



Research article

Global existence and boundedness of HIV chemotaxis model with nonlinear diffusions

Wenjuan Lv*

School of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: lv081996@163.com.

Abstract: This paper investigates a class of reaction-diffusion models that describing the chemotactic movement of cytotoxic T lymphocytes (CTLs) in HIV-1 infection, thereby incorporating nonlinear diffusion mechanisms for $CD4^+$ T cells, infected cells, and CTLs. Assuming that the CTL diffusion coefficient satisfies $D_3(w) \geq \hat{D}(w+1)^\theta$ with $\theta > 1$, and the chemotactic sensitivity function satisfies $|\chi(w)| \leq \hat{\chi}w(1+w)^{\eta-1}$ with $\eta \leq 1$, this study proves, via energy estimates, semigroup techniques, and the Moser iteration, provided $\theta > 1$ and $\eta \leq 1$, that the classical solutions globally-exist and are unique and uniformly bounded for any spatial dimension $n \geq 1$. The results demonstrate that a sufficiently strong nonlinear diffusion (with $\theta > 1$) can effectively suppress chemotaxis-driven instability, thus ensuring the boundedness of the systems long-term behavior. This research provides mathematical theoretical support to understand the regulatory role of immune chemotactic movement in viral dynamics.

Keywords: HIV model; immune chemokines; nonlinear diffusion; global existence

Mathematics Subject Classification: 35A01

1. Introduction

In recent years, modeling of the immune system with chemotaxis and its dynamical behavior has attracted widespread attention (see [1–3]). In particular, Lai and Zou [4, 5] formulated a reaction-diffusion model to explore the effects of cytotoxic T lymphocyte (CTL) chemotactic movement on the

dynamics of HIV-1 infection. The model is written in the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \Delta u + \tilde{h} - d_u u - \tilde{\beta} uv, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = D_2 \Delta v + \tilde{\beta} uv - d_v v - \tilde{p} vw, & x \in \Omega, t > 0, \\ \frac{\partial w}{\partial t} = D_3 \Delta w - \nabla[\chi(w)\nabla v] + \tilde{c} vw - d_w w, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where Ω is a smooth bounded domain in \mathbb{R}^n , ($n \geq 1$), and ν is the normal to the boundary $\partial\Omega$. As in (1), $u(x, t)$, $v(x, t)$, and $w(x, t)$ denote the population densities of uninfected CD4⁺T cells, infected CD4⁺T cells, and CTLs, respectively. Uninfected cells are produced at a constant rate \tilde{h} and become infected at a rate $\tilde{\beta}uv$, infected cells are cleared by CTLs at a rate $\tilde{p}vw$, and CTLs expand through antigen-stimulated proliferation at a rate $\tilde{c}vw$. Diffusion effects (denoted by D_1 , D_2 , and D_3 , respectively) and natural death rates (denoted by d_u , d_v , and d_w , respectively) are considered for all cell types. In the above, all parameters are positive. The movement of CTLs are guided by chemotaxis, which is the directional migration along gradients of chemokines (signaling molecules) produced during infection. The term $-\nabla[\chi(w)\nabla v]$ models this effect: Infected cells (v) secrete chemokines that attract CTLs, so $\chi(w) > 0$ represents chemoattraction, while $\chi(w) < 0$ represents chemorepulsion. They [4, 5] found that chemorepulsion movement of CTLs can destabilize the positive steady state as the strength of the chemotactic sensitivity increases. In addition, for an arbitrary spatial dimension $n \geq 1$, the global existence of solutions to system (1) in the case $D_1 = D_2$ was established via a comparison argument.

In order to better understand the above models, the famous chemotactic model should be mentioned, which was proposed by Keller and Segel [6] and described the aggregation of *Dictyostelium discoideum* as follows:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + f(u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + h(u), & x \in \Omega, \quad t > 0, \end{cases} \quad (2)$$

where $f(u)$ is a source, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. For the case $f(u) = 0$ in (2), if $h(u) = u$, then the asymptotic property of $\frac{S(u)}{D(u)} \approx u^{\frac{2}{n}}$ is the critical condition for blow-up and global boundedness (see [7, 8]); if $0 < h(u) \leq u^{\tilde{\gamma}}$, $D(u) \geq \mathcal{K}_0(u+1)^\iota$, $0 < S(u) \leq \mathcal{K}_1 u(u+1)^{\varrho-1}$ and $\tilde{\gamma} > 1$ for $\mathcal{K}_0, \mathcal{K}_1 > 0$. Hu and Zheng [9] proved the global existence and boundedness of solutions to (2) under the condition $\varrho - \iota + \tilde{\gamma} < 1 + \frac{2}{n}$ and $\iota > -\frac{2}{n}$. For a more detailed qualitative analysis of model (2), please refer to Ref [10–12].

In HIV infection models, the chemotactic movement of CTLs is typically assumed to follow linear diffusion. However, experimental evidence indicates that the migration capacity of CTLs may depend on their local density (e.g., cell crowding effects). Considering that the chemotaxis model with density-dependent diffusion has also become one of the research hotspots of immune system in recent years [13, 14], this paper is concerned with the following quasilinear HIV-1 model:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = D_1 \Delta u + \tilde{h} - d_u u - \tilde{\beta} uv, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = D_2 \Delta v + \tilde{\beta} uv - d_v v - \tilde{p} vw, & x \in \Omega, t > 0, \\ \frac{\partial w}{\partial t} = \nabla(D_3(w)\nabla w) - \nabla(\chi(w)\nabla v) + \tilde{c} vw - d_w w, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega. \end{array} \right. \quad (3)$$

Where $D_3(w) \geq \hat{D}(w+1)^\theta$, $|\chi(w)| \leq \hat{\chi}w(1+w)^{\eta-1}$ for $\hat{D} > 0$, $\hat{\chi} > 0$. Nonlinear diffusion $D_3(w)$ reflects that CTL motility depends on the local cell density due to crowding or adhesion, which is a crucial effect for realistic tissue-scale immune responses. These nonlinear mechanisms capture a wider range of spatiotemporal dynamics compared to linear diffusion models.

The main purpose of this paper is to prove the uniform boundedness of solutions for problem (3). More precisely, we prove the following theorem on the global existence of solutions.

Theorem 1.1. (Global existence) *Suppose that $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with a smooth boundary. Let $(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0, v_0, w_0 \geq, \neq 0$. Assume that the parameters $\theta > 1$ and $\eta \leq 1$. Then, there exists a constant $C > 0$ independent of t such that system (3) admits a unique global-in-time nonnegative classical solution*

$$\begin{aligned} u(x, t) &\in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ v(x, t) &\in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ w(x, t) &\in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \end{aligned} \quad (4)$$

and

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,p}} + \|w(\cdot, t)\|_{L^\infty} \leq C, \quad (5)$$

for any $p > n$.

In recent years, taxis-driven instability has received increasing attention in virus dynamics models. Bellomo et al. [20] pointed out that in the absence of diffusion or under linear diffusion, a chemotactic intensity that exceeds a threshold in the May-Nowak virus model can trigger spatial pattern formation. In contrast, the present paper introduces CTL cells with their nonlinear diffusion $D_3(w)$ and finds that when the nonlinear diffusion is sufficiently strong ($\theta > 1$), the solution remains globally bounded. Based on the work of Lai and Zou [4, 5], this paper introduces nonlinear diffusion, which brings new essential difficulties to the mathematical analysis: The diffusion coefficient of w depends on w itself, thus rendering the standard parabolic regularity theory inapplicable and hindering the derivation of an L^∞ a priori estimate for w . To overcome this, we construct a weighted energy functional $\int (w+1)^\theta$ and combine it with an improved Gagliardo-Nirenberg inequality and Moser iteration. Under the conditions $\theta > 1$ and $\eta \leq 1$, we prove for the first time the global existence and uniform boundedness of solutions to this nonlinear diffusion HIV model, thus resolving the aforementioned difficulties.

2. Preliminaries

Lemma 2.1. [15] Let $T > 0$, $\tau \in (0, T)$, $a_1 > 0$, and $b_1 > 0$. Suppose that $y : [0, T] \rightarrow [0, \infty)$ is absolutely continuous and fulfills

$$y'(t) + a_1 y(t) \leq h_1(t), \quad \forall t \in (0, T),$$

with some nonnegative function $h_1 \in L^1_{\text{loc}}([0, T])$ satisfying $\int_t^{t+\tau} h_1(s) ds \leq b_1$ for all $t \in [0, T - \tau)$. Then,

$$y(t) \leq \max \left\{ y(0) + b_1, \frac{b_1}{a_1 \tau} + 2b_1 \right\}, \quad \forall t \in (0, T).$$

Lemma 2.2. [16] Let $p \in [1, \infty)$. For every $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that every function $\varphi \in C^2(\bar{\Omega})$ with $\partial_\nu \varphi = 0$ on $\partial\Omega$ satisfies the following:

$$\int_{\partial\Omega} |\nabla \varphi|^{2(p-1)} |\partial_\nu |\nabla \varphi|^2| dS \leq \varepsilon \int_{\Omega} |\nabla \varphi|^{2(p-2)} |\nabla |\nabla \varphi|^2|^2 + C_\varepsilon \int_{\Omega} |\nabla \varphi|^{2p}.$$

Lemma 2.3. [17] Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in Ω , and let $\mu_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then, for all $t > 0$, there exist some constants γ_i ($i = 1, 2, 3, 4$) depending only on Ω such that the following hold:

(i) If $2 \leq p < \infty$, then

$$\|\nabla e^{td\Delta} z\|_{L^p} \leq \gamma_1 e^{-\mu_1 dt} \|\nabla z\|_{L^p} \quad (6)$$

for all $z \in W^{1,p}(\Omega)$.

(ii) If $1 \leq q \leq p \leq \infty$, then

$$\|\nabla e^{td\Delta} z\|_{L^p} \leq \gamma_2 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\mu_1 dt} \|z\|_{L^q} \quad (7)$$

for all $z \in L^q(\Omega)$.

(iii) If $1 \leq q \leq p \leq \infty$, then

$$\|e^{td\Delta} z\|_{L^p} \leq \gamma_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) \|z\|_{L^q} \quad (8)$$

for all $z \in L^q(\Omega)$.

(iv) If $1 < q \leq p \leq \infty$, then

$$\|e^{td\Delta} \nabla \cdot z\|_{L^p} \leq \gamma_4 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\mu_1 dt} \|z\|_{L^q} \quad (9)$$

for all $z \in (C_0^\infty(\Omega))^n$.

We note that the result in Lemma 2.3 (iv) also holds true for any $z \in L^q(\Omega)$ with $1 \leq q < \infty$ since $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)$ ($1 \leq q < \infty$) (see also [18]).

Lemma 2.4. (Local existence). Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a smooth bounded domain. Suppose the initial data satisfy $u_0, v_0, w_0 \geq, \neq 0$ with $(u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$. Then, there exists a maximal existence time $T_{max} \in (0, \infty]$ such that problem (3) admits a nonnegative classical solution as follows:

$$(u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times [0, T_{max}))]^3.$$

Moreover, if $T_{max} < \infty$, then

$$\limsup_{t \nearrow T_{max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,p}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)}) = \infty, \quad (10)$$

for any $p > n$.

Proof. By Amann's well-established parabolic theory introduced in Theorem 7.3 of [19], we can obtain the local existence, uniqueness, and blow-up of criterion (3). \square

By using the comparison principle, we can easily obtain the uniform boundedness of u .

Lemma 2.5. If (u, v, w) is a classical solution of the system (3) in $\Omega \times (0, T_{max})$, then u satisfies the following:

$$u \leq K_1, \text{ for } x \in \Omega, t \in (0, T_{max}), \quad (11)$$

where $K_1 := \max\{\|u_0\|_{L^\infty(\Omega)}, \frac{\tilde{h}}{d_u}\}$.

The following lemma gives a uniform L^1 bound for the local solution, which plays an important role in the proof of the main result.

Lemma 2.6. Let (u, v, w) be the solution of system (3) obtained in Lemma 2.4; then, there exists $K_2 > 0$ such that,

$$\|u(\cdot, t)\|_{L^1(\Omega)} + \|v(\cdot, t)\|_{L^1(\Omega)} + \|w(\cdot, t)\|_{L^1(\Omega)} \leq K_2, \text{ for } t \in (0, T_{max}), \quad (12)$$

where K_2 is a constant independent of t .

Proof. By (3) and the formula for integration by parts, we derive the following:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (\tilde{c}u + \tilde{c}v + \tilde{p}w) &= \tilde{c}\tilde{h}|\Omega| - \tilde{c}d_u \int_{\Omega} u - \tilde{c}d_v \int_{\Omega} v - \tilde{p}d_w \int_{\Omega} w \\ &\leq \tilde{c}\tilde{h}|\Omega| - C_1 \int_{\Omega} (\tilde{c}u + \tilde{c}v + \tilde{p}w), \end{aligned}$$

where $C_1 := \min\{d_u, d_v, d_w\}$; thus,

$$\frac{d}{dt} \int_{\Omega} (\tilde{c}u + \tilde{c}v + \tilde{p}w) + C_1 \int_{\Omega} (\tilde{c}u + \tilde{c}v + \tilde{p}w) \leq \tilde{c}\tilde{h}|\Omega| := C_2.$$

This completes the proof by a direct integration. \square

3. Global existence and boundedness

Lemma 3.1. *Let (u, v, w) be a classical solution of the system (3) in $\Omega \times (0, T_{max})$. Then, for any $p > 1$, there exists a constant $K_3 > 0$ independent of t such that*

$$\|v\|_{L^p} \leq K_3, \text{ for } t \in (0, T_{max}). \quad (13)$$

Proof. Multiplying the second equation of (3) by v^{p-1} and integrating over the domain Ω yields the following:

$$\begin{aligned} \int_{\Omega} v_t \cdot v^{p-1} &= \frac{1}{p} \frac{d}{dt} \int_{\Omega} v^p dx \\ &= D_2 \int_{\Omega} (\Delta v) \cdot v^{p-1} dx + \tilde{\beta} \int_{\Omega} uv^p dx - d_v \int_{\Omega} v^p dx - \tilde{p} \int_{\Omega} wv^p dx. \end{aligned}$$

By applying the integration by parts formula, we obtain the following:

$$\begin{aligned} \int_{\Omega} (\Delta v)v^{p-1} dx &= - \int_{\Omega} (\nabla v) \cdot ((p-1)v^{p-2}\nabla v) dx \\ &= - \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla v^{\frac{p}{2}}|^2 dx. \end{aligned}$$

Additionally, we have the following:

$$\tilde{\beta} \int_{\Omega} uv^p \leq K_1 \tilde{\beta} \int_{\Omega} v^p dx.$$

Hence,

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} v^p dx + \frac{4D_2(p-1)}{p^2} \int_{\Omega} |\nabla v^{\frac{p}{2}}|^2 dx + d_v \int_{\Omega} v^p \leq K_1 \tilde{\beta} \int_{\Omega} v^p dx. \quad (14)$$

It follows from the Gagliardo-Nirenberg inequality and Young's inequality that

$$\begin{aligned} K_1 \tilde{\beta} \int_{\Omega} v^p dx &= C_3 \left(\|\nabla v^{\frac{p}{2}}\|_{L^2}^{2b_2} \|v^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^{2(1-b_2)} + \|v^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^2 \right) \\ &\leq C_4 \|\nabla v^{\frac{p}{2}}\|_{L^2}^{2b_2} + C_5 \\ &\leq \frac{4D_2(p-1)}{p^2} \|\nabla v\|_{L^2}^2 + C_6, \end{aligned} \quad (15)$$

where $b_2 := \frac{\frac{np}{2} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{np}{2}} \in (0, 1)$ for any $p > 1$ and $n \geq 1$. Substituting (15) into (14) eliminates the gradient term, and one can immediately obtain (13) by applying Grönwall's inequality. \square

Lemma 3.2. *Let (u, v, w) be a classical solution of system (3) in $\Omega \times (0, T_{max})$. Then, for any $p > \frac{n}{2}$, there exists a constant $K_4 > 0$ independent of t such that*

$$\|v\|_{L^\infty} \leq K_4, \text{ for } t \in (0, T_{max}). \quad (16)$$

Proof. The second equation in (3) admits the following solution representation:

$$\begin{aligned} v(x, t) &= e^{(D_2\Delta - d_v)t} v_0(x) + \int_0^t e^{(D_2\Delta - d_v)(t-s)} (\tilde{\beta}uv - \tilde{p}vw) ds \\ &\leq e^{(D_2\Delta - d_v)t} v_0(x) + \tilde{\beta} \int_0^t e^{(D_2\Delta - d_v)(t-s)} uv ds. \end{aligned}$$

Using the uniform L^p -bound of v and the Neumann heat semigroup estimate (8), we have the following:

$$\begin{aligned} \|v\|_{L^\infty} &\leq \|e^{(D_2\Delta - d_v)t} v_0\|_{L^\infty} + \tilde{\beta} \int_0^t \|e^{(D_2\Delta - d_v)(t-s)} uv\|_{L^\infty} ds \\ &\leq \|v_0\|_{L^\infty} + \tilde{\beta}\gamma_3 \int_0^t [1 + (t-s)^{-\frac{n}{2p}}] e^{-d_v(t-s)} \|uv\|_{L^p} ds \\ &\leq \|v_0\|_{L^\infty} + \tilde{\beta}\gamma_3 K_1 K_3 \int_0^\infty [1 + (t-s)^{-\frac{n}{2p}}] e^{-d_v(t-s)} ds. \end{aligned}$$

We used the fact that $-\frac{n}{2p} > -1$ by taking $p > \frac{n}{2}$ as in Lemma 3.1. This establishes the uniform L^∞ -bound for v . \square

Lemma 3.3. *Let the assumptions in Lemma 2.4 hold. Then, the solution of (3) satisfies the following:*

$$\int_t^{t+\tau} \int_\Omega |\nabla v(\cdot, s)|^2 \leq K_5, \quad \text{for all } t \in (0, \tilde{T}_{max}), \quad (17)$$

where $K_5 > 0$ is a constant independent of t and

$$\tau := \min \left\{ 1, \frac{1}{2} T_{max} \right\} \quad \text{and} \quad \tilde{T}_{max} := \begin{cases} T_{max} - \tau, & \text{if } T_{max} < \infty, \\ \infty, & \text{if } T_{max} = \infty. \end{cases} \quad (18)$$

Proof. By multiplying the second equation in (3) by $2v$, integrating over Ω , and applying Young's inequality, we find the following:

$$\begin{aligned} \frac{d}{dt} \int_\Omega v^2 + \int_\Omega v^2 &= 2 \int_\Omega v (D_2\Delta v + \tilde{\beta}uv - d_v v - \tilde{p}wv) + \int_\Omega v^2 \\ &\leq -2D_2 \int_\Omega |\nabla v|^2 + C_7, \quad \text{for all } t \in (0, T_{max}), \end{aligned} \quad (19)$$

where $C_7 := (2\tilde{\beta}K_1K_4^2 + K_4^2)|\Omega|$. An application of Grönwall's inequality to Eq (19) gives the following:

$$\int_t^{t+\tau} \int_\Omega |\nabla v|^2 \leq K_5, \quad \text{for all } t \in (0, \tilde{T}_{max}). \quad (20)$$

\square

Lemma 3.4. *Suppose the assumptions in Lemma 2.4 hold. If $\theta > 1$ and $\eta \leq 1$, then there exists a constant $K_6 > 0$ independent of t such that*

$$\begin{aligned} \|w(\cdot, t)\|_{L^\theta} &\leq K_6, \quad \text{for all } t \in (0, T_{max}), \\ \int_t^{t+\tau} \int_{\Omega} (w(\cdot, s) + 1)^{2\theta-2} |\nabla w(\cdot, s)|^2 &\leq K_6, \quad \text{for all } t \in (0, \bar{T}_{max}). \end{aligned} \quad (21)$$

Proof. Upon multiplying the third equation of (3) by $(w + 1)^{\theta-1}$, integrating over Ω , and invoking the boundedness of $\|v\|_{L^\infty}$ and the condition $\eta \leq 1$, Young's inequality yields the following:

$$\begin{aligned} \frac{1}{\theta} \frac{d}{dt} \int_{\Omega} (w + 1)^\theta &= \int_{\Omega} (w + 1)^{\theta-1} (\nabla \cdot (D_3(w) \nabla w) - \nabla \cdot (\chi(w) \nabla v)) \\ &\quad + \int_{\Omega} (w + 1)^{\theta-1} (\tilde{c} w v - d_w w) \\ &\leq -\hat{D}(\theta - 1) \int_{\Omega} (w + 1)^{2\theta-2} |\nabla w|^2 + \hat{\chi}(\theta - 1) \int_{\Omega} (w + 1)^{\theta+\eta-2} |\nabla w| |\nabla v| \\ &\quad + \tilde{c} K_4 \int_{\Omega} (w + 1)^\theta \\ &\leq \frac{-\hat{D}(\theta - 1)}{2} \int_{\Omega} (w + 1)^{2\theta-2} |\nabla w|^2 + \frac{\hat{\chi}^2(\theta - 1)}{2\hat{D}} \int_{\Omega} (w + 1)^{2\eta-2} |\nabla v|^2 + \tilde{c} K_4 \int_{\Omega} (w + 1)^\theta \\ &\leq \frac{-\hat{D}(\theta - 1)}{2} \int_{\Omega} (w + 1)^{2\theta-2} |\nabla w|^2 + C_8 \int_{\Omega} |\nabla v|^2 + \tilde{c} K_4 \int_{\Omega} (w + 1)^\theta, \end{aligned} \quad (22)$$

where $C_8 := \frac{\hat{\chi}^2(\theta-1)}{2\hat{D}}$, for all $t \in (0, T_{max})$. By combining the Gagliardo-Nirenberg inequality, Young's inequality, and (12), we obtain the following:

$$\begin{aligned} \left(\frac{1}{\theta} + \tilde{c} K_4\right) \int_{\Omega} (w + 1)^\theta &= \left(\frac{1}{\theta} + \tilde{c} K_4\right) \|(w + 1)^\theta\|_{L^1} \\ &\leq C_9 \left(\|\nabla (w + 1)^\theta\|_{L^2}^{r_1} \|(w + 1)^\theta\|_{L^{\frac{1}{\theta}}}^{(1-r_1)} + \|(w + 1)^\theta\|_{L^{\frac{1}{\theta}}} \right) \\ &\leq C_{10} \|\nabla (w + 1)^\theta\|_{L^2}^{r_1} + C_{10} \\ &\leq \frac{\hat{D}(\theta - 1)}{4} \int_{\Omega} (w + 1)^{2\theta-2} |\nabla w|^2 + C_{11}, \end{aligned}$$

where $r_1 = \frac{\theta n - n}{1 - \frac{n}{2} + \theta n} \in (0, 1)$. Therefore, we have the following:

$$\frac{1}{\theta} \frac{d}{dt} \int_{\Omega} (w + 1)^\theta + \frac{1}{\theta} \int_{\Omega} (w + 1)^\theta + \frac{\hat{D}(\theta - 1)}{4} \int_{\Omega} (w + 1)^{2\theta-2} |\nabla w|^2$$

$$\leq C_8 \int_{\Omega} |\nabla v|^2 + C_{11}, \quad \text{for all } t \in (0, T_{max}). \quad (23)$$

Equation (21) follows from (17) and (23) via Grönwall's inequality. \square

Lemma 3.5. *Let the assumptions in Theorem 1.1 hold. If $p \geq 1$, then there exists a constant C_{23} independent of t such that*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \int_{\Omega} |\nabla v|^{2p} + \frac{D_2 p}{4} \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \\ & \leq C_{23} \int_{\Omega} w^2 |\nabla v|^{2p-2} + C_{23}, \quad \text{for all } t \in (0, T_{max}). \end{aligned} \quad (24)$$

Proof. By the second equation in (3) and an integration by parts, one can obtain the following:

$$\begin{aligned} & \frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} \\ & = \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla \{D_2 \Delta v + \tilde{\beta} uv - d_v v - \tilde{p} w v\} \\ & = \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla [D_2 \Delta v] + \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla [\tilde{\beta} uv - d_v v - \tilde{p} w v] \\ & = D_2 \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla \Delta v + \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla [\tilde{\beta} uv - d_v v - \tilde{p} w v] \\ & =: B_1 + B_2, \quad \text{for all } t \in (0, T_{max}). \end{aligned} \quad (25)$$

From the observation that $\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2$, it follows that

$$\begin{aligned} B_1 &= \frac{D_2}{2} \int_{\Omega} |\nabla v|^{2p-2} \Delta |\nabla v|^2 - D_2 \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \\ &= \frac{D_2}{2} \int_{\partial \Omega} |\nabla v|^{2p-2} \frac{\partial}{\partial \nu} |\nabla v|^2 dS - \frac{D_2(p-1)}{2} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 - D_2 \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2. \end{aligned} \quad (26)$$

To estimate B_1 , we introduce the trace inequality as follows [16]:

$$\int_{\partial \Omega} |\nabla f|^{2p-2} \cdot \partial_{\nu} |\nabla f|^2 dS \leq \varepsilon \int_{\Omega} |\nabla f|^{2p-2} \cdot |\nabla |\nabla f|^2|^2 + C_{\varepsilon} \int_{\Omega} |\nabla f|^{2p} \quad \text{for any } \varepsilon > 0. \quad (27)$$

Combining (26) and (27), we see the following:

$$B_1 \leq -\frac{D_2(p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 - D_2 \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + C_{16} \int_{\Omega} |\nabla v|^{2p}. \quad (28)$$

Similarly, using the fact $|\Delta v| \leq \sqrt{n}|D^2v|$, B_2 can be estimated as follows:

$$\begin{aligned}
 B_2 &= \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla (\tilde{\beta}uv - d_v v - \tilde{p}wv) \\
 &= - \int_{\Omega} \nabla \cdot (|\nabla v|^{2p-2} \nabla v) (\tilde{\beta}uv - d_v v - \tilde{p}wv) \\
 &\leq (\tilde{\beta}K_1 + d_v)K_4(p-1) \int_{\Omega} |\nabla v|^{2p-3} |\nabla |\nabla v|| + (\tilde{\beta}K_1 + d_v)K_4 \sqrt{n} \int_{\Omega} |\nabla v|^{2p-2} |D^2v| \\
 &\quad + K_4 \tilde{p}(p-1) \int_{\Omega} w |\nabla v|^{2p-3} |\nabla |\nabla v|| + K_4 \tilde{p} \sqrt{n} \int_{\Omega} w |\nabla v|^{2p-2} |D^2v|. \tag{29}
 \end{aligned}$$

Substituting (28) and (29) into (25), and using Young's inequality, we can deduce that

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \frac{D_2 p(p-1)}{2} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v||^2 + 2D_2 p \int_{\Omega} |\nabla v|^{2p-2} |D^2v|^2 \\
 &\leq 2(\tilde{\beta}K_1 + d_v)K_4 p(p-1) \int_{\Omega} |\nabla v|^{2p-3} |\nabla |\nabla v|| + 2p(\tilde{\beta}K_1 + d_v)K_4 \sqrt{n} \int_{\Omega} |\nabla v|^{2p-2} |D^2v| \\
 &\quad + 2K_4 \tilde{p} p(p-1) \int_{\Omega} w |\nabla v|^{2p-3} |\nabla |\nabla v|| + 2\tilde{p} p K_4 \sqrt{n} \int_{\Omega} w |\nabla v|^{2p-2} |D^2v| + 2pC_{16} \int_{\Omega} |\nabla v|^{2p} \\
 &\leq \frac{D_2 p(p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v||^2 + D_2 p \int_{\Omega} |\nabla v|^{2p-2} |D^2v|^2 \\
 &\quad + C_{17} \int_{\Omega} |\nabla v|^{2p} + C_{18} \int_{\Omega} |\nabla v|^{2p-2} w^2 + C_{19} \int_{\Omega} |\nabla v|^{2p-2} \\
 &\leq \frac{D_2 p(p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v||^2 + D_2 p \int_{\Omega} |\nabla v|^{2p-2} |D^2v|^2 \\
 &\quad + C_{20} \int_{\Omega} |\nabla v|^{2p} + C_{18} \int_{\Omega} |\nabla v|^{2p-2} w^2 + C_{21},
 \end{aligned}$$

which gives

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \frac{D_2 p(p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v||^2 + D_2 p \int_{\Omega} |\nabla v|^{2p-2} |D^2v|^2 \\
 &\leq C_{20} \int_{\Omega} |\nabla v|^{2p} + C_{18} \int_{\Omega} |\nabla v|^{2p-2} w^2 + C_{21}. \tag{30}
 \end{aligned}$$

On the other hand, we can check that

$$\begin{aligned}
& (2 + C_{20}) \int_{\Omega} |\nabla v|^{2p} \\
&= (2 + C_{20}) \int_{\Omega} |\nabla v|^{2p-2} \nabla v \cdot \nabla v \\
&= -(2 + C_{20})(p-1) \int_{\Omega} v |\nabla v|^{2p-4} \nabla v \cdot \nabla |\nabla v|^2 - (2 + C_{20}) \int_{\Omega} v |\nabla v|^{2p-2} \Delta v \\
&\leq \frac{D_2 p (p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 + \frac{K_4^2 (2 + C_{20})^2 (p-1)}{D_2 p} \int_{\Omega} |\nabla v|^{2p-2} \\
&\quad + \frac{3D_2 p}{4} \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + \frac{nK_4^2 (2 + C_{20})^2}{3D_2 p} \int_{\Omega} |\nabla v|^{2p-2} \\
&\leq \frac{D_2 p (p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 + \frac{3D_2 p}{4} \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + \int_{\Omega} |\nabla v|^{2p} + C_{22},
\end{aligned}$$

hence,

$$(1 + C_{20}) \int_{\Omega} |\nabla v|^{2p} \leq \frac{D_2 p (p-1)}{4} \int_{\Omega} |\nabla v|^{2p-4} |\nabla |\nabla v|^2|^2 + \frac{3D_2 p}{4} \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + C_{22}. \quad (31)$$

Therefore, due to (30) and (31), we can obtain the following:

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \int_{\Omega} |\nabla v|^{2p} + \frac{D_2 p}{4} \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 \\
&\leq C_{23} \int_{\Omega} w^2 |\nabla v|^{2p-2} + C_{23}, \quad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

□

Lemma 3.6. *Under the assumptions of Theorem 1.1, there exists a constant K_7 independent of t such that*

$$\|\nabla v(\cdot, t)\|_{L^2} \leq K_7, \quad \text{for all } t \in (0, T_{max}).$$

Proof. For $q = 2\theta + \frac{2}{n}$, the Gagliardo-Nirenberg inequality together with (21) implies the following:

$$\begin{aligned}
\int_{\Omega} (w+1)^q &= \|(w+1)^\theta\|_{L^{\frac{q}{\theta}}}^q \\
&\leq \left(\|\nabla (w+1)^\theta\|_{L^2}^{r_3} \|(w+1)^\theta\|_{L^{\frac{1}{\theta}}}^{1-r_3} + \|(w+1)^\theta\|_{L^{\frac{1}{\theta}}} \right)^{\frac{q}{\theta}} \\
&\leq C_{24} \|\nabla (w+1)^\theta\|_{L^2}^2 + C_{25},
\end{aligned} \quad (32)$$

where $r_3 = \frac{\theta - \frac{\theta}{q}}{\theta + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$. It follows from (21) that

$$\int_t^{t+\tau} \int_{\Omega} (w+1)^{2\theta + \frac{2}{n}} \leq C_{26}, \quad \text{for all } t \in (0, \tilde{T}_{max}). \quad (33)$$

For the case $p = 1$ in (24), we find the following:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 + \frac{D_2}{4} \int_{\Omega} |D^2 v|^2 \\ \leq C_{23} \int_{\Omega} w^2 + C_{23}, \quad \text{for all } t \in (0, T_{max}). \end{aligned} \quad (34)$$

Upon applying (33) and Lemma 2.1 to (34), this establishes Lemma 3.6. \square

Lemma 3.7. *Let the assumptions in Theorem 1.1 hold. Then, there exists a constant K_8 independent of t such that*

$$\|w(\cdot, t)\|_{L^q} + \|\nabla v(\cdot, t)\|_{L^{2p}} \leq K_8, \quad \text{for all } t \in (0, T_{max}), \quad (35)$$

for any $p \geq 2$ and $q > 1$.

Proof. To begin, we multiply the third equation of (3) by $(w + 1)^{q-1}$ (with $q > 2$), integrate the result over Ω , and invoke the boundedness of $\|v\|_{L^\infty}$. This preparation enables the use of Young's inequality, which leads to the following:

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \int_{\Omega} (w + 1)^q &= \int_{\Omega} (w + 1)^{q-1} \nabla \cdot (D_3(w) \nabla w) - \int_{\Omega} (w + 1)^{q-1} \nabla \cdot (\chi(w) \nabla v) \\ &\quad + \int_{\Omega} (w + 1)^{q-1} \tilde{c}_v w - d_w \int_{\Omega} (w + 1)^q w \\ &\leq -\hat{D}(q-1) \int_{\Omega} (w + 1)^{q-2+\theta} |\nabla w|^2 + \hat{\chi}(q-1) \int_{\Omega} (w + 1)^{q-2+\eta} |\nabla w| |\nabla v| \\ &\quad + \tilde{c} K_4 \int_{\Omega} (w + 1)^q \\ &\leq -\frac{\hat{D}(q-1)}{2} \int_{\Omega} (w + 1)^{q-2+\theta} |\nabla w|^2 + \frac{\hat{\chi}^2(q-1)}{2\hat{D}} \int_{\Omega} (w + 1)^{q-2-\theta+2\eta} |\nabla v|^2 \\ &\quad + \tilde{c} K_4 \int_{\Omega} (w + 1)^q, \quad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (w + 1)^q + \int_{\Omega} (w + 1)^q \\ \leq -\frac{\hat{D}q(q-1)}{2} \int_{\Omega} (w + 1)^{q-2+\theta} |\nabla w|^2 + \frac{\hat{\chi}^2q(q-1)}{2\hat{D}} \int_{\Omega} (w + 1)^{q-2-\theta+2\eta} |\nabla v|^2 \\ + (\tilde{c}qK_4 + 1) \int_{\Omega} (w + 1)^q, \quad \text{for all } t \in (0, T_{max}). \end{aligned} \quad (36)$$

Hölder's inequality applied to (36), together with the condition $r_4 > 1$, gives the following:

$$\frac{\hat{\chi}^2q(q-1)}{2\hat{D}} \int_{\Omega} (w + 1)^{q-2-\theta+2\eta} |\nabla v|^2 \leq \frac{\hat{\chi}^2q(q-1)}{2\hat{D}} \left(\int_{\Omega} (w + 1)^{(q-2-\theta+2\eta)r_4} \right)^{\frac{1}{r_4}} \left(\int_{\Omega} |\nabla v|^{2r_4'} \right)^{\frac{1}{r_4'}}, \quad (37)$$

where $r'_4 = \frac{r_4}{r_4-1}$. By the definition of $r_4 > 1$, we find $(n-2)r_4 < n$. With the facts $\eta \leq 1$, $\theta > 1$, we know

$$\frac{n-2}{n} \left(1 + \frac{2(\eta - \theta - 1)}{q + \theta} \right) < \frac{1}{r_4}. \quad (38)$$

Let $q > \frac{1}{r_4} + \theta + 2 - 2\eta$. With this choice, applying the Gagliardo-Nirenberg inequality to (38) gives the following:

$$\begin{aligned} & \frac{\hat{\chi}^2 q(q-1)}{2\hat{D}} \left(\int_{\Omega} (w+1)^{(q-2-\theta+2\eta)r_4} \right)^{\frac{1}{r_4}} \\ &= \frac{\hat{\chi}^2 q(q-1)}{2\hat{D}} \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2r_4(q-2-\theta+2\eta)}{q+\theta}}}^{\frac{2(q-2-\theta+2\eta)}{q+\theta}} \\ &\leq C_{27} \left(\|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{r_5} \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2}{q+\theta}}}^{1-r_5} + \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2}{q+\theta}}} \right)^{\frac{2(q-2-\theta+2\eta)}{q+\theta}} \\ &\leq C_{28} \|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{\frac{2r_5(q-2-\theta+2\eta)}{q+\theta}} + C_{28}, \end{aligned} \quad (39)$$

where $r_5 = \frac{\frac{q+\theta}{2} - \frac{q+\theta}{2r_4(q-2-\theta+2\eta)}}{\frac{q+\theta}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$. For $p > \frac{(n-2)r'_4}{n}$, by Lemma 3.6, we invoke Gagliardo-Nirenberg's inequality to estimate the following:

$$\begin{aligned} \left(\int_{\Omega} |\nabla v|^{2r'_4} \right)^{\frac{1}{r'_4}} &= \|\nabla v\|_{L^{\frac{2r'_4}{p}}}^{\frac{2}{p}} \\ &\leq C_{29} \left(\|\nabla|\nabla v|^p\|_{L^2}^{r_6} \|\nabla v\|_{L^{\frac{2}{p}}}^{1-r_6} + \|\nabla v\|_{L^{\frac{2}{p}}} \right)^{\frac{2}{p}} \\ &\leq C_{30} \|\nabla|\nabla v|^p\|_{L^2}^{\frac{2r_6}{p}} + C_{30}, \end{aligned} \quad (40)$$

where $r_6 = \frac{\frac{p}{2} - \frac{p}{2r'_4}}{\frac{p}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$. Then, by substituting (39) and (40) into (37), we have the following:

$$\begin{aligned} & \frac{\hat{\chi}^2 q(q-1)}{2\hat{D}} \int_{\Omega} (w+1)^{q-2-\theta+2\eta} |\nabla v|^2 \\ &\leq \left(C_{28} \|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{\frac{2r_5(q-2-\theta+2\eta)}{q+\theta}} + C_{28} \right) \left(C_{30} \|\nabla|\nabla v|^p\|_{L^2}^{\frac{2r_6}{p}} + C_{30} \right), \text{ for all } t \in (0, T_{max}). \end{aligned} \quad (41)$$

Next, we claim that one can find $r_7 > 1$ such that $r_7 > \frac{n}{2}$. With the facts $\eta \leq 1$, $\theta > 1$, we know

$$\frac{1}{r_7} < \frac{2}{n} + \frac{n-2}{n} \cdot \frac{1}{p}. \quad (42)$$

An application of Hölder's inequality to (24) yields the following:

$$C_{23} \int_{\Omega} |\nabla v|^{2p-2} (w+1)^2 \leq C_{23} \left(\int_{\Omega} (w+1)^{2r_7} \right)^{\frac{1}{r_7}} \left(\int_{\Omega} |\nabla v|^{(2p-2)r'_7} \right)^{\frac{1}{r'_7}}, \quad (43)$$

where $r'_7 = \frac{r_7}{r_7-1}$.

Let $q > \max\{\frac{1}{r_4} + \theta + 2 - 2\eta, 2r_7 - \frac{4r_7}{n} - \theta\}$. It follows from the Gagliardo-Nirenberg inequality that

$$\begin{aligned} & C_{23} \left(\int_{\Omega} (w+1)^{2r_7} \right)^{\frac{1}{r_7}} \\ &= C_{23} \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{4r_7}{q+\theta}}}^{\frac{4}{q+\theta}} \\ &\leq C_{23} \left(\|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{r_8} \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2}{q+\theta}}}^{1-r_8} + \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2}{q+\theta}}} \right)^{\frac{4}{q+\theta}} \\ &\leq C_{31} \|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{\frac{4r_8}{q+\theta}} + C_{31}, \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \quad (44)$$

where $r_8 = \frac{\frac{q+\theta}{2} - \frac{q+\theta}{4r_7}}{\frac{q+\theta}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$. By (42), we invoke Gagliardo-Nirenberg's inequality to estimate the following:

$$\begin{aligned} & \left(\int_{\Omega} |\nabla v|^{(2p-2)r'_7} \right)^{\frac{1}{r'_7}} = \|\nabla v\|_{L^{\frac{(2p-2)r'_7}{p}}}^{\frac{2(p-1)}{p}} \\ &\leq C_{32} \left(\|\nabla|\nabla v|^p\|_{L^2}^{r_9} \|\nabla v\|_{L^{\frac{2}{p}}}^{1-r_9} + \|\nabla v\|_{L^{\frac{2}{p}}} \right)^{\frac{2(p-1)}{p}} \\ &\leq C_{33} \|\nabla|\nabla v|^p\|_{L^2}^{\frac{2r_9(p-1)}{p}} + C_{33}, \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \quad (45)$$

where $r_9 = \frac{\frac{p}{2} - \frac{p}{2(p-1)r'_7}}{\frac{p}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$. We substitute (44) and (45) into (43) to obtain the following:

$$\begin{aligned} & C_{23} \int_{\Omega} |\nabla v|^{2p-2} (w+1)^2 \\ &\leq \left(C_{31} \|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{\frac{4r_8}{q+\theta}} + C_{31} \right) \left(C_{33} \|\nabla|\nabla v|^p\|_{L^2}^{\frac{2r_9(p-1)}{p}} + C_{33} \right), \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (46)$$

Now, we prove the inequalities $\frac{r_5(q-2-\theta+2\eta)}{q+\theta} + \frac{r_6}{p} < 1$ and $\frac{2r_8}{q+\theta} + \frac{r_9(p-1)}{p} < 1$. Given $\theta > 1$ and $\eta \leq 1$, take $p > q > \frac{2r_8}{1-r_9} - \theta$ such that

$$\eta - \theta < \frac{1}{n} + \frac{1}{2} + \frac{1}{r_4} \left(\frac{\frac{p}{2} - \frac{q+\theta}{2}}{p + \frac{2}{n} - 1} \right). \quad (47)$$

In fact, (47) is equivalent to the first inequality. The second inequality, $\frac{2r_8}{q+\theta} + \frac{r_9(p-1)}{p} < 1$, directly follows; taking $p > q > \frac{2r_8}{1-r_9} - \theta$ suffices under the same conditions. Hence, applying Young's inequality to (41) and (46) leads to the following:

$$\frac{\hat{\chi}^2 q(q-1)}{2\hat{D}} \int_{\Omega} (w+1)^{q-2-\theta+2\eta} |\nabla v|^2 + C_{23} \int_{\Omega} |\nabla v|^{2p-2} (w+1)^2$$

$$\leq \frac{D_2 p}{4} \int_{\Omega} |\nabla v|^{2p-2} |D^2 v|^2 + \frac{\hat{D}q(q-1)}{4} \int_{\Omega} (w+1)^{q-2+\theta} |\nabla w|^2 + C_{34}, \quad \text{for all } t \in (0, T_{\max}). \quad (48)$$

Conversely, another application of the Gagliardo-Nirenberg inequality provides the estimate for the following:

$$\begin{aligned} & (\tilde{c}qK_4 + 1) \int_{\Omega} (w+1)^q = \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2q}{q+\theta}}}^{\frac{2q}{q+\theta}} \\ & \leq C_{35} \|\nabla(w+1)^{\frac{q+\theta}{2}}\|_{L^2}^{\frac{2qr_{10}}{q+\theta}} \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2}{q+\theta}}}^{\frac{2q(1-r_{10})}{q+\theta}} + C_{35} \|(w+1)^{\frac{q+\theta}{2}}\|_{L^{\frac{2}{q+\theta}}}^{\frac{2q}{q+\theta}} \\ & \leq \frac{\hat{D}q(q-1)}{4} \int_{\Omega} (w+1)^{q-2+\theta} |\nabla w|^2 + C_{36}, \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \quad (49)$$

where $r_{10} = \frac{\frac{q+\theta}{2} - \frac{q+\theta}{2q}}{\frac{q+\theta}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$.

Synthesizing (24), (36), (48), and (49) leads to the following:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \frac{d}{dt} \int_{\Omega} (w+1)^q + \int_{\Omega} |\nabla v|^{2p} + \int_{\Omega} (w+1)^q \\ & \leq C_{37}, \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (50)$$

The proof of Lemma 3.7 is completed by an application of Lemma 2.1.

Based on a modified Moser iteration as in Lemma A.1 of [7], we can obtain $\|w\|_{L^\infty} \leq K_9$ for all $t \in (0, T_{\max})$. \square

Proof of Theorem 1.1. The combination of Lemmas 2.5, 3.2 and 3.7 yields a constant $C > 0$ independent of t such that

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,p}} + \|w(\cdot, t)\|_{L^\infty} \leq C, \quad (51)$$

which, together with the extension criterion in Lemma 2.6, proves Theorem 1.1.

4. Conclusion and discussion

Under the conditions $\theta > 1$ and $\eta \leq 1$, this paper proves the global existence and uniform boundedness of classical solutions to the HIV chemotaxis model (3). It shows that a strong nonlinear diffusion of CTLs ($\theta > 1$) can effectively suppress instabilities induced by chemorepulsion, thereby ensuring the long-term controllability of the immune response. Compared with the May-Nowak model studied by Bellomo et al. [20], the present model incorporates three cell types and nonlinear diffusion, thus complementing the conclusion that “taxis can cause instability in the absence of diffusion” and revealing the compensatory role of density-dependent diffusion in mitigating chemotaxis-driven instability. Our results extend the previous work of Lai and Zou [4,5] (which treated the linear diffusion case) to a more biologically realistic nonlinear setting, and provide new analytical techniques to handle coupled parabolic systems with density-dependent diffusion and cross-diffusion. However, the model

does not consider logistic cell proliferation or free virus, the initial regularity requirement is relatively strong, and steady-state patterns or critical blow-up cases are not discussed. Future work may extend the analysis to weaker initial data, investigate the critical exponents $\theta = 1$ or $\eta > 1$ for possible blow-up, and combine numerical simulations to validate the theoretical results.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflicts of interest.

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