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Research article

Dynamic analysis of stochastic rumor model with education effect

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Abstract: With the development of the economy and society, rumors have become increasingly popular. Therefore, quantitative research on rumors has become essential. In this paper, a new rumor model with age of education is introduced by considering two different types of populations. First, we obtain the existence of global positive solutions by utilizing Itô's formula and the Lyapunov functional method. Then, several sufficient conditions for the disappearance and persistence of rumors are given, where the threshold will be affected by the age of education. Finally, to support the main results of this article, several illustrative numerical simulations are conducted.

Keywords: age-dependent education; rumor model; stochastic model; extinction; persistence in the mean
Mathematics Subject Classification: 60H10, 60H30, 92D30, 93E15

1. Introduction

In the past four decades, it has become increasingly popular to use physics methods to study different societal phenomena. This development has been due to physicists venturing outside of their traditional domains of interest, alongside scientists from other disciplines taking from proven physics methods throughout the 19th and 20th centuries [1]. With the development of the internet, rumors have become increasingly prevalent. The spread and control of rumors is a sociological issue, and its development can be quantified using physical models [2]. Moreover, with the rapid development of self media, information dissemination is faster. Therefore, the spread of rumors is also accelerated. Rumors pose a great threat to the development of society, and thus it is worth studying the dissemination mechanism of rumors. In this paper, we will introduce a new rumor model and study its dynamic properties.

Rumor models have been studied by many authors: See [3, 4], in which people were divided into a susceptible population, rumor carriers, and recovered individuals. The research on rumor propagation models originated in the 1960s. Daley-Kendall [5] first gave the following model:

$$\begin{cases} S'(t) = A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t), \\ I'(t) = \beta S(t)I(t) - (\mu + \eta)I(t) - \alpha I^2(t), \\ R'(t) = \eta I(t) - \nu R(t), \end{cases} \tag{1.1}$$

where the parameters are positive constants, S, I, R denote a susceptible person, spreaders, and the immunization population, respectively. System (1.1) is based on warehouse modeling. Later, Maki-Thompson [6] generalized system (1.1) based on the fact that rumors are spread through two-way contact between spreaders and other people. For research on the dynamic analysis of the rumor propagation model, Zanette [4] gave a spreading threshold for rumor similar to an epidemic model. As research deepens, stochastic models are introduced to characterize real-world situations [7–9]. Additionally, noise was introduced in (1.1) as follows:

$$\begin{cases} dS(t) = [A - \beta S(t)I(t) - \mu S(t) + \alpha I^2(t)]dt + \sigma_1(S(t), I(t))dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu + \eta)I(t) - \alpha I^2(t)]dt + \sigma_2(S(t), I(t))dB(t), \\ dR(t) = [\eta I(t) - \nu R(t)]dt + \sigma_3(S(t), I(t))dB(t), \end{cases}$$

where $B(t)$ denotes the Brownian motion, and σ_i $(i = 1, 2, 3)$ is the diffusion term. Rumor models are similar to epidemic models [10, 11] because of their similar mechanisms (e.g., these mechanisms can be modeled using a compartment model). On the other hand, with the development of higher education, an increasing number of people have the opportunity to attend university, thus improving their academic qualifications. Moreover, the improvement of education means an increase in the recognition of rumors. Wang et al. [12] considered rumor propagation models, which contain different education levels and hesitation mechanisms, while Tong et al. [13] studied the rumor propagation model with media coverage and class-age-dependent education. There is a lot of research that studied the stabilization of stochastic models using control theory, including the following: optimal control of a reaction-diffusion rumor propagation model [14], optimal control of a rumor propagation model with reporting effect [15], optimal control strategies for a new rumor spreading model with comprehensive interventions [16, 17], feedback control of stochastic differential equations [18], and so on. The effect of education is added because highly educated individuals are more likely to identify rumors. Thus, we introduce model (2.1).

From an economic perspective, we need to study the spread of rumors [19–21]. Some rumors can cause economic damage to businesses and individuals, and can lower the reputation of businesses. At present, there are also many organizations or individuals engaged in rumor identification, which we refer to as rumor handlers. In the process of identifying rumors, rumor handlers may be tempted to become weak rumor handlers, which means that the ability to handle rumors has weakened. In addition, to better control rumors, we can take mandatory quarantine measures against some stubborn supporters of rumors, such as sealing off certain people's video accounts and forcibly canceling certain platforms.Therefore, in this paper, we will introduce a new rumor model, which not only covers the effect of education, but also contains the behavior of the rumor handler. It is worth noting that we divided rumor handler into two parts: "Rumor susceptible handler" and "weak rumor handler". Then, we establish some sufficient conditions for the existence and extinction of rumors. Then, we provide some numerical results to verify our theoretical results.

Compared with the previous results, the main contributions of this paper are summarized as follows:

(1) The system considered in this paper is new, where the classification of rumor spreaders is more detailed, thus better aligning with the actual situation, and the impact of the age of education is considered;
(2) the effect on noise is also considered, and the numerical simulations verify our results.

The rest of the article is organized as follows: In Section 2, we establish the stochastic rumor model that considers the age of education, in Section 3, we prove that there exists a unique global positive solution of system (2.8) with any positive initial value, which is fundamental in the dynamics of population modeling; in Sections 4 and 5, we study the extinction and persistence of rumors, respectively; and finally, some numerical simulations are conducted in Section 6 to illustrate the theoretical results.

2. Formulation of the model

In this section, we first propose a new rumor model with five different populations and educational age, namely susceptible population S, rumor spreader I, rumor susceptible handler D, weak rumor handler E, rumor spreaders in quarantine Z, and educational age R, , which are defined as follows

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta_1 S(t)I(t) - \xi S(t) + \displaystyle\int_0^\infty w(\theta)\Upsilon(\theta,t)\mathrm{d}\theta - \mu_1 S(t) + kI^2(t),\\
\dfrac{\partial\Upsilon(\theta,t)}{\partial\theta} + \dfrac{\partial\Upsilon(\theta,t)}{\partial t} = -(\mu + w(\theta))\Upsilon(\theta,t),\\
\dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta_1 S(t)I(t) - kI^2(t) - aI(t)D(t) - arI(t)E(t) - \mu_2 I(t),\\
\dfrac{\mathrm{d}D(t)}{\mathrm{d}t} = \delta P - \beta_2 D(t)I(t) - \beta_3 D(t)E(t) - \mu_3 D(t),\\
\dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta_2 D(t)I(t) + \beta_3 D(t)E(t) - \mu_3 E(t),\\
\dfrac{\mathrm{d}Z(t)}{\mathrm{d}t} = aI(t)D(t) + arI(t)E(t) - \mu_4 Z(t),
\end{cases}
\tag{2.1}
$$

where the parameters' meanings of (2.1) are listed in Table 1. Figure 1 provides the model's schematic diagram. The time unit may be days, hours, or minutes. The time unit should be determined according to specific circumstances. Figure 1 shows that the system can be divided into two parts; thus, model (2.1) is simplified. The second equation of (2.1) shows the evolution for age of education; see [13] for more details. Meanwhile, model (2.1) can also be some social model (crime model [22–24]).

Table 1. Description of parameters used in the model.

parameter	description
Λ	the emigration rate of the susceptible population
μ_1	the immigration rate of the susceptible population
μ_2	the rate of rumor member exit from rumor spreader
μ_3	the rate of rumor susceptible handler becoming to weak rumor handler
μ_4	the rate of death
β_1	the recruitment rate of susceptible population into rumor spreader
β_2	the recruitment rate of rumor susceptible handler by rumor spreader
β_3	the recruitment rate of rumor susceptible handler by weak rumor handler
β_4	the rate of release of quarantine
δ	the number of rumor susceptible handlers
a	the rate of handling rumor
r	weak rumor handler at a reduced rate $r(0 < r < 1)$
k	the yield rate between rumor members
θ	the age of education
$0 \leqslant w(\theta) \leqslant 1$	the forgetting rate of educated population at the age of education θ
$\Upsilon(\theta, t)$	the density of educated population at the age of education θ and time t
$\int_0^\infty w(\theta)\Upsilon(\theta, t)d\theta$	the total educated population at time t

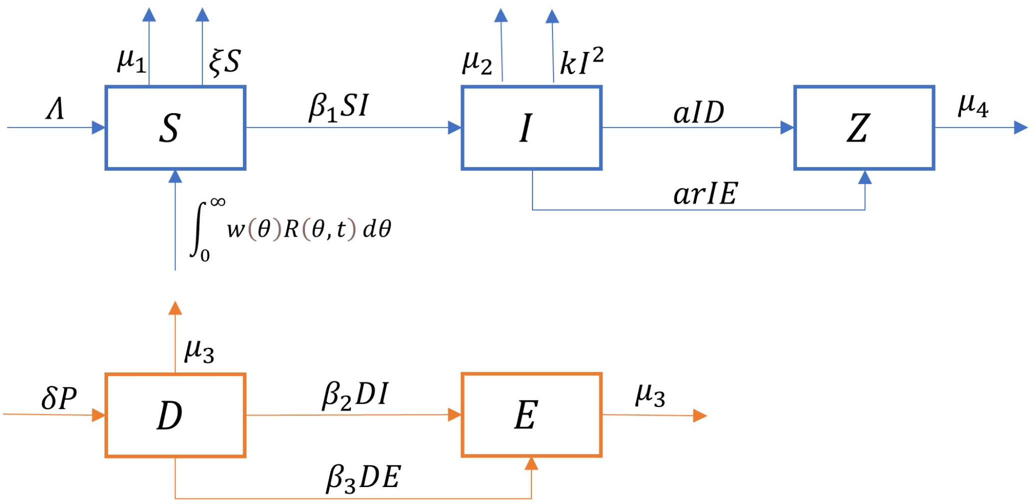


Figure 1. Model schematic diagram.

Considering that the total number of rumor handlers (including weak rumor handler) in a country is basically limited, we use P to represent the total number of rumor handlers; then, susceptible police $D(t)$ can be represented by $P - E(t)$. Therefore, we can simplify model (2.1) as follows:

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta_1 S(t)I(t) - \xi S(t) + \displaystyle\int_0^\infty w(\theta)\Upsilon(\theta,t)\mathrm{d}\theta - \mu_1 S(t) + kI^2(t),\\ \dfrac{\partial\Upsilon(\theta,t)}{\partial\theta} + \dfrac{\partial\Upsilon(\theta,t)}{\partial t} = -(\mu + w(\theta))\Upsilon(\theta,t),\\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta_1 S(t)I(t) - kI^2(t) - aI(t)(P - E(t) + rE(t)) - \mu_2 I(t),\\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta_2(P - E(t))I(t) + \beta_3(P - E(t))E(t) - \mu_3 E(t),\\ \dfrac{\mathrm{d}Z(t)}{\mathrm{d}t} = aI(t)(P - E(t) + rE(t)) - \mu_4 Z(t).\end{cases} \tag{2.2}$$

We need the following initial boundary conditions:

$$S(0) = S_0, I(0) = I_0, E(0) = E_0, \Upsilon(\theta,0) = \Upsilon_0(\theta), \Upsilon(0,t) = \xi S(t), \tag{2.3}$$

and

$$S_0 + I_0 + E_0 + Z_0 + \int_0^\infty w(\theta)\Upsilon_0(\theta)\mathrm{d}\theta = N(0),$$

where $N(0)$ is the initial value of the population, $w(\theta)$ is the forgetting rate of educated individuals at the age of education θ, and $\xi \in (0,1)$ denotes the education rate.

Inspired by [13] , assuming that model (2.2) is stable and does not depend on t, we have

$$\frac{\mathrm{d}\Upsilon(\theta)}{\mathrm{d}\theta} = -(\mu + w(\theta))\Upsilon(\theta),$$

which yields that

$$\varpi_0(a) = e^{-\int_0^a(\mu + w(\tau))\mathrm{d}\tau}.$$

Under the condition that $\int_0^\infty w(\theta)\mathrm{d}\theta = \infty$, we have that if $\theta = \infty$, then there will be no one in the educated compartment. In other words, we have the following:

$$\lim_{\theta\to\infty}\Upsilon(\theta,t) = 0. \tag{2.4}$$

On the other hand, similar to [13,25], we get the basic reproduction number

$$\mathcal{R} = \frac{\Lambda\beta_1}{(aP + \mu_2)(\mu_1 + \xi(1 - \Theta))},$$

and $\Theta = \int_0^\infty w(\theta)\varpi_0(\theta)\mathrm{d}\theta$.

In addition, the rumor handler free equilibrium is the steady state solution of model (2.2), where there is no rumor susceptible handler, weak rumor handler and rumor spreaders in quarantine in the population. Thus, the rumor handler free equilibrium of model (2.2), denoted by X^0,is given by the following:

$$X^0 = (S^0, \Upsilon^0(a), 0, 0, 0),$$

where

$$S^0 = \frac{\Lambda}{\mu_1 + \xi(1 - \Theta)},\ \ R^0(a) = \frac{\xi\Lambda}{\mu_1 + \xi(1 - \Theta)}.$$

Denote $N_0 = \frac{\Lambda}{\mu} + N(0)$. From the meanings of δ, a, k, θ that $0 \le a, k \le 1$ and $\delta, \theta \in [0, \infty)$.

Lemma 2.1. *Let* $\hat{\Omega} = \{(S(t), \Upsilon(t), I(t), E(t), Z(t)) \in \mathbb{R}^5_+ : 0 \leqslant S(t)+I(t)+E(t)+Z(t)+\int_0^\infty \Upsilon(\theta, t)\mathrm{d}\theta \leqslant N_0\}$. *Then,* $\hat{\Omega}$ *is the positive invariant set of model (2.2).*

The proof is similar to [13, Lemma 1] and so we omit the details here.

By using the Volterra formulation and integrating the second equation of model (2.2) along the characteristic line $t - \theta = c$, where c is a constant, we have (see [13] for more details) the following:

$$\Upsilon(\theta, t) = \begin{cases} \xi S(t-\theta)\varpi_0(\theta), & \text{if } t > \theta \geqslant 0, \\ \Upsilon_0(\theta - t)\dfrac{\varpi_0(\theta)}{\varpi_0(\theta - t)}, & \text{if } \theta \geqslant t > 0. \end{cases} \tag{2.5}$$

Using the characteristic line method as in [13] and introducing $\varpi(\theta) = \omega(\theta)\xi\varpi_0(\theta)$, we can obtain the following:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta_1 S(t)I(t) + \beta_4 Z(t) - \xi S(t) + \int_0^\infty \varpi(\theta)S(t-\theta)\mathrm{d}\theta - \mu_1 S(t) + kI^2(t),$$

where $\varpi(\theta) = w(\theta)\xi\varpi_0(\theta)$ and

$$\int_0^\infty \varpi(\theta)\mathrm{d}\theta \leqslant \int_0^\infty \xi e^{-\mu\theta}\mathrm{d}\theta = \frac{\xi}{\mu}. \tag{2.6}$$

Then, we can derive the following equivalent model:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta_1 S(t)I(t) + \displaystyle\int_0^\infty \varpi(\theta)S(t-\theta)\mathrm{d}\theta - (\xi + \mu_1)S(t) + kI^2(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta_1 S(t)I(t) - kI^2(t) - aI(t)(P - E(t) + rE(t)) - \mu_2 I(t), \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta_2(P - E(t))I(t) + \beta_3(P - E(t))E(t) - \mu_3 E(t), \\ \dfrac{\mathrm{d}Z(t)}{\mathrm{d}t} = aI(t)(P - E(t) + rE(t)) - \mu_4 Z(t). \end{cases} \tag{2.7}$$

Obviously, all input variables of system (2.7) follow the law of certainty and are a deterministic function of time t, thereby completely ignoring the randomness of these variables. However, in real life, uncertainty and randomness are ubiquitous, such as the psychology, behavior, and environment of rumor spreaders and rumor handlers. In order to make system (2.7) more realistic and reasonable, this paper establishes a stochastic differential equation model. Note that β_1 is the recruitment rate of the susceptible population into being a rumor spreader, and it is easily susceptible to external interference. In order to better describe real phenomena, parameters should be written in a random form. $E(t)$ is the weak rumor handler, which is easily removed due to external interference, and thus the parameter μ_3 should be random. Similarly, $Z(t)$ are the spreaders in quarantine, and the rate of release is easily disturbed by a random environment. Therefore, we can assume that

$$\beta_1 dt \to \beta_1 dt + \sigma_1 dB_1(t), \ \ \mu_3 dt \to \mu_3 dt + \sigma_3 dB_3(t), \ \ \mu_4 dt \to \mu_4 dt + \sigma_4 dB_4(t);$$

then, the following stochastic model based on model (2.7) is introduced:

$$\begin{cases} dS(t) = [\Lambda - \beta_1 S(t)I(t) + \int_0^{\infty} \varpi(\theta)S(t-\theta)d\theta - (\xi + \mu_1)S(t) + kI^2(t)]dt - \sigma_1 S(t)I(t)dB_1(t), \\ dI(t) = [\beta_1 S(t)I(t) - kI^2(t) - aI(t)(P - E(t) + rE(t)) - \mu_2 I(t)]dt + \sigma_1 S(t)I(t)dB_1(t), \\ dE(t) = [\beta_2(P - E(t))I(t) + \beta_3(P - E(t))E(t) - \mu_3 E(t)]dt + \sigma_3 E(t)dB_3(t), \\ dZ(t) = [aI(t)(P - E(t) + rE(t)) - \mu_4 Z(t)]dt + \sigma_4 Z(t)dB_4(t), \end{cases} \tag{2.8}$$

where $B_i(t), i = 1, 3, 4$ are independent standard Brownian motions, and $\sigma_i > 0, i = 1, 3, 4$ are the intensities of the environmental random disturbance. The similar stochastic models can be founded in [13, 26, 27]. Regarding the noise, there are a lot of other interesting cases; see [28] for more details.

Throughout this article, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ that satisfies the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) [29], and let $B_i(t), i = 1, 3, 4$ be defined on the complete probability space. We introduce the following notations:

$$a \vee b = \max\{a, b\} \text{ for any } a, b \in \mathbb{R}, \quad a \wedge b = \min\{a, b\} \text{ for any } a, b \in \mathbb{R},$$

and

$$\mathbb{R}_+^d = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leqslant i \leqslant d\},$$
$$\bar{\mathbb{R}}_+^d = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_i \geqslant 0, 1 \leqslant i \leqslant d\}.$$

In addition, the meaning of each parameter of the random model (2.8) is contained in Table 1. Next, some interesting conclusions will be drawn in the stochastic rumor propagation model.

3. Existence and uniqueness of the global positive solution

From the meaning of $S(t), I(t), E(t), Z(t)$ in system (2.8) that they are nonnegative. In this section, some sufficient conditions of system (2.8) are given such that system (2.8) has a unique global positive solution.

Theorem 3.1. *For any initial value $(S(0), I(0), E(0), Z(0)) \in \mathbb{R}_+^4$, system (2.8) has a unique positive solution $(S(t), I(t), E(t), Z(t))$ on $t \geqslant 0$, which almost surely remains in $\mathbb{R}_+^4$.*

Proof. The proof is similar to the statement of [13] and we only give the outline of the proof. Note that the coefficients of system (2.8) satisfy the local Lipschitz condition; hence, for any positive initial value, there is a unique local solution $(S(t), I(t), E(t), Z(t))$ on $t \in [0, \tau_e)$, where τ_e represents the explosion time [29]. In order to show that this solution is global, it suffices to show $\tau_e = \infty$ a.s. Let $k_0 \geq 1$ be large enough so that $S(0)$ and $I(0)$ lie within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the following stopping time:

$$\tau_k = inf\{t \in [0, \tau_e) | S(t) \notin (1/k, k) \text{ or } I(t) \notin (1/k, k)\},$$

where we assume that $inf\emptyset = \infty$ ($\emptyset$ represents the empty set). Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = lim_{k\to\infty}\tau_k$; then, we have $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s.

and $(S(t), I(t), E(t), Z(t)) \in \mathbb{R}_+^4$ a.s. for all $t > 0$. In other words, to complete the proof, it suffices to show $\tau_\infty = \infty$ a.s. If this claim is false, then there exists a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ that satisfy the following:

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Thus, from the meaning of the limit, it follows that there exists an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon \tag{3.1}$$

for all $k \geq k_1$. Define the following:

$$V(S, I) = (S - b - b \ln \frac{S}{b}) + (I - 1 - \ln I) + (E - P - \ln \frac{E}{P}) + (Z - 1 - \ln Z),$$

where $b > 0$ is a constant to be determined later. Itô's formula yields the following:

$$\begin{aligned}
dV(S, I, E, Z) &= (1 - \frac{b}{S})dS + (1 - \frac{1}{I})dI + \frac{1}{2}\frac{b}{S^2}(dS)^2 + \frac{1}{2}\frac{1}{I^2}(dI)^2 \\
&\quad +(1 - \frac{1}{E})dE + \frac{1}{2}\frac{1}{E^2}(dE)^2 + (1 - \frac{1}{Z})dZ + \frac{1}{2}\frac{1}{Z^2}(dZ)^2 \\
&= (1 - \frac{b}{S})\left[\Lambda - \beta_1 S I + \int_0^\infty \varpi(\theta) S(t - \theta) \mathrm{d}\theta - (\xi + \mu_1)S + kI^2\right] dt \\
&\quad +\sigma_1(S(t)I(t) - bI(t))dB_1(t) \\
&\quad +(1 - \frac{1}{I})\left[\beta_1 S I - \mu_2 I - aI(P - E + rE) - kI^2\right] dt \\
&\quad +\sigma_1(S(t)I(t) - I(t))dB_1(t) + \frac{1}{2}{\sigma_1}^2 \frac{b}{S^2} S^2 I^2 dt + \frac{1}{2}{\sigma_1}^2 \frac{1}{I^2} S^2 I^2 dt \\
&\quad +(1 - \frac{P}{E})\left[\beta_2(P - E)I + \beta_3(P - E)E - \mu_3 E\right] \mathrm{d}t \\
&\quad +(1 - \frac{1}{Z})[aI(P - E + rE) - \mu_4 Z]\mathrm{d}t \\
&\quad +\sigma_3(E - P)\mathrm{d}B_3(t) + \sigma_4(Z - 1)\mathrm{d}B_4(t) + \frac{1}{2}(\sigma_3^2 + \sigma_4^2)dt \\
&= LV(S, I, E, Z)dt + \sigma_1[2S(t)I(t) - (1 + b)I(t)]dB_1(t) \\
&\quad +\sigma_3(E - P)\mathrm{d}B_3(t) + \sigma_4(Z - 1)\mathrm{d}B_4(t),
\end{aligned}$$

where $LV(S, I, E, Z)$ is defined by the following:

$$\begin{aligned}
LV(S, I, E, Z) &= \Lambda - (\xi + \mu_1)S - \frac{\Lambda b}{S} + b\beta_1 I + (1 - \frac{b}{S}) \int_0^\infty \varpi(\theta) S(t - \theta)\mathrm{d}\theta - \frac{bk}{S} I^2 + b(\xi + \mu_1) \\
&\quad -\beta_1 S + \mu_2 - \mu_2 I + kI - I(P - E + rE) + P - E + rE + \frac{1}{2}{\sigma_1}^2(bI^2 + S^2) \\
&\quad -\beta_2(P - E)^2 I/E - \beta_3(P - E)^2 - \mu_3 E + \mu_3 P + aI(P - E + rE) \\
&\quad -aI(P - E + rE)/Z - \mu_4 Z + \mu_4 \\
&\leq \Lambda + I(b\beta_1 - \mu_2 + k) + \tilde{\Theta} N_0 + a(\xi + \mu_1)
\end{aligned}$$

$$+\mu_2 + P + \frac{1}{2}{\sigma_1}^2(aI^2 + S^2) + \mu_3 P + aI(P + rE) + \mu_4,$$

where we used (2.6) and the facts that $0 < a, r < 1$. Choose $b = \frac{\mu_2 - k}{\beta_1}$ such that $(b\beta_1 - \mu_2 + k)I = 0$; then, we get the following:

$$\begin{aligned} LV(S,I) &\leq \Lambda + \tilde{\Theta} N_0 + a(\xi + \mu_1) + \mu_2 + P + \frac{1}{2}{\sigma_1}^2(aI^2 + S^2) + \mu_3 P + aI(P + rE) + \mu_4 \\ &\leq K, \end{aligned}$$

where $K > 0$ is a constant. Consequently, we get that

$$\begin{aligned} dV(S,I,E,Z) &\leq Kdt + \sigma_1[2S(t)I(t) - (1+b)I(t)]dB_1(t) \\ &\quad + \sigma_3(E - P)\mathrm{d}B_3(t) + \sigma_4(Z - 1)\mathrm{d}B_4(t). \end{aligned} \tag{3.2}$$

Integrating (3.2) from 0 to $\tau_k \wedge T$ and taking the expectations yield the following:

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T), E(\tau_k \wedge T), Z(\tau_k \wedge T)) \leq V(S(0), I(0), E(0), Z(0)) + K\mathbb{E}(\tau_k \wedge T).$$

Therefore,

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T), E(\tau_k \wedge T), Z(\tau_k \wedge T)) \leq V(S(0), I(0), E(0), Z(0)) + KT. \tag{3.3}$$

Let $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$, and by virtue of (3.1), we obtain $\mathbb{P}(\Omega_k) \geq \epsilon$. Note that for each $\omega \in \Omega_k$, there exists $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ equal to either k or $\frac{1}{k}$. Therefore, $V(S(\tau_k, \omega), I(\tau_k, \omega), E(\tau_k, \omega), Z(\tau_k, \omega))$ is no less than either

$$k - 1 - \ln k \quad \text{or} \quad \frac{1}{k} - 1 - \ln\frac{1}{k} = \frac{1}{k} - 1 + \ln k.$$

Consequently, we obtain

$$(S(\tau_k, \omega), I(\tau_k, \omega), E(\tau_k, \omega), Z(\tau_k, \omega)) \geq [k - 1 - lnk] \wedge [\frac{1}{k} - 1 + lnk].$$

It follows from (3.3) that

$$\begin{aligned} V(S(0), I(0), E(0), Z(0)) + KT &\geq \mathbb{E}[I_{\Omega_k}(\omega)V(S(\tau_k, \omega), I(\tau_k, \omega), E(\tau_k, \omega), Z(\tau_k, \omega))] \\ &\geq \epsilon[k - 1 - \ln k] \wedge [\frac{1}{k} - 1 + \ln k], \end{aligned}$$

where I_{Ω_k} denotes the indicator function of Ω_k. Then, by letting $k \to \infty$, we have the following:

$$\infty > V(S(0), I(0), E(0), Z(0)) + KT = \infty,$$

which yields the contradiction, and $\tau_\infty = \infty$. This means that $S(t)$, $I(t)$, $E(t)$, and $Z(t)$ almost surely will not explode in a finite time. This completes the proof. □

4. The extinction of rumor spreaders

For convenience, we define the following:

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)\mathrm{d}s.$$

Let $\langle W,W\rangle_t$ be the second variation of $W(t)$. For example, if $W(t) = \int_0^t g(s)dB(s)$, then $\langle W,W\rangle_t = \int_0^t g^2(s)ds$, where $B(t)$ is a standard Brownian motion, and $g(s)$ is a adaptive process with respect to $\mathcal{F}_t := \sigma\{B(u), 0 \le u \le t\}$.

Lemma 4.1. *[29] Let $W(t)$ be a local martingale and $W(0) = 0$, then the following conclusion is ture.*

$$\limsup_{t\to\infty}\frac{\langle W,W\rangle_t}{t} < \infty\ a.s \Rightarrow \lim_{t\to\infty}\frac{W(t)}{t} = 0\ a.s..$$

Lemma 4.2. *Let $(S(t), I(t), E(t), Z(t))$ be the solution of system (2.8) with any initial value $S(0) > 0$ and $I(0) > 0$; then,*

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\int_0^t S(s)\mathrm{d}B_1(s)}{t} = 0,\ \ \lim_{t\to\infty}\frac{\int_0^t I(s)\mathrm{d}B_2(s)}{t} = 0,\\
&\lim_{t\to\infty}\frac{\int_0^t E(s)\mathrm{d}B_3(s)}{t} = 0,\ \ \lim_{t\to\infty}\frac{\int_0^t Z(s)\mathrm{d}B_4(s)}{t} = 0\ \ a.s..
\end{aligned} \tag{4.1}$$

Proof. According to Lemma 2.1, for any $\vartheta > 0$, there is a constant T_0 that satisfies the following:

$$S(t), I(t) \leqslant N_0 + \vartheta,\ t \geqslant T_0.$$

On the other hand, define $W(t) = \int_0^t S(s)\mathrm{d}B_1(s)$; then, we have that $W(t)$ is a local martingale and $W(0) = 0$; for $t \geqslant T_0$,

$$\langle W(t), W(t)\rangle = \int_0^t S^2(s)\mathrm{d}s \leqslant \int_0^{T_0} S^2(s)\mathrm{d}s + (t - T_0)(N_0 + \vartheta)^2,$$

then

$$\limsup_{t\to\infty}\frac{\langle W(t), W(t)\rangle}{t} \leqslant (N_0 + \vartheta)^2 < \infty.$$

According to Lemma 4.1, we obtain the conclusion. Similarly, by using Lemma 2.1 again, we can obtain the other equations for (4.1). □

Theorem 4.1. *Let $(S(t), I(t), E(t), Z(t))$ be the solution of system (2.8) with any initial value $S(0) > 0$ and $I(0) > 0$.*

(1) If $(\beta_1 + \tau)^2 < 2\sigma_1^2\mu_2$, then

$$\limsup_{t\to\infty}\frac{\log I(t)}{t} \leqslant \frac{1}{2}\Big(\frac{\beta_1 + \tau}{\sigma_1}\Big)^2 - \mu_2 < 0\ \ a.s..$$

(2) If $\tilde{\mathcal{R}} < 1$ and $N_0 \leqslant \frac{\beta_1}{\sigma_1^2}$, then

$$\limsup_{t\to\infty}\frac{\log I(t)}{t} \leqslant \mu_2(\tilde{\mathcal{R}} - 1) < 0\ \ a.s.,$$

where

$$\tilde{\mathcal{R}} = \frac{2\beta_1 N_0 - \sigma_1^2 N_0^2}{2\mu_2}.$$

In addition,

$$\limsup_{t\to\infty}\langle S(t)\rangle \leqslant \frac{\Lambda\mu + \xi N_0}{\mu(\xi+\mu)}.$$

Therefore, $I(t)$ tends to zero exponentially (i.e., the rumor spreaders disappear with a probability of one).

Proof. From Lemma 2.1 and (2.6), we can get that

$$\int_0^\infty \varpi(\theta)S(t-\theta)\mathrm{d}\theta \leqslant \frac{\xi}{\mu}N_0. \tag{4.2}$$

Notice that

$$\begin{aligned} \mathrm{d}(S(t)+I(t)+Z(t)) =&[\Lambda - (\xi+\mu_1)S(t) + \int_0^\infty \varpi(\theta)S(t-\theta)\mathrm{d}\theta - \mu_2 I(t) - \mu_4 Z(t)]\mathrm{d}t \\ &+ \sigma_1 S(t)\mathrm{d}B_1(t) + \sigma_2 I(t)\mathrm{d}B_2(t) + \sigma_4 Z(t)\mathrm{d}B_4(t). \end{aligned}$$

Then, we can get that

$$\begin{aligned} &\frac{S(t)+I(t)+Z(t)}{t} - \frac{S(0)+I(0)+Z(0)}{t} \\ =&\Lambda - \frac{\xi+\mu_1}{t}\int_0^t S(s)\mathrm{d}s + \frac{1}{t}\int_0^t\int_0^\infty \varpi(\theta)S(t-\theta)\mathrm{d}\theta\mathrm{d}s - \frac{\mu_2}{t}\int_0^t I(s)\mathrm{d}s \\ &- \frac{\mu_4}{t}\int_0^t Z(s)\mathrm{d}s + \frac{\int_0^t \sigma_4 Z(t)\mathrm{d}B_4(t)}{t} \\ \leqslant&\Lambda - (\xi+\mu_1)\langle S(t)\rangle + \frac{\xi}{\mu}N_0 + \frac{\int_0^t \sigma_4 Z(t)\mathrm{d}B_4(t)}{t}. \end{aligned}$$

Thus, we obtain the following:

$$\begin{aligned} &\langle S(t)\rangle \\ \leqslant&\frac{\Lambda}{\xi+\mu_1} + \frac{\xi N_0}{\mu(\xi+\mu_1)} - [\frac{S(t)+I(t)+Z(t)}{(\xi+\mu_1)t} - \frac{S(0)+I(0)+Z(0)}{(\xi+\mu_1)t} - \frac{\int_0^t \sigma_4 Z(t)\mathrm{d}B_4(t)}{(\xi+\mu_1)t}] \\ =&\frac{\Lambda}{\xi+\mu_1} + \frac{\xi N_0}{\mu(\xi+\mu_1)} - \psi(t), \end{aligned} \tag{4.3}$$

where

$$\psi(t) = \frac{S(t)+I(t)+Z(t)}{(\xi+\mu_1)t} - \frac{S(0)+I(0)+Z(0)}{(\xi+\mu_1)t} + \frac{\int_0^t \sigma_4 Z(t)\mathrm{d}B_4(t)}{(\xi+\mu_1)t}.$$

By Lemmas 2.1 and 4.2, we can easily get

$$\lim_{t\to\infty}\psi(t) = 0 \;\; a.s., \tag{4.4}$$

and

$$\limsup_{t\to\infty}\langle S(t)\rangle \leqslant \frac{\Lambda}{\xi+\mu_1}+\frac{\xi N_0}{\mu(\xi+\mu_1)}.$$

On the other hand, applying Itô's formula to the second equation of model (2.8) yields the following:

$$\begin{aligned}\mathrm{d}\log I(t) =&\frac{1}{I(t)}\{[\beta_1 S(t)I(t)-kI^2(t)-aI(t)(P-E(t)+rE(t))-\mu_2 I(t)]\mathrm{d}t\\&+\sigma_1 S(t)I(t)\mathrm{d}B_2(t)\}-\frac{\sigma_2^2}{2}S^2(t)\mathrm{d}t\\=&[\beta_1 S(t)-kI(t)-a(P-E(t)+rE(t))-\mu_2-\frac{\sigma_2^2}{2}S^2(t)]\mathrm{d}t+\sigma_1 S(t)\mathrm{d}B_1(t);\end{aligned}$$

then,

$$\begin{aligned}\frac{\log I(t)-\log I(0)}{t}=&\frac{\beta_1}{t}\int_0^t S(s)\mathrm{d}s-\frac{k}{t}\int_0^t I(s)\mathrm{d}s-\frac{a}{t}\int_0^t(P-E(s)+rE(s))\mathrm{d}s-\mu_2\\&-\frac{\sigma_1^2}{2t}\int_0^t S^2(s)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)\\\leqslant&\frac{1}{t}\int_0^t(\beta_1 S(s)-\frac{\sigma_1^2}{2}S^2(s)-\mu_2)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s).\end{aligned}\tag{4.5}$$

Therefore, by substituting (4.3) into (4.5), we can obtain the following:

$$\frac{\log I(t)}{t}\leqslant\frac{\log I(0)}{t}+\frac{1}{t}\int_0^t\Big((\beta_1+\tau)S(s)-\frac{\sigma_1^2}{2}S^2(s)-\mu_2\Big)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s).$$

Define the following:

$$H(x)=-\frac{\sigma_1^2}{2}x^2+(\beta_1+\tau)x-\mu_2.$$

Then, $H(x)$ is monotonically increasing for $[0,\frac{\beta_1+\tau}{\sigma_1^2})$ and monotonically decreasing for $[\frac{\beta_1+\tau}{\sigma_1^2},\infty)$. Therefore, when $x=\frac{\beta_1+\tau}{\sigma_1^2}$, we get the following ($H_{max}(x)$ denotes the maximum value of $H(x)$):

$$H_{max}(x)=-\frac{\sigma_1^2}{2}\Big(\frac{\beta_1+\tau}{\sigma_1^2}\Big)^2+\frac{(\beta_1+\tau)^2}{\sigma_1^2}-\mu_2=\frac{1}{2}\Big(\frac{\beta_1+\tau}{\sigma_1}\Big)^2-\mu_2.$$

Therefore, we can obtain the following:

$$\frac{\log I(t)}{t}\leqslant\frac{\log I(0)}{t}+\frac{1}{2}\Big(\frac{\beta_1+\tau}{\sigma_1}\Big)^2-\mu_2+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s).\tag{4.6}$$

If $(\beta_1+\tau)^2<2\sigma_1^2\mu_2$, then from Lemma 4.2, we know that

$$\limsup_{t\to\infty}\frac{\log I(t)}{t}\leqslant\frac{1}{2}\Big(\frac{\beta_1+\tau}{\sigma_1}\Big)^2-\mu_2<0\ \ a.s.,$$

On the other hand, it follows from the monotonicity of the function on the interval that if $N_0 \leqslant \frac{\beta_1}{\sigma_1^2}$ is satisfied, then it holds that $0 \leqslant \vartheta \leqslant \tau$, such that $N_0+\vartheta \leqslant \frac{\beta_1+\tau}{\sigma_1^2}$, which suggests that $H(S(t)) \leqslant H(N_0+\vartheta)$, where $S(t) \leqslant N_0 + \vartheta$. Therefore, if $t \geqslant T_0$, then we can obtain the following:

$$\begin{aligned}\frac{\log I(t)}{t} \leqslant& \frac{\log I(0)}{t} + \frac{1}{t}\int_0^t \Big((\beta_1+\tau)S(s) - \frac{\sigma_1^2}{2}S^2(s) - \mu_2\Big)\mathrm{d}s + \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)\\ =&\frac{\log I(0)}{t} + \frac{1}{t}\int_0^t H(S(s))\mathrm{d}s + \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)\\ =&\frac{\log I(0)}{t} + \frac{1}{t}\int_0^{T_0} H(S(s))\mathrm{d}s + \frac{1}{t}\int_{T_0}^t H(S(s))\mathrm{d}s + \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)\\ \leqslant&\frac{\log I(0)}{t} + \frac{1}{t}\int_0^{T_0} H(S(s))\mathrm{d}s + \frac{t-T_0}{t}H(N_0+\vartheta) + \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s).\end{aligned} \tag{4.7}$$

From Lemma 4.2, we know the following:

$$\begin{aligned}\limsup_{t\to\infty}\frac{\log I(t)}{t} \leqslant& H(N_0+\vartheta)\\ =&-\frac{\sigma_1^2}{2}(N_0+\vartheta)^2 + (\beta_1+\tau)(N_0+\vartheta) - \mu_2.\end{aligned}$$

By using the arbitrariness of ϑ and τ, let $\vartheta = \tau = 0$; then, we have the following:

$$\begin{aligned}\limsup_{t\to\infty}\frac{\log I(t)}{t} \leqslant& -\frac{\sigma_1^2}{2}N_0^2 + \beta_1 N_0 - \mu_2\\ =&\mu_2(\tilde{\mathcal{R}} - 1) < 0 \ \ a.s.,\end{aligned}$$

where

$$\tilde{\mathcal{R}} = \frac{2\beta_1 N_0 - \sigma_1^2 N_0^2}{2\mu_2}. \tag{4.8}$$

This completes the proof. □

5. The persistence of rumor

In this section, we consider the persistence of system (2.8). First, we give the definition of persistence in the mean as follows.

Definition 5.1. *System (2.8) is said to be persistent in the mean if*

$$\liminf_{t\to\infty}\langle S(t)\rangle > 0,\ \ \liminf_{t\to\infty}\langle I(t)\rangle > 0,\ \ \liminf_{t\to\infty}\langle E(t)\rangle > 0,\ \ \liminf_{t\to\infty}\langle Z(t)\rangle > 0 \ \ a.s..$$

Theorem 5.1. *Let $(S(t), I(t), E(t), Z(t))$ be the solution of system (2.8) with an initial value*

$S(0), I(0), E(0), Z(0) > 0$. *If* $\bar{\mathcal{R}} > 1$, *then*

$$\begin{aligned}
\liminf_{t\to\infty}\langle S(t)\rangle &\geqslant \frac{\Lambda}{\beta_1 N_0 + \xi + \mu_1} > 0,\\
\liminf_{t\to\infty}\langle I(t)\rangle &\geqslant \frac{aP + \mu_2}{k}(\bar{\mathcal{R}} - 1) > 0,\\
\liminf_{t\to\infty}\langle E(t)\rangle &\geqslant \frac{\beta_2 P(aP + \mu_2)}{k(\beta_2 N_0 + \beta_3 P + \mu_3)}(\bar{\mathcal{R}} - 1) > 0,\\
\liminf_{t\to\infty}\langle Z(t)\rangle &\geqslant \frac{arP(aP + \mu_2)}{k\mu_4}(\bar{\mathcal{R}} - 1) > 0 \ \ a.s.,
\end{aligned}$$

where

$$\bar{\mathcal{R}} = \frac{(2\beta_1 - \sigma_1^2 N_0)(\Lambda - \beta_1 N_0^2)}{2(\xi + \mu_1)(aP + \mu_2)}. \tag{5.1}$$

The constant $\bar{\mathcal{R}}$, which is given based on the next-generation matrix method, is different from $\tilde{\mathcal{R}}$ in Theorem 4.1. $\bar{\mathcal{R}}$ is the threshold for whether rumors can spread persistently; $\tilde{\mathcal{R}}$ is the threshold for whether rumors almost surely disappear.

Proof. Notice that

$$\begin{aligned}
\mathrm{d}S(t) =&[\Lambda - \beta_1 S(t)I(t) + \int_0^\infty \varpi(\theta)S(t-\theta)\mathrm{d}\theta - (\xi + \mu_1)S(t) + kI^2(t)]\mathrm{d}t + \sigma_1 S(t)I(t)\mathrm{d}B_1(t)\\
\geqslant&[\Lambda - \beta_1 S(t)I(t) - (\xi + \mu_1)S(t)]\mathrm{d}t + \sigma_1 S(t)I(t)\mathrm{d}B_1(t)\\
\geqslant&[\Lambda - \beta_1 N_0^2 - (\xi + \mu_1)S(t)]\mathrm{d}t + \sigma_1 S(t)I(t)\mathrm{d}B_1(t);
\end{aligned}$$

then,

$$\begin{aligned}
\frac{S(t) - S(0)}{t} \geqslant&\Lambda - \beta_1 N_0^2 - (\xi + \mu_1)\frac{\int_0^t S(s)\mathrm{d}s}{t} + \sigma_1\frac{\int_0^t S(s)I(s)\mathrm{d}B_1(s)}{t}\\
=&\Lambda - \beta_1 N_0^2 - (\xi + \mu_1)\langle S(t)\rangle + \sigma_1\frac{\int_0^t S(s)I(s)\mathrm{d}B_1(s)}{t}.
\end{aligned}$$

Therefore,

$$\langle S(t)\rangle \geqslant \frac{\Lambda - \beta_1 N_0^2}{\xi + \mu_1} + \frac{\sigma_1\int_0^t S(s)I(s)\mathrm{d}B_1(s)}{(\xi + \mu_1)t} - \frac{S(t) - S(0)}{(\xi + \mu_1)t}.$$

This, together with Lemmas 2.1 and 4.2, leads to the following:

$$\liminf_{t\to\infty}\langle S(t)\rangle \geqslant \frac{\Lambda - \beta_1 N_0^2}{\xi + \mu_1}. \tag{5.2}$$

By the first equality of (4.5), we can obtain the following:

$$\begin{aligned}
\frac{\log I(t) - \log I(0)}{t} =&\frac{\beta_1}{t}\int_0^t S(s)\mathrm{d}s - \frac{k}{t}\int_0^t I(s)\mathrm{d}s - \frac{a}{t}\int_0^t (P - E(s) + rE(s))\mathrm{d}s\\
&- \mu_2 - \frac{\sigma_1^2}{2t}\int_0^t S^2(s)\mathrm{d}s + \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s)\\
\geqslant&\beta_1\langle S(t)\rangle - k\langle I(t)\rangle - (aP + \mu_2) - \frac{\sigma_1^2}{2}N_0\langle S(t)\rangle + \frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}B_1(s),
\end{aligned}$$

then,

$$\langle I(t)\rangle \geqslant (\frac{\beta_1}{k}-\frac{\sigma_1^2}{2k}N_0)\langle S(t)\rangle - \frac{aP+\mu_2}{k}+\frac{\sigma_1}{kt}\int_0^t S(s)\mathrm{d}B_1(s)-\frac{\log I(t)-\log I(0)}{kt}.$$

Together with (5.2) and Lemma 2.1, one can see that

$$\begin{aligned}\liminf_{t\to\infty}\langle I(t)\rangle &\geqslant \frac{(2\beta_1-\sigma_2^2N_0)(\Lambda-\beta_1N_0^2)}{2(\xi+\mu_1)k}-\frac{aP+\mu_2}{k}\\ &=\frac{aP+\mu_2}{k}\Big[\frac{(2\beta_1-\sigma_2^2N_0)(\Lambda-\beta_1N_0^2)}{2(\xi+\mu_1)(aP+\mu_2)}-1\Big]\\ &=\frac{aP+\mu_2}{k}(\bar{\mathcal{R}}-1)>0.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{d}E(t) =&[\beta_2(P-E(t))I(t)+\beta_3(P-E(t))E(t)-\mu_3E(t)]\mathrm{d}t+\sigma_3E(t)\mathrm{d}B_3(t)\\ \geqslant&[\beta_2PI(t)-(\beta_2N_0+\beta_3P+\mu_3)E(t)]\mathrm{d}t+\sigma_3E(t)\mathrm{d}B_3(t);\end{aligned}$$

then,

$$\frac{E(t)-E(0)}{t}\geqslant\beta_2P\langle I(t)\rangle-(\beta_2N_0+\beta_3P+\mu_3)\langle E(t)\rangle+\sigma_3\frac{\int_0^t E(s)\mathrm{d}B_3(s)}{t}.$$

Thus,

$$\langle E(t)\rangle\geqslant\frac{\beta_2P}{\beta_2N_0+\beta_3P+\mu_3}\langle I(t)\rangle+\frac{\sigma_3\int_0^t E(s)\mathrm{d}B_3(s)}{(\beta_2N_0+\beta_3P+\mu_3)t}-\frac{E(t)-E(0)}{(\beta_2N_0+\beta_3P+\mu_3)t},$$

$$\liminf_{t\to\infty}\langle E(t)\rangle\geqslant\frac{\beta_2P(aP+\mu_2)}{k(\beta_2N_0+\beta_3P+\mu_3)}(\bar{\mathcal{R}}-1)>0.$$

By the last equation of (2.8),

$$\begin{aligned}\mathrm{d}Z(t) =&[aI(t)(P-E(t)+rE(t))-\mu_4Z(t)]\mathrm{d}t+\sigma_4Z(t)\mathrm{d}B_4(t)\\ \geqslant&[arPI(t)-\mu_4Z(t)]\mathrm{d}t+\sigma_4Z(t)\mathrm{d}B_4(t),\end{aligned}$$

then,

$$\frac{Z(t)-Z(0)}{t}\geqslant arP\langle I(t)\rangle-\mu_4\langle Z(t)\rangle+\sigma_4\frac{\int_0^t Z(s)\mathrm{d}B_4(s)}{t}.$$

Thus,

$$\begin{aligned}\langle Z(t)\rangle &\geqslant \frac{arP}{\mu_4}\langle I(t)\rangle+\frac{\sigma_4\int_0^t Z(s)\mathrm{d}B_4(s)}{\mu_4t}-\frac{Z(t)-Z(0)}{\mu_4t},\\ \liminf_{t\to\infty}\langle Z(t)\rangle &\geqslant \frac{arP(aP+\mu_2)}{k\mu_4}(\bar{\mathcal{R}}-1)>0.\end{aligned}$$

The proof is complete. □

6. Numerical simulations

In this section, numerical simulations are given in order to verify the main results. First, inspired by [13, 30], define

$$w(\theta) = \begin{cases} 0, & \text{if } 0 < \theta \leqslant 10; \\ 0.6667(\theta - 10)^2 e^{-0.6(\theta-10)}, & \text{if } 10 < \theta \leqslant 30; \\ 0.0185, & \text{if } 30 < \theta. \end{cases} \tag{6.1}$$

By combing the discretization method of partial differential equations mentioned in Anita et al. [31] as well as the discretization method of stochastic differential equations mentioned in Higham [32], we write the discretization equation of model (2.8) as follows

$$\begin{cases} S_{j+1} = S_j + [\Lambda - \beta_1 S_j I_j - (\xi + \mu_1) S_j + \int_0^{\infty} \varpi(\theta) S(t-\theta) \mathrm{d}\theta]\Delta t - \sigma_1 S_j I_j \sqrt{\Delta t}\zeta_i \\ \qquad + k I_j^2 - \frac{\sigma_1^2}{2} S_j I_j (\zeta_i^2 - 1)\Delta t, \\ I_{j+1} = I_j + [\beta_1 S_j I_j - k I_j^2 - a I_j (P - E_j + r E_j) - \mu_2 I_j]\Delta t + \sigma_1 S_j I_j \sqrt{\Delta t}\zeta_i + \frac{\sigma_1^2}{2} S_j I_j (\zeta_i^2 - 1)\Delta t, \\ E_{j+1} = E_j + [\beta_2 (P - E_j) I_j + \beta_3 (P - E_j) E_j - \mu_3 E_j]\Delta t + \sigma_3 E_j \sqrt{\Delta t} + \frac{\sigma_3^2}{2} E_j (\eta_i^2 - 1)\Delta t, \\ Z_{j+1} = Z_j + [a I_j (P - E_j + r E_j) - \mu_4 Z_j]\Delta t + \sigma_4 Z_j \sqrt{\Delta t} + \frac{\sigma_4^2}{2} Z_j (\nu_i^2 - 1)\Delta t, \end{cases}$$

and $\varpi(\theta) = \xi w(\theta) \exp\{-\int_0^{\theta} (\mu + w(\tau)) \mathrm{d}\tau\}$,which will be replaced by the complex trapezoidal formula. In addition, $\zeta_i, \eta_i, \nu_i (i = 1, 2, \cdots n)$ is an independent Gaussian random variable $N(0, 1)$. Here, we use the Milstein-type format to deal with model (2.8), but more advanced methods are available for such systems with superlinearly growing coefficients, such as the following: Hutzenthaler et al. [33] considered the suplinear case using Euler method; Wang and Gan [34] studied the commutative stochastic differential equations with non-globally Lipschitz continuous coefficients using tamed Milstein method; Beyn et al. [35] established the stochastic C-stability and B-consistency of explicit and implicit Milstein-type schemes; and Izgi and Cetin [36] developed Milstein-type versions of semi-implicit split-step methods for numerical solutions of non-linear stochastic differential equations with locally Lipschitz coefficients. For simplicity, we only consider S, I, E, Z in one dimension and the unit of time is second.

First, take $\Lambda = 1, \beta_1 = 0.3, \xi = 0.67, \mu = \mu_i = 0.6 (i = 1, 2, 3, 4), \sigma_i = 0.01 (i = 1, 2, 3, 4), a = 0.02, P = 2, r = 0.25, \beta_2 = 0.105, \beta_3 = 0.1, k = 0.15$, $\Delta t = 0.01$. The initial values are $S_0 = 1.2, I_0 = 0.5, E_0 = 0.6$, and $Z_0 = 0.5$. Simple calculations show that $\tilde{\mathcal{R}} \approx 0.810185$, $\sigma_1^2 - \frac{\beta_1}{N_0} = -0.17 < 0$, which indicates that condition (ii) of Theorem 4.1 is satisfied. Consequently, we have that the rumor spreaders will disappear with a probability of one, as $t \to \infty$, which can be seen in Figure 2(a).

Second, take $\Lambda = 0.35, \beta_1 = 0.2, \xi = 0.65, \mu_i = 0.2 (i = 1, 2, 3, 4), \sigma_i = 0.01 (i = 1, 3, 4), a = 0.01, P = 1, r = 0.25, \beta_2 = 0.105, \beta_3 = 0.05, k = 0.11$, $\Delta t = 0.01$. Simple calculations shows that $\bar{R} \approx 1.0104 > 1$. Thus, the conditions of Theorem 5.1 are satisfied. Therefore, the rumor spreaders will persist in the mean, as shown in Figure 2(b). Moreover, the expression of $\bar{R}$ has an effect on the age of education, that is to say, the effect on the value of ξ.

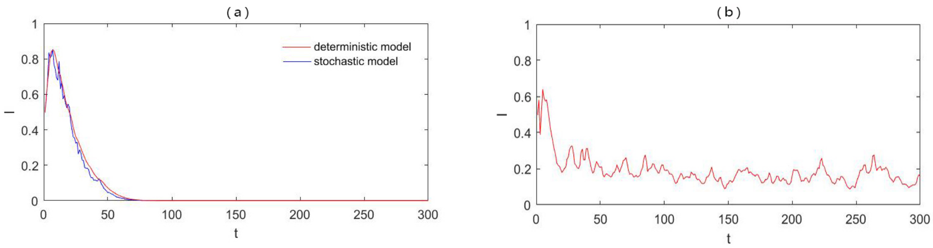
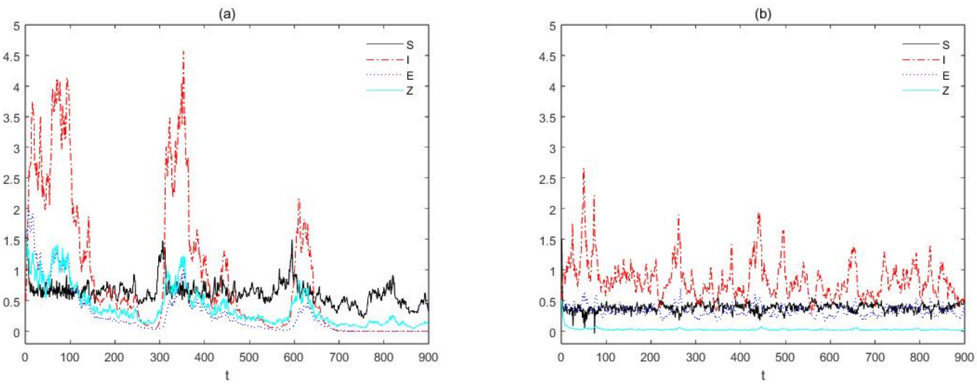


Figure 2. The dynamic behavior of equilibria of model (2.8).



(a) (b)

Figure 3. (a), (b) denote the solutions of system (2.8) with two different cases.

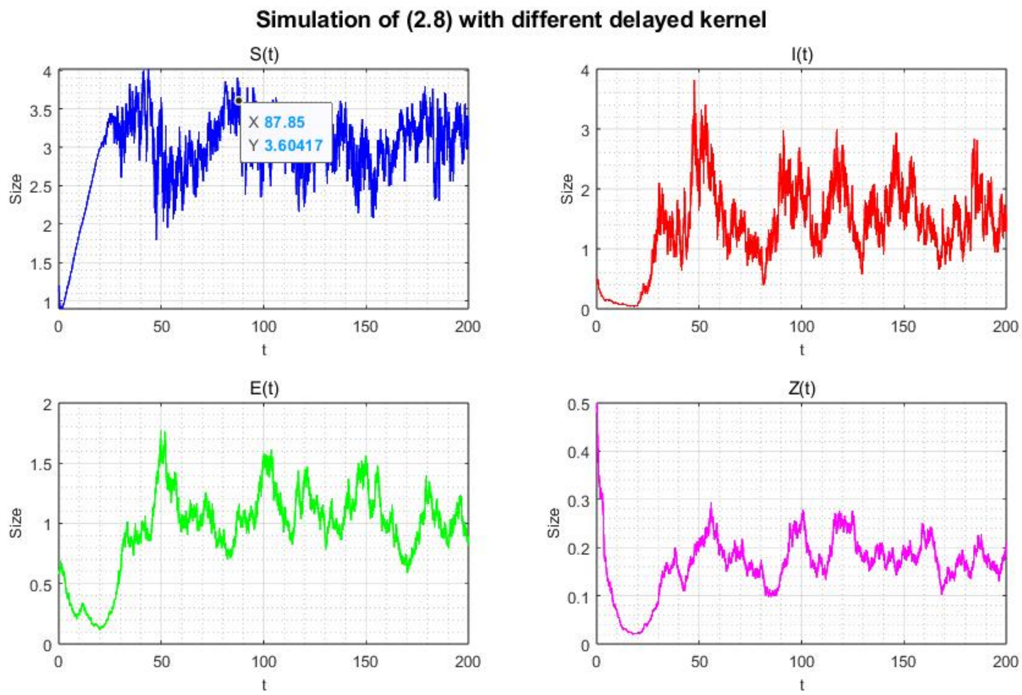


Figure 4. Different delayed kernels of (2.8).

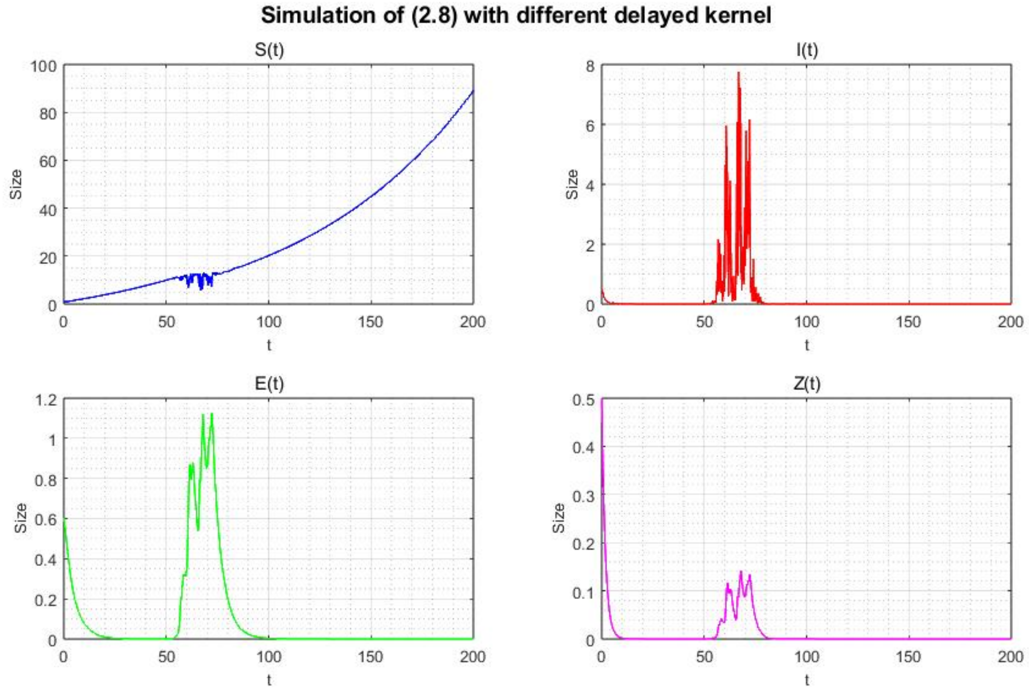


Figure 5. Rumor disappears of (2.8) with different delayed kernels.

Furthermore, the profiles of S, I, E, Z are given in Figure 3, where Figure 3(a) shows that the spreaders will disappear with a probability of one and Figure 3(b) shows that the rumor spreaders will persist in the mean (we only choose one path).

Lastly, we consider the effect of a delayed kernel. In Figures 2 and 3, we take the kernel as in (6.1). If we take $w(\theta) = 0.2 \times e^{-0.1\theta}$, then the profiles of S, I, E, and Z are as in Figure 4, where the rumor will not disappear because $\int_0^\infty \varpi(\theta)\mathrm{d}\theta = 2\xi$ and $\tilde{\mathcal{R}} = \frac{3}{2} > 1$. Moreover, if we take $\beta_1 = 0.2$ and $\mu_2 = 0.9$, then the rumor will disappear, as shown in Figure 5.

7. Conclusions

In this paper, we introduced a new stochastic rumor model, which contains a susceptible population, rumor spreader, rumor susceptible handler, weak rumor handler, rumor spreaders in quarantine, and the educational age. By using Itô's formula and Lyapunov function, the existence of global positive solutions for the model was first obtained. Using the strong law of large numbers, several sufficient conditions for the disappearance and persistence of rumor were derived. Finally, to support the main results of this article, several illustrative numerical simulations were conducted.

It follows from Theorems 4.1 and 5.1 that the basic reproduction numbers ($\tilde{\mathcal{R}}$ and $\bar{\mathcal{R}}$) are different and this result is consistent with the existing results in [13]. Our results give the effect on the age of education.

By comparing (1.1) with (2.8), we added two different rumor susceptible handler and rumor spreaders in quarantine in (2.8), which concluded that $\tilde{\mathcal{R}}$ and $\bar{\mathcal{R}}$ are different. In this paper, the choice of parameters was simple, with the aim of satisfying the assumptions of the theorem. In future work, we will utilize real data to verify the theory results, which is similar to [37].

Author contributions

Hui Zhu: Conceptualization, method, investigation, writing-original draft; Zonghe Guo: Validation, writing-original draft, writing-review and editing; Yunping Liu: Conceptualization, investigation, software, writing-review and editing; Wei Ou: Validation, software, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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