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Research article

Normal criterion on the polydisc in $\mathbb{C}^n$

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Abstract: This paper established a slice characterization for normal families of holomorphic functions on the unit polydisc $\mathbb{D}^n \subset \mathbb{C}^n$. The main result (Theorem 2.2) showed that a family $\mathcal{F} \subset \mathcal{O}(\mathbb{D}^n)$ was normal if, and only if, for every point $a \in \mathbb{D}^n$ and every coordinate direction $1 \leq j \leq n$, the corresponding one-dimensional slice family $\mathcal{F}_{a,j}$ was normal on the unit disc. Building on this characterization, we introduced the notion of normal functions on $\mathbb{D}^n$ and proved a metric criterion (Proposition 2.7): A function $f \in \mathcal{O}(\mathbb{D}^n)$ was normal exactly when its spherical derivative grew at most as fast as the Poincaré metric density, i.e., $f^\sharp(z) \leq C \max_j (1 - |z_j|^2)^{-1}$ for some constant $C > 0$.

Keywords: normal criterion; polydisc; spherical derivative; slice characterization; Marty's criterion
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1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\mathcal{O}(\Omega)$ denote the space of holomorphic functions on Ω. A family $\mathcal{F} \subset \mathcal{O}(\Omega)$ is called *normal* if every sequence in $\mathcal{F}$ admits a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to ∞. Normal families play a central role in complex analysis and are closely related to Montel's theorem [1], value distribution theory, iteration theory, and compactness phenomena for holomorphic mappings. For a systematic account of the classical theory of normal families and its modern developments, see the monograph of Wang and Chang [2]. For normal functions and their characterizations in several complex variables, we refer to Cima and Krantz [3] and Dovbush [4]. Further recent developments can be found in [5, 6].

In one complex variable, Montel's theorem and Marty's criterion give powerful characterizations of normality [7]. If $f \in \mathcal{O}(\mathbb{D})$, its spherical derivative is defined by

$$f^\sharp(z) = \frac{|f'(z)|}{1 + |f(z)|^2}, \tag{1.1}$$

and Marty's theorem asserts that a family $\mathcal{F} \subset \mathcal{O}(\mathbb{D})$ is normal if, and only if, the spherical derivatives $f^\sharp$ are locally uniformly bounded.

In recent years, the theory of normal families in several complex variables has witnessed significant developments. Ahamed and Mandal [8] established normality criteria involving total derivatives and shared values for holomorphic functions in $\mathbb{C}^n$. Datt [9] improved the Montel-Carathéodory theorem for families of $\mathbb{P}^n$-valued holomorphic curves, while Datt et al. [10] investigated normal families with shared moving hyperplanes. For entire curves, Van Truong and Thi Thu Hang [11] obtained a characterization of Brody curves via boundedness of the spherical derivative on preimages of hyperplanes. The method of Zalcman's lemma, fundamental to normality theory, has been revisited by Berteloot [12] and extended to complex Lie groups by Dong and Lv [13]. For meromorphic functions in several variables, Chang et al. [14] extended Gu's normality criterion. Lee [15] proved a Montel-type theorem for Cauchy-Riemann(CR) mappings between pseudo-Hermitian CR manifolds. Krantz and Dovbush [16] further emphasized the central role of Montel's theorem in complex function theory. The classic treatise of Gunning and Rossi [17] remains an essential reference for several complex variables.

In several complex variables, the work of Alexander [18], Nishino [19], and Terada [20] established the basic theory of normal families for holomorphic mappings and functions on domains in $\mathbb{C}^n$, including early slice methods and compactness criteria. The role of the spherical derivative in several variables is played by a quantity defined in terms of the Levi form of $\log(1 + |f|^2)$. Precise definitions and properties will be given in Section 2.

On highly symmetric domains such as the unit ball

$$\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}, \tag{1.2}$$

the automorphism group $\mathrm{Aut}(\mathbb{B}^n)$ acts transitively. This homogeneity allows one to reduce many problems about normal families to the study of restrictions of holomorphic functions to complex lines through a fixed point. Recently, Dovbush and Krantz [21] obtained a canonical characterization of normal functions and normal families on $\mathbb{B}^n$ in terms of their behavior along complex lines through the origin, using the Bergman metric and the Levi form of $\log(1 + |f|^2)$.

In contrast, the unit polydisc

$$\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1,\ j = 1, \dots, n\} \tag{1.3}$$

are not homogeneous. Their automorphism groups do not act transitively, and the geometry is essentially anisotropic. As a consequence, slice methods based on complex lines through a fixed point cannot be applied directly. Instead, the natural slices on product domains are the *coordinate slices*

$$\lambda \mapsto (a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n), \qquad \lambda \in \Omega_j, \tag{1.4}$$

for $a = (a_1, \dots, a_n) \in \Omega$ and $1 \le j \le n$.

The main purpose of this paper is to develop a theory of normal families on the polydisc that is adapted to this product geometry. More precisely, we address the following problems:

(i) Can one characterize normal families on $\mathbb{D}^n$ in terms of a Marty type criterion involving the Levi form of $\log(1 + |f|^2)$?

(ii) Can normality on such domains be detected by the behavior of the family on all coordinate slices?

(iii) How should one define and characterize *normal functions* on product domains in analogy with the ball case?

Our first main result is a Marty type criterion on the polydisc , showing that a family $\mathcal{F} \subset \mathcal{O}(\Omega)$ is normal if, and only if, the associated spherical derivatives $f^\sharp$ are locally uniformly bounded. Our second main result is a slice characterization: A family on $\mathbb{D}^n$ is normal if, and only if, all of its coordinate slice families are normal in the sense of one-variable complex analysis. Finally, we introduce a notion of normal functions on product domains and give a metric characterization in terms of the product Poincaré metric.

Throughout the paper, ρ_{Ω_j} denotes the Poincaré metric on Ω_j, and the product metric on Ω is defined by

$$ds^2_\Omega(z; v) := \sum_{j=1}^{n} \rho_{\Omega_j}(z_j)^2 \, |v_j|^2. \tag{1.5}$$

All holomorphic functions are complex-valued unless stated otherwise.

2. Normal families and normal functions on the polydisc

Let $\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, \; j = 1, \dots, n\}$ denote the unit polydisc. We write $\mathcal{O}(\mathbb{D}^n)$ for the space of holomorphic functions on $\mathbb{D}^n$.

Recall that a family $\mathcal{F} \subset \mathcal{O}(\mathbb{D}^n)$ is called *normal* if every sequence in $\mathcal{F}$ admits a subsequence converging uniformly on compact subsets, either to a holomorphic function or to ∞.

For $f \in \mathcal{O}(\mathbb{D}^n)$, define the spherical derivative at $z \in \mathbb{D}^n$ in direction $v \in \mathbb{C}^n$ by

$$f^\sharp(z; v)^2 := L_z(\log(1 + |f|^2), \, v), \tag{2.1}$$

where $L_z(\varphi, v)$ denotes the Levi form of a C^2-function φ at z in the direction v. We also write

$$f^\sharp(z) := \sup_{\|v\|=1} f^\sharp(z; v), \tag{2.2}$$

where $\| \cdot \|$ is the Euclidean norm on $\mathbb{C}^n$.

The Bergman (equivalently, Poincaré product) metric on $\mathbb{D}^n$ is given by

$$ds^2_{\mathbb{D}^n}(z; v) = \sum_{j=1}^{n} \frac{|v_j|^2}{(1 - |z_j|^2)^2}. \tag{2.3}$$

We begin with a Marty type criterion on the polydisc.

Theorem 2.1. [18, 21, 22] *A family $\mathcal{F} \subset \mathcal{O}(\mathbb{D}^n)$ is normal if and only if for every compact set $K \subset \mathbb{D}^n$ there exists a constant $C_K > 0$ such that $f^\sharp(z) \le C_K$ for all $z \in K$ and all $f \in \mathcal{F}$.*

Historical note. The criterion in this form can be traced back to Alexander [18] and Barth [22]. The formulation using the Levi form of $\log(1 + |f|^2)$ appears in [21]. Our contribution here is to state it explicitly for the polydisc geometry, which will be used in the subsequent slice analysis.

Adaptation to the polydisc geometry. While the Marty-type criterion in the form of Theorem 2.1 is known for general domains in $\mathbb{C}^n$ [18, 22], its application to the polydisc requires careful consideration of the anisotropic geometry. Unlike the unit ball, where the automorphism group acts transitively and the Kobayashi metric is isotropic, the polydisc $\mathbb{D}^n$ is a product domain with a distinguished coordinate structure. The spherical derivative $f^\sharp(z)$ defined in Eq (2.2) involves a supremum over all directions, but the bound $f^\sharp(z) \le C_K$ obtained from Theorem 2.1 does not reveal how the estimate depends on the individual coordinates. The novelty of our approach lies in connecting this global bound to coordinate-wise estimates via the slice families $\mathcal{F}_{a,j}$. As shown in the proof of Theorem 2.2, the polydisc geometry allows us to decompose the Levi form into a sum of coordinate contributions, leading to the explicit bound

$$f^\sharp(z) \le \sqrt{n}\, M_K \max_j \frac{1}{1-|z_j|^2},$$

which reflects the product structure. Thus, Theorem 2.1 serves not merely as a known result, but as a crucial bridge between the one-dimensional Marty theory on slices and the full several-variables normality on the polydisc.

For $a = (a_1, \dots, a_n) \in \mathbb{D}^n$ and $1 \le j \le n$, define the coordinate slice family

$$\mathcal{F}_{a,j} := \Big\{\lambda \mapsto f(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) : f \in \mathcal{F}\Big\} \subset \mathcal{O}(\mathbb{D}). \tag{2.4}$$

Having established a differential–geometric characterization of normality, we next turn to a slice description that reflects the product structure of the polydisc.

Theorem 2.2. Let $\mathcal{F} \subset \mathcal{O}(\mathbb{D}^n)$. Then, $\mathcal{F}$ is normal on $\mathbb{D}^n$ if, and only if, for every $a \in \mathbb{D}^n$ and every $1 \le j \le n$, the slice family $\mathcal{F}_{a,j}$ is normal on $\mathbb{D}$.

Proof. Necessity. If $\mathcal{F}$ is normal on $\mathbb{D}^n$, then its restriction to any complex submanifold is normal.

Sufficiency. Assume that for every $a \in \mathbb{D}^n$ and every $1 \le j \le n$, the slice family

$$\mathcal{F}_{a,j} = \{\lambda \mapsto f(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) : f \in \mathcal{F}\} \subset \mathcal{O}(\mathbb{D}) \tag{2.5}$$

is normal on $\mathbb{D}$.

Let $K \subset \mathbb{D}^n$ be an arbitrary compact set. Choose $r < 1$ such that

$$K \subset \{z \in \mathbb{D}^n : |z_j| \le r,\ j = 1, \dots, n\}. \tag{2.6}$$

For each $j \in \{1, \dots, n\}$ and each $a \in K$, the one-variable Marty theorem applied to the normal family $\mathcal{F}_{a,j}$ yields a constant $M_{a,j}$ such that for all $f \in \mathcal{F}$ and all $\lambda \in \mathbb{D}$ with $(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) \in K$,

$$(1-|\lambda|^2)\frac{\left|\partial f/\partial z_j(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n)\right|}{1+|f(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n)|^2} \le M_{a,j}. \tag{2.7}$$

For each $(a, j) \in K \times \{1, \dots, n\}$, the one-variable Marty theorem provides a constant $M_{a,j}$ satisfying Eq (2.7). The function $(a, j) \mapsto M_{a,j}$ is upper semicontinuous, hence, it attains a maximum on the compact set $K \times \{1, \dots, n\}$. Let M_K be this maximum. Then, Eq (2.7) holds with M_K for all $a \in K$, all $1 \le j \le n$, all $f \in \mathcal{F}$, and all λ with $(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) \in K$.

Now fix $z = (z_1, \dots, z_n) \in K$ and $f \in \mathcal{F}$. Applying Eq (2.7) with $a = (z_1, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n)$ and $\lambda = z_j$, we obtain

$$\frac{\left|\partial f/\partial z_j(z)\right|}{1+|f(z)|^2} \le \frac{M_K}{1-|z_j|^2}, \qquad j = 1, \dots, n. \tag{2.8}$$

Let $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ be a unit vector ($\|v\| = 1$). By the formula for the Levi form of $\log(1+|f|^2)$,

$$L_z(\log(1 + |f|^2), v) = \frac{\left|\sum_{j=1}^{n} (\partial f/\partial z_j)(z)\, v_j\right|^2}{(1 + |f(z)|^2)^2}. \tag{2.9}$$

Using the Cauchy–Schwarz inequality together with Eq (2.8), we estimate

$$\begin{aligned}
\left|\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\, v_j\right| &\le \sum_{j=1}^{n} \left|\frac{\partial f}{\partial z_j}(z)\right| |v_j| \\
&\le \sum_{j=1}^{n} \frac{M_K}{1 - |z_j|^2} (1 + |f(z)|^2)\, |v_j| \\
&\le M_K (1 + |f(z)|^2) \Big(\sum_{j=1}^{n} \frac{|v_j|^2}{(1 - |z_j|^2)^2}\Big)^{1/2} \Big(\sum_{j=1}^{n} 1\Big)^{1/2} \\
&= M_K \sqrt{n}\, (1 + |f(z)|^2) \Big(\sum_{j=1}^{n} \frac{|v_j|^2}{(1 - |z_j|^2)^2}\Big)^{1/2}.
\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.9) yields

$$L_z(\log(1 + |f|^2), v) \le n M_K^2 \sum_{j=1}^{n} \frac{|v_j|^2}{(1 - |z_j|^2)^2} = n M_K^2\, ds^2_{\mathbb{D}^n}(z; v). \tag{2.11}$$

Hence,

$$f^\sharp(z; v)^2 = L_z(\log(1 + |f|^2), v) \le n M_K^2\, ds^2_{\mathbb{D}^n}(z; v),$$

and taking the supremum over all $\|v\| = 1$ gives

$$f^\sharp(z) \le \sqrt{n}\, M_K \max_{1 \le j \le n} \frac{1}{1 - |z_j|^2}, \qquad z \in K. \tag{2.12}$$

Therefore, $f^\sharp$ is locally uniformly bounded on $\mathbb{D}^n$, and by Theorem 2.1 the family $\mathcal{F}$ is normal on $\mathbb{D}^n$. □

The slice characterization allows us to introduce a natural notion of a normal *function* on the polydisc, in analogy with the ball case.

Definition 2.3. A function $f \in \mathcal{O}(\mathbb{D}^n)$ is called *normal* if the family

$$\mathcal{F}_f := \{f \circ \phi : \phi \in \mathrm{Aut}(\mathbb{D}^n)\}$$

is normal on $\mathbb{D}^n$. Here, $\mathrm{Aut}(\mathbb{D}^n)$ is the group of biholomorphic self-maps of $\mathbb{D}^n$. For $\phi \in \mathrm{Aut}(\mathbb{D}^n)$, the composition $f \circ \phi$ is defined by

$$(f \circ \phi)(z) = f(\phi(z)), \qquad z \in \mathbb{D}^n. \tag{2.13}$$

Remark 2.4. Two different spherical derivatives appear in this paper, each serving a distinct purpose in the proofs:

- The *Euclidean spherical derivative* $f^\sharp(z)$ defined in Eq (2.2) uses the Euclidean norm $\|v\|$ in the supremum. It is convenient for pointwise estimates and appears in the final characterization Eq (2.19).
- The *Bergman spherical derivative* $f^\sharp_{\mathrm{Berg}}(z)$ defined in Eq (2.15) normalizes by the Poincaré metric length $ds_{\mathbb{D}^n}(z;v)$. Its key property is automorphism invariance (Lemma 2.6), which allows us to transfer estimates from arbitrary points to the origin.

The logical flow of the proof of Proposition 2.7 illustrates the interplay between the two: We first use the invariance of the Bergman derivative to obtain $f^\sharp_{\mathrm{Berg}}(z) \le C_1$ for all z, then convert this bound to the Euclidean derivative via the equivalence of norms at each point. This two-step strategy—invariant Bergman estimate followed by Euclidean conversion—is essential for obtaining the sharp growth condition (2.19).

We now give a metric characterization of normal functions on $\mathbb{D}^n$ in terms of the spherical derivative and the product Poincaré metric.

Lemma 2.5. [23, 24] Let $\Omega, \Omega' \subset \mathbb{C}^n$ be domains, let $\phi\colon \Omega \to \Omega'$ be a holomorphic mapping, and let $\psi \in C^2(\Omega')$. Then, for every $z \in \Omega$ and every $v \in \mathbb{C}^n$, we have

$$L_z(\psi \circ \phi,\ v) = L_{\phi(z)}(\psi,\ \phi'(z)v). \tag{2.14}$$

Lemma 2.6. Let $\phi \in \mathrm{Aut}(\mathbb{D}^n)$ and let $f \in \mathcal{O}(\mathbb{D}^n)$. Define the Bergman spherical derivative of f by

$$f^\sharp_{\mathrm{Berg}}(z) := \sup_{v\neq 0} \frac{\sqrt{L_z(\log(1+|f|^2), v)}}{ds_{\mathbb{D}^n}(z;v)}. \tag{2.15}$$

Then

$$(f\circ\phi)^\sharp_{\mathrm{Berg}}(z) = f^\sharp_{\mathrm{Berg}}(\phi(z)), \qquad z \in \mathbb{D}^n. \tag{2.16}$$

Proof. Let $z \in \mathbb{D}^n$ and $v \in \mathbb{C}^n$, $v \neq 0$. By the chain rule for the Levi form (Lemma 2.5), we have

$$L_z\left(\log(1+|f\circ\phi|^2), v\right) = L_{\phi(z)}\left(\log(1+|f|^2), \phi'(z)v\right). \tag{2.17}$$

On the other hand, the Bergman (product Poincaré) metric on the polydisc is invariant under automorphisms:

$$ds_{\mathbb{D}^n}(z;v) = ds_{\mathbb{D}^n}(\phi(z);\ \phi'(z)v), \qquad z \in \mathbb{D}^n,\ v \in \mathbb{C}^n. \tag{2.18}$$

Combining (2.17) and (2.18), we obtain

$$\frac{\sqrt{L_z(\log(1+|f\circ\phi|^2), v)}}{ds_{\mathbb{D}^n}(z;v)} = \frac{\sqrt{L_{\phi(z)}(\log(1+|f|^2), \phi'(z)v)}}{ds_{\mathbb{D}^n}(\phi(z);\phi'(z)v)}.$$

Taking the supremum over all $v \neq 0$ yields

$$(f\circ\phi)^\sharp_{\mathrm{Berg}}(z) = f^\sharp_{\mathrm{Berg}}(\phi(z)).$$

This proves the invariance. □

The invariance established in Lemma 2.6 is the key property that makes the Bergman spherical derivative useful: It allows us to reduce estimates at arbitrary points to the origin, where computations are simpler.

Proposition 2.7. A function $f \in \mathcal{O}(\mathbb{D}^n)$ is normal if, and only if, there exists $C > 0$ such that

$$f^\sharp(z) \le C \max_{1\le j\le n} \frac{1}{1-|z_j|^2}, \qquad z \in \mathbb{D}^n. \tag{2.19}$$

Proof. Sufficiency. Assume that f is normal, i.e., the family $\mathcal{F}_f = \{f \circ \phi : \phi \in \mathrm{Aut}(\mathbb{D}^n)\}$ is normal. By Theorem 2.1, for the compact set $K = \{0\}$, there exists a constant $C_0 > 0$ such that

$$g^\sharp(0) \le C_0 \qquad \text{for all } g \in \mathcal{F}_f,$$

where $g^\sharp$ denotes the spherical derivative with respect to the Euclidean norm. By Theorem 2.1, there exists $C_0 > 0$ such that $g^\sharp(0) \le C_0$ for all $g \in \mathcal{F}_f$. At the origin, the Euclidean norm $\|v\|$ and the Bergman metric length $\sqrt{ds^2_{\mathbb{D}^n}(0; v)}$ are equivalent: there exists $A > 0$ such that

$$\frac{1}{A}\|v\| \le \sqrt{ds^2_{\mathbb{D}^n}(0; v)} \le A\|v\|, \qquad v \in \mathbb{C}^n. \tag{2.20}$$

Here we use the equivalence of norms at the origin to convert the Euclidean bound $g^\sharp(0) \le C_0$ into a Bergman bound $g^\sharp_{\mathrm{Berg}}(0) \le C_1$. This conversion is possible because at $z = 0$, the Euclidean norm and the Poincaré metric are comparable, as noted in Eq (2.20). The Bergman bound then propagates to all points via invariance, while the final estimate (2.19) is expressed in terms of the Euclidean derivative for convenience. Consequently, the two spherical derivatives satisfy $g^\sharp_{\mathrm{Berg}}(0) \le A\, g^\sharp(0) \le AC_0$. Setting $C_1 = AC_0$, we obtain

$$g^\sharp_{\mathrm{Berg}}(0) \le C_1 \qquad \text{for all } g \in \mathcal{F}_f, \tag{2.21}$$

where for any $h \in \mathcal{O}(\mathbb{D}^n)$, we define

$$h^\sharp_{\mathrm{Berg}}(z) = \sup_{v\neq 0} \frac{\sqrt{L_z(\log(1+|h|^2), v)}}{ds_{\mathbb{D}^n}(z; v)}.$$

Since the family $\mathcal{F}_f$ is normal, by the Marty type criterion there exists a constant $C_1 > 0$ such that

$$h^\sharp_{\mathrm{Berg}}(0) \le C_1 \quad \text{for all } h \in \mathcal{F}_f.$$

Now fix an arbitrary point $z \in \mathbb{D}^n$. Choose a Möbius transformation ϕ_j of the unit disc with $\phi_j(z_j)$=0 for each $j = 1, \ldots, n$ and set $\phi = (\phi_1, \ldots, \phi_n)$. Then, $\phi \in \mathrm{Aut}(\mathbb{D}^n)$ and $\phi(z) = 0$. Put $g = f \circ \phi^{-1} \in \mathcal{F}_f$. Using the invariance of the Bergman spherical derivative under automorphisms, we obtain

$$f^\sharp_{\mathrm{Berg}}(z) = (f \circ \phi^{-1})^\sharp_{\mathrm{Berg}}(0) = g^\sharp_{\mathrm{Berg}}(0) \le C_1.$$

Consequently, for every $v \in \mathbb{C}^n$,

$$L_z(\log(1+|f|^2), v) \le C_1^2\, ds^2_{\mathbb{D}^n}(z; v) = C_1^2 \sum_{j=1}^n \frac{|v_j|^2}{(1-|z_j|^2)^2}.$$

In particular, taking v to be the j-th coordinate direction gives

$$\frac{|\partial f/\partial z_j|^2}{(1+|f|^2)^2} \le C_1^2 \frac{1}{(1-|z_j|^2)^2},$$

hence,

$$\frac{|\partial f/\partial z_j|}{1+|f|^2} \le C_1 \frac{1}{1-|z_j|^2}, \qquad j = 1, \ldots, n. \tag{2.22}$$

Now, let $v = (v_1, \ldots, v_n)$ be a unit vector in the Euclidean norm. Using Eq (2.22) and the Cauchy–Schwarz inequality,

$$\begin{aligned}\frac{|f'(z)v|}{1+|f|^2} &\le \sum_{j=1}^{n} \frac{|\partial f/\partial z_j|\,|v_j|}{1+|f|^2} \le C_1 \sum_{j=1}^{n} \frac{|v_j|}{1-|z_j|^2} \\ &\le C_1 \Big(\sum_{j=1}^{n} \frac{1}{(1-|z_j|^2)^2}\Big)^{1/2} \Big(\sum_{j=1}^{n} |v_j|^2\Big)^{1/2} \\ &\le C_1 \sqrt{n} \max_{1\le j\le n} \frac{1}{1-|z_j|^2}. \end{aligned} \tag{2.23}$$

Taking the supremum over all unit vectors v yields

$$f^\sharp(z) \le C_1 \sqrt{n} \max_{1\le j\le n} \frac{1}{1-|z_j|^2},$$

so we may take

$$C = C_1 \sqrt{n}.$$

Necessity. Assume that there exists a constant $C > 0$ such that for all $z \in \mathbb{D}^n$,

$$f^\sharp(z) \le C \max_{1\le j\le n} \frac{1}{1-|z_j|^2}. \tag{2.24}$$

We have to show that $\mathcal{F}_f$ is normal. According to Theorem 2.1, it suffices to prove that for every compact set $K \subset \mathbb{D}^n$, there is a constant M_K such that

$$g^\sharp(z) \le M_K$$

for all $g \in \mathcal{F}_f,\ z \in K$.

Let K be compact. Choose $r < 1$ with $K \subset \{z : |z_j| \le r,\ j = 1, \ldots, n\}$. Fix $z \in K$ and $\phi \in \mathrm{Aut}(\mathbb{D}^n)$ and put $g = f \circ \phi$. Write $\phi = (\phi_1, \ldots, \phi_n)$ with each ϕ_j, a Möbius transformation of the unit disc. Then, the derivative $\phi'(z)$ is a diagonal matrix with entries $\phi_j'(z_j)$, and its Euclidean operator norm equals

$$\|\phi'(z)\| = \max_{1\le j\le n} |\phi_j'(z_j)| = \max_{1\le j\le n} \frac{1-|\phi_j(z_j)|^2}{1-|z_j|^2}, \tag{2.25}$$

where we used the well-known derivative formula for Möbius transformations of the disc.

Now estimate $g^\sharp(z)$ using the chain rule and (2.24):

$$g^\sharp(z) = \frac{|(f\circ\phi)'(z)|}{1+|f(\phi(z))|^2} \le \frac{|f'(\phi(z))|\,\|\phi'(z)\|}{1+|f(\phi(z))|^2} = f^\sharp(\phi(z))\,\|\phi'(z)\|$$

$$\le C\Big(\max_{1\le j\le n}\frac{1}{1-|\phi_j(z_j)|^2}\Big)\Big(\max_{1\le j\le n}\frac{1-|\phi_j(z_j)|^2}{1-|z_j|^2}\Big)$$

$$= C\max_{1\le j\le n}\frac{1}{1-|z_j|^2}. \tag{2.26}$$

Because $z\in K$, we have $|z_j|\le r$ for every j, whence

$$\frac{1}{1-|z_j|^2}\le\frac{1}{1-r^2}.$$

Thus,

$$g^\sharp(z)\le\frac{C}{1-r^2}$$

for all $g\in\mathcal{F}_f,\ z\in K$, and the righthand side is a constant independent of g and $z\in K$. Hence, $\mathcal{F}_f$ is normal by Theorem 2.1, i.e., f is a normal function. □

This shows that normal functions on the polydisc are precisely those whose spherical derivatives grow at most at the rate dictated by the product Poincaré geometry.

3. Examples

3.1. Example illustrating Theorem 2.2

We present two simple families on $\mathbb{D}^2$ to illustrate the slice characterization (Theorem 2.2) and to contrast normal versus non-normal behavior.

Example 3.1. (A non-normal family) Consider

$$\mathcal{F}=\{f_k(z_1,z_2)=kz_1z_2 : k\in\mathbb{N}\}.$$

Fix $a=(a_1,a_2)\in\mathbb{D}^2$. The coordinate slice families are

$$\mathcal{F}_{a,1}=\{\lambda\mapsto ka_2\lambda : k\in\mathbb{N}\},\qquad \mathcal{F}_{a,2}=\{\lambda\mapsto ka_1\lambda : k\in\mathbb{N}\}.$$

If $a_2\ne 0$, the slice family $\mathcal{F}_{a,1}$ is not normal on $\mathbb{D}$. Indeed, for any fixed $\lambda\ne 0$,

$$|ka_2\lambda|\to\infty\quad(k\to\infty),$$

while $ka_2\cdot 0 = 0$ for all k. Hence, the sequence $\{ka_2\lambda\}$ does not admit a subsequence converging uniformly on compact subsets of $\mathbb{D}$ either to a holomorphic function or to ∞. Thus, $\mathcal{F}_{a,1}$ is not normal.

Choosing, for instance, $a=(1/2,1/2)$, we obtain a non-normal slice family. By Theorem 2.2, this implies that $\mathcal{F}$ is not normal on $\mathbb{D}^2$.

Example 3.2. (A normal family) Consider

$$\mathcal{G} = \{g_k(z_1, z_2) = z_1^k : k \in \mathbb{N}\}.$$

Let $a = (a_1, a_2) \in \mathbb{D}^2$. The slice families are

$$\mathcal{G}_{a,1} = \{\lambda \mapsto \lambda^k : k \in \mathbb{N}\}, \qquad \mathcal{G}_{a,2} = \{\lambda \mapsto a_1^k : k \in \mathbb{N}\}.$$

The family $\mathcal{G}_{a,1}$ is the classical sequence of monomials on the unit disc. For every compact set $K \subset \mathbb{D}$ there exists $r < 1$ such that $|\lambda| \le r$ on K, and, hence,

$$|\lambda^k| \le r^k \to 0.$$

Therefore, $\lambda^k \to 0$ uniformly on compact subsets of $\mathbb{D}$, so $\mathcal{G}_{a,1}$ is normal.

The slice family $\mathcal{G}_{a,2}$ consists of constant functions a_1^k, and every family of constants is trivially normal.

Thus, every slice family of $\mathcal{G}$ is normal. By Theorem 2.2, the family $\mathcal{G}$ is normal on $\mathbb{D}^2$.

3.2. Example illustrating Proposition 2.7

We now present two functions on $\mathbb{D}^2$ to demonstrate the metric criterion for normal functions established in Proposition 2.7.

Example 3.3. (A normal function) Consider the function $f(z_1, z_2) = z_1$, which depends only on the first variable. Then, $\partial f/\partial z_1 = 1$, $\partial f/\partial z_2 = 0$, and for any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$f^\sharp(z) = \sup_{\|v\|=1} \frac{|v_1|}{1 + |z_1|^2} = \frac{1}{1 + |z_1|^2}.$$

Since $1 + |z_1|^2 \ge 1 - |z_1|^2$, we have

$$f^\sharp(z) \le \frac{1}{1 - |z_1|^2} \le \max_{j=1,2} \frac{1}{1 - |z_j|^2},$$

so the inequality (2.19) holds with $C = 1$. Hence f is normal on $\mathbb{D}^2$ by Proposition 2.7 (indeed, one may verify directly that the family $\{f \circ \phi : \phi \in \mathrm{Aut}(\mathbb{D}^2)\}$ is normal).

Example 3.4. (A non-normal function) Let

$$h(z_1, z_2) = \exp\left(\frac{i}{1 - z_1}\right).$$

Since the function $(1 - z_1)^{-1}$ is holomorphic on $\mathbb{D}$, the function h is holomorphic on $\mathbb{D}^2$. Moreover, $|h(z)| \equiv 1$.

A direct computation gives

$$\frac{\partial h}{\partial z_1} = \frac{i}{(1 - z_1)^2} h(z_1), \qquad \frac{\partial h}{\partial z_2} = 0.$$

Hence,

$$|h'(z_1)| = \frac{1}{|1 - z_1|^2},$$

and therefore the spherical derivative is

$$h^\sharp(z) = \frac{|h'(z_1)|}{1 + |h(z_1)|^2} = \frac{1}{2|1 - z_1|^2}.$$

Choose

$$z^{(k)} = \Big(1 - \frac{1}{k}, 0\Big), \quad k \geq 2.$$

Then,

$$h^\sharp(z^{(k)}) = \frac{k^2}{2}.$$

On the other hand,

$$1 - |z_1^{(k)}|^2 = 1 - \left(1 - \frac{1}{k}\right)^2 = \frac{2}{k} - \frac{1}{k^2},$$

so

$$\max_{j=1,2} \frac{1}{1 - |z_j^{(k)}|^2} = \frac{1}{\frac{2}{k} - \frac{1}{k^2}} \sim \frac{k}{2} \quad (k \to \infty).$$

Consequently,

$$\frac{h^\sharp(z^{(k)})}{\max_j (1 - |z_j^{(k)}|^2)^{-1}} \sim k \to \infty.$$

Thus, no constant C can satisfy

$$h^\sharp(z) \leq C \max_j \frac{1}{1 - |z_j|^2}$$

for all $z \in \mathbb{D}^2$. By Proposition 2.7, h is not a normal function on $\mathbb{D}^2$.

4. Conclusions

In this paper, we developed a theory of normal families and normal functions on the unit polydisc $\mathbb{D}^n$ that fully respects its product geometry. Our main results are threefold.

First, we established a slice characterization (Theorem 2.2): A family $\mathcal{F} \subset O(\mathbb{D}^n)$ is normal if and only if for every point $a \in \mathbb{D}^n$ and every coordinate direction j, the one-dimensional slice family $\mathcal{F}_{a,j}$ is normal on the unit disc. This reduces normality in several variables to normality in one variable along all coordinate lines, and it reflects the anisotropic nature of the polydisc where automorphisms do not act transitively.

Second, we introduced a natural notion of normal functions on $\mathbb{D}^n$ via invariance under the full automorphism group $\mathrm{Aut}(\mathbb{D}^n)$. We then proved a metric criterion (Proposition 2.7): A holomorphic function f on $\mathbb{D}^n$ is normal if and only if its spherical derivative $f^\sharp(z)$ satisfies the growth estimate

$$f^\sharp(z) \leq C \max_{1 \leq j \leq n} \frac{1}{1 - |z_j|^2}, \quad z \in \mathbb{D}^n,$$

for some constant $C > 0$. This criterion is sharp, as shown by the examples in Section 3, and it provides a practical tool to test normality.

Third, our approach highlights the interplay between two different spherical derivatives—the Euclidean one and the Bergman (Poincaré) one—and shows how the product Poincaré metric on $\mathbb{D}^n$ naturally appears in the estimates. The invariance of the Bergman spherical derivative under automorphisms (Lemma 2.6) plays a crucial role in reducing the problem to the origin.

These results extend the classical one-variable theory to a higher-dimensional setting where the domain is not homogeneous but has a product structure. Potential future directions include: (i) extending the slice characterization to other product domains such as bounded symmetric domains of tube type; (ii) studying normal families of holomorphic mappings between polydiscs; and (iii) applying the metric criterion to investigate value distribution and uniqueness problems for normal functions on $\mathbb{D}^n$.

Author contributions

Qing Li: wrote the initial draft of the paper; Guobin Lin: provided the main research ideas of the paper and further improved the paper quality. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

During the preparation of this work, the authors used generative artificial intelligence (AI) tools solely for language polishing. No AI tool was used to generate mathematical content, proofs, theorems, or original research ideas. All mathematical reasoning, derivations, examples, and conclusions are the sole work of the authors. The authors take full responsibility for the accuracy, originality, and integrity of the content presented herein.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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