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Research article

Spectra of quasi-corona and multiple Q-complemented-vertex join graphs

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Abstract: The Q-complemented graph of a graph G, denoted by $CT(G)$, was obtained by adding a new vertex for each edge $uv \in E(G)$, where this new vertex was adjacent to all vertices of G except u and v. In this paper, we introduced two new graph constructions based on $CT(G)$, namely, the quasi-corona Q-complemented-vertex join $G_1 \odot G_2$ and the multiple Q-complemented-vertex join $G_1 \triangledown G_2$. We determined the adjacency, Laplacian, and signless Laplacian spectra of these composite graphs. Using these spectral results, we constructed infinite families of A-, L-, and Q-cospectral graphs. Moreover, we derived explicit formulas for several key invariants of the new graphs, including the number of spanning trees, the Kirchhoff index, the signless Laplacian energy-like invariant, and graph energy. Furthermore, numerical experiments were provided to illustrate the structural properties of the proposed graphs, suggesting that the multiple Q-complemented-vertex join construction may exhibit certain small-world-like characteristics.

Keywords: Laplacian spectrum; cospectral graph; Kirchhoff index; spanning tree; graph energy
Mathematics Subject Classification: 05C50, 05C12, 15A18

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with n vertices and m edges, where the vertex set is $V(G) = \{v_1, v_2, \ldots, v_n\}$, and the edge set is denoted by $E(G)$. The adjacency matrix of G, denoted by $A(G)$ or simply A, is the $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent, and 0 otherwise. Let $D(G) = \mathrm{diag}\,(d_1, d_2, \ldots, d_n)$, where d_i is the degree of vertex v_i. The Laplacian matrix and signless

Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, or simply L and Q, respectively. Let I_n be the identity matrix of order n.

For an $n \times n$ matrix M associated with a graph G, let $\Phi_G(M, x) = \det(xI_n - M)$ denote its characteristic polynomial. In particular, $\Phi_G(A, x), \Phi_G(L, x)$, and $\Phi_G(Q, x)$ are called the adjacency, Laplacian, and signless Laplacian characteristic polynomials of G, respectively. Let $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$, $\nu_1(G) \geq \nu_2(G) \geq \ldots \geq \nu_n(G)$ be the adjacency, Laplacian, and signless Laplacian eigenvalues of G, respectively. The multiset of eigenvalues of M (counting multiplicities) is called the M-spectrum of G. Two graphs are M-cospectral if they share the same M-spectrum. This terminology applies to the A-, L-, and Q-spectra [2, 6].

Computing characteristic polynomials and spectra is a fundamental task in spectral graph theory. Many graphs have been studied in the literature [31]. Known spectra enable the construction of infinite families of A-, L-, and Q-cospectral graphs [8, 14]. Beyond cospectrality, graph eigenvalues have numerous applications. For example, the number of spanning trees of a graph G [6] is given by

$$t(G) = \frac{\prod_{i=2}^{n} \mu_i(G)}{n}.$$

For a connected graph G, the Kirchhoff index [17] is defined as the sum of resistance distances between all pairs of vertices, which is given by

$$Kf(G) = n \sum_{i=2}^{n} \frac{1}{\mu_i(G)}.$$

Another invariant based on the signless Laplacian spectrum is the signless Laplacian energy-like invariant (SLEL) [5], which is defined as

$$SLEL(G) = \sum_{i=1}^{n} \sqrt{\nu_i(G)}.$$

In addition to applications of Laplacian matrix, the adjacency spectrum also plays an important role in network analysis. In particular, adjacency spectral methods based on main eigenvalue and its corresponding eigenvector (principal eigenvector) can be applied to graph partitioning and community detection [10–12, 22]. Another important invariant derived from the adjacency spectrum is the graph energy [6], defined by

$$GE(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

Further studies on spectral-based graph invariant can be found in [4, 20, 23, 34].

Recently, significant progress has been made on the spectra of graph operations such as complemen [1], union [1], join [28, 29], corona [15, 27], and various graph products [18, 24]. For instance, Barik et al. studied the Laplacian spectrum of the corona of two graphs [33], Indulal determined the spectra of subdivision-vertex and subdivision-edge joins of two graphs [16], Dai et al. derived the spectra for weighted corona graphs [7], Lan and Zhou determined the spectra of operations based on R-graphs [24], and Das and Panigrahi analyzed the spectra of R-vertex and R-edge join graphs [9]. Additional related work appears in [13]. Rajkumar and Gayathri introduced the Q-complemented graph, denoted by $CT(G)$ [32]. It is obtained by adding a new vertex for each edge

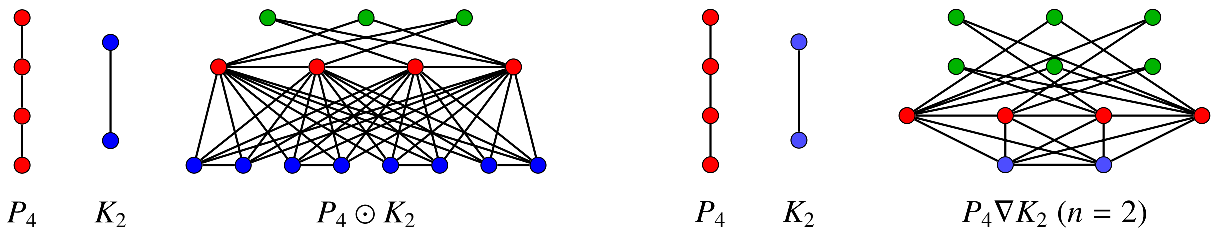
$uv \in E(G)$, where the new vertex is adjacent to all vertices of G except u and v [21]. The original edges of G are retained, and no edges are added between the new vertices. We generalize this concept to the multiple Q-complemented graph, denoted by $CT^n(G)$, where n is a positive integer, and it is obtained by inserting n such new vertices for every edge of G. When $n = 1$, $CT^1(G) = CT(G)$. Building on this, we define two new graph operations.

Let G_1 and G_2 be graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively.

Definition 1.1. *The quasi-corona Q-complemented-vertex join of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph formed from one copy of $CT(G_1)$ and n_1 copies of G_2 by connecting every vertex of G_1 to all vertices of each copy of G_2.*

Definition 1.2. *The multiple Q-complemented-vertex join of G_1 and G_2, denoted by $G_1 \triangledown G_2$, is a graph constructed from $CT^n(G_1)$ (where n is a positive integer) and G_2 by choosing one copy of $CT^n(G_1)$ and one copy of G_2, and then connecting each vertex of G_1 to every vertex of G_2.*

The graph $G_1 \odot G_2$ has $n_1n_2 + n_1 + m_1$ vertices and $m_1(n_1 - 1) + n_1m_2 + n_1^2n_2$ edges. The graph $G_1 \nabla G_2$ has $nm_1 + n_1 + n_2$ vertices and $m_1 + n(n_1 - 2)m_1 + m_2 + n_1n_2$ edges. Examples for $G_1 = P_4$ and $G_2 = K_2$ are illustrated in Figure 1(a) ($P_4 \odot K_2$) and Figure 1(b) ($P_4 \nabla K_2$ with $n = 2$).



(a) Construction method of the $P_4 \odot K_2$ **(b)** Construction method of the $P_4 \nabla K_2$ $(n = 2)$

Figure 1. Two constructions of composite graphs.

The above two graph operations may be viewed as deterministic complex network models with different structural roles. The graph $G_1 \odot G_2$ is essentially a coronal product type deterministic network obtained by attaching copies of G_2 to the base structure associated with G_1. This construction produces a natural modular organization and hence reflects certain community properties [25, 26]. In contrast, $G_1 \triangledown G_2$, yields a more strongly interconnected structure through iterative Q-complemented transformation together with additional join operations. Although deterministic rather than random, it may exhibit some typical features of small world networks, such as enhanced local connectivity and relatively short paths between different parts of the graph [3]. Therefore, these two operations provide useful models for deterministic modular and partially small-world-like networks.

The M-Coronal [30] of $n \times n$ matrix M is defined as

$$\Gamma_M(x) = J_{1\times n}(xI_n - M)^{-1}J_{n\times 1}, \tag{1.1}$$

where $J_{n\times 1}$ is the all-ones column vector of size n. If every row sum of M equals a constant c, then $\Gamma_M(x) = \frac{n}{x-c}$. In particular, for an r-regular graph G,

$$\Gamma_{A(G)}(x) = \frac{n}{x-r}, \quad \Gamma_{L(G)}(x) = \frac{n}{x}, \quad \Gamma_{Q(G)}(x) = \frac{n}{x-2r}.$$

We recall two lemmas as follows.

Lemma 1.1. *[28] If M is a real matrix of $n \times n$, then*

$$\det\left(xI_n - M - \alpha J_{n\times n}\right) = \left(1 - \alpha\Gamma_M(x)\right)\det\left(xI_n - M\right). \tag{1.2}$$

Lemma 1.2. *[36] Let M_1, M_2, M_3, and M_4 be four matrices, where M_1 and M_4 are non-singular square matrices, then*

$$\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3M_1^{-1}M_2) = \det(M_4)\det(M_1 - M_2M_4^{-1}M_3). \tag{1.3}$$

The Kronecker product of the matrix $M = \left(m_{ij}\right)$ of order $p \times q$ and the matrix B of order $r \times s$, denoted by $M \otimes B$, is the $pr \times qs$ matrix obtained by replacing each entry m_{ij} with $m_{ij}B$ [19]. For any four matrices with compatible dimensions,

$$\left(M_1 \otimes M_2\right)\left(M_3 \otimes M_4\right) = \left(M_1M_3\right) \otimes \left(M_2M_4\right).$$

If M_1 and M_2 are invertible, $\left(M_1 \otimes M_2\right)^{-1} = M_1^{-1} \otimes M_2^{-1}$. Moreover, if M_1 and M_2 are square matrices with order p and r, respectively, then we have $\det\left(M_1 \otimes M_2\right) = \left(\det M_1\right)^r \left(\det M_2\right)^p$.

Let $R = \left(r_{ij}\right)$ be the $n \times m$ incidence matrix of G, where $r_{ij} = 1$ if vertex v_i is incident with edge e_j, and 0 otherwise. Then,

$$RR^{\top} = A(G) + D(G) = Q(G).$$

The organization of this paper is as follows, Section 2 focuses on the spectral properties of the quasi-corona Q-complemented-vertex join graph $G_1 \odot G_2$, including adjacency, Laplacian, and signless Laplacian spectra. Section 3 presents the spectral analysis of the multiple Q-complemented-vertex join graph $G_1 \triangledown G_2$. Section 4 discusses the applications of the spectral results in counting the number of spanning trees, the Kirchhoff index, the signless Laplacian energy-like invariant, and graph energy. Section 5 provides a numerical illustration to investigate the structural properties of the proposed graphs and to demonstrate their potential small-world-like characteristics. Finally, the conclusions are summarized in Section 6.

2. Spectra of quasi-corona Q-complemented-vertex join $G_1 \odot G_2$

In this section, we determine the adjacency, Laplacian, and signless Laplacian spectra of $G_1 \odot G_2$. Let G_1 and G_2 be graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Let R_1 be the incidence matrix of G_1.

Theorem 2.1. *Let G_1 be an r_1-regular graph and G_2 be an any graph. Then,*

$$\Phi_{G_1\odot G_2}(A, x) = x^{m_1-n_1}\prod_{i=1}^{n_2}\left(x - \lambda_i(G_2)\right)^{n_1}\left(x^2 - r_1x - 2r_1 - n_1\left(x\Gamma_{A(G_2)\otimes I_{n_1}}(x) + m_1 - 2r_1\right)\right)$$
$$\cdot\prod_{i=2}^{n_1}\left(x^2 - r_1 - (x+1)\,\lambda_i(G_1)\right).$$

Proof. By assigning appropriate vertex labels, the adjacency matrix of $G_1 \odot G_2$ can be expressed as

$$A(G_1 \odot G_2) = \begin{pmatrix} A(G_1) & J_{n_1\times m_1} - R_1 & J_{n_1\times n_2} \otimes J_{1\times n_1} \\ J_{m_1\times n_1} - R_1^\top & 0_{m_1\times m_1} & 0_{m_1\times n_2} \otimes J_{1\times n_1} \\ J_{n_2\times n_1} \otimes J_{n_1\times 1} & 0_{n_2\times m_1} \otimes J_{n_1\times 1} & A(G_2) \otimes I_{n_1} \end{pmatrix}.$$

The adjacency characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned} \Phi_{G_1\odot G_2}(A, x) &= \det\left(xI_{n_1n_2+m_1+n_1} - A(G_1 \odot G_2)\right) \\ &= \det\begin{pmatrix} xI_{n_1} - A(G_1) & -J_{n_1\times m_1} + R_1 & -J_{n_1\times n_2} \otimes J_{1\times n_1} \\ -J_{m_1\times n_1} + R_1^\top & xI_{m_1} & 0_{m_1\times n_2} \otimes J_{1\times n_1} \\ -J_{n_2\times n_1} \otimes J_{n_1\times 1} & 0_{n_2\times m_1} \otimes J_{n_1\times 1} & (xI_{n_2} - A(G_2)) \otimes I_{n_1} \end{pmatrix} \\ &= \det\left(xI_{n_2} - A(G_2)\right)^{n_1} \det(S), \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -J_{n_1\times m_1} + R_1 \\ -J_{m_1\times n_1} + R_1^\top & xI_{m_1} \end{pmatrix} \\ &\quad - \begin{pmatrix} -J_{n_1\times n_2} \otimes J_{1\times n_1} \\ 0_{m_1\times n_2} \otimes J_{1\times n_1} \end{pmatrix} \left((xI_{n_2} - A(G_2))^{-1} \otimes I_{n_1}\right) \begin{pmatrix} -J_{n_2\times n_1} \otimes J_{n_1\times 1} & 0_{n_2\times m_1} \otimes J_{n_1\times 1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2)\otimes I_{n_1}}(x) J_{n_1\times n_1} & -J_{n_1\times m_1} + R_1 \\ -J_{m_1\times n_1} + R_1^\top & xI_{m_1} \end{pmatrix}. \end{aligned}$$

Since G_1 is an r_1-regular graph, then $R_1R_1^\top = A(G_1) + r_1 I_{n_1}$. So, we can obtain

$$(R_1 - J_{n_1\times m_1})(R_1^\top - J_{m_1\times n_1}) = (m_1 - 2r_1)J_{n_1\times n_1} + r_1 I_{n_1} + A(G_1).$$

By Lemmas 1.3 and 1.4 and $\Gamma_{(x+1)A(G_1)}(x^2 - r_1) = \frac{n_1}{x^2 - r_1 - (x+1)r_1}$, we have

$$\begin{aligned} \det(S) =& x^{m_1} \det\left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)\otimes I_{n_1}}(x) J_{n_1\times n_1} - \frac{1}{x}(R_1 - J_{n_1\times m_1})(R_1^\top - J_{m_1\times n_1})\right) \\ =& x^{m_1} \det\left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)\otimes I_{n_1}}(x) J_{n_1\times n_1} - \frac{(m_1 - 2r_1)J_{n_1\times n_1} + r_1 I_{n_1} + A(G_1)}{x}\right) \\ =& x^{m_1-n_1} \det\left(\left(x^2 - r_1\right) I_{n_1} - (x+1)A(G_1) - \left(x\Gamma_{A(G_2)\otimes I_{n_1}}(x) + m_1 - 2r_1\right) J_{n_1\times n_1}\right) \\ =& x^{m_1-n_1} \left(1 - \left(x\Gamma_{A(G_2)\otimes I_{n_1}}(x) + m_1 - 2r_1\right)\Gamma_{(x+1)A(G_1)}(x^2 - r_1)\right) \det\left(\left(x^2 - r_1\right) I_{n_1} - (x+1)A(G_1)\right) \\ =& x^{m_1-n_1} \left(1 - \left(x\Gamma_{A(G_2)\otimes I_{n_1}}(x) + m_1 - 2r_1\right) \frac{n_1}{x^2 - r_1 - (x+1)r_1}\right) \det\left(\left(x^2 - r_1\right) I_{n_1} - (x+1)A(G_1)\right). \end{aligned}$$

Thus,

$$\begin{aligned} \Phi_{G_1\odot G_2}(A, x) = x^{m_1-n_1} \prod_{i=1}^{n_2} (x - \lambda_i(G_2))^{n_1} \left(x^2 - r_1 x - 2r_1 - n_1\left(x\Gamma_{A(G_2)\otimes I_{n_1}}(x) + m_1 - 2r_1\right)\right) \\ \cdot \prod_{i=2}^{n_1} \left(x^2 - r_1 - (x+1)\lambda_i(G_1)\right). \end{aligned}$$

□

Corollary 2.1. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then,*

$$\Phi_{G_1 \odot G_2}(A, x) = x^{m_1 - n_1} \prod_{i=2}^{n_2} (x - \lambda_i(G_2))^{n_1} \prod_{i=2}^{n_1} \left(x^2 - r_1 - (x+1)\,\lambda_i(G_1)\right)(x - r_2)^{n_1 - 1}\, t,$$

where $t = x^3 - (r_1 + r_2)x^2 + (r_1 r_2 + 2n_1 r_1 - 2r_1 - m_1 n_1 - n_1^2 n_2)x + r_2\,(2r_1 + n_1 m_1 - 2n_1 r_1)$.

Proof. For (1.1), we have $\Gamma_{A(G_2)\otimes I_{n_1}}(x) = \frac{n_1 n_2}{x - r_2}$. From Theorem 2.1, we have

$$\begin{aligned}\Phi_{G_1 \odot G_2}(A, x) = x^{m_1 - n_1} &\prod_{i=2}^{n_2} (x - \lambda_i(G_2))^{n_1} \prod_{i=2}^{n_1} \left(x^2 - r_1 - (x+1)\,\lambda_i(G_1)\right)(x - r_2)^{n_1 - 1} \\ &\cdot \left(x^3 - (r_1 + r_2)x^2 + (r_1 r_2 + 2n_1 r_1 - 2r_1 - m_1 n_1 - n_1^2 n_2)x + r_2\,(2r_1 + n_1 m_1 - 2n_1 r_1)\right).\end{aligned}$$

□

Corollary 2.2. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then, the A-spectrum of $G_1 \odot G_2$ consists of:*

(1) 0*, repeated* $m_1 - n_1$ *times;*
(2) $\frac{\lambda_i(G_1) \pm \sqrt{\lambda_i^2(G_1) + 4(r_1 + \lambda_i(G_1))}}{2}$*, for* $i = 2, 3, \ldots, n_1$*;*
(3) $\lambda_i(G_2)$*, for* $i = 2, 3, \ldots, n_2$*, repeated* n_1 *times;*
(4) r_2*, repeated* $n_1 - 1$ *times;*
(5) the roots of the cubic equation
$x^3 - (r_1 + r_2)x^2 + (r_1 r_2 - 2r_1 - n_1^2 n_2 - n_1 m_1 + 2r_1 n_1)x + 2r_1 r_2 + n_1 m_1 r_2 - 2n_1 r_1 r_2$.

Next, we determine the Laplacian polynomial of $G_1 \odot G_2$.

Theorem 2.2. *Let G_1 be an r_1-regular graph and G_2 be an any graph. Then,*

$$\begin{aligned}\Phi_{G_1 \odot G_2}(L, x) = &\prod_{i=2}^{n_2} (x - n_1 - \mu_i(G_2))^{n_1} (x - n_1 + 2)^{m_1 - n_1}\, x \left(x^2 + tx + s + n_1^2 - 2n_1\right) \\ &\cdot \prod_{i=2}^{n_1} \left(x^2 + tx + (s - r_1 n_1) - (x - n_1 + 1)\,\mu_i(G_1)\right),\end{aligned}$$

where $t = -m_1 - n_1 n_2 - 2n_1 + r_1 + 2$ *and* $s = n_1^2 n_2 + m_1 n_1 - 2n_1 n_2 - 2m_1$.

Proof. With proper vertex labeling, the Laplacian matrix of $G_1 \odot G_2$ is

$$L(G_1 \odot G_2) = \begin{pmatrix} (m_1 + n_1 n_2 - r_1)I_{n_1} + L(G_1) & R_1 - J_{n_1 \times m_1} & -J_{n_1 \times n_2} \otimes J_{1 \times n_1} \\ R_1^\top - J_{m_1 \times n_1} & (n_1 - 2)I_{m_1} & 0_{m_1 \times n_2} \otimes J_{1 \times n_1} \\ -J_{n_2 \times n_1} \otimes J_{n_1 \times 1} & 0_{n_2 \times m_1} \otimes J_{n_1 \times 1} & (L(G_2) + n_1 I_{n_2}) \otimes I_{n_1} \end{pmatrix}.$$

The Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned}\Phi_{G_1 \odot G_2}(L, x) &= \det\left(x I_{n_1 n_2 + m_1 + n_1} - L(G_1 \odot G_2)\right) \\ &= \det \begin{pmatrix} (x - m_1 - n_1 n_2 + r_1)I_{n_1} - L(G_1) & -R_1 + J_{n_1 \times m_1} & J_{n_1 \times n_2} \otimes J_{1 \times n_1} \\ -R_1^\top + J_{m_1 \times n_1} & (x - n_1 + 2)I_{m_1} & 0_{m_1 \times n_2} \otimes J_{1 \times n_1} \\ J_{n_2 \times n_1} \otimes J_{n_1 \times 1} & 0_{n_2 \times m_1} \otimes J_{n_1 \times 1} & ((x - n_1)I_{n_2} - L(G_2)) \otimes I_{n_1} \end{pmatrix} \\ &= \det\left((x - n_1)I_{n_2} - L(G_2)\right)^{n_1} \det(S),\end{aligned}$$

where

$$
\begin{aligned}
S &= \begin{pmatrix} (x-m_1-n_1n_2+r_1)I_{n_1}-L(G_1) & -R_1+J_{n_1\times m_1} \\ -R_1^\top+J_{m_1\times n_1} & (x-n_1+2)I_{m_1} \end{pmatrix} \\
&\quad -\begin{pmatrix} J_{n_1\times n_2}\otimes J_{1\times n_1} \\ 0_{m_1\times n_2}\otimes J_{1\times n_1} \end{pmatrix}\left(((x-n_1)I_{n_2}-L(G_2))^{-1}\otimes I_{n_1}\right)\left(J_{n_2\times n_1}\otimes J_{n_1\times 1} \quad 0_{n_2\times m_1}\otimes J_{n_1\times 1}\right) \\
&= \begin{pmatrix} (x-m_1-n_1n_2+r_1)I_{n_1}-L(G_1) & -R_1+J_{n_1\times m_1} \\ -R_1^\top+J_{m_1\times n_1} & (x-n_1+2)I_{m_1} \end{pmatrix}-\begin{pmatrix} \Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & 0_{n_1\times m_1} \\ 0_{m_1\times n_1} & 0_{m_1\times m_1} \end{pmatrix} \\
&= \begin{pmatrix} (x-m_1-n_1n_2+r_1)I_{n_1}-L(G_1)-\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & J_{n_1\times m_1}-R_1 \\ J_{m_1\times n_1}-R_1^\top & (x-n_1+2)I_{m_1} \end{pmatrix}.
\end{aligned}
$$

Since G_1 is an r_1-regular graph, then $R_1R_1^\top = 2r_1I_{n_1} - L(G_1)$. So, we can obtain

$$
(R_1 - J_{n_1\times m_1})(R_1^\top - J_{m_1\times n_1}) = (m_1 - 2r_1)J_{n_1\times n_1} + 2r_1I_{n_1} - L(G_1). \tag{2.1}
$$

Let $K = x - n_1 + 2$ and $N = (x - m_1 - n_1n_2 + r_1)\,K - 2r_1$. By (1), we have $\Gamma_{(K-1)L(G_1)}(N) = \frac{n_1}{N}$. From Lemmas 1.3, 1.4 and (2.1), we have

$$
\begin{aligned}
\det(S) &= K^{m_1}\det\left(\frac{N+2r_1}{K}I_{n_1}-L(G_1)-\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1}-\frac{1}{K}(R_1-J_{n_1\times m_1})(R_1^\top-J_{m_1\times n_1})\right) \\
&= K^{m_1}\det\left(\frac{N+2r_1}{K}I_{n_1}-L(G_1)-\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1}-\frac{(m_1-2r_1)J_{n_1\times n_1}+2r_1I_{n_1}-L(G_1)}{K}\right) \\
&= K^{m_1-n_1}\det\left(NI_{n_1}-(K-1)\,L(G_1)-\left(K\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)+m_1-2r_1\right)J_{n_1\times n_1}\right) \\
&= K^{m_1-n_1}\left(1-\left(K\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)+m_1-2r_1\right)\Gamma_{(K-1)L(G_1)}(N)\right)\det\left(NI_{n_1}-(K-1)\,L(G_1)\right) \\
&= K^{m_1-n_1}\left(1-\left(K\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)+m_1-2r_1\right)\frac{n_1}{N}\right)\det\left(NI_{n_1}-(K-1)\,L(G_1)\right).
\end{aligned}
$$

Since $\mu_1(G_i) = 0$ and $\Gamma_{L(G_2)\otimes I_{n_1}}(x - n_1) = \frac{n_1n_2}{x-n_1}$, it follows that

$$
\begin{aligned}
&\Phi_{G_1\odot G_2}(L,x) \\
&=\prod_{i=1}^{n_2}(x-n_1-\mu_i(G_2))^{n_1}K^{m_1-n_1}\left(1-\left(K\Gamma_{L(G_2)\otimes I_{n_1}}(x-n_1)+m_1-2r_1\right)\frac{n_1}{N}\right)\det\left(NI_{n_1}-(K-1)\,L(G_1)\right) \\
&=\prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2))^{n_1}(x-n_1)^{n_1-1}K^{m_1-n_1}\left(N(x-n_1)-(Kn_1n_2+(m_1-2r_1)(x-n_1))\cdot n_1\right) \\
&\quad\cdot\prod_{i=2}^{n_1}\left(N-(K-1)\,\mu_i(G_1)\right) \\
&=\prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2))^{n_1}(x-n_1)^{n_1-1}(x-n_1+2)^{m_1-n_1}x\Big(x^2+(-m_1-n_1n_2-2n_1+r_1+2)x \\
&\quad+m_1n_1+n_1^2n_2+n_1^2-2n_1n_2-2n_1-2m_1\Big) \\
&\quad\cdot\prod_{i=2}^{n_1}\left(x^2+(-m_1-n_1n_2-n_1+r_1+2)x+(n_1^2n_2-2n_1n_2+m_1n_1-2m_1-r_1n_1)-(x-n_1+1)\,\mu_i(G_1)\right).
\end{aligned}
$$

□

According to Theorem 2.2, we determine the Laplacian spectrum of $G_1 \odot G_2$.

Corollary 2.3. *Let G_1 be an r_1-regular graph with n_1 vertices and let G_2 be an any graph with n_2 vertices. Then, the L-spectrum of $G_1 \odot G_2$ consists of:*

(1) $n_1 + \mu_i(G_2)$, *for* $i = 2, 3, \ldots, n_2$, *repeated* n_1 *times;*
(2) n_1, *repeated* $n_1 - 1$ *times;*
(3) $n_1 - 2$, *repeated* $m_1 - n_1$ *times;*
(4) $\frac{-(-m_1-n_1n_2-n_1+r_1+2-\mu_i(G_1))\pm\sqrt{(-m_1-n_1n_2-n_1+r_1+2-\mu_i(G_1))^2-4(n_1^2n_2-2n_1n_2+m_1n_1-2m_1-r_1n_1+n_1\mu_i(G_1)-\mu_i(G_1))}}{2}$, *for* $i = 2, 3, \ldots, n_1$*;*
(5) $\frac{(m_1+n_1n_2+2n_1-r_1-2)\pm\sqrt{(m_1+n_1n_2+2n_1-r_1-2)^2-4(m_1n_1+n_1^2n_2+n_1^2-2n_1n_2-2n_1-2m_1)}}{2}$*;*
(6) 0.

Next, we determine the signless Laplacian polynomial of $G_1 \odot G_2$.

Theorem 2.3. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then,*

$$\begin{aligned}\Phi_{G_1\odot G_2}(Q,x) =&(x-n_1+2)^{m_1-n_1}\cdot(x-n_1-2r_2)^{n_1-1}\prod_{i=2}^{n_2}(x-n_1-\nu_i(G_2))^{n_1}\\ &\cdot\prod_{i=2}^{n_1}\left(x^2+(-m_1-n_1n_2-n_1+r_1+2)\,x+(n_1-2)(n_1n_2+m_1-r_1)-(x-n_1+3)\nu_i(G_1)\right)\\ &(x^3+(-m_1-n_1n_2-2n_1-r_1+2-2r_2)x^2+((n_1+n_1n_2+m_1)(n_1+2r_2-2)+2r_1(2n_1\\ &+r_2-2)-4r_2)\,x-n_1^2(3r_1+2n_2r_2)+2n_1(m_1+2r_1-3r_1r_2+2n_2r_2)+4r_2(m_1+2r_1)).\end{aligned}$$

Proof. By proper labeling of the vertices, the signless Laplacian matrix of $G_1 \odot G_2$ is

$$Q(G_1\odot G_2)=\begin{pmatrix}(m_1+n_1n_2-r_1)I_{n_1}+Q(G_1) & J_{n_1\times m_1}-R_1 & J_{n_1\times n_2}\otimes J_{1\times n_1}\\ J_{m_1\times n_1}-R_1^{\top} & (n_1-2)\,I_{m_1\times m_1} & 0_{m_1\times n_2}\otimes J_{1\times n_1}\\ J_{n_2\times n_1}\otimes J_{n_1\times 1} & 0_{n_2\times m_1}\otimes J_{n_1\times 1} & (n_1I_{n_2}+Q(G_2))\otimes I_{n_1}\end{pmatrix}.$$

The signless Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned}\Phi_{G_1\odot G_2}(Q,x)&=\det\left(xI_{n_1n_2+m_1+n_1}-Q(G_1\odot G_2)\right)\\ &=\det\begin{pmatrix}(x-m_1-n_1n_2+r_1)I_{n_1}-Q(G_1) & R_1-J_{n_1\times m_1} & -J_{n_1\times n_2}\otimes J_{1\times n_1}\\ R_1^{\top}-J_{m_1\times n_1} & (x-n_1+2)I_{m_1} & 0_{m_1\times n_2}\otimes J_{1\times n_1}\\ -J_{n_2\times n_1}\otimes J_{n_1\times 1} & 0_{n_2\times m_1}\otimes J_{n_1\times 1} & ((x-n_1)I_{n_2}-Q(G_2))\otimes I_{n_1}\end{pmatrix}\\ &=\det\left((x-n_1)I_{n_2}-Q(G_2)\right)^{n_1}\det(S),\end{aligned}$$

where

$$\begin{aligned}S=&\begin{pmatrix}(x-m_1-n_1n_2+r_1)I_{n_1}-Q(G_1) & R_1-J_{n_1\times m_1}\\ R_1^{\top}-J_{m_1\times n_1} & (x-n_1+2)I_{m_1}\end{pmatrix}\\ &-\begin{pmatrix}-J_{n_1\times n_2}\otimes J_{n_1\times 1}\\ 0_{m_1\times n_2}\otimes J_{n_1\times 1}\end{pmatrix}\left(((x-n_1)I_{n_2}-Q(G_2))^{-1}\otimes I_{n_1}\right)\left(-J_{n_2\times n_1}\otimes J_{n_1\times 1}\quad 0_{n_2\times m_1}\otimes J_{n_1\times 1}\right)\\ =&\begin{pmatrix}(x-m_1-n_1n_2+r_1)I_{n_1}-Q(G_1) & R_1-J_{n_1\times m_1}\\ R_1^{\top}-J_{m_1\times n_1} & (x-n_1+2)I_{m_1}\end{pmatrix}-\begin{pmatrix}\Gamma_{Q(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & 0_{n_1\times m_1}\\ 0_{m_1\times n_1} & 0_{m_1\times m_1}\end{pmatrix}\\ =&\begin{pmatrix}(x-m_1-n_1n_2+r_1)I_{n_1}-Q(G_1)-\Gamma_{Q(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & R_1-J_{n_1\times m_1}\\ R_1^{\top}-J_{m_1\times n_1} & (x-n_1+2)I_{m_1}.\end{pmatrix}\end{aligned}$$

Since G_1 is a regular graph, then $R_1R_1^\top = Q(G_1)$. So, we can obtain

$$(R_1 - J_{n_1\times m_1})(R_1^\top - J_{m_1\times n_1}) = (m_1 - 2r_1)J_{n_1\times n_1} + Q(G_1).$$

Let $K = x - n_1 + 2$ and $N = (x - m_1 - n_1n_2 + r_1)K$. By Lemmas 1.3 and 1.4, we have

$$\begin{aligned}\det(S) &= K^{m_1}\det\left(\frac{N}{K}I_{n_1} - Q(G_1) - \Gamma_{Q(G_2)\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} - \frac{1}{K}(R_1 - J_{n_1\times m_1})(R_1^\top - J_{m_1\times n_1})\right)\\ &= K^{m_1-n_1}\det\left(NI_{n_1} - (K+1)Q(G_1) - \left(K\Gamma_{Q(G_2)\otimes I_{n_1}}(x-n_1) + (m_1-2r_1)\right)J_{n_1\times n_1}\right)\\ &= K^{m_1-n_1}\left(1 - (K\Gamma_{Q(G_2)\otimes I_{n_1}}(x-n_1) + (m_1-2r_1))\Gamma_{(K+1)Q(G_1)}(N)\right)\prod_{i=1}^{n_1}(N - (K+1)\nu_i(G_1)).\end{aligned}$$

Since G_1 and G_2 are regular graphs, it follows that $\Gamma_{Q(G_2)\otimes I_{n_1}}(x-n_1) = \frac{n_1n_2}{x-n_1-2r_2}$. and $\Gamma_{(K+1)Q(G_1)}(N) = \frac{n_1}{N-(K+1)2r_1}$. Thus,

$$\begin{aligned}\Phi_{G_1\odot G_2}(Q,x) =&(x-n_1+2)^{m_1-n_1}\cdot(x-n_1-2r_2)^{n_1-1}\prod_{i=2}^{n_2}(x-n_1-\nu_i(G_2))^{n_1}\\ &\cdot\prod_{i=2}^{n_1}\left(x^2 + (-m_1-n_1n_2-n_1+r_1+2)x + (n_1-2)(n_1n_2+m_1-r_1) - (x-n_1+3)\nu_i(G_1)\right)\\ &(x^3 + (-m_1-n_1n_2-2n_1-r_1+2-2r_2)x^2 + ((n_1+n_1n_2+m_1)(n_1+2r_2-2) + 2r_1(2n_1\\ &+r_2-2) - 4r_2)x - n_1^2(3r_1+2n_2r_2) + 2n_1(m_1+2r_1-3r_1r_2+2n_2r_2) + 4r_2(m_1+2r_1)).\end{aligned}$$

□

According to Theorem 2.3, we determine the signless Laplacian spectrum of $G_1 \odot G_2$.

Corollary 2.4. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then, the Q-spectrum of $G_1 \odot G_2$ consists of:*

(1) $n_1 + \nu_i(G_2)$, for $i = 2, 3, \ldots, n_2$, repeated n_1 times;
(2) $n_1 + 2r_2$, repeated $n_1 - 1$ times;
(3) $n_1 - 2$, repeated $m_1 - n_1$ times;
(4) $\frac{-a\pm\sqrt{a^2-4(n_1^2n_2-2n_1n_2+m_1n_1-2m_1-r_1n_1+n_1\nu_i(G_1)-3\nu_i(G_1))}}{2}$, for $i = 2, 3, \ldots, n_1$, where $a = -m_1 - n_1n_2 - n_1 + r_1 + 2 - \nu_i(G_1)$;
(5) the roots of the cubic equation $x^3+(-m_1-n_1n_2-2n_1-r_1+2-2r_2)x^2+(n_1m_1+n_1^2n_2+4n_1r_1-2m_1-2n_1n_2-4r_1+n_1^2-2n_1+2m_1r_2+2n_1n_2r_2+2n_1r_2+2r_1r_2-4r_2)x+(-3n_1^2r_1+2n_1m_1+4n_1r_1-2n_1^2n_2r_2-6n_1r_1r_2+4m_1r_2+4n_1n_2r_2+8r_1r_2 = 0$.

3. Spectra of multiple Q-complemented-vertex join $G_1 \triangledown G_2$

In this section, we further studied the spectrum of the adjacency, Laplacian, and signless Laplacian spectra of $G_1 \triangledown G_2$. Let G_1 and G_2 be graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Let R_1 be the incidence matrix of G_1.

Theorem 3.1. *Let G_1 be an r_1-regular graph and G_2 be an arbitrary graph, then*

$$\begin{aligned}\Phi_{G_1 \triangledown G_2}(A,x) = &\prod_{i=1}^{n_2}(x-\lambda_i(G_2))x^{nm_1-n_1}\left(x^2-nr_1-(x+n)r_1-n_1(x\Gamma_{A(G_2)}(x)+n(m_1-2r_1))\right)\\ &\cdot\prod_{i=2}^{n_1}\left(x^2-nr_1-(x+n)\lambda_i(G_1)\right).\end{aligned}$$

Proof. By proper labeling of the vertices, the adjacency matrix of $G_1 \triangledown G_2$ is

$$A(G_1 \triangledown G_2) = \begin{pmatrix} A(G_1) & (J_{n_1\times m_1}-R_1)\otimes J_{1\times n} & J_{n_1\times n_2} \\ (J_{m_1\times n_1}-R_1^\top)\otimes J_{n\times 1} & 0_{m_1\times m_1}\otimes 0_{n\times n} & 0_{m_1\times n_2}\otimes J_{n\times 1} \\ J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n} & A(G_2)\end{pmatrix}.$$

The adjacency characteristic polynomial of $G_1 \triangledown G_2$ is

$$\begin{aligned}\Phi_{G_1 \triangledown G_2}(A,x) &= \det(xI_{nm_1+n_1+n_2}-A(G_1 \triangledown G_2))\\ &= \det\begin{pmatrix} xI_{n_1}-A(G_1) & -(J_{n_1\times m_1}-R_1)\otimes J_{1\times n} & -J_{n_1\times n_2} \\ -(J_{m_1\times n_1}-R_1^\top)\otimes J_{n\times 1} & xI_{m_1}\otimes I_n & 0_{m_1\times n_2}\otimes J_{n\times 1} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n} & xI_{n_2}-A(G_2)\end{pmatrix}\\ &= \det(xI_{n_2}-A(G_2))\det(S),\end{aligned}$$

where

$$\begin{aligned}S &= \begin{pmatrix} xI_{n_1}-A(G_1) & -(J_{n_1\times m_1}-R_1)\otimes J_{1\times n} \\ -(J_{m_1\times n}-R_1^\top)\otimes J_{n\times 1} & xI_{m_1}\otimes I_n\end{pmatrix}\\ &\quad-\begin{pmatrix}-J_{n_1\times n_2}\\ 0_{m_1\times n_2}\otimes J_{n\times 1}\end{pmatrix}(xI_{n_2}-A(G_2))^{-1}\begin{pmatrix}-J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n}\end{pmatrix}\\ &= \begin{pmatrix} xI_{n_1}-A(G_1)-\Gamma_{A(G_2)}(x)J_{n_1\times n_1} & -(J_{n_1\times m_1}-R_1)\otimes J_{1\times n} \\ -(J_{m_1\times n_1}-R_1^\top)\otimes J_{n\times 1} & xI_{m_1}\otimes I_n\end{pmatrix}.\end{aligned}$$

Let $N = x^2-nr_1$. By Lemmas 1.3 and 1.4, and $\Gamma_{(x+n)A(G_1)}(N) = \frac{n_1}{N-(x+n)r_1}$, we have

$$\begin{aligned}\det(S) &= x^{nm_1}\left(xI_{n_1}-A(G_1)-\Gamma_{A(G_2)}(x)J_{n_1\times n_1}-\frac{1}{x}(J_{n_1\times m_1}-R_1)(J_{m_1\times n_1}-R_1^\top)\otimes n\right)\\ &= x^{nm_1-n_1}\left((x^2-nr_1)I_{n_1}-(x+n)A(G_1)-(x\Gamma_{A(G_2)}(x)+n(m_1-2r_1))J_{n_1\times n_1}\right)\\ &= x^{nm_1-n_1}\left(1-(x\Gamma_{A(G_2)}(x)+n(m_1-2r_1))\Gamma_{(x+n)A(G_1)}(N)\right)\det\left(NI_{n_1}-(x+n)A(G_1)\right)\\ &= x^{nm_1-n_1}\left(1-(x\Gamma_{A(G_2)}(x)+n(m_1-2r_1))\frac{n_1}{N-(x+n)r_1}\right)\prod_{i=1}^{n_1}(N-(x+n)\lambda_i(G_1))\\ &= x^{nm_1-n_1}\left(N-(x+n)r_1-n_1(x\Gamma_{A(G_2)}(x)+n(m_1-2r_1))\right)\prod_{i=2}^{n_1}(N-(x+n)\lambda_i(G_1)).\end{aligned}$$

Thus,

$$\begin{aligned}\Phi_{G_1 \triangledown G_2}(A,x) = &\prod_{i=1}^{n_2}(x-\lambda_i(G_2))x^{nm_1-n_1}\left(x^2-nr_1-(x+n)r_1-n_1(x\Gamma_{A(G_2)}(x)+n(m_1-2r_1))\right)\\ &\cdot\prod_{i=2}^{n_1}\left(x^2-nr_1-(x+n)\lambda_i(G_1)\right).\end{aligned}$$

□

Corollary 3.1. *Let G_i be an r_i-regular graph with n_i vertices for $i = 1, 2$. Then,*

$$\Phi_{G_1 \triangledown G_2}(A, x) = \left(x^3 - (r_1 + r_2)x^2 + (r_1 r_2 - 2nr_1 - n_1 n_2 - nn_1(m_1 - 2r_1))\, x + nr_2\, (2r_1 + n_1(m_1 - 2r_1))\right) \cdot x^{nm_1 - n_1} \prod_{i=2}^{n_2} (x - \lambda_i(G_2)) \prod_{i=2}^{n_1} \left(x^2 - nr_1 - (x + n)\lambda_i(G_1)\right).$$

Proof. According Theorem 3.1 and $\Gamma_{A(G_2)}(x) = \frac{n_2}{x - r_2}$, we can obtain

$$\begin{aligned}
&\Phi_{G_1 \triangledown G_2}(A, x) \\
&= \left(x^2 - nr_1 - (x + n)r_1 - n_1(x\frac{n_2}{x - r_2} + n(m_1 - 2r_1))\right) x^{nm_1 - n_1} \prod_{i=1}^{n_2} (x - \lambda_i(G_2)) \prod_{i=2}^{n_1} \left(x^2 - nr_1 - (x + n)\lambda_i(G_1)\right) \\
&= \left(x^3 - (r_1 + r_2)x^2 + (r_1 r_2 - 2nr_1 - n_1 n_2 - nn_1(m_1 - 2r_1))\, x + nr_2\, (2r_1 + n_1(m_1 - 2r_1))\right) \\
&\quad \cdot x^{nm_1 - n_1} \prod_{i=2}^{n_2} (x - \lambda_i(G_2)) \prod_{i=2}^{n_1} \left(x^2 - nr_1 - (x + n)\lambda_i(G_1)\right).
\end{aligned}$$

□

According to Corollary 3.1, we determine the adjacency spectrum of $G_1 \triangledown G_2$.

Corollary 3.2. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then, the A-spectrum of $G_1 \triangledown G_2$ consists of:*

(1) 0*, repeated* $nm_1 - n_1$ *times;*
(2) $\lambda_i(G_2)$*, repeated for* $i = 2, 3, \ldots, n_2$ *times;*
(3) $\frac{\lambda_i(G_1) \pm \sqrt{\lambda_i^2(G_1) + 4n(r_1 + \lambda_i(G_1))}}{2}$*, repeated for* $i = 2, 3, \ldots, n_1$ *times;*
(4) the roots of the cubic equation
$x^3 - (r_1 + r_2)x^2 + (r_1 r_2 - 2nr_1 - n_1 n_2 - nn_1(m_1 - 2r_1))\, x + nr_2\, (2r_1 + n_1(m_1 - 2r_1))$.

Next, we determine the Laplacian polynomial of $G_1 \triangledown G_2$.

Theorem 3.2. *Let G_1 be an r_1-regular graph and G_2 be an any graph, then*

$$\Phi_{G_1 \triangledown G_2}(L, x) = (x - n_1 + 2)^{nm_1 - n_1} \prod_{i=2}^{n_1} \Big((x^2 - tx + (n_1 - 2)(nm_1 + n_2) - nn_1 r_1) - (x - n_1 + 2 - n)\mu_i(G_1)\Big) \prod_{i=2}^{n_2} (x - n_1 - \mu_i(G_2)) \cdot \left(x^3 - (t + n_1)x^2 + (n_1 - 2)(n_1 + n_2 - nm_1)x\right),$$

where $t = n(m_1 - r_1) + n_1 + n_2 - 2$.

Proof. By proper labeling of the vertices, the Laplacian matrix of $G_1 \triangledown G_2$ is

$$L(G_1 \triangledown G_2) = \begin{pmatrix} (n(m_1 - r_1) + n_2)\, I_{n_1} + L(G_1) & (-J_{n_1 \times m_1} + R_1) \otimes J_{1 \times n} & -J_{n_1 \times n_2} \\ (-J_{m_1 \times n_1} + R_1^{\top}) \otimes J_{n \times 1} & (n_1 - 2)I_{nm_1} & 0_{m_1 \times n_2} \otimes J_{n \times 1} \\ -J_{n_2 \times n_1} & 0_{n_2 \times m_1} \otimes J_{1 \times n} & n_1 I_{n_2} + L(G_2) \end{pmatrix}.$$

The Laplacian characteristic polynomial of $G_1 \triangledown G_2$ is

$$\begin{aligned}\Phi_{G_1\triangledown G_2}(L,x) &= \det(xI_{nm_1+n_1+n_2} - L(G_1 \triangledown G_2))\\ &= \det\begin{pmatrix}(x-n(m_1-r_1)-n_2)\,I_{n_1}-L(G_1) & (J_{n_1\times m_1}-R_1)\otimes J_{1\times n} & J_{n_1\times n_2}\\ (J_{m_1\times n_1}-R_1^{\top})\otimes J_{n\times 1} & (x-n_1+2)I_{nm_1} & 0_{m_1\times n_2}\otimes J_{n\times 1}\\ J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n} & (x-n_1)I_{n_2}-L(G_2)\end{pmatrix}\\ &= \det((x-n_1)I_{n_2}-L(G_2))\det(S),\end{aligned}$$

where

$$\begin{aligned}S =&\begin{pmatrix}(x-n(m_1-r_1)-n_2)\,I_{n_1}-L(G_1) & (J_{n_1\times m_1}-R_1)\otimes J_{1\times n}\\ (J_{m_1\times n_1}-R_1^{\top})\otimes J_{n\times 1} & (x-n_1+2)I_{nm_1}\end{pmatrix}\\ &-\begin{pmatrix}J_{n_1\times n_2}\\ 0_{m_1\times n_2}\otimes J_{n\times 1}\end{pmatrix}((x-n_1)I_{n_2}-L(G_2))^{-1}\begin{pmatrix}J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n}\end{pmatrix}\\ =&\begin{pmatrix}(x-n(m_1-r_1)-n_2)\,I_{n_1}-L(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} & (J_{n_1\times m_1}-R_1)\otimes J_{1\times n}\\ (J_{m_1\times n_1}-R_1^{\top})\otimes J_{n\times 1} & (x-n_1+2)I_{nm_1}\end{pmatrix}.\end{aligned}$$

Let $K = x - n_1 + 2$ and $N = x - n(m_1 - r_1) - n_2$. By (1.2) and (1.3), we have

$$\begin{aligned}\det(S) &= K^{nm_1}\det\left(NI_{n_1}-L(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1}-\frac{n}{K}((m_1-2r_1)J_{n_1\times n_1}+2r_1I_{n_1}-L(G_1))\right)\\ &= K^{nm_1-n_1}\det\left((KN-2nr_1)I_{n_1}-(K-n)L(G_1)-(K\Gamma_{L(G_2)}(x-n_1)+n(m_1-2r_1))J_{n_1\times n_1}\right)\\ &= K^{nm_1-n_1}\det\left((KN-2nr_1)I_{n_1}-(K-n)L(G_1)\left(1-(K\Gamma_{L(G_2)}(x-n_1)+n(m_1-2r_1))\right.\right.\\ &\quad\cdot\left.\left.\Gamma_{(K-n)L(G_1)}(KN-2nr_1)\right)\right).\end{aligned}$$

By (1.1), we have $\Gamma_{(K-n)L(G_1)}(KN-2nr_1) = \frac{n_1}{KN-2nr_1}$. and $\Gamma_{L(G_2)}(x-n_1) = \frac{n_2}{x-n_1}$. Thus,

$$\begin{aligned}\Phi_{G_1\triangledown G_2}(L,x) &= K^{nm_1-n_1}\prod_{i=2}^{n_1}((KN-2nr_1)I_{n_1}-(K-n)\mu_i(G_1))\prod_{i=1}^{n_2}(x-n_1-\mu_i(G_2))\\ &\quad\cdot((KN-2nr_1)-n_1(K\Gamma_{L(G_2)}(x-n_1)+n(m_1-2r_1)))\\ &= (x-n_1+2)^{nm_1-n_1}\prod_{i=2}^{n_1}\Big((x^2-(n(m_1-r_1)+n_1+n_2-2)x+n(n_1-2)m_1-nn_1r_1+n_1n_2-2n_2)\\ &\quad-(x-n_1+2-n)\mu_i(G_1))\prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2))\\ &\quad\cdot\left(x^3-(n(m_1-r_1)+2n_1+n_2-2)x^2+(n_1-2)(n_1+n_2-nm_1)x\right).\end{aligned}$$

According to Theorem 3.2, we determine the Laplacian spectrum of $G_1 \triangledown G_2$.

Corollary 3.3. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then, the L-spectrum of $G_1 \triangledown G_2$ consists of:*

(1) $n_1 + \mu_i(G_2)$, *for* $i = 2, 3, \ldots, n_2$;
(2) $n_1 - 2$, *repeated* $nm_1 - n_1$ *times;*
(3) $\frac{n(m_1-r_1)+n_1+n_2-2+\mu_i(G_1)\pm\sqrt{(n(m_1-r_1)+n_1+n_2-2+\mu_i(G_1))^2-4(n(n_1-2)m_1-nn_1r_1+n_1n_2-2n_2+(n_1+n-2)\mu_i(G_1))}}{2}$, *for* $i = 2, 3, \ldots, n_1$;

(4) $\frac{(n(m_1-r_1)+2n_1+n_2-2)\pm\sqrt{(n(m_1-r_1)+2n_1+n_2-2)^2-4(n_1-2)(n_1+n_2-nm_1)}}{2}$

(5) 0.

□

Next, we determine the signless Laplacian polynomial of $G_1 \triangledown G_2$.

Theorem 3.3. *Let G_i be an r_i-regular graph for $i = 1, 2$. Then,*

$$\Phi_{G_1\triangledown G_2}(Q, x) = (x - n_1 + 2)^{nm_1-n_1}(x^3 + C_2x^2 + C_1x + C_0)\prod_{i=2}^{n_2}(x - n_1 - \nu_i(G_2))$$
$$\cdot\prod_{i=2}^{n_1}((x - n(m_1 - r_1) - n_2)(x - n_1 + 2) - (x - n_1 + 2 + n)\nu_i(G_1)).$$

where $C_2 = -n(m_1 - r_1) - n_2 - 2n_1 - 2(r_1 + r_2) + 2$, $C_1 = -n_1n_2 - nn_1m_1 + 2nn_1r_1 + n(m_1 - r_1)(n_1 - 2) + n_2(n_1 - 2) + 2r_1(n_1 - 2) - 2r_1n + (n_1 + 2r_2)(n_1 - 2) + n(m_1 - r_1)(n_1 + 2r_2) + (n_1 + 2r_2)(n_2 + 2r_1)$, $C_0 = n_1n_2(n_1 - 2) + nn_1(m_1 - 2r_1)(n_1 + 2r_2) - (n_1 + 2r_2)(n(m_1 - r_1)(n_1 - 2) + n_2(n_1 - 2) + 2r_1(n_1 - 2) - 2r_1n)$.

Proof. By proper labeling of the vertices, the signless Laplacian matrix of $G_1 \triangledown G_2$ is

$$Q(G_1 \triangledown G_2) = \begin{pmatrix} (n(m_1 - r_1) + n_2)I_{n_1} + Q(G_1) & (J_{n_1\times m_1} - R_1)\otimes J_{1\times n} & J_{n_1\times n_2} \\ (J_{m_1\times n_1} - R_1^\top)\otimes J_{n\times 1} & (n_1 - 2)I_{nm_1} & 0_{m_1\times n_2}\otimes J_{n\times 1} \\ J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n} & n_1I_{n_2} + Q(G_2)\end{pmatrix}.$$

The signless Laplacian characteristic polynomial of $G_1 \triangledown G_2$ is

$$\begin{aligned}\Phi_{G_1\triangledown G_2}(Q, x) &= \det(xI_{nm_1+n_1+n_2} - Q(G_1 \triangledown G_2)) \\ &= \det\begin{pmatrix} (x - n(m_1 - r_1) - n_2)I_{n_1} - Q(G_1) & -(J_{n_1\times m_1} - R_1)\otimes J_{1\times n} & -J_{n_1\times n_2} \\ -(J_{m_1\times n_1} - R_1^\top)\otimes J_{n\times 1} & (x - n_1 + 2)I_{nm_1} & 0_{m_1\times n_2}\otimes J_{n\times 1} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n} & (x - n_1)I_{n_2} - Q(G_2)\end{pmatrix} \\ &= \det((x - n_1)I_{n_2} - Q(G_2))\det(S),\end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x - n(m_1 - r_1) - n_2)I_{n_1} - Q(G_1) & -(J_{n_1\times m_1} - R_1)\otimes J_{1\times n} \\ -(J_{m_1\times n} - R_1^\top)\otimes J_{n\times 1} & (x - n_1 + 2)I_{nm_1}\end{pmatrix} \\ &\quad - \begin{pmatrix} -J_{n_1\times n_2} \\ 0_{m_1\times n_2}\otimes J_{n\times 1}\end{pmatrix}((x - n_1)I_{n_2} - Q(G_2))^{-1}\begin{pmatrix} -J_{n_2\times n_1} & 0_{n_2\times m_1}\otimes J_{1\times n}\end{pmatrix} \\ &= \begin{pmatrix} (x - n(m_1 - r_1) - n_2)I_{n_1} - Q(G_1) - \Gamma_{Q(G_2)}(x - n_1)J_{n_1\times n_1} & -(J_{n_1\times m_1} - R_1)\otimes J_{1\times n} \\ -(J_{m_1\times n_1} - R_1^\top)\otimes J_{n\times 1} & (x - n_1 + 2)I_{nm_1}\end{pmatrix}.\end{aligned}$$

Let $K = x - n_1 + 2$ and $N = (x - n(m_1 - r_1) - n_2)K$. Clearly, $\Gamma_{(K+n)Q(G_1)}(N) = \frac{n_1}{N - 2r_1(K+n)}$ and $(J_{n_1\times m_1} - R_1)(J_{m_1\times n_1} - R_1^\top) = (m_1 - 2r_1)J_{n_1\times n_1} + Q(G_1)$, and we have

$$\det(S) = K^{nm_1} \det\Big((x - n(m_1 - r_1) - n_2)\, I_{n_1} - Q(G_1) - \Gamma_{Q(G_2)}(x - n_1)J_{n_1\times n_1} - \frac{1}{K}(J_{n_1\times m_1} - R_1)(J_{m_1\times n_1} - R_1^{\top}) \otimes n\Big)$$
$$= K^{nm_1-n_1} \det\left(NI_{n_1} - (K+n)Q(G_1) - ((x - n_1 + 2)\Gamma_{Q(G_2)}(x - n_1) + n(m_1 - 2r_1))J_{n_1\times n_1}\right)$$
$$= K^{nm_1-n_1}(1 - (K\Gamma_{Q(G_2)}(x - n_1) + n(m_1 - 2r_1))\Gamma_{(K+n)Q(G_1)}(N)) \cdot \det(NI_{n_1} - (K+n)Q(G_1))$$
$$= K^{nm_1-n_1}(-n_1(K\Gamma_{Q(G_2)}(x - n_1) + n(m_1 - 2r_1)) + (N - (K+n)2r_1)) \cdot \prod_{i=2}^{n_1}(N - (K+n)\nu_i(G_1)).$$

By (1.1), we can obtain $\Gamma_{Q(G_2)}(x - n_1) = \frac{n_2}{x-n_1-2r_2}$. Thus,

$$\Phi_{G_1\triangledown G_2}(Q, x) = K^{nm_1-n_1}(-n_1(n_2(x - n_1 + 2) + n(m_1 - 2r_1)(x - n_1 - 2r_2)) + (x - n_1 - 2r_2) ((x - n(m_1 - r_1) - n_2)\,(x - n_1 + 2) - (x - n_1 + 2 + n)2r_1)) \prod_{i=2}^{n_1}((x - n(m_1 - r_1) - n_2)\,(x - n_1 + 2) - (x - n_1 + 2 + n)\nu_i(G_1)) \prod_{i=2}^{n_2} (x - n_1 - \nu_i(G_2))$$
$$= (x - n_1 + 2)^{nm_1-n_1}(x^3 + C_2x^2 + C_1x + C_0) \prod_{i=2}^{n_1}((x - n(m_1 - r_1) - n_2)\,(x - n_1 + 2) - (x - n_1 + 2 + n)\nu_i(G_1)) \prod_{i=2}^{n_2} (x - n_1 - \nu_i(G_2))$$

where $C_2 = -n(m_1 - r_1) - n_2 - 2n_1 - 2(r_1 + r_2) + 2$, $C_1 = -n_1n_2 - nn_1m_1 + 2nn_1r_1 + n(m_1 - r_1)(n_1 - 2) + n_2(n_1-2) + 2r_1(n_1-2) - 2r_1n + (n_1+2r_2)(n_1-2) + n(m_1-r_1)(n_1+2r_2) + (n_1+2r_2)(n_2+2r_1)$, $C_0 = n_1n_2(n_1 - 2) + nn_1(m_1 - 2r_1)(n_1 + 2r_2) - (n_1 + 2r_2)\,(n(m_1 - r_1)(n_1 - 2) + n_2(n_1 - 2) + 2r_1(n_1 - 2) - 2r_1n)$. □

According to Theorem 3.3, we determine the signless Laplacian spectrum of $G_1 \triangledown G_2$.

Corollary 3.4. *Let G_1 be an r_1-regular graph with n_1 vertices and let G_2 be an r_2-regular graph with n_2 vertices. Then, the Q-spectrum of $G_1 \triangledown G_2$ consists of:*

(1) $n_1 + \nu_i(G_2)$, *for* $i = 2, 3, \ldots, n_2$, *repeated* n_1 *times;*
(2) $n_1 - 2$, *repeated* $nm_1 - n_1$ *times;*
(3) $\frac{(n_1+n_2+n(m_1-r_1)+\nu_i(G_1))-2\pm\sqrt{(n_1+n_2+n(m_1-r_1)+\nu_i(G_1)-2)^2-4((n(m_1-r_1)+n_2)(n_1-2)-n\nu_i(G_1))}}{2}$, *for* $i = 2, 3, \ldots, n_1$;
(4) the roots of the cubic equation
$x^3 + C_2x^2 + C_1x + C_0 = 0$, *where* $C_2 = -n(m_1 - r_1) - n_2 - 2n_1 - 2(r_1 + r_2) + 2$, $C_1 = -n_1n_2 - nn_1m_1 + 2nn_1r_1 + n(m_1 - r_1)(n_1 - 2) + n_2(n_1 - 2) + 2r_1(n_1 - 2) - 2r_1n + (n_1 + 2r_2)(n_1 - 2) + n(m_1 - r_1)(n_1 + 2r_2) + (n_1 + 2r_2)(n_2 + 2r_1)$, $C_0 = n_1n_2(n_1 - 2) + nn_1(m_1 - 2r_1)(n_1 + 2r_2) - (n_1 + 2r_2)\,(n(m_1 - r_1)(n_1 - 2) + n_2(n_1 - 2) + 2r_1(n_1 - 2) - 2r_1n)$.

4. Applications

As applications, based on the adjacency, Laplacian, and signless Laplacian spectra of the quasi-corona Q-complemented-vertex join $G_1 \odot G_2$ and the multiple Q-complemented-vertex join $G_1 \triangledown G_2$, we construct infinite families of A-, L- and Q-cospectral graphs, and derive explicit calculation formulas for the number of spanning trees, the Kirchhoff index, the SLEL, and graph energy of the composite graphs.

Theorem 4.1. *(a) If G_1 and G_2 are A-cospectral (L-cospectral, Q-cospectral) regular graphs, and H is a regular graph, then $G_1 \odot H$ and $G_2 \odot H$ are A-cospectral (L-cospectral, Q-cospectral).*

(b) If G is a regular graph, and H_1 and H_2 are A-cospectral (L-cospectral, L-cospectral) regular graphs, then $G \odot H_1$ and $G \odot H_2$ are A-cospectral (L-cospectral, Q-cospectral).

(c) If G_1 and G_2 are A-cospectral (L-cospectral, Q-cospectral) regular graphs, and H is a regular graph, then $G_1 \triangledown H$ and $G_2 \triangledown H$ are A-cospectral (L-cospectral, Q-cospectral).

(d) If G is a regular graph, and H_1 and H_2 are A-cospectral (L-cospectral, L-cospectral) regular graphs, then $G \triangledown H_1$ and $G \triangledown H_2$ are A-cospectral (L-cospectral, Q-cospectral).

Proof. From Corollaries 2.2–2.4 and 3.2–3.4, it follows that the above result holds. □

Theorem 4.2. *Let G_1 be an r_1-regular graph and G_2 be an arbitrary graph, then*

$$t(G_1 \odot G_2) = \frac{\prod_{i=2}^{n_2}(n_1 + \mu_i(G_2))n_1^{n_1-1}(n_1-2)^{m_1-n_1}\prod_{i=2}^{n_1}(n_1^2 n_2 - 2n_1n_2 + m_1n_1 - 2m_1 - r_1n_1 + n_1\mu_i(G_1) - \mu_i(G_1))}{n_1n_2 + n_1 + m_1} \cdot \left(m_1n_1 + n_1^2n_2 + n_1^2 - 2n_1n_2 - 2n_1 - 2m_1\right).$$

$$t(G_1 \triangledown G_2) = \frac{\prod_{i=1}^{n_2}(n_1 + \mu_i(G_2))(n_1-2)^{nm_1-n_1}\prod_{i=2}^{n_1}(n(n_1-2)m_1 - nn_1r_1 + n_1n_2 - 2n_2 + (n_1+n-2)\mu_i(G_1))}{nm_1 + n_1 + n_2} \cdot (n_1-2)(n_1+n_2-nm_1).$$

Proof. Using the formula $t(G) = \frac{\prod_{i=2}^{n}\mu_i(G)}{n}$, we substitute the corresponding Laplacian spectra derived in Corollaries 2.3 and 3.3 to obtain the desired results. □

Theorem 4.3. *Let G_1 be an r_1-regular graph and G_2 be an any graph, then*

$$\begin{aligned} Kf(G_1 \odot G_2) = (n_1n_2 + n_1 + m_1)\Bigg(&\sum_{i=2}^{n_2}\frac{1}{n_1 + \mu_i(G_2)} + \frac{n_1 - 1}{n_1} + \frac{m_1 - n_1}{n_1 - 2} \\ &- \sum_{i=2}^{n_1}\frac{-m_1 - n_1n_2 - n_1 + r_1 + 2 - \mu_i(G_1)}{n_1^2n_2 - 2n_1n_2 + m_1n_1 - 2m_1 - r_1n_1 + n_1\mu_i(G_1) - \mu_i(G_1)} \\ &+ \frac{m_1 + n_1n_2 + 2n_1 - r_1 - 2}{m_1n_1 + n_1^2n_2 + n_1^2 - 2n_1n_2 - 2n_1 - 2m_1}\Bigg). \end{aligned}$$

$$Kf(G_1 \triangledown G_2) = (nm_1 + n_1 + n_2)\Big(\sum_{i=2}^{n_2} \frac{1}{n_1 + \mu_i(G_2)} + \frac{nm_1 - n_1}{n_1 - 2} + \sum_{i=2}^{n_1} \frac{n(m_1 - r_1) + n_1 + n_2 - 2 + \mu_i(G_1)}{n(n_1 - 2)m_1 - nn_1r_1 + n_1n_2 - 2n_2 + (n_1 + n - 2)\mu_i(G_1)} + \frac{n(m_1 - r_1) + 2n_1 + n_2 - 2}{(n_1 - 2)(n_1 + n_2 - nm_1)} \Big).$$

Proof. Using the formula $Kf(G) = n\sum_{i=2}^{n} \frac{1}{\mu_i(G)}$, we substitute the Laplacian spectra from Corollaries 2.3 and 3.3 and simplify. □

Theorem 4.4. *Let G_1 be an r_1-regular graph and G_2 be an r_2-regular graph, then*

$$SLEL(G_1 \odot G_2) = \sum_{i=2}^{n_2} (n_1 + \nu_i(G_2))^{\frac{n_1}{2}} + (n_1 - 1)\sqrt{n_1 + 2r_2} + (m_1 - n_2)\sqrt{n_1 - 2} + \sum_{i=2}^{n_1} \sqrt{\frac{S_1 \pm \sqrt{S_1^2 - 4(n_1^2n_2 - 2n_1n_2 + m_1n_1 - 2m_1 - r_1n_1 + n_1\nu_i(G_1) - 3\nu_i(G_1))}}{2}} + \sum_{i=1}^{3} \sqrt{\theta_i},$$

where $S_1 = m_1 + n_1n_2 + n_1 - r_1 - 2 + \nu_i(G_1)$, θ_i (i=1,2,3) is the root of the cubic equation $x^3 + (-m_1 - n_1n_2 - 2n_1 - r_1 + 2 - 2r_2)x^2 + (n_1m_1 + n_1^2n_2 + 4n_1r_1 - 2m_1 - 2n_1n_2 - 4r_1 + n_1^2 - 2n_1 + 2m_1r_2 + 2n_1n_2r_2 + 2n_1r_2 + 2r_1r_2 - 4r_2)x + (-3n_1^2r_1 + 2n_1m_1 + 4n_1r_1 - 2n_1^2n_2r_2 - 6n_1r_1r_2 + 4m_1r_2 + 4n_1n_2r_2 + 8r_1r_2 = 0.$

$$SLEL(G_1 \triangledown G_2) = \sum_{i=2}^{n_2} (n_1 + \nu_i(G_2))^{\frac{n_1}{2}} + (nm_1 - n_1)\sqrt{n_1 - 2} + \sum_{i=2}^{n_1} \sqrt{\frac{S_2 \pm \sqrt{S_2^2 - 4\left((n(m_1 - r_1) + n_2)(n_1 - 2) - n\nu_i(G_1)\right)}}{2}} + \sum_{i=1}^{3} \sqrt{\gamma_i},$$

where $S_2 = n_1 + n_2 + n(m_1 - r_1) + \nu_i(G_1) - 2$, $\gamma_i (i = 1, 2, 3)$ is the root of the cubic equation $x^3 + C_2x^2 + C_1x + C_0 = 0$, where $C_2 = -n(m_1 - r_1) - n_2 - 2n_1 - 2(r_1 + r_2) + 2$, $C_1 = -n_1n_2 - nn_1m_1 + 2nn_1r_1 + n(m_1 - r_1)(n_1-2) + n_2(n_1-2) + 2r_1(n_1-2) - 2r_1n + (n_1+2r_2)(n_1-2) + n(m_1-r_1)(n_1+2r_2) + (n_1+2r_2)(n_2+2r_1)$, $C_0 = n_1n_2(n_1 - 2) + nn_1(m_1 - 2r_1)(n_1 + 2r_2) - (n_1 + 2r_2)\left(n(m_1 - r_1)(n_1 - 2) + n_2(n_1 - 2) + 2r_1(n_1 - 2) - 2r_1n\right)$.

Proof. The invariant $SLEL(G)$ is defined as the sum of the square roots of the signless Laplacian eigenvalues. Substituting the Q-spectra from Corollaries 2.4 and 3.4 gives the desired expressions. □

Theorem 4.5. *Let G_1 be an r_1-regular graph and G_2 be an r_2-regular graph, then*

$$GE(G_1 \odot G_2) = n_1GE(G_2) - r_2 + \sum_{i=2}^{n_1} \sqrt{\lambda_i^2(G_1) + 4(r_1 + \lambda_i(G_1))} + \sum_{i=1}^{3} |\chi_i|,$$

where χ_i (i=1,2,3) is the root of the cubic equation $x^3 - (r_1 + r_2)x^2 + (r_1r_2 - 2r_1 - n_1^2n_2 - n_1m_1 + 2r_1n_1)x + 2r_1r_2 + n_1m_1r_2 - 2n_1r_1r_2 = 0.$

$$GE(G_1 \triangledown G_2) = GE(G_2) - r_2 + \sum_{i=2}^{n_1} \sqrt{\lambda_i^2(G_1) + 4n(r_1 + \lambda_i(G_1))} + \sum_{i=1}^{3} |\xi_i|,$$

where $\xi_i (i = 1, 2, 3)$ *is the root of the cubic equation* $x^3 - (r_1 + r_2)x^2 + (r_1 r_2 - 2nr_1 - n_1 n_2 - nn_1(m_1 - 2r_1))\, x + nr_2\,(2r_1 + n_1(m_1 - 2r_1)) = 0.$

Proof. The graph energy $GE(G)$ is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of a graph. Substituting the A-spectra from Corollaries 2.2 and 3.2 gives the desired expressions. □

Remark 4.1. *The matrix* $L(G_2) + n_1 I$ *is a shifted Laplacian matrix of* G_2 *[35], obtained by adding a uniform scalar* n_1 *to all Laplacian eigenvalues. Such matrices naturally arise in extensions of the matrix Tree theorem and are closely related to weighted spanning forests. In particular, the term* $\prod_{i=2}^{n_2}(n_1 + \mu_i(G_2))$ *can be interpreted as a spanning tree-like quantity associated with the shifted Laplacian* $L(G_2) + n_1 I$*, reflecting a weighted modification of the combinatorial structure of* G_2*. Moreover, the quantity* $\sum_{i=2}^{n_2} \frac{1}{n_1+\mu_i(G_2)}$ *can be regarded as a Kirchhoff-like index, corresponding to the trace of the inverse of the shifted Laplacian matrix. Therefore, the formulas in Theorems 4.2 and 4.3 explicitly express the spanning tree number and the Kirchhoff index of the composite graphs as functions of the Laplacian eigenvalues of the subgraph* G_2*. This establishes a direct and explicit relationship between the spectral invariants of the subgraphs and the topological indices of the composite graphs.*

For example, when G_2 *is a complete graph or a cycle graph, whose Laplacian spectra are explicitly known, the above expressions reduce to closed form formulas.*

5. Numerical illustration

In this section, we provide a numerical illustration to further explore the structural properties of the multiple Q-complemented-vertex join graph $G_1 \triangledown G_2$. Our purpose is not to give a complete theoretical characterization, but to offer empirical evidence supporting the potential applicability of the proposed graph construction in network modeling.

We fix the $G_1 = P_{10}$ and $G_2 = K_2$, and investigate how the structural properties of the composite graph $G_1 \triangledown G_2$ evolve as the parameter n increases. Specifically, we consider $n = 1, 2, 3, 4, 5, 10$, and compute the average path length, clustering coefficient, and Kirchhoff index for each case. The numerical results are summarized in Table 1.

Table 1. Structural properties of $G_1 \triangledown G_2$ with $G_1 = P_{10}$ and $G_2 = K_2$ for different values of n.

n	Average path length	Clustering coefficient	Kirchhoff index
1	1.5238	0.2649	44.1031
2	1.6092	0.2144	86.4784
3	1.6761	0.2036	150.4356
4	1.7252	0.2014	235.0822
5	1.7619	0.2016	340.1781
10	1.8583	0.2070	1170.4043

From Table 1, it can be observed that, as n increases, the average path length grows slowly, while remaining relatively small. In fact, the average path length stays below 2 even for larger values of n, indicating that the constructed graphs maintain a highly compact structure. Meanwhile, the clustering coefficient slightly decreases at first and then stabilizes around a moderate level, reflecting the persistence of local connectivity in the network. In addition, the Kirchhoff index increases rapidly with n, which is consistent with the growth of the graph size and reflects the change of global resistance properties in the network.

These observations suggest that the proposed construction $G_1 \triangledown G_2$ may exhibit certain small-world-like characteristics, such as relatively short average path lengths together with nontrivial clustering structure. Therefore, the multiple Q-complemented-vertex join graph provides a potentially useful deterministic model for studying structural properties of complex networks.

6. Conclusions

Computing graph spectra is a central topic in spectral graph theory. In this paper, we analyze the spectral properties of two new composite graph operations, the quasi-corona Q-complemented-vertex join ($G_1 \odot G_2$) and the multiple Q-complemented-vertex join ($G_1 \nabla G_2$), and derive explicit adjacency, Laplacian, and signless Laplacian spectra for these graphs. The research findings cover two key applications, namely constructing infinite families of cospectral graphs and deriving direct calculation approaches for core graph invariants based on the obtained spectral results. In addition, a numerical illustration is provided to explore the structural properties of the proposed graphs, which suggests that the multiple Q-complemented-vertex join construction may exhibit certain small-world-like characteristics. It is worth noting that most of the results are obtained under the assumption that one or both of the subgraphs are regular in this paper. Extending the present results to non-regular graphs would be an interesting direction for future research.

Author contributions

Yang Yang: Writing-review & editing, writing-original draft, visualization, validation, supervision, software, project administration, methodology, investigation, formal analysis, data curation, conceptualization; Yanyan Song: Writing-review & editing; Zhanjun Si: Writing-review & editing, writing-original draft, methodology, investigation, funding acquisition; Yi Liu: Writing-review & editing, funding acquisition. Yang Yang, Yanyan Song and Zhanjun Si contribute equally to the article. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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