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Research article

On total edge irregularity strength of arithmetic ladder graphs

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Abstract: Let G be a simple, connected, and undirected graph. An edge irregular total κ-labeling of G is a mapping that assigns each vertex and each edge an integer from $\{1, 2, \dots, \kappa\}$ such that distinct edges receive distinct weights, where the weight of an edge is defined as the sum of its label and the labels of its end vertices. The smallest such κ is called the total edge irregularity strength of G, denoted by $tes(G)$. In this paper, we determine the exact value of $tes(G)$ for the class of arithmetic ladder graphs with l levels, k columns, and difference d, where $l \geq 2$, $k \geq 2$, and $d \geq 1$. The results are obtained via explicit constructions of optimal total labelings.

Keywords: graph labeling; total edge irregularity strength; arithmetic ladder graph
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1. Introduction

Let $G = (V(G), E(G))$ be a simple, finite, and undirected graph. According to Wallis [1], a labeling (or valuation) of a graph is a mapping that assigns numbers to elements of the graph, namely vertices, edges, or their union. A labeling defined on $V(G) \cup E(G)$ is called a total labeling. As noted by Gallian [2], various types of graph labelings have been studied.

Baća et al. [3] introduced the concept of total irregular labeling. A total κ-irregular labeling of a graph $G = (V(G), E(G))$ is a mapping $f\colon V(G) \cup E(G) \to \{1, 2, \dots, \kappa\}$ such that for any two distinct edges xy and $x'y'$, their weights $\omega_f(xy) = f(x) + f(xy) + f(y)$ and $\omega_f(x'y') = f(x') + f(x'y') + f(y')$ are distinct. The minimum such κ is called the total edge irregularity strength of G, denoted by $tes(G)$. Baća et al. [3] also established the lower bound

$$tes(G) \geq \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\},$$

where $\Delta(G)$ denotes the maximum degree of G.

Ivanco and Jendrol' in [4] proposed the following conjecture determining the exact value of $tes(G)$ for arbitrary graphs except for the complete graph K_5 of order 5.

Conjecture 1. ([4]) *For any graph G different from K_5,*

$$tes(G) = \max\left\{\left\lceil\frac{|E(G)| + 2}{3}\right\rceil, \left\lceil\frac{\Delta(G) + 1}{2}\right\rceil\right\}.$$

The Ivančo–Jendrol' conjecture has been verified for various fundamental graph classes, which can be broadly categorized into two groups. The first group includes dense and symmetric graphs, such as complete graphs and complete bipartite graphs [5], where the edge density often dictates the total edge irregularity strength. The second, more relevant group consists of structured and sparse graphs characterized by repetitive subunits. This class includes trees, as studied in [4]. Other examples of structured graphs are book graphs and double book graphs, as discussed in [6], as well as arithmetic book graphs investigated in [7]. In addition, several "snake-like" graphs have been examined, including quinted snake graphs and related variants in [8], and uniform theta snake graphs in [9].

Among these, ladder-related graphs [10] and staircase graphs [11, 12] represent a specific lineage of planar graphs formed by augmenting paths or cycles. While these previous studies addressed basic ladder or staircase structures, they often focused on fixed or simpler growth patterns. Arithmetic ladder graphs, characterized by parameters (l, k, d), generalize these concepts by introducing an arithmetic progression in their construction. This added complexity, specifically the variable difference d and levels l, presents a more rigorous test for the Ivančo–Jendrol' conjecture than the simpler ladder variants previously studied.

These results indicate that the conjecture holds for graph families with structured and repetitive patterns. However, despite this growing body of evidence, the total edge irregularity strength of arithmetic ladder graphs with general parameters has not yet been established. In this paper, we determine the exact value of the total edge irregularity strength of arithmetic ladder graphs with l levels, k columns, and difference d, thereby confirming the Ivanco–Jendrol' conjecture for this graph family.

2. Result

In this section, the arithmetic ladder graph with l levels, k columns, and difference d is defined. The vertices in the arithmetic ladder graph with l levels, k columns, and difference d are indexed by i, r, and j, which respectively represent the level number, the column number, and the vertex number within a column.

Definition 2.1. Suppose we have three arbitrary positive integers $l \geq 2, k \geq 2$, and $d \geq 1$. By arithmetic ladder graph $T_{l,k,d}$ of l levels, k columns, and difference d, we define a graph with a vertex set and an edge set as follows:

$$\begin{aligned}
V(T_{l,k,d}) &= \{t^0_{i,0} | 1 \leq i \leq l\} \cup \{t^r_{i,j} | 1 \leq i \leq l, 1 \leq j \leq 1 + (r-1)d, 1 \leq r \leq k\},\\
E(T_{l,k,d}) &= \{t^0_{i,0}t^1_{i,1} | 1 \leq i \leq l\} \cup \{t^r_{i,j}t^{r+1}_{i,1} | 1 \leq i \leq l, j = 1 + (r-1)d, 1 \leq r \leq k-1\}\\
&\cup \{t^r_{i,j}t^r_{i,j+1} | 1 \leq i \leq l, 1 \leq j \leq (r-1)d, 2 \leq r \leq k\} \cup \{t^0_{i,0}t^0_{i+1,0} | 1 \leq i \leq l-1\}\\
&\cup \{t^r_{i,j}t^r_{i+1,j} | 1 \leq i \leq l-1, j = 1 + (r-1)d, 1 \leq r \leq k\}.
\end{aligned}$$

The arithmetic ladder graph of l levels, k columns, and difference d, as in Definition 2.1, is shown in Figure 1.

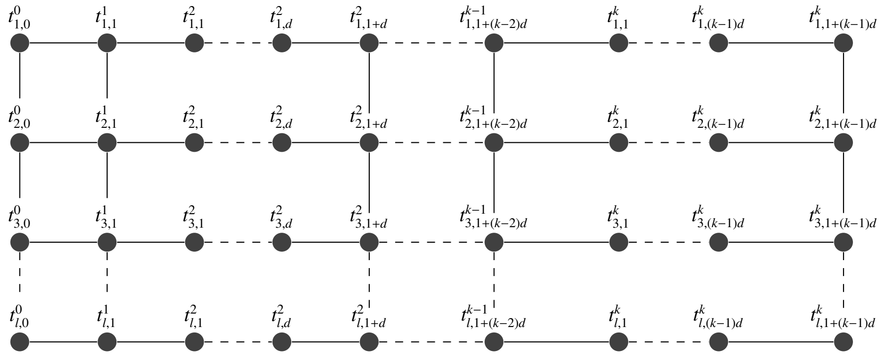

Figure 1. Arithmetic ladder graph $T_{l,k,d}$.

By a simple calculation we have that

i. $|V(T_{l,k,d})| = \frac{l(dk^2+2k-dk+2)}{2}$.
ii. $|E(T_{l,k,d})| = l\left(\frac{dk^2+2k-dk}{2}\right) + (l-1)(k+1)$.

The construction of the total labeling in the following lemmas and theorem is fundamentally driven by the grid-like and arithmetic nature of $T_{l,k,d}$. To ensure the edge weights are distinct and satisfy the lower bound of Ivančo–Jendrol', we employ a sequential column-wise labeling strategy.

It is intuitive to treat each level l and column r as a unit of progression. We assign vertex labels as base values that increase linearly from the first column ($r = 1$) to the last ($r = k$), while edge labels serve as offsets to ensure no two weights overlap. The parameter α is then determined as the minimum value that accommodates the growth rate d while meeting the lower bound.

Before presenting the exact value of $tes(T_{l,k,d})$ for general parameters, we first establish several lemmas for fixed values of d. These special cases are crucial as they reveal the underlying sequential labeling patterns and serve as the foundational building blocks for the proof of the main theorem.

Lemma 2.2. *Let $d = 1$ be fixed. Then, for any $l \geq 2$ and $k \geq 2$, we have*

$$tes(T_{l,k,1}) = \left\lceil \frac{k^2 l + 3kl - 2k + 2l + 2}{6} \right\rceil.$$

Proof. An arithmetic ladder graph $T_{l,k,1}$ has maximum degree $\Delta(T_{l,k,1}) = 4$. Therefore, by

$$tes(G) \geq max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\},$$

see [3], we have

$$tes(T_{l,k,1}) \geq \left\lceil \frac{k^2 l + 3kl - 2k + 2l + 2}{6} \right\rceil.$$

It is sufficient to show that there is an edge irregular total κ-labeling with $\kappa = \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$. First, we determine the biggest positive number α such that

$$\frac{\alpha^2 l + 2\alpha l - 2\alpha + 4}{4} \leq \left\lceil \frac{k^2 l + 3kl - 2k + 2l + 2}{6} \right\rceil.$$

Several values of l, k, and α are listed in Table 1.

Table 1. List of several l, k, and α for $T_{l,k,1}$.

l	3	200	750	2	2	2	3	3	3	99	250	399	100	45678	987654
k	2	2	2	3	10	100	3	50	200	99	300	50	100	123	10000
α	2	2	2	2	8	81	2	41	163	81	245	41	81	100	8165

We define a total κ-labeling

$$\delta : V(T_{l,k,1}) \cup E(T_{l,k,1}) \to \left\{1, 2, 3, \ldots, \left\lceil \frac{k^2 l + 3kl - 2k + 2l + 2}{6} \right\rceil \right\}$$

to label the graph elements as given in Table 2.

Table 2. Vertex and edge labels under δ for $\alpha = k$.

Vertex and edge label	$i, j,$ and r
$\delta(t_{i,0}^0) = 1$	$1 \le i \le l$
$\delta(t_{i,j}^r) = \frac{lr^2+2lr-2r+4}{4}$	$1 \le i \le l, j = r, 1 \le r \le \alpha, r$ is even
$\delta(t_{i,j}^r) = \frac{lr^2+2lr-2r+l+2}{4}$	$1 \le i \le l, j = r, 1 \le r \le \alpha, r$ is odd
$\delta(t_{i,j}^r) = \frac{lr^2-2r+4}{4} + \frac{lj}{2}$	$1 \le i \le l, 1 \le j \le r-1, j$ is even, $1 \le r \le \alpha, r$ is even
$\delta(t_{i,j}^r) = \frac{lr^2-2r+4l}{4} + \frac{l(j-1)}{2}$	$1 \le i \le l, 1 \le j \le r-1, j$ is odd, $1 \le r \le \alpha, r$ is even
$\delta(t_{i,j}^r) = \frac{lr^2-2r-l+6}{4} + \frac{lj}{2}$	$1 \le i \le l, 1 \le j \le r-1, j$ is even, $1 \le r \le \alpha, r$ is odd
$\delta(t_{i,j}^r) = \frac{lr^2-2r+3l+2}{4} + \frac{l(j-1)}{2}$	$1 \le i \le l, 1 \le j \le r-1, j$ is odd, $1 \le r \le \alpha, r$ is odd
$\delta(t_{i,0}^0 t_{i+1,0}^0) = i$	$1 \le i \le l-1$
$\delta(t_{i,j}^r t_{i+1,j}^r) = \frac{lr}{2} + i$	$1 \le i \le l-1, j = r, 1 \le r \le \alpha, r$ is even
$\delta(t_{i,j}^r t_{i+1,j}^r) = \frac{lr-l+2}{2} + i$	$1 \le i \le l-1, j = r, 1 \le r \le \alpha, r$ is odd
$\delta(t_{i,0}^0 t_{i,1}^1) = i$	$1 \le i \le l$
$\delta(t_{i,j}^r t_{i,1}^{r+1}) = \frac{lr}{2} + i$	$1 \le i \le l, j = r, 1 \le r \le \alpha-1, r$ is even
$\delta(t_{i,j}^r t_{i,1}^{r+1}) = \frac{lr-l+2}{2} + i$	$1 \le i \le l, j = r, 1 \le r \le \alpha-1, r$ is odd
$\delta(t_{i,j}^r t_{i,j+1}^r) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l, 1 \le j \le r-1, 1 \le r \le \alpha, r$ is even
$\delta(t_{i,j}^r t_{i,j+1}^r) = \frac{lr-l}{2} + i$	$1 \le i \le l, 1 \le j \le r-1, 1 \le r \le \alpha, r$ is odd

The procedure terminates whenever $\alpha = k$. Otherwise, if $\alpha < k$, we define the parameters P and Q as

$$P = \left\lceil \frac{k^2 l + 3kl - 2k + 2l + 2}{6} \right\rceil - \frac{lr^2 - l - 2r + 6}{4}$$

and

$$Q = \left\lceil \frac{k^2 l + 3kl - 2k + 2l + 2}{6} \right\rceil - \frac{lr^2 - 2r + 4}{4}.$$

With these values of P and Q, the process can be continued accordingly, as given in Tabel 3.

Table 3. Vertex and edge labels under δ for $\alpha < k$.

Vertex and edge label	i	j	r
$\delta(t^r_{i,j}) = \frac{lr^2-l-2r+6}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le \bar{j}$, j even	$r = \alpha + 1$, α even
$\delta(t^r_{i,j}) = \frac{lr^2-l-2r+6}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le \bar{j}$, j odd	
$\delta(t^r_{i,j}) = \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > \bar{j}$	
$\delta(t^r_{i,j}) = \frac{lr^2-2r+4}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le \bar{\bar{j}}$, j even	$r = \alpha + 1$, α odd
$\delta(t^r_{i,j}) = \frac{lr^2-2r+4}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le \bar{\bar{j}}$, j odd	
$\delta(t^r_{i,j}) = \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > \bar{\bar{j}}$	
$\delta(t^r_{i,j}) = \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$1 \le j \le r$	$\alpha + 1 < r \le k$
For $P < l$			
$\delta(t^{r-1}_{i,j} t^r_{i,1}) = \frac{lr^2+2lr+l-2r+2}{4} + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = \alpha$	$r = \alpha + 1, \alpha$ even
$\delta(t^r_{i,j}, t^r_{i,j+1}) = \frac{lr^2+lr-2r+4}{2} + lj + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$1 \le j \le \alpha$	$r = \alpha + 1, \alpha$ even
For $P \ge l$			
$\delta(t^{r-1}_{i,j} t^r_{i,1}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$j = \alpha$	$r = \alpha + 1, \alpha$ even
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$1 \le j < \bar{j}$	$r = \alpha + 1, \alpha$ even
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$j = \frac{2P}{l} - 1$	$r = \alpha + 1, \alpha$ even
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr^2+2lr+l-2r+2}{4} + \frac{lj}{2} + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 2\left\lfloor \frac{P}{l} \right\rfloor$	$r = \alpha + 1, \alpha$ even
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr^2+lr-2r+4}{2} + lj + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$j > \bar{j}$	$r = \alpha + 1, \alpha$ even
For $Q < l$			
$\delta(t^{r-1}_{i,j} t^r_{i,1}) = \frac{lr^2+2lr-2r+4}{4} + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = \alpha$	$r = \alpha + 1, \alpha$ odd
$\delta(t^r_{i,j}, t^r_{i,j+1}) = \frac{lr^2+lr-2r+4}{2} + lj + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$1 \le j \le \alpha$	$r = \alpha + 1, \alpha$ odd
For $Q \ge l$			
$\delta(t^{r-1}_{i,j} t^r_{i,1}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$j = \alpha$	$r = \alpha + 1, \alpha$ odd
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$1 \le j < \bar{\bar{j}}$	$r = \alpha + 1, \alpha$ odd
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$j = \frac{2Q}{l} - 1$	$r = \alpha + 1, \alpha$ odd
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr^2+2lr-2r+4}{4} + \frac{lj}{2} + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 2\left\lfloor \frac{Q}{l} \right\rfloor$	$r = \alpha + 1, \alpha$ odd
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr^2+lr-2r+4}{2} + lj + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$j > \bar{\bar{j}}$	$r = \alpha + 1, \alpha$ odd
$\delta(t^r_{i,j} t^r_{i+1,j}) = \frac{lr^2+3lr-2r+4}{2} + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l - 1$	$j = r$	$\alpha + 1 \le r \le k$
$\delta(t^r_{i,j} t^{r+1}_{i,1}) = \frac{lr^2+3lr-2r+2l+2}{2} + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$j = r$	$\alpha + 1 \le r \le k - 1$
$\delta(t^r_{i,j} t^r_{i,j+1}) = \frac{lr^2+lr-2r+4}{2} + lj + i - \left\lceil \frac{k^2l+3kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$1 \le j \le r - 1$	$\alpha + 1 < r \le k$

Where

$$\bar{j} = \begin{cases} \frac{2P}{l} - 1, & \text{if } P \equiv 0 \pmod{l}, \\ 2\left\lfloor \frac{P}{l} \right\rfloor, & \text{if } P \not\equiv 0 \pmod{l}, \end{cases}$$

and

$$\bar{\bar{j}} = \begin{cases} \frac{2Q}{l} - 1, & \text{if } Q \equiv 0 \pmod{l}, \\ 2\left\lfloor \frac{Q}{l} \right\rfloor, & \text{if } Q \not\equiv 0 \pmod{l}. \end{cases}$$

We then have the weights of the edges as shown in Table 4.

Table 4. Weight of edges under δ.

Weights	i	j	r
$wt_\delta(t^0_{i,0}t^0_{i+1,0}) = 2 + i$	$1 \le i \le l-1$	–	–
$wt_\delta(t^0_{i,0}t^1_{i,1}) = 1 + i + l$	$1 \le i \le l$	–	–
$wt_\delta(t^r_{i,j}t^r_{i+1,j}) = \frac{lr^2+3lr-2r+4}{2} + i$	$1 \le i \le l-1$	$j = r$	$1 \le r \le k$
$wt_\delta(t^r_{i,j}t^{r+1}_{i,1}) = \frac{lr^2+3lr-2r+2l+2}{2} + i$	$1 \le i \le l$	$j = r$	$1 \le r \le k-1$
$wt_\delta(t^r_{i,j}t^r_{i,j+1}) = \frac{lr^2+lr-2r+4}{2} + lj + i$	$1 \le i \le l$	$1 \le j \le r-1$	$2 \le r \le k$

To show that the weights of $E(T_{l,k,d})$ constitute all integers from 3 up to $\frac{k^2l+3kl-2k+2l+2}{2}$, we examine the range of weights produced by each edge set in Table 4:

(1) The weights $wt_\delta(t^0_{i,0}t^0_{i+1,0}) = 2 + i$ for $1 \le i \le l-1$ yield the set $\{3, 4, \ldots, l+1\}$.

(2) The weights $wt_\delta(t^0_{i,0}t^1_{i,1}) = 1 + i + l$ for $1 \le i \le l$ yield the set $\{l+2, l+3, \ldots, 2l+1\}$.
Notice that the maximum of the first set $(l+1)$ is immediately followed by the minimum of the second set $(l+2)$. We continue this analysis for the remaining edges involving parameters r and j.

(3) For a fixed r, the weights $wt_\delta(t^r_{i,j}t^r_{i,j+1})$ for $1 \le j \le r-1$ and $wt_\delta(t^r_{i,j}t^r_{i+1,j})$ for $j = r$ are designed to fill the interval between columns. Specifically, when $r = 1$, the sequence continues from $2l + 2$.

(4) By substituting the boundary values of $i \in \{1, l\}$, $j \in \{1, r\}$, and $r \in \{1, k\}$ into the formulas in Table 4, it can be shown that the minimum weight occurs at $wt_\delta(t^0_{1,0}t^0_{2,0}) = 2 + 1 = 3$, and the maximum weight is $\frac{k^2l+3kl-2k+2l+2}{2}$, which occurs when $i = l-1$ and $j = r = k$.

Since each subsequent range of weights begins exactly at the maximum of previous weights plus 1, the union of all edge weights forms a complete set of consecutive integers. Thus, the labeling is a total edge irregular labeling. This completes the proof. □

Lemma 2.3. *Let $l = 2$ be fixed. Then, for any $k \ge 2$ and $d \ge 1$, we have*

$$tes(T_{2,k,d}) = \left\lceil \frac{dk^2 - dk + 3k + 3}{3} \right\rceil.$$

Proof. An arithmetic ladder graph $T_{2,k,d}$ has maximum degree $\Delta(T_{2,k,d}) = 3$. Therefore, by

$$tes(G) \ge max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\},$$

see [3], we have

$$tes(T_{2,k,d}) \ge \left\lceil \frac{dk^2 - dk + 3k + 3}{3} \right\rceil.$$

It is sufficient to show that there is an edge irregular total κ-labeling with

$$\kappa = \left\lceil \frac{dk^2 - dk + 3k + 3}{3} \right\rceil.$$

First, we determine the biggest positive number α such that

$$\frac{\alpha^2 d - \alpha d + 2\alpha + 2}{2} \leq \left\lceil \frac{dk^2 - dk + 3k + 3}{3} \right\rceil.$$

Several values k, d, and α for $T_{2,k,d}$ are listed in Table 5.

Table 5. List of several k, d, and α for $T_{2,k,d}$.

k	2	3	36	78	196	56	625	1024	78896	16000	1000000
d	2	1	20	2	3	77	25	2048	1	100	100000
α	2	2	29	63	160	45	510	836	64418	13064	816496

We define a total κ-labeling

$$\sigma : V(T_{2,k,d}) \cup E(T_{2,k,d}) \to \left\{1, 2, \ldots, \left\lceil \frac{dk^2 - dk + 3k + 3}{3} \right\rceil\right\}$$

to label the graph elements as given in Table 6.

Table 6. Vertex and edge labels under σ for $\alpha = k$.

Vertex and edge label	$i, j,$ and r
$\sigma(t^0_{i,0}) = 1$	$1 \leq i \leq 2$
$\sigma(t^r_{i,j}) = \frac{dr^2 - dr + 2r + 2}{2}$	$1 \leq i \leq 2, j = 1 + (r-1)d, 1 \leq r \leq \alpha$
$\sigma(t^r_{i,j}) = \frac{dr^2 - 3dr + 2r + 2d}{2} + j$	$1 \leq i \leq 2, 1 \leq j \leq (r-1)d, 1 \leq r \leq \alpha$
$\sigma(t^0_{1,0}t^0_{2,0}) = 1$	–
$\sigma(t^r_{1,j}t^r_{2,j}) = r + i$	$j = 1 + (r-1)d, 1 \leq r \leq \alpha$
$\sigma(t^0_{i,0}t^1_{i,1}) = i$	$1 \leq i \leq 2$
$\sigma(t^r_{i,j}t^{r+1}_{i,1}) = r + i$	$1 \leq i \leq 2, j = 1 + (r-1)d, 1 \leq r \leq \alpha - 1$
$\sigma(t^r_{i,j}t^r_{i,j+1}) = r - 1 + i$	$1 \leq i \leq 2, 1 \leq j \leq (r-1)d, 1 \leq r \leq \alpha$

We stop the process if $\alpha = k$. If

$$\alpha < k, \quad S = \left\lceil \frac{dk^2 - dk + 3k + 3}{3} \right\rceil - \frac{dr^2 - 3dr + 2d + 2r}{2},$$

then we continue to label the vertices and the edges as given in Table 7.

Table 7. Vertex and edge labels under σ for $\alpha < k$.

Vertex and edge label	i	j	r
If S is even			$r = \alpha + 1$
$\sigma(t^r_{i,j}) = \frac{dr^2-3dr+2d+2r}{2} + j$	$1 \le i \le 2$	$1 \le j \le S-1$,	
$\sigma(t^r_{i,j}) = \left\lceil \frac{dk^2-dk+3k+3}{3} \right\rceil$	$1 \le i \le 2$	$j > S-1$	
If S is odd			
$\sigma(t^r_{i,j}) = \frac{dr^2-3dr+2d+2r}{2} + j$	$1 \le i \le 2$	$1 \le j \le 2\left\lfloor \frac{S}{2} \right\rfloor$,	
$\sigma(t^r_{i,j}) = \left\lceil \frac{dk^2-dk+3k+3}{3} \right\rceil$	$1 \le i \le 2$	$j > 2\left\lfloor \frac{S}{2} \right\rfloor$	
$\sigma(t^r_{i,j}) = \left\lceil \frac{dk^2-dk+3k+3}{3} \right\rceil$	$1 \le i \le 2$	$1 \le j \le 1+(r-1)d$	$\alpha + 1 < r \le k$
If $S < 2$			
$\sigma(t^{r-1}_{i,j}, t^r_{i,1}) = \frac{dr^2-3dr+2d+4r}{2} + i - \left\lceil \frac{dk^2-dk+3k+3}{3} \right\rceil$	$1 \le i \le 2$	$j = 1+(\alpha-1)d$	$r = \alpha + 1$
$\sigma(t^r_{i,j}, t^r_{i,j+1}) = dr^2 - 3dr + 2d + 3r + 2j + i - \left\lceil \frac{2dk^2-2dk+6k+6}{3} \right\rceil$	$1 \le i \le 2$	$1 \le j \le \alpha d$	$r = \alpha + 1$
If $S \ge 2$			
$\sigma(t^{r-1}_{i,j} t^r_{i,1}) = r - 1 + i$	$1 \le i \le 2$	$j = 1+(\alpha-1)d$	–
$\sigma(t^r_{i,j} t^r_{i,j+1}) = r - 1 + i$	$1 \le i \le 2$	$1 \le j < \hat{j}$	$r = \alpha + 1$
$\sigma(t^r_{i,j} t^r_{i,j+1}) = r - 1 + i$	$1 \le i \le 2$	$j = S-1$	$r = \alpha + 1$
$\sigma(t^r_{i,j} t^r_{i,j+1}) = \frac{dr^2-3dr+2d+4r}{2} + j + i - \left\lceil \frac{dk^2-dk+3k+3}{3} \right\rceil$	$1 \le i \le 2$	$j = 2\lfloor \frac{S}{2} \rfloor$	$r = \alpha + 1$
$\sigma(t^r_{i,j} t^r_{i,j+1}) = dr^2 - 3dr + 2d + 3r + 2j + i - \left\lceil \frac{2dk^2-2dk+6k+6}{3} \right\rceil$	$1 \le i \le 2$	$j > \hat{j}$	$r = \alpha + 1$
$\sigma(t^r_{1,j} t^r_{2,j}) = dr^2 - dr + 3r + 3 - \left\lceil \frac{2dk^2-2dk+6k+6}{3} \right\rceil$	–	$j = 1+(r-1)d$	$\alpha + 1 \le r \le k$
$\sigma(t^r_{i,j} t^{r+1}_{i,1}) = dr^2 - dr + 3r + 3 + i - \left\lceil \frac{2dk^2-2dk+6k+6}{3} \right\rceil$	$1 \le i \le 2$	$j = 1+(r-1)d$	$\alpha + 1 \le r \le k-1$
$\sigma(t^r_{i,j} t^r_{i,j+1}) = dr^2 - 3dr + 3r + 2d + 2j + i - \left\lceil \frac{2dk^2-2dk+6k+6}{3} \right\rceil$	$1 \le i \le 2$	$1 \le j \le (r-1)d$	$\alpha + 1 < r \le k$

Where

$$\hat{j} = \begin{cases} S-1, & \text{if } S \text{ even}, \\ 2\left\lfloor \frac{S}{l} \right\rfloor, & \text{if } S \text{ odd}. \end{cases}$$

The weights of the edges are as shown in Table 8.

Table 8. Weight of edges under σ

Weights	i	j	r
$wt_\sigma(t^0_{1,0} t^0_{2,0}) = 3$	–	–	–
$wt_\sigma(t^0_{i,0} t^1_{i,1}) = 3 + i$	$1 \le i \le 2$	–	–
$wt_\sigma(t^r_{1,j} t^r_{2,j}) = dr^2 - dr + 3r + 3$	–	$j = 1+(r-1)d$	$1 \le r \le k$
$wt_\sigma(t^r_{i,j} t^{r+1}_{i,1}) = dr^2 - dr + 3r + 3 + i$	$1 \le i \le 2$	$j = 1+(r-1)d$	$1 \le r \le k-1$
$wt_\sigma(t^r_{i,j} t^r_{i,j+1}) = dr^2 - 3dr + 3r + 2d + 2j + i$	$1 \le i \le 2$	$1 \le j \le (r-1)d$	$2 \le r \le k$

To verify that the weights of edges in $E(T_{2,k,d})$ under σ constitute all consecutive integers from 3 up to the required upper bound, we evaluate the range of weights for each sub-collection of edges presented in Table 8.

(1) The weight of the starting edge $wt_\sigma(t^0_{1,0}t^0_{2,0}) = 3$ provides the minimum value.

(2) For $1 \le i \le 2$, the weights $wt_\sigma(t^0_{i,0}t^1_{i,1}) = 3 + i$ yield the set $\{4, 5\}$.

(3) For the horizontal and vertical edges associated with columns $1 \le r \le k$, the weights $wt_\sigma(t^r_{1,j}t^r_{2,j})$ for $j = 1 + (r-1)d$ yield the value $dr^2 - dr + 3r + 3$ and the weights $wt_\sigma(t^r_{i,j}t^{r+1}_{i,1})$ and $wt_\sigma(t^r_{i,j}t^r_{i,j+1})$ fill the remaining gaps.

By substituting the boundary values of $i \in \{1, 2\}$, $j \in \{1, \ldots, 1 + (r-1)d\}$, and $r \in \{1, \ldots, k\}$, we observe a seamless transition between the sets of weights. Specifically, the maximum weight of one edge category is exactly $w - 1$, where w is the minimum weight of the succeeding category in the sequence. The maximum weight $dk^2 - dk + 3k + 3$ is achieved for the edge $t^k_{1,j}t^k_{2,j}$. Since these weights form a set of consecutive integers without any gaps or repetitions, we conclude that the labeling σ is a total edge irregular labeling that attains the lower bound. □

The above two lemmas reveal the underlying labeling patterns for $T_{l,k,d}$ when one parameter is fixed. These patterns serve as building blocks that are combined and generalized to obtain the exact value of $tes(T_{l,k,d})$ in the general case.

Theorem 2.4. *For arbitrary positive integers $l \ge 2, k \ge 2$, and $d \ge 1$, it follows that*

$$tes(T_{l,k,d}) = \left\lceil \frac{dk^2l - dkl + 4kl - 2k + 2l + 2}{6} \right\rceil.$$

Proof. The proof proceeds by first identifying a maximal integer α such that the induced labeling on the first α columns guarantees distinct edge weights. The remaining columns are then labeled using a saturation technique to avoid collisions with previously assigned weights. It is already clear that

$$tes(T_{l,k,d}) \ge \left\lceil \frac{dk^2l - dkl + 4kl - 2k + 2l + 2}{6} \right\rceil.$$

Hence, it is sufficient to show that there is a total edge irregularity κ-labeling on $T_{l,k,d}$ with

$$\kappa = \left\lceil \frac{dk^2l - dkl + 4kl - 2k + 2l + 2}{6} \right\rceil.$$

First, we determine the biggest positive number α depending on whether d is even or odd.

(1) If d is even, we determine the biggest positive number α such that

$$\frac{dl\alpha^2 - dl\alpha + 4l\alpha - 4\alpha + 4}{4} \le \left\lceil \frac{dlk^2 - dlk + 4lk - 2k + 2l + 2}{6} \right\rceil.$$

(2) If d is odd, we determine the biggest positive number p and q such that

$$\frac{dlp^2 - dlp + 3lp - 2p + 4}{4} \le \left\lceil \frac{dlk^2 - dlk + 4lk - 2k + 2l + 2}{6} \right\rceil$$

and

$$\frac{dlq^2 - dlq + 3lq - 2q + l + 2}{4} \le \left\lceil \frac{dlk^2 - dlk + 4lk - 2k + 2l + 2}{6} \right\rceil.$$

Then, we have $\alpha = \max\{p, q\}$.

Several values of l, k, d, and α are listed in Table 9.

Table 9. List of several d, l, k, and α for $T_{l,k,d}$

d	1	1	1	2	2	3	3	3	18	27	55	99	233	500	1000	10000
l	2	2	2	2	2	3	3	4	23	444	555	999	3	777	45	20000
k	2	3	50	3	100	7	150	233	15	60	100	70	555	999	999	1000000
α	2	2	41	2	81	5	122	190	12	49	81	57	453	815	815	816496

We define a total κ-labeling

$$\psi : V(T_{l,k,d}) \cup E(T_{l,k,d}) \to \left\{1, 2, \ldots, \left\lceil \frac{ldk^2 - ldk + 4lk - 2k + 2l + 2}{6} \right\rceil\right\}$$

to label the graph elements in the following way:

(1) If the difference d is an even integer, then the labels are given in Table 10.

Table 10. Vertex and edge labels under ψ when d is an even integer and $\alpha = k$

Vertex and edge label	i	j	r
$\psi(t_{i,0}^0) = 1$	$1 \le i \le l$	–	–
$\psi(t_{i,j}^r) = \frac{dlr^2 - dlr + 4lr - 4r + 4}{4}$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le \alpha$
$\psi(t_{i,j}^r) = \frac{dlr^2 - 3dlr + 4lr + 2dl - 4l - 4r + 8}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le (r-1)d$, j is even	$1 \le r \le \alpha$
$\psi(t_{i,j}^r) = \frac{dlr^2 - 3dlr + 4lr + 2dl - 4r + 4}{4} + \frac{l(j-1)}{2}$	$1 \le i \le l$	$1 \le j \le (r-1)d$, j is odd	$1 \le r \le \alpha$
$\psi(t_{i,0}^0 t_{i+1,0}^0) = i$	$1 \le i \le l-1$	–	–
$\psi(t_{i,j}^r t_{i+1,j}^r) = r + i$	$1 \le i \le l-1$	$j = 1 + (r-1)d$	$1 \le r \le \alpha$
$\psi(t_{i,0}^0 t_{i,1}^1) = i$	$1 \le i \le l-1$	–	–
$\psi(t_{i,j}^r t_{i,1}^{r+1}) = r + i$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le \alpha - 1$
$\psi(t_{i,j}^r t_{i,j+1}^r) = r - 1 + i$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$1 \le r \le \alpha$

We stop the process if $\alpha = k$. If

$$\alpha < k, \quad C = \left\lceil \frac{ldk^2 - ldk + 4lk - 2k + 2l + 2}{6} \right\rceil - \frac{dlr^2 - 3dlr + 2dl + 4lr - 4l - 4r + 8}{4},$$

then we continue the labeling as given in Table 11.

Table 11. Vertex and edge labels under ψ when d is an even integer and $\alpha < k$.

Vertex and edge label	i	j	r
If $C \equiv 0 \pmod{l}$			$r = \alpha + 1$
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+4lr-4l-4r+8}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le \frac{2C}{l} - 1$, j is even	$r = \alpha + 1$
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+4lr-4l-4r+8}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le \frac{2C}{l} - 1$, j is odd	$r = \alpha + 1$
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > \frac{2C}{l} - 1$	$r = \alpha + 1$
If $C \not\equiv 0 \pmod{l}$			$r = \alpha + 1$
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+4lr-4l-4r+8}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le 2\left\lfloor \frac{C}{l} \right\rfloor$, j is even	$r = \alpha + 1$
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+4lr-4l-4r+8}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le 2\left\lfloor \frac{C}{l} \right\rfloor$, j is odd	$r = \alpha + 1$
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > 2\left\lfloor \frac{C}{l} \right\rfloor$	$r = \alpha + 1$
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$1 \le j \le 1 + (r-1)d$	$\alpha + 1 < r \le k$
If $C < l$			
$\psi(t^{r-1}_{i,j}, t^r_{i,1}) = \frac{r^2dl-3rdl+2dl+4rl}{4} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 1 + (\alpha-1)d$	$r = \alpha + 1$
$\psi(t^r_{i,j}, t^r_{i,j+1}) = \frac{dlr^2-3dlr+2dl+4lr-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$1 \le j \le \alpha d$	$r = \alpha + 1$
If $C \ge l$			
$\psi(t^{r-1}_{i,j} t^r_{i,1}) = r - 1 + i$	$1 \le i \le l$	$j = 1 + (\alpha-1)d$	$r = \alpha + 1$
$\psi(t^r_{i,j} t^r_{i,j+1}) = \alpha + i$	$1 \le i \le l$	$1 \le j < j^*$	$r = \alpha + 1$
$\psi(t^r_{i,j} t^r_{i,j+1}) = \alpha + i$	$1 \le i \le l$	$j = \left\lfloor \frac{C}{l} \right\rfloor + 1$	$r = \alpha + 1$
$\psi(t^r_{i,j} t^r_{i,j+1}) = \frac{dlr^2-3dlr+2dl+4lr}{4} + \frac{lj}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 2\left\lfloor \frac{C}{l} \right\rfloor$	$r = \alpha + 1$
$\psi(t^r_{i,j} t^r_{i,j+1}) = \frac{dlr^2-3dlr+2dl+4lr-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$j > j^*$	$r = \alpha + 1$
$\psi(t^r_{i,j} t^r_{i+1,j}) = \frac{dlr^2-dlr+4lr-2r+4}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l-1$	$j = 1 + (r-1)d$	$\alpha + 1 \le r \le k$
$\psi(t^r_{i,j} t^{r+1}_{i,1}) = \frac{dlr^2-dlr+4lr-2r+2l+2}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$j = 1 + (r-1)d$	$\alpha + 1 \le r \le k$
$\psi(t^r_{i,j} t^r_{i,j+1}) = \frac{dlr^2-3dlr+4lr+2dl-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$\alpha + 1 < r \le k$

Where

$$j^* = \begin{cases} \frac{2C}{l} - 1, & \text{if } C \equiv 0 \pmod{l}, \\ 2\left\lfloor \frac{C}{l} \right\rfloor, & \text{if } C \not\equiv 0 \pmod{l}. \end{cases}$$

(2) If the difference d is an odd integer, then the labels are given in Table 12.

Table 12. Vertex and edge labels under ψ when d is an odd integer and $\alpha = k$.

Vertex and edge label	i	j	r
$\psi(t^0_{i,0}) = 1$	$1 \le i \le l$	–	–
$\psi(t^r_{i,j}) = \frac{dlr^2-dlr+3lr-2r+4}{4}$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le \alpha$, r is even
$\psi(t^r_{i,j}) = \frac{dlr^2-dlr+3lr-2r+l+2}{4}$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le \alpha$, r is odd
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+3lr-2r+2dl-2l+4}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le (r-1)d$, j is even	$1 \le r \le \alpha$, r is even
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+3lr-2r+2dl}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le (r-1)d$, j is odd	$1 \le r \le \alpha$, r is even
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+3lr-2r+2dl-3l+6}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le (r-1)d$, j is even	$1 \le r \le \alpha$, r is odd
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+3lr-2r+2dl-l+2}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le (r-1)d$, j is odd	$1 \le r \le \alpha$, r is odd

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Vertex and edge label	i	j	r
$\psi(t^0_{i,0}t^0_{i+1,0}) = i$	$1 \le i \le l-1$	–	–
$\psi(t^r_{i,j}t^r_{i+1,j}) = \frac{lr}{2} + i$	$1 \le i \le l-1$	$j = 1 + (r-1)d$	$1 \le r \le \alpha, r$ is even
$\psi(t^r_{i,j}t^r_{i+1,j}) = \frac{lr-l+2}{2} + i$	$1 \le i \le l-1$	$j = 1 + (r-1)d$	$1 \le r \le \alpha, r$ is odd
$\psi(t^0_{i,0}t^1_{i,1}) = i$	$1 \le i \le l$	–	–
$\psi(t^r_{i,j}t^{r+1}_{i,1}) = \frac{lr}{2} + i$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le \alpha - 1, r$ is even
$\psi(t^r_{i,j}t^{r+1}_{i,1}) = \frac{lr-l+2}{2} + i$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le \alpha - 1, r$ is odd
$\psi(t^r_{i,j}t^r_{i,j+1}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$1 \le r \le \alpha, r$ is even
$\psi(t^r_{i,j}t^r_{i,j+1}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$1 \le r \le \alpha, r$ is odd

The labeling process is terminated whenever $\alpha = k$. Otherwise, for $\alpha < k$, define

$$D = \left\lceil \frac{ldk^2 - ldk + 4lk - 2k + 2l + 2}{6} \right\rceil - \frac{dlr^2 - 3dlr + 2dl + 3lr - 3l - 2r + 6}{4}$$

and

$$E = \left\lceil \frac{ldk^2 - ldk + 4lk - 2k + 2l + 2}{6} \right\rceil - \frac{dlr^2 - 3dlr + 2dl + 3lr - 2l - 2r + 4}{4}.$$

In this case, the continuation of the labeling construction is presented in Table 13.

Table 13. Vertex and edge labels under ψ when d is an odd integer and $\alpha < k$.

Vertex and Edge Label	i	j	r
For $D \equiv 0 \pmod{l}$			$r = \alpha + 1$, α is even
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-3l-2r+6}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le \frac{2D}{l} - 1$, j is even	
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-3l-2r+6}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le \frac{2D}{l} - 1$, j is odd	
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > \frac{2D}{l} - 1$	
For $D \not\equiv 0 \pmod{l}$			
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-3l-2r+6}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le 2\left\lfloor \frac{D}{l} \right\rfloor$, j is even	
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-3l-2r+6}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le 2\left\lfloor \frac{D}{l} \right\rfloor$, j is odd	
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > 2\left\lfloor \frac{D}{l} \right\rfloor$	
For $E \equiv 0 \pmod{l}$			$r = \alpha + 1$, α is odd
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-2l-2r+4}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le \frac{2E}{l} - 1$, j is even	
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-2l-2r+4}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le \frac{2E}{l} - 1$, j is odd	
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > \frac{2E}{l} - 1$	
For $E \not\equiv 0 \pmod{l}$			
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-2l-2r+4}{4} + \frac{lj}{2}$	$1 \le i \le l$	$1 \le j \le 2\left\lfloor \frac{E}{l} \right\rfloor$, j is even	
$\psi(t^r_{i,j}) = \frac{dlr^2-3dlr+2dl+3lr-2l-2r+4}{4} + \frac{l(j-1)}{2} + l - 1$	$1 \le i \le l$	$1 \le j \le 2\left\lfloor \frac{E}{l} \right\rfloor$, j is odd	
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > 2\left\lfloor \frac{E}{l} \right\rfloor$	
$\psi(t^r_{i,j}) = \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$1 \le j \le 1 + (r-1)d$	$\alpha + 1 < r \le k$

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Vertex and edge label	i	j	r
For $D < l$			
$\psi(t_{i,j}^{r-1} t_{i,1}^{r}) = \frac{dlr^2-3dlr+2dl+5lr-l-2r+2}{4} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 1 + (\alpha-1)d$	$r = \alpha+1$, α even
$\psi(t_{i,j}^{r}, t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+2dl+4lr-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$r = \alpha+1$, α even
For $D \ge l$			
$\psi(t_{i,j}^{r-1} t_{i,1}^{r}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$j = 1 + (\alpha-1)d$	$r = \alpha+1$, α is even
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$1 \le j < \tilde{j}$	$r = \alpha+1$, α is even
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{lr-l}{2} + i$	$1 \le i \le l$	$1 \le j < \frac{2D}{l} - 1$	$r = \alpha+1$, α is even
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+2dl+5lr-l-2r+2}{4} + \frac{lj}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 2\left\lfloor \frac{D}{l} \right\rfloor$	$r = \alpha+1$, α is even
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+2dl+4lr-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > \tilde{j}$	$r = \alpha+1$, α is even
For $E < l$			
$\psi(t_{i,j}^{r-1} t_{i,1}^{r}) = \frac{dlr^2-3dlr+2dl+5lr-2l-2r+4}{4} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 1 + (\alpha-1)d$	$r = \alpha+1$, α is odd
$\psi(t_{i,j}^{r}, t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+2dl+4lr-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$1 \le j \le (\alpha-1)d$	$r = \alpha+1$, α is odd
For $E \ge l$			
$\psi(t_{i,j}^{r-1} t_{i,1}^{r}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$j = 1 + (\alpha-1)d$	$r = \alpha+1$, α is odd
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$1 \le j < j^{**}$	$r = \alpha+1$, α is odd
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{lr-2l+2}{2} + i$	$1 \le i \le l$	$1 \le j < \frac{2E}{l} - 1$	$r = \alpha+1$, α is odd
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+2dl+5lr-2l-2r+4}{4} + \frac{lj}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j = 2\left\lfloor \frac{E}{l} \right\rfloor$	$r = \alpha+1$, α is odd
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+2dl+4lr-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{6} \right\rceil$	$1 \le i \le l$	$j > j^{**}$	$r = \alpha+1$, α is odd
$\psi(t_{i,j}^{r} t_{i+1,j}^{r}) = \frac{dlr^2-dlr+4lr-2r+4}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l-1$	$j = 1 + (r-1)d$	$\alpha+1 \le r \le k$
$\psi(t_{i,j}^{r} t_{i,1}^{r+1}) = \frac{dlr^2-dlr+4lr-2r+2l+2}{2} + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$j = 1 + (r-1)d$	$\alpha+1 \le r \le k-1$
$\psi(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+4lr+2dl-2l-2r+4}{2} + lj + i - \left\lceil \frac{dk^2l-dkl+4kl-2k+2l+2}{3} \right\rceil$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$\alpha+1 < r \le k$

Where

$$\tilde{j} = \begin{cases} \frac{2D}{l} - 1, & \text{if } D \equiv 0 \pmod{l}, \\ 2\left\lfloor \frac{D}{l} \right\rfloor, & \text{if } D \not\equiv 0 \pmod{l}, \end{cases} \quad \text{and} \quad j^{**} = \begin{cases} \frac{2E}{l} - 1, & \text{if } E \equiv 0 \pmod{l}, \\ 2\left\lfloor \frac{E}{l} \right\rfloor, & \text{if } E \not\equiv 0 \pmod{l}. \end{cases}$$

We then have the weights of the edges as listed in Table 14.

Table 14. Weight of edges under ψ for $T_{l,k,d}$.

Weights	i	j	r
$wt_{\psi}(t_{i,0}^{0} t_{i+1,0}^{0}) = 2 + i$	$1 \le i \le l-1$	–	–
$wt_{\psi}(t_{i,0}^{0} t_{i,1}^{1}) = 1 + i + l$	$1 \le i \le l$	–	–
$wt_{\psi}(t_{i,j}^{r} t_{i+1,j}^{r}) = \frac{dlr^2-dlr+4lr-2r+4}{2} + i$	$1 \le i \le l-1$	$j = 1 + (r-1)d$	$1 \le r \le k$
$wt_{\psi}(t_{i,j}^{r} t_{i,1}^{r+1}) = \frac{dlr^2-dlr+4lr-2r+2l+2}{2} + i$	$1 \le i \le l$	$j = 1 + (r-1)d$	$1 \le r \le k-1$
$wt_{\psi}(t_{i,j}^{r} t_{i,j+1}^{r}) = \frac{dlr^2-3dlr+4lr+2ld-2l-2r+4}{2} + lj + i$	$1 \le i \le l$	$1 \le j \le (r-1)d$	$2 \le r \le k$

To rigorously verify that the weights of $E(T_{l,k,d})$ under ψ constitute a complete set of consecutive integers from 3 up to $\frac{dk^2l-dkl+4kl-2k+2l+2}{2}$, we analyze the intervals produced by each edge sub-family in Table 14:

(1) The initial segment of weights is generated by the vertical and connection edges of the base structure ($r = 0$ and the first transition to $r = 1$). These yield the set $\{3, 4, \ldots, 2l + 1\}$.
(2) For the general case where $1 \leq r \leq k$ and $1 \leq j \leq 1 + (r - 1)d$, the labeling ψ is constructed such that the weights follow the arithmetic progression of the graph's levels and columns.
(3) By examining the transition between columns, we observe that for any column r, the maximum weight achieved by the horizontal edges $wt_\psi(t^r_{i,j}t^r_{i,j+1})$ at $i = l$, $j = (r - 1)d$ is exactly one unit less than the minimum weight of the next set of edges (either vertical edges within the same column or the transition to column $r + 1$).
(4) Substituting the extremal indices $i = l - 1$, $r = k$, and $j = 1 + (k - 1)d$ into the formula:

$$wt_\psi(t^r_{i,j}t^r_{i+1,j}) = \frac{dlr^2 - dlr + 4lr - 2r + 4}{2} + i \tag{2.1}$$

confirms that the sequence reaches the upper bound defined by the lower bound of Ivanco–Jendrol' for $tes(T_{l,k,d})$.

The meticulous alignment of these weight intervals, ensures that no integer is omitted and no weight is duplicated. This completes the proof. □

It is straightforward to verify that the obtained value agrees with the lower bound of Ivanco–Jendrol', and hence confirms the conjecture for the family of arithmetic ladder graphs.

To illustrate the labeling construction that is described in Theorem 2.4, we provide explicit examples of total edge irregular labelings for the graphs $T_{2,3,1}$, $T_{3,3,2}$, and $T_{4,2,6}$, which are depicted in the Figures 2–4, respectively. In each figure, the labels assigned to vertices are displayed inside the corresponding circles, whereas the edge labels are indicated by numbers placed at the midpoints of the edges. For these examples, the corresponding values of the parameter α are 2, 2, and 1, respectively.

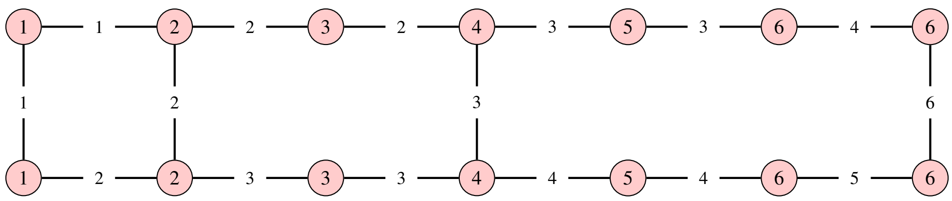

Figure 2. Total edge irregular 6-labeling of graph $T_{2,3,1}$.

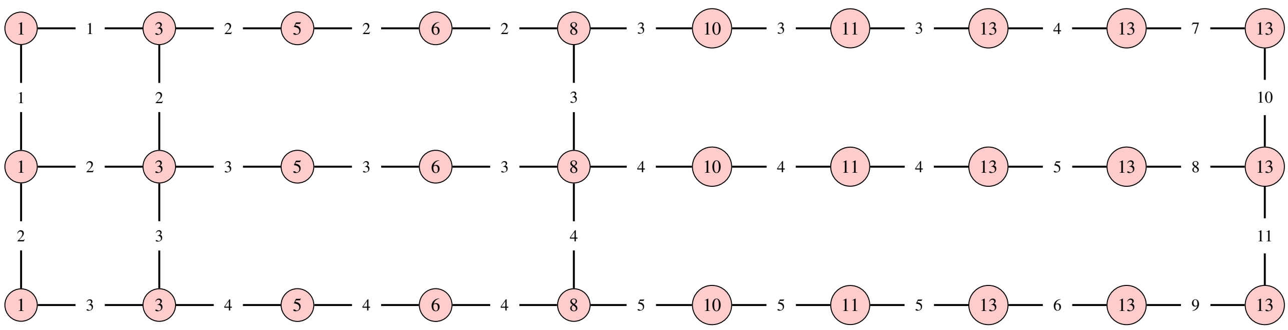

Figure 3. Total edge irregular 13-labeling of graph $T_{3,3,2}$.

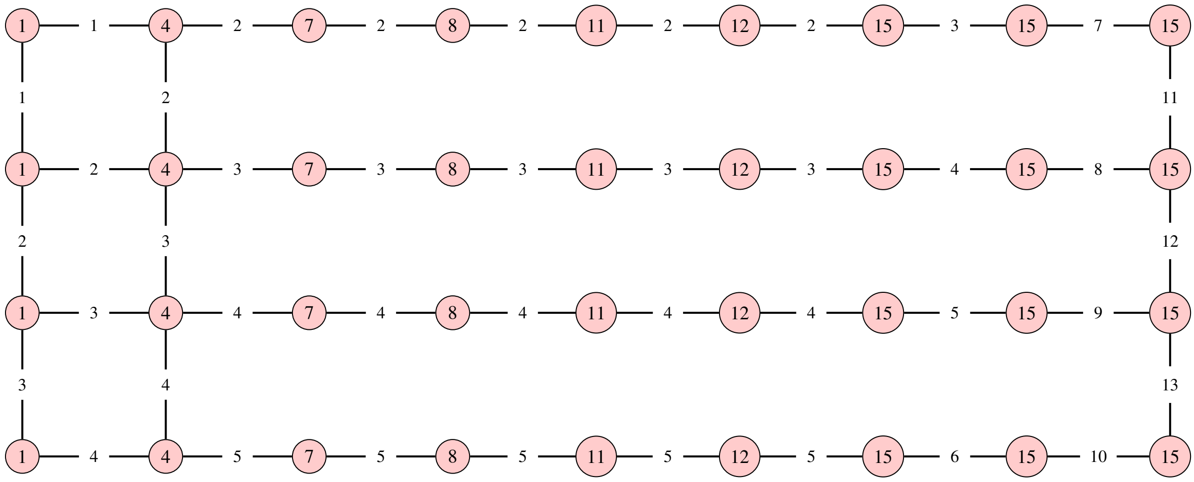

Figure 4. Total edge irregular 15-labeling of graph $T_{4,2,6}$.

3. Conclusions

In this paper, we determined the exact value of the total edge irregularity strength of arithmetic ladder graphs $T_{l,k,d}$. For all integers $l \geq 2$, $k \geq 2$, and $d \geq 1$, it is shown that

$$tes(T_{l,k,d}) = \left\lceil \frac{dk^2l - dkl + 4kl - 2k + 2l + 2}{6} \right\rceil.$$

This result confirms that the total edge irregularity strength of arithmetic ladder graphs attains the lower bound proposed by Ivanco and Jendrol', and thus supports the validity of the conjecture for this family of graphs.

Author contributions

E. Wulandari: identified the underlying patterns, developed the labeling constructions, and verified the results through Python-based simulations; Y. Susanti: conceived the research framework, supervised the study, and refined the manuscript. All authors have read and approved the final version of the manuscript.

Use of generative AI tools declaration

The authors used generative AI tools solely for language improvement, including paraphrasing and grammar checking. No AI tools were used in the development of the mathematical results, proofs, or conclusions. All outputs were carefully reviewed and verified by the authors, who take full responsibility for the content of this manuscript.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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