1)t - \frac{((r+1)(n-r))^2}{2r+1}t^2 - \frac{(r(n-r+1))^2}{2r-1}}{2t((n-r+1)-(n-r)t)^2},$$

$$t \in (0,1).$$

The following theorem provides an expression for the extropy of $T_{r|n:F}$ using the previously discussed transformations and Lemma 2.1.

Theorem 2.1 Suppose $T_{r|n:F}$ is the lifetime of a linear consecutive r-out-of-n: F system having common iid component lifetimes with pdf h and cdf H. Let Y denote the transformed random variable defined in the preceding representation, whose support is contained in an interval such that, after scaling by the constant C, we have $CY \in [0,1]$ almost surely. Consequently, the argument of the quantile function $H^{-1}(\cdot)$ remains within its natural domain. Then, for $n/2 \leq r < n$, we have

$$\bar{J}(T_{r|n:F};t) = \bar{J}(U_{r|n:F};H(t))\mathbb{E}\left[h\left(H^{-1}\left(\tfrac{r(n-r+1)}{(r+1)(n-r)}Y_{r|n}\right)\right)\right], \tag{11}$$

in which $Y_{r|n} \sim \mathcal{B}\left(\frac{(r+1)(n-r)}{r(n-r+1)}H(t);2r-1,3\right)$, for all $t>0$.

Proof. Using the definitions of A and B in the proof of Lemma 2.1, and considering (1), (2), and (10), a change of variable $u = H(x)$ yields

$$\bar{J}(T_{r|n:F};t) = -\frac{1}{2}\int_0^t \left(\frac{h_{r|n:F}(x)}{H_{r|n:F}(t)}\right)^2 dx$$

$$= -\frac{1}{2H^2_{r|n:F}(t)}\int_0^t h^2(x)(AH^{r-1}(x) - BH^r(x))^2 dx$$

$$= -\frac{1}{2H_{r|n:F}^2(t)}\int_0^{H(t)} h(H^{-1}(u))(Au^{r-1}-Bu^r)^2du$$

$$= -\frac{A^2}{2H_{r|n:F}^2(t)}\int_0^{H(t)} h(H^{-1}(u))u^{2r-2}\left(1-\frac{B}{A}u\right)^2 du,$$

$$= -\frac{A^{2r+1}}{2B^{2r-1}G_{r|n:F}^2\big(H(t)\big)}\int_0^{\frac{B}{A}H(t)} h\left(H^{-1}\left(\frac{A}{B}w\right)\right)w^{2r-2}(1-w)^2dw,$$

$$\left(\text{taking } \mathrm{w}=\frac{\mathrm{B}}{\mathrm{A}}\mathrm{u}\right)$$

$$= -\frac{A^{2r+1}\mathcal{B}(\frac{B}{A}H(t);2r-1,3)}{2B^{2r-1}G_{r|n:F}^2(H(t))}\int_0^{\frac{B}{A}H(t)} h\left(H^{-1}\left(\frac{A}{B}w\right)\right)\frac{w^{2r-2}(1-w)^2}{\mathcal{B}(\frac{B}{A}H(t);2r-1,3)}dw$$

$$= \bar{J}(U_{r|n:F};H(t))\mathbb{E}\left[h\left(H^{-1}\left(\tfrac{r(n-r+1)}{(r+1)(n-r)}Y_{r|n}\right)\right)\right], \quad (12)$$

where the last equality follows from Lemma 2.1, which completes the proof of the theorem.

The following theorem provides a formula for the parallel system.

Theorem 2.2. If $U_{n|n:F}$ represents the lifetime of the parallel system having the common iid component lifetimes with pdf h and cdf H, then, for all $n \geq 1$, we have

$$\bar{J}(T_{n|n:F};t) = \bar{J}(U_{n|n:F};H(t))\mathbb{E}\left[h\left(H^{-1}(Y_{n|n})\right)\right], \quad (13)$$

in which $Y_{n|n} \sim \mathcal{B}(H(t);2n-1,1)$, for all $t > 0$.

The age of components in a consecutive r-out-of-n: F system significantly impacts its past extropy. We explore this relationship for systems with DRFR parent distributions. Recall that T has the decreasing reversed failure rate (DRFR) property if the reversed hazard rate function of X, i.e., $\tau(x) = h(x)/H(x)$, is decreasing in x. The following theorem establishes the relationship between the DRFR property of the parent distribution and the past extropy of the lifetime of the consecutive r-out-of-n: F system. As a preliminary result, we demonstrate that the DRFR property is preserved under the r-out-of-n: F system.

Theorem 2.3. If $2r \geq n$ and the component lifetimes are DRFR, then $T_{r|n:F}$ is DRFR.

Proof. It is evident that the reversed hazard rate function of $T_{r|n:F}$ can be expressed as

$$\tau_{r|n:F}(x) = \psi(H(x))\tau(x),$$

where

$$\psi(z) = \frac{r(n-r+1) - (r+1)(n-r)z}{(n-r+1) - (n-r)z}, \qquad 0 < z < 1.$$

As $\psi'(z) < 0$ for $2r \geq n$ and $0 < z < 1$, it follows that $\psi(z)$ is a monotonically decreasing function of z. Consequently, $\psi(H(x)) > 0$ is positive and decreasing in x. Therefore, the DRFR property of the component lifetimes implies that $\tau_{r|n:F}(x)$ is decreasing in x, and hence $\tau_{r|n:F}(x)$ is DRFR. This completes the proof.

Remark 2.2. Since representation (1) can be interpreted as a distortion representation of the system lifetime distribution, the preservation of the DRFR property may also be obtained directly from general results on distortion systems (see Theorem 4.1 in Navarro [13]).

Next, we examine the monotonicity properties of past extropy for consecutive systems.

Theorem 2.4. If T is DRFR, then for all $2r \geq n$, $\bar{J}(T_{r|n:F}; t)$ is increasing in t.

Proof. By Theorem 2.3, $T_{r|n:F}$ is DRFR whenever $2r \geq n$ and the component lifetimes are DRFR. The desired monotonicity property of $\bar{J}(T_{r|n:F}; t)$ then follows directly from Theorem 5.6 of Toomaj et al. [17].

The result of Theorem 2.4 is applicable to samples drawn from gamma and Weibull distributions with shape parameters less than 1, Fréchet, power, and Pareto distributions, as well as from mixtures of exponential distributions. The following example illustrates the application of Theorems 2.1, 2.2, and 2.4.

Example 2.1. Suppose $\mathrm{T}_{\mathrm{r|n:F}}$ represents the lifetime of a linear consecutive r-out-of-n:F system with iid component lifetimes following a Fréchet distribution with cdf

$$H(t) = e^{-t^{-2}}, \quad t > 0.$$

The system lifetime is defined as $T_{r|n:F} = min(T^{[1:r]}, T^{[2:r+1]}, \ldots, T^{[n-r+1:n]})$, where $T^{[j:m]} = max(T_j, \ldots, T_m)$ for $1 \leq j < m \leq n$. The Fréchet distribution, a special case of the generalized extreme value distribution (also known as the inverse Weibull distribution), satisfies $h(H^{-1}(v)) = 2v(-\log v)^{\frac{3}{2}}$ for all $0 < v < 1$. So, by Lemma 2.1, we have

$$\bar{J}(U_{r|n:F}; H(t)) = -\frac{(r(n-r+1))^{2r+1}\mathcal{B}\left(\frac{(r+1)(n-r)}{r(n-r+1)}e^{-t^{-2}}; 2r-1,3\right)}{2((r+1)(n-r))^{2r-1}\left(G_{r|n:F}\left(e^{-t^{-2}}\right)\right)^2}, \qquad t > 0.$$

Alternatively, we have

$$\mathbb{E}\left[h\left(H^{-1}\left(\frac{r(n-r+1)}{(r+1)(n-r)}Y_{r|n}\right)\right)\right]$$

$$=\frac{2r(n-r+1)}{(r+1)(n-r)}\times\int_0^{\frac{(r+1)(n-r)}{r(n-r+1)}e^{-t^{-2}}}\left(-\log\left(\frac{r(n-r+1)}{(r+1)(n-r)}w\right)\right)^{\frac{3}{2}}$$

$$\times\frac{w^{2r-1}(1-w)^2}{\mathcal{B}\left(\frac{(r+1)(n-r)}{r(n-r+1)}e^{-t^{-2}};2r-1,3\right)}dw.$$

Therefore, for $n/2 \le r < n$, (11) and the preceding expressions imply that

$$\bar{J}(T_{r|n:F};t)=-\frac{(r(n-r+1))^{2r+2}}{((r+1)(n-r))^{2r}\left((n-r+1)e^{-kt^{-2}}-(n-r)e^{-(r+1)t^{-2}}\right)^2}$$

$$\times\int_0^{\frac{(r+1)(n-r)}{r(n-r+1)}e^{-t^{-2}}}w^{2r-1}(1-w)^2\left(-log\left(\frac{r(n-r+1)}{(r+1)(n-r)}w\right)\right)^{\frac{3}{2}}dw,$$

for all $t>0$. For the special case $r=n$, the system reduces to the parallel system $T_{n:n}$. Using Theorem 2.2 together with the representation $h(H^{-1}(v))=2v(-logv)^{\frac{3}{2}}$ for the Fréchet model $H(t)=e^{-t^{-2}}$, the corresponding past extropy can be expressed as

$$\bar{J}(T_{n|n:F};t)=-\frac{e^{\frac{2n}{t^2}}}{2^{\frac{5}{2}}\sqrt{n}}\Gamma\left(\frac{5}{2},\frac{2n}{t^2}\right),t>0,$$

where $\Gamma(a,x)$ is the incomplete gamma function.

As deriving an explicit analytical expression is a challenging task, we employ the numerical methods to examine the temporal behavior of $\bar{J}(T_{r|7:F};t)$.

Figure 1 depicts the numerical results, illustrating the relationship between $\bar{J}(T_{r|7:F};t)$ and t for $r=4,5,6,7$. These trends corroborate the result in Theorem 2.4, which establishes the increasing nature of past extropy over time for DRFR random variables.

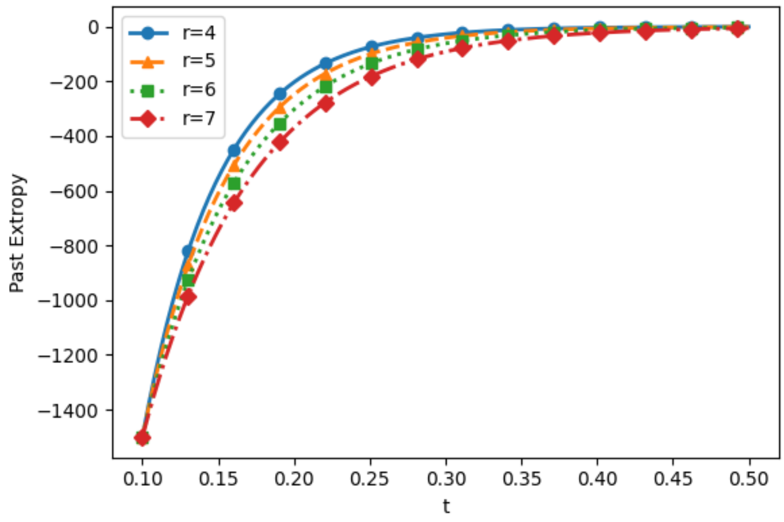

Figure 1. Exact values of $\bar{J}(T_{r|7:F};t)$ as a function of t for the Fréchet distribution, corresponding to $r=4,5,6,7$. The curves illustrate the effect of varying r on the behavior of the past extropy. Distinct line styles and markers are employed to ensure clarity and readability in grayscale reproduction.

Since there are no-closed form expressions for the past extropy in complex, consecutive multicomponent systems, approximate methods are essential to characterize their behavior. This challenge has necessitated the search for bounds on the past extropy of consecutive r-out-of-n: F systems. For this reason, we present a theorem that establishes a lower bound for the past extropy of consecutive r-out-of-n: F systems, which depends on the past extropy of consecutive r-out-of-n: F systems derived from the uniform distribution on [0,1] and the mode value of the original distribution.

Theorem 2.5. Let us consider the conditions of Theorem 2.1 such that $\bar{J}(T_{r|n:F};t)<\infty$ for all $t>0$.

(i) Let $h_r(y)$ be the pdf of $Y_{r|n}$. Define $M^\star=h_r(y^\star)$, where $y^\star=min\left\{\frac{(r+1)(n-r)}{r(n-r+1)}H(t),\frac{r-1}{r}\right\}$. Then, for $2r\geq n$, we have

$$\bar{J}(T_{r|n:F};t)\geq D_r(t)\bar{J}(T;t),$$

where

$$D_r(t)=\frac{M^\star H^2(t)(r(n-r+1))^{2r+1}\mathcal{B}(\frac{(r+1)(n-r)}{r(n-r+1)}H(t);2r-1,3)}{((r+1)(n-r))^{2r-1}G^2_{r|n:F}(H(t))},\ \text{for all}\ t>0;$$

(ii) Suppose we have $M=h(m)<\infty$, where m is the mode of the pdf h. Then, for $2r\geq n$, we have

$$\bar{J}(T_{r|n:F};t)\geq M\bar{J}\left(U_{r|n:F};H(t)\right),\ \text{for all}\ t>0.$$

Proof. (i) With A and B as defined in Lemma 2.1, the mode of $Y_{r|n}$ is $y^\star = min\left\{\frac{B}{A}H(t), \frac{r-1}{r}\right\}$. Let $M^\star = h_r(y^\star)$, where

$$h_r(y) = \frac{y^{2r-2}(1-y)^2}{\mathcal{B}((B/A)H(t); 2r-1,3)}, \qquad 0 < y < (B/A)H(t), \text{for } t > 0.$$

It then follows from the proof of Theorem 2.1 that

$$\bar{J}(T_{r|n:F}; t) = -\frac{A^{2r+1}\mathcal{B}\left(\frac{B}{A}H(t); 2r-1,3\right)}{2B^{2r-1}G^2_{r|n:F}(H(t))}\int_0^{\frac{B}{A}H(t)} h\left(H^{-1}\left(\frac{A}{B}w\right)\right)\frac{w^{2r-2}(1-w)^2}{\mathcal{B}\left(\frac{B}{A}H(t); 2r-1,3\right)}dw$$

$$\geq -\frac{M^\star H^2(t)A^{2r+1}\mathcal{B}\left(\frac{B}{A}H(t); 2r-1,3\right)}{2B^{2r-1}G^2_{r|n:F}(H(t))H^2(t)}\int_0^{\frac{B}{A}H(t)} h\left(H^{-1}\left(\frac{A}{B}w\right)\right)dw$$

$$= -\frac{M^\star H^2(t)A^{2r+1}\mathcal{B}\left(\frac{B}{A}H(t); 2r-1,3\right)}{2B^{2r-1}G^2_{r|n:F}\big(H(t)\big)}\int_0^{H(t)} \frac{h\big(H^{-1}(z)\big)}{H^2(t)}dz, \qquad \left(taking\ z = \frac{A}{B}w\right)$$

$$= \frac{M^\star H^2(t)A^{2r+1}\mathcal{B}\left(\frac{B}{A}H(t); 2r-1,3\right)}{B^{2r-1}G^2_{r|n:F}\big(H(t)\big)}\bar{J}(T; t).$$

The final equality is a direct consequence of (4), and this concludes the proof of part (i).
(ii) For $n/2 \leq r < n$, we have

$$h\left(H^{-1}\left(\frac{r(n-r+1)}{(r+1)(n-r)}u\right)\right) \leq M, \qquad 0 < u < 1.$$

Hence,

$$\mathbb{E}\left[h\left(H^{-1}\left(\frac{r(n-r+1)}{(r+1)(n-r)}Y_{r|n}\right)\right)\right] \leq M.$$

The required result is now easily obtained from (11), and this completes the proof of the theorem.

Proceeding along the same lines and starting from Theorem 2.2, we can systematically determine the lower bound for the parallel system. We have established two lower bounds for the past extropy of $T_{r|n:F}$ with respect to the past extropy of a uniform r-out-of-n: F system and the mode M of the original distribution. Table 1 provides lower bounds for some common distributions based on Theorem 2.5. In Table 1, $\mathcal{B}(t; a, b)$ denotes the lower incomplete beta function (defined in (9)), $\text{erf}(z)$ is the error function, and $\gamma(s, x)$ is the lower incomplete gamma function. These functions are given, respectively, as

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du \quad and \quad \gamma(s,z) = \int_0^z u^{s-1}e^{-u}du, \qquad s,z > 0.$$

Table 1. Bounds on $\bar{J}(T_{r|n:F};t)$ derived from Theorem 2.5 (Parts (i) and (ii)).

Probability density function	Lower bounds
Beta distribution $h_1(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathcal{B}(1;a,b)}, 0<x<1, \alpha,\beta>0,$	$-\frac{D_r(t)\mathcal{B}(t;2\alpha-1,2\beta-1)}{2\mathcal{B}^2(t;a,b)}$ $\bar{J}(U_{r\vert n:F};H(t))h\left(\frac{\alpha-1}{\alpha+\beta-2}\right), \alpha,\beta>1$
Half-normal distribution $h_2(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-x^2/2\sigma^2}, x>0,$	$-\frac{D_r(t)\mathrm{erf}\left(\frac{t}{\sigma}\right)}{2\sigma^2\sqrt{\pi}\left(\mathrm{erf}\left(\frac{t}{\sigma\sqrt{2}}\right)\right)^2}$ $\bar{J}(U_{r\vert n:F};H(t))\left(\frac{2}{\sigma\sqrt{2}\pi}\right)$
Generalized exponential distribution $h_3(x) = \frac{\lambda}{\beta}e^{-\frac{x}{\beta}}\left(1-e^{-\frac{x}{\beta}}\right)^{\lambda-1}, x>0,$	$-\frac{\lambda^2 D_r(t)\mathcal{B}\left(1-e^{-\frac{t}{\beta}};2\lambda-1,2\right)}{2\beta\left(1-e^{-\frac{t}{\beta}}\right)^{2\lambda}}$ $(U_{r\vert n:F};H(t))\beta^{-1}\left(1-\frac{1}{\lambda}\right)^{\lambda-1}$
Generalized gamma distribution $h_4(x) = \frac{b^c}{\Gamma(c)}x^{c-1}e^{-bx}, x>0,$	$-\frac{b^2 D_r(t)\gamma(2c-1,2bt)}{2^{2(c-1)}(\gamma(c,bt))^2}$ $\bar{J}(U_{r\vert n:F};H(t))\left(\frac{b(c-1)^{c-1}e^{1-c}}{\Gamma(c)}\right)$

The following theorem establishes lower and upper bounds for the extropy of consecutive r-out-of-n: F systems. These bounds depend on the pdf of parent distribution and the past extropy of consecutive r-out-of-n: F systems with uniformly distributed component lifetimes on $[0,1]$.

Theorem 2.6. Under the assumptions of Theorem 2.1, $H(0)=0$, and $h(0)>0$, for $2r \geq n$, if $h(H^{-1}(x))$ is decreasing in x, then

$$h(0)\bar{J}\left(U_{r|n:F};H(t)\right) \leq \bar{J}\left(T_{r|n:F};t\right) \leq h(t)\bar{J}\left(U_{r|n:F};H(t)\right);$$

if $h(H^{-1}(x))$ is increasing in x, then

$$h(t)\bar{J}\left(U_{r|n:F};H(t)\right) \leq \bar{J}\left(T_{r|n:F};t\right) \leq h(0)\bar{J}\left(U_{r|n:F};H(t)\right), \text{for } t>0.$$

Proof. We just prove Part (i), as the proof for Part (ii) can be done in a similar manner. As $H^{-1}(0)=0$ and $h(H^{-1}(x))$ is decreasing in x, then $h(0) > h(H^{-1}(u)) \geq h(t)$ for all $0<u<H(t)$.

Therefore, for all $t > 0$,

$$h(0)\int_0^{H(t)} (Au^{r-1} - Bu^r)^2 du \geq \int_0^{H(t)} h(H^{-1}(u))(Au^{r-1} - Bu^r)^2 du$$

$$\geq h(t)\int_0^{H(t)} (Au^{r-1} - Bu^r)^2 du.$$

Combining this with Eq (12) and using the representation

$$\bar{J}\left(T_{r|n:F}; H(t)\right) = -C\int_0^{H(t)} h\big(H^{-1}(u)\big) g^2_{r|n:F}(u) du, \ \ C > 0,$$

together with

$$\bar{J}\left(U_{r|n:F}; H(t)\right) = -C\int_0^{H(t)} g^2_{r|n:F}(u) du,$$

and the fact that $g_{r|n:F}(u) = Au^{r-1} - Bu^r$ for $0 < u < 1$, we obtain the stated bounds.

The theoretical lower bounds obtained from Theorem 2.6 for various common probability distributions are summarized in Table 1. These results illustrate how the past extropy of a system can be bounded in terms of its parent distribution and the corresponding parameters.

Next, we investigate the monotonicity properties of past extropy for consecutive systems and establish characterization results. For the first result, we study the closure properties of past extropy in the formation of consecutive systems.

Theorem 2.7 For $2r \geq n$, if $\bar{J}(T_{n:n}; t)$ is increasing in t, then $\bar{J}(T_{r|n:F}; t)$ is also increasing in $t > 0$.

Proof. As the reversed hazard rate function of $T_{n:n}$ is given by $\tau_{n:n}(t) = n\tau(t)$, $t > 0$, where $\tau(t)$ represents the reversed hazard rate function of T, we can write from (1) and (2) that the following relation holds:

$$\frac{\tau_{r|n:F}(t)}{\tau_{n:n}(t)} = \Phi_{r|n:F}(H(t)), \tag{14}$$

where

$$\Phi_{r|n:F}(x) = \frac{r(n-r+1) - (r+1)(n-r)x}{n(n-r+1) - n(n-r)x}, \qquad 0 < x < 1.$$

As $\Phi_{r|n:F}(x) > 0$ is strictly decreasing in $x \in (0,1)$ for $2r \geq n$, $\Phi_{r|n:F}(H(t))$ is strictly decreasing in t. Consequently, the ratio $\tau_{r|n:F}(t)/\tau_{n:n}(t)$ is also strictly decreasing in t. Moreover, the relation $T_{r|n:F} \leq_{lr} T_{n:n}$ is readily evident because $f_{n:n}(t) = nh(t)H^{n-1}(t)$, and hence, recalling (1), we have

$$h_{r|n:F}(t) = h(t)[r(n-r+1)H^{r-1}(t) - (\mathrm{r}+1)(\mathrm{n}-\mathrm{r})H^{r}(t)],$$

and so

$$\frac{h_{r|n:F}(t)}{h_{n:n}(t)} = \frac{H^{r-n}(t)}{n}[r(n-r+1) - (\mathrm{r}+1)(\mathrm{n}-\mathrm{r})H(t)].$$

It follows that this ratio is decreasing in *t* for $1 \leq r \leq n$.

Thus, Theorem 4.1 of Kayid [25] directly implies the increasing monotonicity of $\bar{J}(T_{r|n:F}; t)$ in t, and this completes the proof of the theorem.

Determining the nature of the underlying distribution is a problem of great importance. Baratpour et al. [26] have shown that the Rényi entropy of order statistics uniquely identifies the underlying distribution. Subsequent studies have shown that this property also holds for the cumulative residual entropy Baratpour [27] and extropy [18, 20]. Here, we investigate how the underlying distribution can be characterized by the past extropy of consecutive systems. The derivative of the past extropy can be expressed as

$$\frac{d\bar{J}(Z;t)}{dt} = -\left(2\tau(t)\bar{J}(Z;t) + \frac{\tau^2(t)}{2}\right).$$

Since we use the $\bar{J}$ for past extropy, which is defined by a negative factor, this convention leads to the next differential identity

$$\frac{\tau^2(t)}{2} = -\left(2\tau(t)\bar{J}(Z;t) + \frac{d\bar{J}(Z;t)}{dt}\right), \text{for all } t > 0. \tag{15}$$

Remark 2.3. Equation (15) yields a quadratic equation in τ. When $\bar{J}(Z;t)$ is decreasing, this equation admits a unique positive solution, implying that $\bar{J}(Z;t)$ uniquely determines τ(t). However, when $\bar{J}(Z;t)$ is increasing, the quadratic equation may admit two positive solutions, in which case τ(t) is not uniquely determined by $\bar{J}(Z;t)$. In the characterization results that follow, we therefore restrict our attention to the case where $\bar{J}(Z;t)$ is decreasing to ensure uniqueness.

For our purpose, we recall the problem of finding a sufficient condition for the unique solution of the initial value problem (IVP) as given in (16)

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0, \tag{16}$$

where f is a function of two variables whose domain is a region $D \cap R^2$, (x_0, y_0) is a point in D, and y is the unknown function.

The following theorem and lemma, due to Gupta and Kirmani [28], will be used to prove our characterization results.

Theorem 2.8. Let $f(x,y)$ be a continuous function defined on a domain $D \subset R^2$. Assume that f satisfies a Lipschitz condition with respect to its second argument, that is, $|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$, for all $(x,y_1),(x,y_2) \in D$, where $L > 0$ is a constant. Then the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$, admits a unique solution in a neighborhood of x_0.

Lemma 2.3. Suppose the function f is continuous in a convex region $D \cap R^2$, and $\frac{\partial f}{\partial y}$ exists and is continuous in D. Then, f satisfies the Lipschitz condition in D.

We conclude this section by characterizing the underlying distribution of the past extropy for r-out-of-n: F systems.

Theorem 2.9. Let $\bar{J}(T^Z_{r|n:F};t)$ and $\bar{J}(T^Y_{r|n:F};t)$ denote the lifetimes of the linear consecutive r-out-of-n:F systems having common iid component lifetimes Z_i and Y_i with pdfs h_Z and h_Y, and cdfs H_Z and H_Y, respectively. Fix r such that $2r \geq n$, and assume that $t \mapsto \bar{J}(T^Z_{r|n:F};t)$ and $t \mapsto \bar{J}(T^Y_{r|n:F};t)$ are well-defined and decreasing on $(0,\infty)$. Then, $Z \overset{d}{=} Y$ if and only if $\bar{J}(T^Z_{r|n:F};t) = \bar{J}(T^Y_{r|n:F};t)$ for all $t > 0$.

Proof. We just prove the sufficiency part as the necessity is trivial. From (15), we have

$$2\tau_{Z,k|n:F}(t)\bar{J}(T^Z_{r|n:F};t) - \frac{d\bar{J}(T^Z_{r|n:F};t)}{dt} = \frac{\tau^2_{Z,k|n:F}(t)}{2}, \qquad t > 0.$$

Taking the derivative of the above equation with regard to t, we have

$$\frac{d\tau_{Z,k|n:F}(t)}{dt} = \frac{\frac{d^2\bar{J}(T^Z_{r|n:F};t)}{dt^2} - 2\tau_{Z,k|n:F}(t)\frac{d\bar{J}(T^Z_{r|n:F};t)}{dt}}{2\bar{J}(T^Z_{r|n:F};t) - \tau_{Z,k|n:F}(t)}, \qquad t > 0.$$

Assume that $\bar{J}(T^Z_{r|n:F};t) = \bar{J}(T^Y_{r|n:F};t) = w(t)$ for all $t \geq 0$, and $2r \geq n$. Since $w(t)$ is assumed to be decreasing, Eq (15) yields a unique solution for $\tau(t)$. Then, for all $t \geq 0$, we get

$$\frac{d\tau_{Z,r|n:F}(t)}{dt} = \Psi(t, \tau_{Z,r|n:F}(t)), \quad \frac{d\tau_{Y,r|n:F}(t)}{dt} = \Psi(t, \tau_{Y,r|n:F}(t)),$$

where

$$\Psi(t,y) = \frac{w''(t) - 2yw'(t)}{2w(t) - y}, \qquad t > 0.$$

Hence, the corresponding differential equation satisfies the uniqueness condition. It then follows from Theorem 2.8 and Lemma 2.3 that for all $t \geq 0$, $H^Z_{r|n:F}(t) = H^Y_{r|n:F}(t)$. In view of $H_Z(t) =$

$G^{-1}_{r|n:F}(H^{Z}_{r|n:F}(t))$ and $H_Y(t) = G^{-1}_{r|n:F}(H^{Y}_{r|n:F}(t))$ for all $t \geq 0$, where $G_{r|n:F}(\cdot)$ is defined in (7), we have $H_Z(t) = H_Y(t)$ for all $t \geq 0$. Hence, we obtain the desired result.

3. Conditional past extropy of consecutive systems

In this section, we analyze the past extropy of the elapsed time since the system failure, that is, the random variable $t - T_{r|n:F} \mid T_{r|n:F} \leq t,\ t > 0$. For the numerical evaluations reported in this section and in Section 4, the involved integrals were all computed using standard numerical integration procedures implemented in R.

Recall that the system lifetime $T_{r|n:F} = min\left(T^{[1:r]}, T^{[2:r+1]}, \dots, T^{[n-r+1:n]}\right)$ is defined as the minimum of the maximum lifetimes of r consecutive components, i.e., $T^{[j:m]} = max\left(T_j, \dots, T_m\right)$ for $1 \leq j < m \leq n$. It should be noted that the cdf of the system in (1) can also be expressed as

$$H_{r|n:F}(x) = P(T_{r|n:F} \leq x) = \sum_{i=r}^{n} \left[P(T^{[i-r+1:i]} \leq x) - P(T^{[i-r+1:i+1]} \leq x)\right],$$

where $P\left(T^{[n-r+1:n+1]} \leq x\right) = 0$ as a convention (see Eryılmaz [6], (2)). The survival function of $T^{r}_{n,F}(t) = [t - T_{r|n:F} | T_{n:n} \leq t]$ can then be expressed as:

$$\overline{H}^{r}_{n,F}(x;t) = P\left(t - T_{r|n:F} > x \middle| T_{n:n} \leq t\right) = P\left(T_{r|n:F} \leq t - x \middle| T_{n:n} \leq t\right)$$

$$= \sum_{i=r}^{n} \left[P(T^{[i-r+1:i]} \leq t - x | T_{n:n} \leq t) - P(T^{[i-r+1:i+1]} \leq t - x | T_{n:n} \leq t)\right],$$

for $0 \leq x \leq t$. As $T_1, \dots, T_n$ are iid random variables, for $r \leq i \leq n$, we have

$$P\left(T^{[i-r+1:i]} \leq t - x \middle| T_{n:n} \leq t\right) = P(T_{i-r+1} \leq t - x, \dots, T_i \leq t - x| \ \frac{P(T_{i-r+1} \leq t - x) \dots P(T_i \leq t - x)}{P(T_{i-r+1} \leq t) \dots P(T_i \leq t)},$$

(r times)

$$= \left(\frac{P(T \leq t - x)}{P(T \leq t)}\right)^{r} = \left(\frac{H(t - x)}{H(t)}\right)^{r}, \qquad \text{for } 0 \leq x \leq t.$$

Similarly,

$$P(T^{[i-r+1:i+1]} \leq (t - x | T_{n:n} \leq t) = P(T_{i-r+1} \leq t - x, \dots, T_{i+1} \leq t - x | T_1 \leq t, \dots, T_n \leq t)$$

$$= \frac{P(T_{i-r+1} \leq t - x) \dots P(T_{i+1} \leq t - x)}{P(T_{i-r+1} \leq t) \dots P(T_{i+1} \leq t)}, \qquad (r + 1 \text{ times})$$

$$= \left(\frac{P(T \le t - x)}{P(T \le t)}\right)^{r+1} = \left(\frac{H(t-x)}{H(t)}\right)^{r+1}, \qquad \text{for } 0 \le x \le t.$$

Hence, we have

$$\overline{H}^{r}_{n,F}(x;t) = (n-r+1)\left(\frac{H(t-x)}{H(t)}\right)^{r} - (n-r)\left(\frac{H(t-x)}{H(t)}\right)^{r+1}, \qquad \text{for all } 0 \le x \le t.$$

Assuming that H is absolutely continuous with pdf h, the pdf of $T^{r}_{n,F}(t)$ for $0 \le x \le t$ is then given by

$$h^{r}_{n,F}(x;t) = \frac{h(t-x)}{H(t)}\left[r(n-r+1)\left(\frac{H(t-x)}{H(t)}\right)^{r-1} - (r+1)(n-r)\left(\frac{H(t-x)}{H(t)}\right)^{r}\right]. \tag{17}$$

We now study the extropy of $T^{r}_{n,F}(t)$, a measure of uncertainty in the density of $[t - T_{r|n:F} | T_{n:n} \le t]$.

For this purpose, we employ the probability integral transform, defining $U_{r|n:F} = H_t(T^{r}_{n,F}(t))$. Its pdf is given in (6). The following theorem expresses the extropy of $T^{r}_{n,F}(t)$ using these transforms.

Theorem 3.1. Let $T_1, \dots, T_n$ be iid random variables representing the lifetimes of the components of the consecutive r-out-of-n:F system having the common cdf H and pdf h. Then, for all $2r \ge n$, the extropy of $T^{r}_{n,F}(t)$ can be expressed as

$$\bar{J}(T^{r}_{n,F}(t)) = -\frac{1}{2}\int_0^1 g^2_{r|n:F}(u) h_t(H_t^{-1}(u))du, \ t > 0, \tag{18}$$

where $h_t(x) = \frac{h(x)}{H(t)}$, and $H_t^{-1}(u) = inf\{x; H_t(x) \ge u\}$ is the quantile function of $H_t(x) = \frac{H(x)}{H(t)}$ for all $0 \le x \le t$.

Proof. Using (3) and (17), we obtain

$$\bar{J}\left(T^{r}_{n,F}(t)\right) = -\frac{1}{2}\int_0^\infty \left(h^{r}_{n,F}(x;t)\right)^2 dx$$

$$= -\frac{1}{2}\int_0^t \left(\frac{h(t-x)}{H(t)}\right)^2 \left[r(n-r+1)\left(\frac{H(t-x)}{H(t)}\right)^{r-1} - (r+1)(n-r)\left(\frac{H(t-x)}{H(t)}\right)^{r}\right]^2 dx$$

$$= -\frac{1}{2}\int_0^t \left(\frac{h(z)}{H(t)}\right)^2 \left[r(n-r+1)\left(\frac{H(z)}{H(t)}\right)^{r-1} - (r+1)(n-r)\left(\frac{H(z)}{H(t)}\right)^{r}\right]^2 dz$$

$$= -\frac{1}{2}\int_0^1 (r(n-r+1)u^{r-1} - (r+1)(n-r)u^r)^2 h_t(H_t^{-1}(u))dx$$

$$= -\frac{1}{2}\int_0^1 g^2_{r|n:F}(u) h_t(H_t^{-1}(u))du. \tag{19}$$

The second equality in (19) is obtained by taking $z = t - x$, the third equality is derived by applying the change of variable $u = H_t(z)$, and the last equality is obtained by recognizing that $g_{r|n:F}(u)$ represents the pdf of $U_{r|n:F}$, as defined in (6). This completes the proof of the theorem.

The next theorem explores the relationship between the past extropy of the consecutive r-out-of-n: F system and the aging properties of its components.

Theorem 3.2. For all $2r \geq n$, if T is DRFR, then $\bar{J}(T_{n,F}^{r}(t))$ is increasing in t.

Proof. Using the identity $h_t(H_t^{-1}(x)) = x\tau_t(H_t^{-1}(x))$, (18) can be rewritten as

$$\bar{J}(T_{n,F}^{r}(t)) = -\frac{1}{2}\int_0^1 g_{r|n:F}^2(u)h_t(H_t^{-1}(u))du$$

$$= -\frac{1}{2}\int_0^1 g_{r|n:F}^2(u)u\tau_t\big(H_t^{-1}(u)\big)du, \text{for all } t > 0.$$

The relation $H_t^{-1}(u) = H^{-1}(uH(t))$ can be easily obtained for all $0 < u < 1$. Consequently, we obtain

$$\tau_t(H_t^{-1}(u)) = \tau(H^{-1}(uH(t))), \qquad 0 < u < 1.$$

Since $t_1 \leq t_2$ implies $H^{-1}(uH(t_1)) \leq H^{-1}(uH(t_2))$, and T has the DRFR property, we have the following inequality:

$$\int_0^1 g_{r|n:F}^2(u)u\tau_{t_1}(H_{t_1}^{-1}(u))du = \int_0^1 g_{r|n:F}^2(u)u\tau(H^{-1}(uH(t_1)))du$$

$$\geq \int_0^1 g_{r|n:F}^2(u)u\tau(H^{-1}(uH(t_2)))du$$

$$= \int_0^1 g_{r|n:F}^2(u)u\tau_{t_2}(H_{t_2}^{-1}(u))du,$$

which shows that $\bar{J}(T_{n,F}^{r}(t_1)) \leq \bar{J}(T_{n,F}^{r}(t_2))$ for all $t_1 \leq t_2$. Hence, the theorem.

The following example illustrates Theorems 3.1 and 3.2.

Example 3.1. Consider a linear consecutive 3-out-of-4: *F* system, where the system fails if and only if three consecutive components fail. The lifetime of this system, denoted by $T_{3|4:F}$ is expressed as

$$T_{3|4:F} = min(max(Z_1, Z_2, Z_3), max(Z_2, Z_3, Z_4)),$$

and the visual representation of this system configuration is shown in Figure 2.

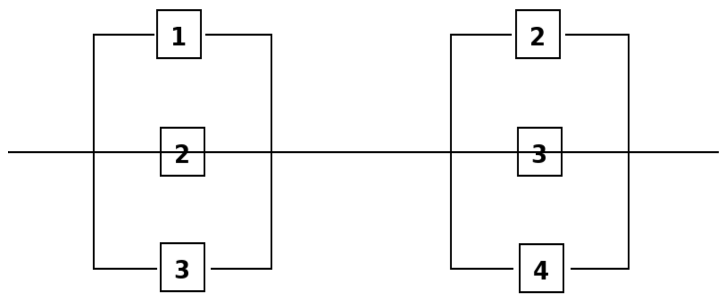

Figure 2. Structure of the linear consecutive 3-out-of-4:F system, illustrating the arrangement of components and the consecutive configurations required for system operation.

Assuming the following component lifetime distributions, we can calculate the exact value of $\bar{J}(T_{4,F}^{3}(t))$ using (18).

(i) Suppose the iid component lifetimes follow a power distribution with cdf $\mathrm{H(t)=t^{\beta}},\ \beta>0$, for $0\leq \mathrm{t}\leq 1$. In this case, we obtain

$$\bar{J}(T_{4,F}^{3}(t))=-\frac{\beta}{2t}\int_{0}^{1}u^{\frac{\beta-1}{\beta}}g_{r|n:F}^{2}(u)du,\qquad 0<t<1\,.$$

As $T_{4,F}^{3}(t)$ is monotonically increasing in t, the uncertainty associated with the system's past lifetime increases with time t. Furthermore, the DRFR property of the component lifetime distribution is evident.

(ii) Suppose T follows a Fréchet distribution with cdf

$$H(t)=e^{-t^{-\alpha}},\qquad t>0,\qquad \alpha>0.$$

In this case, some algebraic calculations yield the following expression:

$$\bar{J}(T_{4,F}^{3}(t))=-\frac{\alpha}{2}\int_{0}^{1}(t^{-\alpha}-logu)^{(1+\frac{1}{\alpha})}ug_{r|n:F}^{2}(u)du,\qquad t>0.$$

An explicit expression for the aforementioned relationship is intractable, necessitating a numerical analysis. Figure 3 shows $\bar{J}(T_{4,F}^{3}(t))$ as a function of t for various values of α. Consistent with Theorem 3.2, $\bar{J}(T_{4,F}^{3}(t))$ increases with t due to the DRFR property of T, as illustrated in Figure 3.

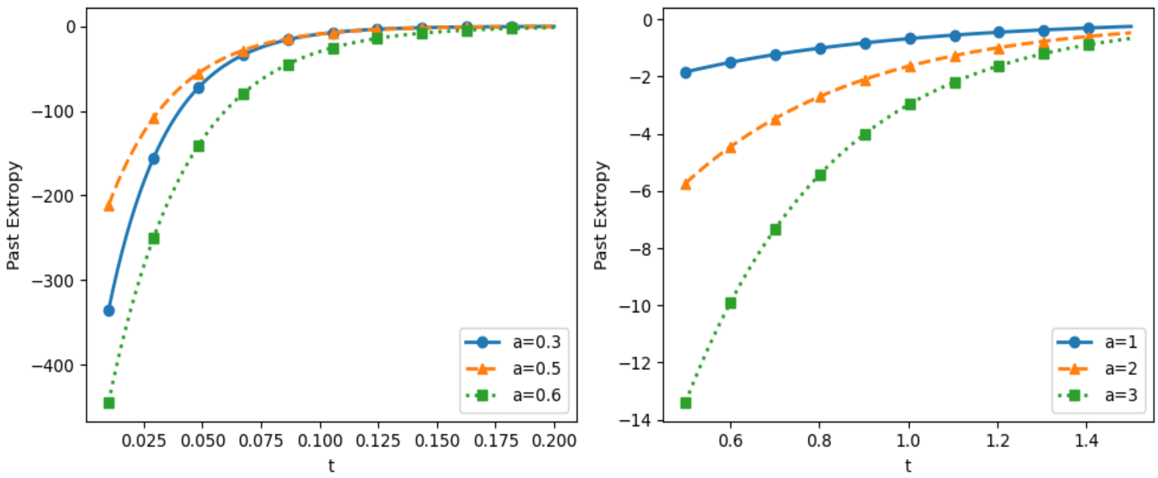

Figure 3. Exact values of the past extropy $\bar{J}(T^3_{4,F}(t))$ in Example 3.1 for the Fréchet distribution for different values of the parameter a. The left panel corresponds to (α = 0.3, 0.5, 0.6) over $t \in [0, 0.2]$, while the right panel corresponds to (α =1, 2, 3) over $t \in [0.5, 1.5]$. Different line styles and markers are used to ensure clarity in grayscale printing.

Figure 3 displays the exact values of the past extropy $\bar{J}(T^3_{4,F}(t))$ for the Fréchet distribution with different values of the shape parameter α. The curves show that the past extropy increases with t and approaches zero as time grows. In addition, larger values of α correspond to smaller past extropy values, indicating a lower level of uncertainty in the elapsed lifetime of the system.

The subsequent theorem establishes a lower bound for the extropy of $T^r_{n,F}(t)$ in terms of the past extropy $\bar{J}(T;t)$ of the parent distribution function.

Theorem 3.3 For all $2r \geq n$, a lower bound for the extropy of $T^r_{n,F}(t)$ is given by the following inequality (20)

$$\bar{J}(T^r_{n,F}(t)) \geq [r(n-r+1)]^2 \bar{J}(T;t), \ for\ t > 0. \tag{20}$$

Proof. Setting $A = r(n-r+1)$ and $B = (r+1)(n-r)$, we have $-Bu^r \leq 0$, for $0 < u < 1$, since $B > 0$. Applying relation (6) now yields

$$g^2_{r|n:F}(u) = [Au^{r-1} - Bu^r]^2 \leq A^2 u^{2r-2} \leq A^2,$$

where the last inequality follows from $0 < u^m \leq 1$ for all $m \geq 0$ and $0 < u < 1$. From (18) and the above inequality, we obtain a lower bound as

$$\bar{J}(T^r_{n,F}(t)) = -\frac{1}{2}\int_0^1 g^2_{r|n:F}(u) h_t(H_t^{-1}(u))du \geq -\frac{A^2}{2}\int_0^1 h_t(H_t^{-1}(u))du$$

$$= A^2 \bar{J}(T;t), \qquad t > 0,$$

where the final equality follows from (5). This completes the proof of the theorem.

An application of the bound provided in Theorem 3.3 is illustrated in the following example.

Example 3.2. Let $T_{5|8:F}$ denote the lifetime of a linear consecutive 5-out-of-8:F system. This system comprises 8 components arranged linearly and failing if and only if at least 5 consecutive components fail. Assuming component lifetimes to be iid Lomax (Pareto type II) distribution with cdf $H(t) = \frac{t}{1+t}$ for $t > 0$, it follows that

$$h_t(H_t^{-1}(u)) = \left(\frac{1 + t(1-u)}{1+t}\right)^2, \qquad 0 < u < 1.$$

Consequently, the past extropy of T is given by

$$\bar{J}(T;t) = -\frac{3 + 3t + t^2}{6t(1+t)}, \qquad t > 0.$$

Then Theorem 3.3 yields a lower bound for the extropy as

$$\bar{J}\left(T_{8,F}^{5}(t)\right) \geq -66.7\frac{(3+3t+t^2)}{t(1+t)}, \text{for all } t > 0. \tag{21}$$

The lower bound obtained from Theorem 3.3 and the exact value computed from (18) are illustrated in Figure 4.

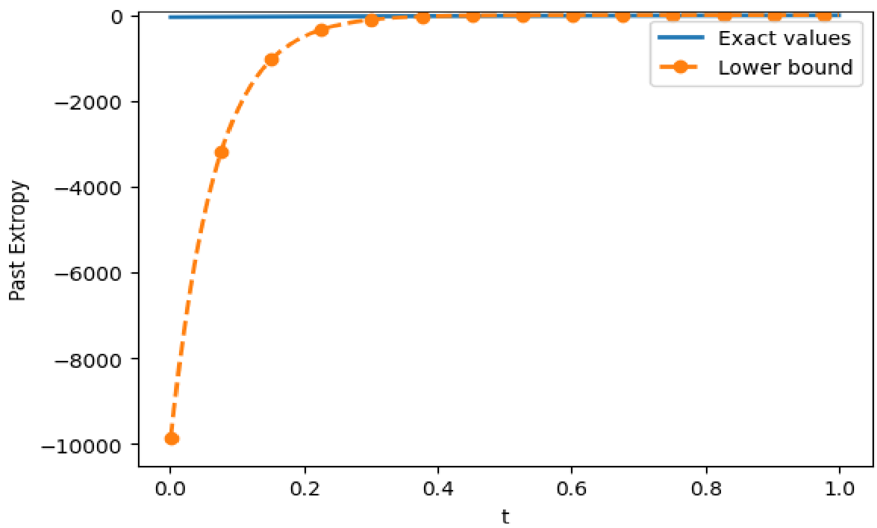

Figure 4. Exact values and associated lower bound of the past extropy in Example 3.2, expressed as a function of $\bar{J}(T_{8,F}^{5}(t))$. The exact values are depicted by a solid line, whereas the lower bound is represented by a dashed line with markers to enhance clarity and readability in grayscale reproduction.

Figure 4 compares the exact values of $\bar{J}(T_{8,F}^{5}(t))$ with the corresponding derived lower bound. The exact values are represented by a solid line, while the lower bound is depicted using a dashed line with markers to ensure clear distinction in grayscale printing. It is evident that both curves increase with time *t* and approach zero as *t* grows. Furthermore, the lower bound consistently lies below the exact values, thereby confirming the theoretical result and indicating that the proposed bound provides an accurate and tight approximation of the past extropy.

Finally, we show that the extropy of the conditional lifetimes of consecutive (*n*-*i*)-out-of-*n*: *F* systems for all $i = 0,1,\dots,[\frac{n}{2}]$, where $[x]$ is the greatest integer less than or equal to x, when all components have failed before time t, uniquely determines the distribution function. To this end, we recall the Müntz-Szász theorem from Higgins [29], which is crucial for the proof of the main result.

Lemma 3.1. Let $\psi(x)$ be an integrable function on the finite interval (a,b). Suppose that $\int_a^b x^{n_j}\psi(x)dx = 0,\ j \geq 1,$ where $\{n_j,\ j \geq 1\}$ is a strictly increasing sequence of positive integers satisfying $\sum_{j=1}^{\infty}\frac{1}{n_j} = \infty$. Then, $\psi(x) = 0$ for almost all $x \in (a,b)$.

It is important to highlight that Lemma 3.1 is a fundamental result in functional analysis, demonstrating the completeness of the set of functions $\{x^{n_1}, x^{n_2}, \dots; 1 \leq n_1 < n_2 < \cdots\}$. Hwang and Lin [30] generalized the Müntz–Szász theorem to the functions $\{\phi^{n_j}(x),\ n_j \geq 1\}$, where $\phi(x)$ is both absolutely continuous and monotonic over the interval (a,b). This lemma plays a pivotal role in uniquely characterizing the parent distribution based on $T_{n,F}^{n-i}(t)$ as in the following theorem.

Theorem 3.4. Let $T_{n,F}^{n-i,Z}(t) = [t - T_{n-i|n:F}^{Z} | Z_{n:n} \leq t]$ and $T_{n,F}^{n-i,Y}(t) = [t - T_{n-i|n:F}^{Y} | Y_{n:n} \leq t]$ denote past lifetimes of consecutive $(n-i)$-out-of-n:*F* systems having n iid component lifetimes Z_i and Y_i from cdfs H_Z and H_Y, only if

$$\bar{J}\left(T_{n,F}^{n-i,Z}(t)\right) = \bar{J}\left(T_{n,F}^{n-i,Y}(t)\right),\ for\ all\ n \geq 2i \text{ and } i = 0,1,\dots,\lfloor n/2 \rfloor.$$

Proof. We prove the sufficiency part, as the necessity is straightforward. Given that $n \geq 2i$ and $i = 0,1,\dots,\lfloor n/2 \rfloor$, (18) can be rewritten as

$$\bar{J}\left(T_{n,F}^{n-i,Z}(t)\right) = -\frac{1}{2}\int_0^1 g_{n-i|n:F}^2(u) h_{Z,t}\left(H_{Z,t}^{-1}(u)\right)du, \text{for all } t > 0, \tag{22}$$

where

$$g_{n-i|n:F}(u) = (n-i)(i+1)u^{n-i-1} - (n-i+1)iu^{n-i}, 0 < u < 1.$$

A similar argument holds for Y. Assuming $\bar{J}(T_{n,F}^{n-i,Z}(t)) = \bar{J}(T_{n,F}^{n-i,Y}(t))$ and using (22), we have

$$\int_0^1 g_{n-i|n:F}^2(u)\left(h_{Z,t}\left(H_{Z,t}^{-1}(u)\right) - h_{Y,t}\left(H_{Y,t}^{-1}(u)\right)\right)du = 0, \qquad \text{for all } t > 0.$$

Thus, we obtain

$$\int_0^1 u^{n-2i}\phi_{i,n}(u)\big(h_{Z,t}(H_{Z,t}^{-1}(u)) - h_{Y,t}(H_{Y,t}^{-1}(u))\big)du = 0,$$

where

$$\phi_{i,n}(u) = (n-i)(i+1)u^{n-2} - (n-i+1)iu^n, \qquad 0 < u < 1.$$

Applying now Lemma 3.1 to $\phi_{i,n}(u)$ and considering the complete sequence $\{u^{n-2i},\ n \geq 2i\}$, we obtain

$$h_{Z,t}\left(H_{Z,t}^{-1}(u)\right) = h_{Y,t}\left(H_{Y,t}^{-1}(u)\right), \qquad \text{for all } u \in (0,1).$$

As $dH_t^{-1}(u) = -1/h_t(H_t^{-1}(u))du$, we have $H_{Z,t}^{-1}(u) = H_{Y,t}^{-1}(u)$ for all $0 < u < 1$. Letting $z = H_{Z,t}^{-1}(u)$, we obtain $H_{Z,t}(z) = u$. Given $H_{Z,t}(z) = \frac{H_Z(z)}{H_Z(t)}$, and similarly for Y, we find $\frac{H_Z(z)}{H_Z(t)} = \frac{H_Y(z)}{H_Y(t)}$. As $t \to \infty$, we conclude that $H_Z(z) = H_Y(z)$, $z > 0$, implying that Z and Y are identically distributed, thus completing the proof of the theorem.

4. Nonparametric estimation

In this section, we assume that a random sample from the component lifetimes distribution is available. Such data may arise from component-level reliability tests, laboratory experiments, or historical maintenance records where the lifetimes of individual components are observed independently of the system configuration. This assumption is commonly used in reliability analysis when the objective is to estimate system characteristics based on the underlying component lifetime distribution. Under this framework, the estimation problem reduces to estimating the component lifetime distribution and the associated reliability function, from which the past extropy of the corresponding consecutive system can be obtained through the functional representation derived in the previous sections.

Here, we introduce a nonparametric estimator for the past extropy of consecutive r-out-of-n: F systems, as defined in (11). Suppose we have a sequence of iid random variables $T_1, T_2, \cdots, T_N$, representing individual component lifetimes with pdf $h(x)$ and cdf $H(x)$. Following Silverman [30], we estimate $h(x)$ using the kernel density estimator as

$$\hat{h}(x) = \frac{1}{Nb_N}\sum_{i=1}^{N} k\left(\frac{x-T_i}{b_N}\right), \ \ x \in R,$$

where b_N is the bandwidth, and $k(\cdot)$ is the kernel function. We choose b_N such that $b_N \to 0$ and $Nb_N \to \infty$ as $N \to \infty$, ensuring convergence of $\hat{h}(x)$ to $h(x)$ as the sample size increases. The kernel function $k(\cdot)$ is a symmetric pdf with finite variance, with common examples including the normal (Gaussian), Epanechnikov, and tricube kernels. Here, we use the normal kernel. We also consider the kernel-smoothed cdf estimator

$$\widehat{H}(x)=\int_{-\infty}^{x}\hat{h}(u)du=\frac{1}{N}\sum_{i=1}^{N}K\left(\frac{x-T_i}{b_N}\right),\ \ x\in R,$$

where $K(\cdot)$ is the cdf of $k(\cdot)$ defined as $K(x)=\int_{-\infty}^{x}k(t)\,dt$. For bandwidth selection, we employ the heuristic formula

$$b_N=0.9\sigma N^{-\frac{1}{5}},$$

derived by minimizing the integrated mean squared error with a Gaussian kernel. Typically, σ is estimated as $\min\left\{s,\frac{Q}{1.34}\right\}$, where s is the sample standard deviation, and Q is the interquartile range. This normal reference rule (Silverman [30]) performs well for smooth densities. Before presenting the nonparametric estimator, we recall that the pdf and cdf of $T_{r|n:F}$ can be expressed as

$$h_{r|n:F}(x)\ =\ h(x)g_{r|n:F}(H(x))\text{ and }H_{r|n:F}(x)\ =\ G_{r|n:F}(H(x)),$$

for all $x>0$, using relations (6) and (7). Thus, with the kernel density estimator $\hat{h}(x)$, the nonparametric kernel-based estimator for the past extropy $\bar{J}(T_{r|n:F};t)$ can be presented as

$$\widehat{\bar{J}}(T_{r|n:F};t)=-\frac{1}{2}\int_{0}^{t}\left(\frac{\hat{h}_{r|n:F}(x)}{\widehat{H}_{r|n:F}(t)}\right)^{2}dx$$

$$=-\frac{1}{2}\int_{0}^{t}\left(\frac{\hat{h}(x)g_{r|n:F}(\widehat{H}(x))}{G_{r|n:F}(\widehat{H}(x))}\right)^{2}dx,\ \text{ for all } t>0. \quad (23)$$

Since the lifetime distributions considered in this work are supported on [0, ∞), the use of a symmetric Gaussian kernel may introduce boundary bias near zero. In the present study, this effect is limited because the past extropy is evaluated through integration over (0, t) with t > 0. Nevertheless, boundary-correction techniques could be incorporated to further improve estimation accuracy near the boundary.

4.1. Simulation studies

In the simulation study, samples are generated from the assumed component lifetime distribution in order to evaluate the performance of the proposed estimator under controlled conditions. To examine the behavior and performance of the proposed nonparametric extropy estimator in (23), we carried out a Monte Carlo simulation study. For evaluating the kernel-based estimator of the PEX of consecutive r-out-of-n: F systems, we use the exponential distribution with parameter λ, having pdf $h(x)=\lambda e^{-\lambda x}$ and cdf $H(\mathrm{x})=1-e^{-\lambda x}$, x > 0, as the underlying data generating process, i.e.,

$$\bar{J}(T_{r|n:F};t) = -\frac{1}{2}\int_0^t \left(\frac{\lambda e^{-\lambda x} g_{r|n:F}(H(\mathrm{x}))}{G_{r|n:F}(H(\mathrm{x}))}\right)^2 dx, \text{ for all } t > 0.$$

For the estimation of $\bar{J}(T_{r|n:F};t)$, we generate random samples from exponential distribution with parameter $\lambda = 0.5, 1.0, 2.0$. A total of 1,000 iterations were performed, and the obtained results are presented in Tables 2–4. These results offer valuable insights into the finite-sample performance of the estimator across varying sample sizes ($N = 40, 50, 100, 200$) and different values of r and t. The values of bias and root mean squared error (RMSE) are reported, allowing a thorough assessment of both the accuracy and precision of the estimator.

Table 2. Bias and RMSE of the estimator of $\bar{J}(T_{r|3:F};t)$ for an exponential distribution (λ =0.5) with varying t, r, and N.

			N = 40		N = 50		N = 100		N = 200	
n	r	t	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
3	2	0.5	-0.465484	0.633022	-0.201711	0.454374	-0.112885	0.136406	-0.075306	0.084833
		0.9	-0.092047	0.110814	-0.075303	0.090910	-0.048539	0.056794	-0.036167	0.039901
		1.0	-0.075659	0.097224	-0.061499	0.081457	-0.042888	0.047819	-0.029563	0.034398
		2.0	-0.020343	0.026277	-0.017525	0.021520	-0.012037	0.014247	-0.008374	0.009764
	3	0.5	-0.354287	0.371482	-0.287368	0.297284	-0.171754	0.203613	-0.124453	0.132974
		0.9	-0.135503	0.180536	-0.117933	0.139389	-0.084560	0.095574	-0.064234	0.071779
		1.0	-0.120285	0.145836	-0.099206	0.122640	-0.075226	0.085113	-0.055287	0.066078
		2.0	-0.038318	0.055870	-0.034487	0.049700	-0.022263	0.034263	-0.016270	0.023181

Table 3. Bias and RMSE of the estimator of $\bar{J}(T_{r|3:F};t)$ for an exponential distribution ($\lambda = 1$) with varying t, r, and N.

			N = 40		N = 50		N = 100		N = 200	
n	r	t	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
3	2	0.5	-0.154658	0.198186	-0.123190	0.158914	-0.083604	0.096432	-0.061391	0.067818
		0.9	-0.051350	0.064786	-0.043647	0.050976	-0.029105	0.034949	-0.020154	0.024719
		1.0	-0.043669	0.050797	-0.034790	0.045171	-0.023613	0.028233	-0.016400	0.019222
		2.0	-0.013850	0.034770	-0.010711	0.029187	-0.005497	0.019795	-0.004489	0.014207
	3	0.5	-0.230284	0.305693	-0.201810	0.245701	-0.147047	0.175044	-0.111521	0.128064
		0.9	-0.091214	0.132614	-0.081696	0.110546	-0.054550	0.081263	-0.040795	0.057775
		1.0	-0.078592	0.108102	-0.069220	0.101276	-0.047042	0.069280	-0.032217	0.050086
		2.0	-0.027305	0.037485	-0.023717	0.032148	-0.014155	0.020083	-0.009653	0.013360

Table 4. Bias and RMSE of the estimator of $\bar{J}(T_{r|3:F};t)$ for an exponential distribution $(\lambda = 2)$ with varying t, r, and N.

			N = 40		N = 50		N = 100		N = 200	
n	r	t	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
3	2	0.5	-0.086220	0.106091	-0.070826	0.090802	-0.047851	0.055454	-0.032778	0.036562
		0.9	-0.032406	0.063334	-0.024426	0.055908	-0.015369	0.035558	-0.009878	0.024261
		1.0	-0.026929	0.074554	-0.022132	0.058622	-0.012381	0.039811	-0.006550	0.028337
		2.0	-0.015107	0.080265	-0.005539	0.073605	-0.002422	0.050565	-0.002752	0.033907
	3	0.5	-0.166086	0.223918	-0.141487	0.196756	-0.092657	0.126618	-0.068328	0.099283
		0.9	-0.067124	0.084225	-0.055979	0.073652	-0.034722	0.048270	-0.023380	0.032641
		1.0	-0.056763	0.074605	-0.049613	0.064726	-0.030351	0.040951	-0.021167	0.027207
		2.0	-0.040928	0.073928	-0.033420	0.066368	-0.017551	0.041721	-0.010447	0.028621

As expected, both bias and RMSE consistently decrease as the sample size N increases across different combinations of r and t. While simulation studies cannot establish asymptotic properties theoretically, the results provide empirical evidence supporting the good finite-sample performance of the proposed estimator. These empirical findings indicate an improvement in estimation accuracy as the sample size increases. The number of consecutive failed components of the system r also plays a crucial role in influencing bias and RMSE. The results indicate that, for a fixed n, increasing r from 2 to 3 leads to a decrease in RMSE when the scale parameter λ is less than one, but an increase in RMSE when λ is greater than one. This suggests that the estimator tends to perform better under smaller-scale parameters as r increases, whereas its performance deteriorates for larger -scale parameters when n is fixed.

5. Concluding remarks

This paper examines the role of past extropy as an information-theoretic measure for uncertainty quantification in lifetime analysis, with particular emphasis on linear consecutive r-out-of-n: F reliability systems. A comprehensive theoretical framework has been developed to analyze the behavior of past extropy under this important class of system structures. In particular, explicit analytical representations have been derived, and several fundamental properties have been established, including monotonicity results, probabilistic bounds, and characterization theorems. These results provide new structural insight into how reliability configurations influence information-based measures of uncertainty. The investigation has been further extended to conditional past extropy, which captures the uncertainty associated with elapsed lifetimes under partial system information. This perspective provides a more refined description of uncertainty dynamics in reliability environments where degradation, clustered failures, or time-dependent operational conditions may affect system performance. From a methodological standpoint, a nonparametric kernel-based estimator for past extropy has been proposed in order to facilitate practical implementation. Its performance has been examined through Monte Carlo simulation experiments, which indicate stable finite-sample behavior and reliable estimation accuracy across a range of parameter settings. Although closed-form expressions for past extropy may become analytically challenging for highly complex system structures or heavily parameterized lifetime models, the probabilistic bounds, preservation properties, and characterization results established in this study will provide effective analytical tools for understanding its behavior. In particular, the derived stochastic ordering relationships highlight the sensitivity of past extropy to aging characteristics and structural variations in lifetime distributions, thereby enhancing its interpretability in reliability modeling and uncertainty assessment.

The results obtained in this work also suggest several promising directions for future research. First, the present analysis focuses on systems with independent and identically distributed component lifetimes. Extending the framework to dependent structures such as those modeled by Archimedean, hierarchical, or vine copulas would significantly broaden the scope of applicability. Second, many engineering systems exhibit dynamic reliability features including load-sharing mechanisms, repairable components, and failure interactions. Developing past extropy formulations for such dynamic systems represents a natural and important extension. Third, modern engineering applications often involve spatially distributed networks and multivariate degradation processes, motivating further investigation of past extropy in spatial stochastic networks and multivariate lifetime models. Fourth, information-based measures such as past extropy may play a valuable role in reliability-oriented decision-making, including optimal maintenance scheduling, sensor allocation, and system design, where uncertainty quantification is central. Finally, while a kernel-based estimator has been introduced, further investigation of its asymptotic properties, including consistency, asymptotic distribution, bias correction, and confidence interval construction, remains an important topic for future methodological development. Exploring connections between past extropy and other information measures, including past Rényi entropy, Tsallis entropy, cumulative extropy, and Fisher information, may also reveal deeper theoretical relationships and contribute to the broader development of information-based reliability analysis.

Overall, the findings of this study highlight the potential of past extropy as a flexible and informative framework for analyzing uncertainty in lifetime distributions and reliability systems. By integrating rigorous theoretical results with practical estimation methodology, this work contributes to the expanding literature on information measures in reliability theory and opens several avenues for further theoretical and applied research.

Author Contributions

Conceptualization, F. A. and M. K.; methodology, F. A., B. A, N. B. and M. K.; software, F. A. and M. K.; validation, N. B. and M. K.; formal analysis, F. A. N. B, B. A. and M. K.; investigation, N. B.; resources, F. A.; data curation, F. A, N. B. and B. A.; writing—original draft preparation, F. A and B. A.; writing—review and editing, N. B.; visualization, F. A. and N. B; supervision, M. K.; project administration, F. A.; funding acquisition, B. A. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Conflict of interest

The authors declare no conflict of interest.

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