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Research article

Linear codes arising from geometrical operations

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Abstract: We construct linear codes over the finite field $\mathbb{F}_q$ from arbitrary simplicial complexes, establishing a connection between topological properties and fundamental coding parameters. First, we study the behaviour of the weights of codewords from a geometric point of view, interpreting them in terms of the combinatorial structure of the associated simplicial complex. This approach allows us to describe the minimum distance of the codes in terms of certain geometric features of the complex. Subsequently, we analyze how various topological operations on simplicial complexes affect the classical parameters of the codes. This study leads to the formulation of geometric criteria that make it possible to explicitly control and manipulate these parameters. Finally, as an application of the obtained results, we construct several families of optimal linear codes over $\mathbb{F}_2$ using these geometric methods. Thanks to the previously established geometric properties, we can precisely determine the parameters of these families.

Keywords: linear codes; simplicial complex; minimum distance; geometric methods
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1. Introduction

In recent decades, the interaction between coding theory and different areas of mathematics has given rise to new constructions and approaches of great interest. In particular, since the construction of linear codes from simplicial complexes was formalized in the binary case in [3], this topic has been the subject of study in numerous works. In [2, 11] the authors extend the study of the binary case whereas, [8–10] simplicial codes are defined on different alphabets. The optimal case is treated in [4, 6, 13]. In a more general setting simplicial codes can be considered as a special case of down-sets [5,12]. In these articles, the weights of the codewords have been analyzed mainly through formulas derived from the one introduced by Adamaszek [1], based on the inclusion–exclusion principle, which allows one to obtain multivariable expressions for counting faces. This type of approach is largely disconnected from the geometry of the simplicial complex. The motivation of this work is to propose an alternative approach to the study of codewords and their weights, with the aim of providing the first geometric interpretation of weights in coding theory.

In [6], it is assumed that every maximal face of the simplicial complex must contain an isolated vertex; that is,

$$A_i \setminus \bigcup_{j\in[s]\setminus\{i\}} A_j \neq \emptyset.$$

On the other hand, in [5], although the theoretical construction is generalized, in the study of concrete cases, attention is restricted to complexes with a unique maximal face in order to control the weight function and analyze its properties. In general, in previous works, the imposed hypotheses prevent addressing the study of codes associated with simplicial complexes with a richer geometric or topological structure, such as, for example, triangulations of (pseudo)manifolds. The approach of this article aims to extend the knowledge of the code parameters to the case of arbitrary simplicial complexes.

To this end, we begin by presenting a series of preliminary notions in Section 2 that establish how the code will be defined to reflect the topological structure of the complex. Likewise, the notation that will be used throughout the article is introduced. Subsequently, in Section 3, one of the key results of this work is presented: Theorem 3.1, which makes it possible to understand the weight in terms of intersections of the faces of the complex. From this result and the geometric approach to the weights of codewords, the remaining results are developed. Previously, the optimality of linear codes has been studied from the perspective of weight distribution, for example in [7]. However, this approach becomes very complex when working with arbitrary simplicial complexes, hence the importance of this result, which uses the additional structure of the code to simplify the computation of the distance. Following this perspective, a series of topological operations on complexes and how they affect the code are studied in Section 4. Finally, working over the field $\mathbb{F}_2$ to further facilitate geometric intuition, the construction of some families of codes is presented based on the results obtained previously.

From a broader point of view, this work can be interpreted as a first step toward a possible interaction between coding theory, algebraic geometry, and ring theory, mediated by topological and geometric tools. The results obtained suggest that the structure of simplicial complexes can serve as a natural setting to explore new connections between both disciplines. In this sense, the present article aims to contribute to the construction of a common language that allows the transfer of intuitions and techniques between topology, geometry, and coding theory.

2. Preliminaries

The aim of this section is to introduce the main notions and notation that will be used throughout the paper.

Definition 2.1. A simplicial complex Δ is a family of subsets of $[n] = \{1, \dots, n\}$, such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

Throughout this paper, we assume that q is prime. A linear code over $\mathbb{F}_q$ can be constructed from a simplicial complex Δ. Let $\sigma \in \Delta$. We can identify σ with its characteristic vector in $\mathbb{F}_q^n$, that is,

$\sigma \sim \chi_\sigma \in \{0,1\}^n$, where

$$\chi_\sigma(i) = \begin{cases} 1, & \text{if } i \in \sigma, \\ 0, & \text{if } i \notin \sigma, \end{cases} \quad \forall i \in [n].$$

We then define $D := \{\chi_\sigma \in \mathbb{F}_q^n : \sigma \in \Delta\} \subseteq \mathbb{F}_q^n$. The linear code associated with Δ can be defined as follows:

$$C_\Delta = \{c_\Delta(u) = (u \cdot x)_{x \in D} : u \in \mathbb{F}_q^n\},$$

where $u \cdot x$ denotes the Euclidean inner product of u and x. In addition, we define

$$C_{\Delta^*} = \{(u \cdot x)_{0 \neq x \in D} : u \in \mathbb{F}_q^n\},$$

that is, the code obtained by evaluating only at nonzero elements of D.

Equivalently, we can describe the definition matrix G of the code as one whose rows are the elements of D; in that case,

$$C_\Delta = \{uG^T : u \in \mathbb{F}_q^n\}.$$

Note that G^T is a generator matrix of the code C_Δ.

Definition 2.2. A maximal simplex A of Δ is a simplex that is not properly contained within any other simplex of the complex.

Thus, if $A_1, \ldots, A_s$ are the maximal simplices of Δ, we may write

$$\Delta = \langle A_1, \ldots, A_s \rangle = \{B \subseteq \{1, \ldots, n\} : B \subseteq A_i \text{ for some } i \in \{1, \ldots, s\}\}.$$

Definition 2.3. We define the complement of Δ in the power set of $[n]$ by

$$\Delta^c = \{\sigma \subseteq [n] : \sigma \notin \Delta\}.$$

Note that, in general, Δ^c is not a simplicial complex.

The linear code C_{Δ^c}, called the anticode of Δ, is defined as

$$C_{\Delta^c} = \{(u \cdot x)_{x \notin D} : u \in \mathbb{F}_q^n\}.$$

Definition 2.4. Let Δ be a simplicial complex. The link of a vertex, $\mathrm{lk}_\Delta(e_i)$, is a collection of subsets satisfying $\mathrm{lk}_\Delta(e_i) = \{\tau \subset \Delta : \tau \cap \{e_i\} = \varnothing \text{ and } \tau \cup \{e_i\} \in \Delta\}$.

3. Geometrical and topological meaning of code parameters

Let Δ be a simplicial complex and let C_Δ be its associated code,

$$C_\Delta = \{(u \cdot x)_{x \in D} : u \in \mathbb{F}_q^n\}.$$

From a geometric point of view, the product $(u \cdot x)$ can be understood as

$$\langle u, \chi_\sigma \rangle = \sum_{v \in \sigma} u_v,$$

where χ_σ is the characteristic vector of σ. Following this approach, it is possible to define

$$l_\sigma(u) = \sum_{v\in\sigma} u_v,$$

so that

$$c_\Delta(u) = (l_\sigma(u))_{\emptyset\neq\sigma\in\Delta}$$

and

$$w(c_\Delta(u)) = \#\{\sigma \in \Delta : l_\sigma(u)) \not\equiv 0 \pmod q\}.$$

That is, the weight of a codeword can be interpreted as the number of simplices of the complex for which the sum over their vertices does not vanish on u. Based on this perspective, the following result is presented:

Theorem 3.1. *Let Δ be a simplicial complex on a vertex set $V = \{1, \ldots, k\}$ and let C_Δ be the associated linear code over $\mathbb{F}_q$. For any $u \in \mathbb{F}_q^k$ with $|\operatorname{supp}(u)| \geq 2$, there exists u' such that $\operatorname{supp}(u') \subset \operatorname{supp}(u)$, $|\operatorname{supp}(u')| = 1$, and*

$$w(c_\Delta(u')) \leq w(c_\Delta(u)).$$

Proof. Fix $u \in \mathbb{F}_q^k : u = (u_1, ..., u_k)$ and $u_i \in \mathbb{F}_q^*$. For $\sigma \in \Delta$, we can extend

$$\tilde{l}_\sigma(u) = \sum_{v\in\sigma} \tilde{u}_v \in \mathbb{Z},$$

where $\tilde{u}_v$ is the lifting of u_v into $\mathbb{Z}$.

Recall $w(c_\Delta(u)) = \#\{\sigma \in \Delta : l_\sigma(u)) \not\equiv 0 \pmod q)\}$ and $c_\Delta(u) = (l_\sigma(u))_{\emptyset\neq\sigma\in\Delta} \in \mathbb{F}_q^n$, where $n = |\Delta| - 1$. We define

$$\Delta^a = \{\tau \in \Delta : \sum_{v\in\tau} \tilde{u}_v = a\},$$

and

$$\Delta^a_{v_i} = \{\tau \in \Delta : \sum_{v\in\tau} \tilde{u}_v = a \quad \wedge \quad v_i \in \tau\}.$$

Therefore, $w(c_\Delta(u)) = \sum_{a\notin q\mathbb{Z}} |\Delta^a|$. Let $u' \in \mathbb{F}_q^n$ be the vector supported on $\{v_i\}$, where $v_i \in \operatorname{supp}(u)$. Since u' is supported in a single vertex v_i, for any face τ containing v_i, we have $\tilde{l}_\tau(u') = \tilde{u}_i \notin q\mathbb{Z}$, hence all such faces contribute to the weight.

$$w(c_\Delta(u')) = \sum_{a\in\mathbb{Z}} |\Delta^a_{v_i}|.$$

We use the decomposition

$$|\Delta^a| = |\Delta^a_{v_i}| + |\Delta^a_{\neg v_i}| \qquad \text{for all } a \in \mathbb{Z}.$$

Therefore,

$$\sum_{a\notin q\mathbb{Z}} |\Delta^a| = \sum_{a\notin q\mathbb{Z}} |\Delta^a_{v_i}| + \sum_{a\notin q\mathbb{Z}} |\Delta^a_{\neg v_i}|.$$

Since

$$w(c_\Delta(u')) = \sum_{a\in\mathbb{Z}} |\Delta^a_{v_i}|, \qquad w(c_\Delta(u)) = \sum_{a\notin q\mathbb{Z}} |\Delta^a|,$$

the inequality $w(c_\Delta(u')) \leq w(c_\Delta(u))$ is equivalent to

$$\sum_{a\in q\mathbb{Z}} |\Delta^a_{v_i}| \leq \sum_{a\notin q\mathbb{Z}} |\Delta^a_{\neg v_i}|.$$

Now, for each $t = pq$, define a map

$$\Phi_t : \Delta^{pq}_{v_i} \longrightarrow \Delta^{pq-\tilde{u}_i}_{\neg v_i}, \qquad \tau \mapsto \tau \setminus \{v_i\}.$$

This map is well defined as $\tau \in \Delta$ implies $\tau \setminus \{v_i\} \in \Delta$. Moreover, $pq - \tilde{u}_i \notin q\mathbb{Z}$ since $\tilde{u}_i \notin q\mathbb{Z}$, and $p_1 q - \tilde{u}_i = p_2 q - \tilde{u}_i$ if and only if $p_1 = p_2$. The map is also injective. Therefore, $|\Delta^{(pq)}_{v_i}| \leq |\Delta^{(pq-\tilde{u}_i)}_{\neg v_i}|$.

Summing over all t, we obtain

$$\sum_{j\in q\mathbb{Z}} |\Delta^j_{v_i}| \leq \sum_{j\in q\mathbb{Z}} |\Delta^{j-\tilde{u}_i}_{\neg v_i}| \leq \sum_{j\notin q\mathbb{Z}} |\Delta^j_{\neg v_i}|.$$

Hence, $w(c_\Delta(u)) \geq w(c_\Delta(u'))$. □

Corollary 3.2. *The minimum weight of the code C_{Δ^*}, and hence its minimum distance, is determined by vectors $u \in \mathbb{F}_q^n$ whose support consists of a single vertex of the complex.*

Another important consequence that can be drawn from this result is that the distance of the code does not depend on the chosen value of q. Assuming that $|\operatorname{supp}(u)| = 1$, one can see that, since

$$l_\sigma(u) \not\equiv 0 \pmod{q} \iff \operatorname{supp}(u) \cap \sigma \neq \emptyset,$$

it follows that

$$w(c_\Delta(u)) = \#\{\sigma \in \Delta : \operatorname{supp}(u) \cap \sigma \neq \emptyset\}.$$

Therefore, the value of the weight depends only on the support of u.

Corollary 3.3. *For a simplicial complex Δ on the vertex set $[k]$, whose set of maximal elements is $\{A_1, \ldots, A_s\}$ and such that $\bigcup_{i=1}^s A_i = [k]$, if each A_i satisfies $A_i \setminus \bigcup_{j\in[s]\setminus\{i\}} A_j \neq \emptyset$, then the parameters of the associated code C_{Δ^*} over $\mathbb{F}_q$ are*

$$[|\Delta| - 1, k, \min\{2^{|A_i|-1} : i \in [s]\}].$$

Proof. The weight of a codeword corresponding to a vertex is determined by the number of faces of Δ containing it. Let A_i be a maximal face. By assumption, there exists a vertex $v \in A_i$ such that $v \notin A_j$ for all $j \neq i$. Therefore, the faces of Δ containing v are precisely the subsets of A_i that contain v, and their number is $2^{|A_i|-1}$.

Thus, the weight of the codeword associated with v is $2^{|A_i|-1}$. For any other vertex, since it belongs to at least one maximal face (and possibly more), its weight is greater than or equal to the number of subsets of at least one maximal face containing it, and hence is at least $2^{|A_i|-1}$ for some i.

Therefore, the minimum weight among all nonzero codewords is $\min\{2^{|A_i|-1} : i \in [s]\}$, which coincides with the minimum distance of C_{Δ^*}. □

In general, for an arbitrary simplicial complex Δ, the parameters of C_{Δ^*} are

$$[|\Delta| - 1, k, \min\{|\operatorname{lk}_\Delta(i)| : i \in [k]\}],$$

where the distance is given by the minimum of $\{|\operatorname{lk}_\Delta(i)| : i \in [k]\}$, which equals the number of faces of Δ containing the vertex i.

Remark 3.1. Theorem 3.1 provides a geometric interpretation of the code weights associated with simplicial complexes. Specifically, it shows that the minimum code distance can be calculated by considering only the elementary words, which leads to a simple description in terms of the number of simplices containing a given vertex.

This should be contrasted with previous approaches to similar families of codes, where the computation of weights typically relies on explicit formulas that can be quite involved. Such formulas often become difficult to handle when the underlying simplicial complex does not satisfy strong structural assumptions. This result avoids these difficulties by reducing the problem to a direct combinatorial count.

Moreover, this perspective also simplifies the proofs of further properties of these codes, as illustrated by Corollary 3.3, and can be viewed as a useful tool in situations where standard weight formulas become difficult or intractable to deal with.

4. Construction of new codes via topological operations

Proposition 4.1. *Let $\Delta = \Delta_1 \sqcup \Delta_2$ be a simplicial complex over $\mathbb{F}_q$ with two disjoint connected components, and let $\widetilde{\Delta}$ be the simplicial complex obtained by identifying a face $F_1 \subset \Delta_1$ with a face $F_2 \subset \Delta_2$ of the same dimension. Then the minimum distance of the associated linear code does not decrease under the identification, that is,*

$$d(C_{\widetilde{\Delta}^*}) \geq d(C_{\Delta^*}).$$

Proof. We regard $\widetilde{\Delta}$ as the quotient of Δ obtained by identifying the faces F_1 and F_2, and for each simplex $\sigma \in \Delta$, we denote by $\widetilde{\sigma}$ its image in the quotient.

By Corollary 3.2, the minimum-weight codewords of the code associated to a simplicial complex are precisely those whose support consists of a single vertex. There are two possible cases.

First, suppose that there exists a minimum-weight codeword whose support is a vertex that is not identified in the quotient. In this case, the identification does not affect the set of simplices intersecting the support; therefore, the weight of the corresponding codeword remains unchanged.

Otherwise, every vertex supporting a minimum-weight codeword belongs to one of the faces that are identified. Since such codewords have support consisting of a single vertex, we have

$$w(c_{\widetilde{\Delta}^*}(\widetilde{u})) = \#\left\{\widetilde{\sigma} \in \widetilde{\Delta} : \operatorname{supp}(\widetilde{u}) \cap \widetilde{\sigma} \neq \emptyset\right\} \geq \#\{\sigma \in \Delta : \operatorname{supp}(u) \cap \sigma \neq \emptyset\} = w(c_{\Delta^*}(u)).$$

Indeed, each simplex $\sigma \in \Delta$, such that $\operatorname{supp}(u) \cap \sigma \neq \emptyset$ maps to a simplex $\widetilde{\sigma} \in \widetilde{\Delta}$ satisfying $\operatorname{supp}(\widetilde{u}) \cap \widetilde{\sigma} \neq \emptyset$. Since $\operatorname{supp}(u)$ consists of a single vertex and Δ_1 and Δ_2 are disjoint connected components, no simplex contains vertices from both components. Therefore, the identification does not destroy any existing intersection and may only create new ones. This shows that the minimum weight cannot decrease, and hence $d(C_{\widetilde{\Delta}^*}) \geq d(C_{\Delta^*})$. □

Remark 4.1. In the situation of the previous proposition, suppose that every minimum-weight codeword of C_{Δ^*} is supported on a vertex belonging to one of the faces that are identified. Let u be such a codeword and let $\widetilde{u}$ denote its image in the quotient.

In this case, the increase of the weight produced by the identification can be described explicitly. Writing $\widetilde{\Delta} = \widetilde{\Delta}_1 \cup \widetilde{\Delta}_2$, we have

$$w(c_{\widetilde{\Delta}^*}(\widetilde{u})) = w(c_{\widetilde{\Delta}^*}(\widetilde{u}_{|\widetilde{\Delta}_1})) + w(c_{\widetilde{\Delta}^*}(\widetilde{u}_{|\widetilde{\Delta}_2})) - w(c_{\widetilde{\Delta}^*}(\widetilde{u}_{|F_1})),$$

and since $w(c_{\widetilde{\Delta}^*}(\widetilde{u}_{|\widetilde{\Delta}_1})) = d(C_{\Delta^*})$, the contribution to the increase comes exclusively from simplices in Δ_2 containing the identified face. More precisely,

$$w(c_{\widetilde{\Delta}^*}(\widetilde{u})) = d(C_{\Delta^*}) + \sum_{S\subset\Delta_2\,:\,F_2\subset S} \Big(w(c_{\widetilde{\Delta}^*}(\widetilde{u}_{|S})) - w(c_{\widetilde{\Delta}^*}(\widetilde{u}_{|F_2}))\Big).$$

Example 4.2. Let Δ be a simplicial complex over $\mathbb{F}_2$ whose set of maximal faces is $\{\{0,1\},\{0,2,3\},\{4,5,6,7\}\}$. To determine the length of the code, it suffices to count the number of non-empty simplices of Δ, since the generator matrix has one column for each such simplex. Therefore, multiplying a vector of dimension k by this matrix yields a codeword whose length coincides with the number of columns. The dimension of the code is equal to the number of vertices of Δ, which is equal to the number of rows in the generating matrix. To compute the minimum distance, we rely on Corollary 3.2. Vertex $\{1\}$ is the one contained in the smallest number of simplices, as it only appears in the simplex $\{1\}$ and in the edge $\{0,1\}$. Hence, the minimum distance of the code is 2. Collecting these results, the code C_{Δ^*} has parameters $[24,8,2]$. Meanwhile, if we identify the vertices according to the relations $1\sim 4$, $2\sim 5$, and $3\sim 6$, we obtain a simplicial complex $\tilde{\Delta}$ as shown in Figure 1. By an argument analogous to the one above, we conclude that the associated code $C_{\tilde{\Delta}^*}$ has parameters $[20,5,5]$.

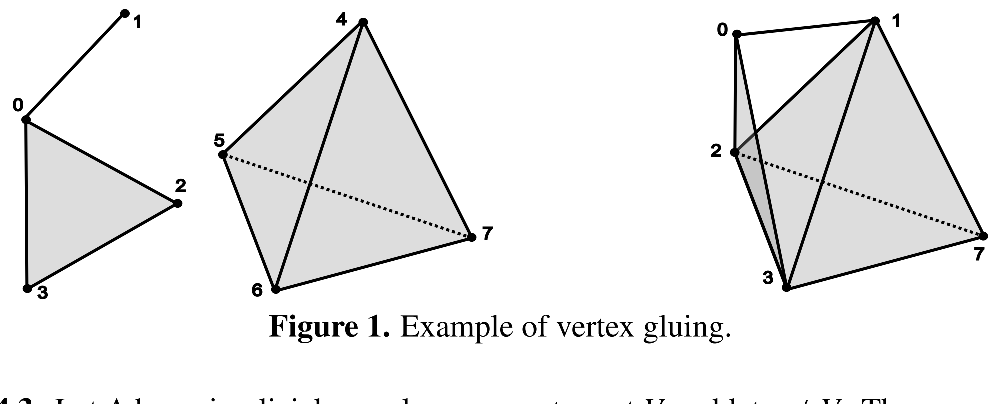

Figure 1. Example of vertex gluing.

Definition 4.3. Let Δ be a simplicial complex on a vertex set V and let $c \notin V$. The cone over Δ is the simplicial complex

$$\operatorname{cone}(\Delta) = \Delta \cup \{\sigma \cup \{c\} : \sigma \in \Delta\}.$$

The vertex c is called the apex of the cone.

Proposition 4.4. *Let Δ be a simplicial complex and let* cone(Δ) *denote the cone over Δ with apex c. Let u be a nonzero vector of the code associated to* cone(Δ)*. If $c \notin$ supp(u), then $w(c_{\operatorname{cone}(\Delta)^*}(u)) = 2\,w(c_{\Delta^*}(u_{|\Delta}))$. If $c \in$ supp(u), then $w(c_{\operatorname{cone}(\Delta)^*}(u)) \geq |\Delta|$.*

Proof. By definition, the cone over Δ is given by $\text{cone}(\Delta) = \{\langle \mathcal{F}_i \cup \{c\} \rangle\}$, where $\mathcal{F}_i$ are the maximal elements of Δ.

Assume first that $c \notin \text{supp}(u)$. For any simplex $\sigma \in \Delta$, we have

$$\sum_{v\in\sigma} u_v \equiv \sum_{v\in\sigma\cup\{c\}} u_v \pmod q.$$

Therefore, σ contributes to the weight of u if and only if $\sigma \cup \{c\}$ does. It follows that each contributing simplex in Δ gives rise to exactly two contributing simplices in $\text{cone}(\Delta)$, and hence

$$w(c_{\text{cone}(\Delta)^*}(u)) = 2\, w(c_{\Delta^*}(u_{|\Delta})).$$

Assume now that $c \in \text{supp}(u)$. For every simplex $\sigma \in \Delta$, at least one of the two simplices σ or $\sigma \cup \{c\}$ satisfies

$$\sum_{v\in\tau} u_v \not\equiv 0 \pmod q.$$

Consequently, at least one simplex in each pair $\{\sigma, \sigma \cup \{c\}\}$ contributes to the weight. Since Δ has $|\Delta|$ simplices, we obtain

$$w(c_{\text{cone}(\Delta)^*}(u)) \geq |\Delta|.$$

□

Corollary 4.5. *Let the code associated to* Δ *have parameters* $[n, k, d]$*. Then the code associated with* $\text{cone}(\Delta)$ *has minimum distance* $d' = 2d$*. Furthermore, its length is* $n' = 2n + 1$*, and its dimension is* $k' = k + 1$.

Proof. Let $u \neq 0$ be a vector. We distinguish two cases.

If $c \notin \text{supp}(u)$, then by Proposition 4.4, we have

$$w(c_{\text{cone}(\Delta)^*}(u)) = 2\, w(c_{\Delta^*}(u_{|\Delta})) \geq 2d.$$

Assume now that $c \in \text{supp}(u)$. By Proposition 4.4, we have $w(c_{\text{cone}(\Delta)^*}(u)) \geq |\Delta|$. We claim that $d \leq \frac{|\Delta|}{2}$. Indeed, let u' be a vector whose support consists of a single vertex v. Since at most half of the simplices of Δ contain v, it follows that

$$w(c_{\Delta^*}(u')) \leq \frac{|\Delta|}{2}.$$

Therefore, $d \leq w(c_{\Delta^*}(u')) \leq \frac{|\Delta|}{2}$.

It follows that

$$w(c_{\text{cone}(\Delta)^*}(u)) \geq |\Delta| \geq 2d.$$

Hence, in all cases, every nonzero codeword has weight of at least $2d$, and equality is attained when $c \notin \text{supp}(u)$. Thus, $d' = 2d$. □

Example 4.6. Let Δ be a simplicial complex whose set of maximal faces is $\{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}$, the 2-skeleton of a 3-simplex. The associated code C_{Δ^*} has parameters $[14, 4, 7]$. If we take the cone over Δ, we obtain a new code with parameters $[29, 5, 14]$.

If all maximal faces contain a common vertex c, it is possible to define an operator that acts as an inverse of the operation cone as the composition of elementary collapses with respect to that vertex. More precisely, given a simplicial complex containing a maximal face $A_i = \langle v_1, \ldots, v_n, c\rangle$, that maximal face and the free faces containing the common vertex c are removed,

$$\langle v_1, \ldots, v_n, c\rangle \longrightarrow \bigcup_{i\in[n]} \langle v_1, \ldots, \widehat{v_i}, \ldots, v_n\rangle,$$

where the symbol $\widehat{v_i}$ denotes the omission of the vertex v_i. Repeating this process for all maximal faces of the complex yields a simplicial complex that can be interpreted as the inverse of the operation cone.

In the general case, when there is no vertex common to all maximal faces, one can define an analogous operator that reduces the dimension of the simplicial complex, namely, the boundary operator. Given an arbitrary maximal face, one can apply the inverse operation of the operator cone by removing the free faces containing each of its vertices. By performing this process for all vertices of the maximal face, that is, by means of a combination of elementary collapses, the boundary of the maximal face is obtained. Repeating this procedure for all maximal faces of the complex yields the boundary of the simplicial complex.

Proposition 4.7. *Let Δ be a simplicial complex and let $\partial(\Delta)$ denote its boundary complex, that is, the subcomplex consisting of all non-maximal simplices of Δ. Passing from Δ to $\partial(\Delta)$ does not increase the minimum distance of the associated linear code.*

Indeed, if Δ has s maximal faces and no isolated vertices, the associated codes satisfy

$$d(C_{\Delta^*}) \geq d(C_{\partial(\Delta)^*}),$$

the length decreases from n to $n' = n - s$, and the dimension remains unchanged.

Proof. Let u' be a vector supported on $\partial(\Delta)$. Its weight is given by

$$c_{\partial(\Delta)^*}(u')) = \#\left\{\partial(\sigma) : \sigma \in \Delta \text{ and } \sum_{v\in\partial(\sigma)} u_v \not\equiv 0 \pmod q\right\},$$

or equivalently,

$$w(c_{\partial(\Delta)^*}(u')) = \#\left\{\sigma \in \Delta : \sigma \text{ is not maximal and } \sum_{v\in\sigma} u_v \not\equiv 0 \pmod q\right\}.$$

When passing from Δ to $\partial(\Delta)$, the only contributions that may be lost correspond to maximal simplices. Since there are s such simplices, the loss in weight is bounded by

$$w(c_{\Delta^*}(u)) - w(c_{\partial(\Delta)^*}(u')) \leq s,$$

which implies

$$d(C_{\Delta^*}) \geq d(C_{\partial(\Delta)^*}).$$

Finally, removing the s maximal simplices reduces the length of the code from n to $n - s$. The dimension remains unchanged provided that Δ has no isolated vertices. ◻

Remark 4.2. When passing from Δ to its boundary $\partial(\Delta)$, the maximal faces of Δ become elements of the complement $\partial(\Delta)^c$. Consequently, the generator matrix of the associated anticode $C_{\partial(\Delta)^c}$ acquires s additional columns, corresponding to these maximal faces, compared to that of C_{Δ^c}. Since none of the original columns are removed, the weight can only increase.

Lemma 4.8. *Let $k > 1$, and let $u \in \mathbb{F}_q^k$ be a nonzero vector. Then*

$$\#\{x \in \mathbb{F}_q^k : \langle u, x\rangle = 0\} = q^{k-1},$$

and consequently,

$$\#\{x \in \mathbb{F}_q^k : \langle u, x\rangle \neq 0\} = (q-1)q^{k-1}.$$

Proof. Since $u \neq 0$, the map

$$\varphi_u : \mathbb{F}_q^k \longrightarrow \mathbb{F}_q, \qquad x \longmapsto \langle u, x\rangle$$

is a nonzero linear functional. Its kernel is therefore a subspace of codimension 1, and hence has cardinality q^{k-1}. The result follows. □

Theorem 4.9. *Let $\{\Delta_k\}_{k>1}$ be a family of simplicial complexes on vertex set $[k]$, and assume that there exists $n_0 \in \mathbb{N}$ such that*

$$\dim \Delta_k \leq n_0 \quad \textit{for all } k.$$

Let $\Delta_k^c = \mathcal{P}([k]) \setminus \Delta_k$ be the complement of Δ_k, and let $C_{\Delta_k^c}$ be the associated linear code. Then, for every nonzero $u \in \mathbb{F}_q^k$,

$$w(c_{\Delta_k^c}(u)) = \#\{x \in \mathbb{F}_q^k \setminus D_k : \langle u, x\rangle \neq 0\} = (q-1)q^{k-1} + O(k^{n_0+1}),$$

where the implicit constant is independent of u. In particular,

$$\frac{d(C_{\Delta_k^c})}{|\mathbb{F}_q^k \setminus D_k|} \longrightarrow \frac{q-1}{q} \qquad \textit{as } k \to \infty.$$

The relative minimum distance converges to $(q-1)/q$, which is the maximal possible value for the relative distance of a q-ary code.

Proof. Fix k and let $u \in \mathbb{F}_q^k$ be nonzero. By the previous lemma,

$$\#\{x \in \mathbb{F}_q^k : \langle u, x\rangle \neq 0\} = (q-1)q^{k-1}.$$

We identify each subset $\sigma \subseteq [k]$ with its characteristic vector $\chi_\sigma \in \mathbb{F}_q^k$, and define $D_k = \{\chi_\sigma : \sigma \in \Delta_k\}$. Since $\dim \Delta_k \leq n_0$, every simplex $\sigma \in \Delta_k$ has cardinality at most $n_0 + 1$. Hence,

$$|D_k| = |\Delta_k| \leq \sum_{i=0}^{n_0} \binom{k}{i+1}.$$

In particular, $|D_k| = O(k^{n_0+1})$. Recall that

$$w(c_{\Delta_k^c}(u)) = \#\{x \in \mathbb{F}_q^k \setminus D_k : \langle u, x\rangle \neq 0\}.$$

Removing the set D_k can affect the weight by at most $|D_k|$, and hence

$$w(c_{\Delta_k^c}(u)) = (q-1)q^{k-1} - \#\{x \in D_k : \langle x, u \rangle \neq 0\} = (q-1)q^{k-1} + O(k^{n_0+1}).$$

Since $|\mathbb{F}_q^k \setminus D_k| = q^k - O(k^{n_0+1})$, dividing by $|\mathbb{F}_q^k \setminus D_k|$ yields

$$\frac{d(C_{\Delta_k^c})}{|\mathbb{F}_q^k \setminus D_k|} \xrightarrow[k\to\infty]{} \frac{q-1}{q}.$$

□

Example 4.10. Let $\Delta_0 = \langle \{1,2,3\}, \{3,4,5\}, \{1,3,5\} \rangle$ be a simplicial complex on the vertex set $\{1,2,3,4,5\}$. By performing suitable subdivisions (see Figure 2) of the simplices of Δ_0, we obtain a refined triangulation

$$\Delta = \langle \{1,2,8\}, \{1,3,8\}, \{2,3,8\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}, \{1,5,7\}, \{1,3,7\}, \{3,5,7\} \rangle$$

on the vertex set $\{1,2,3,4,5,6,7,8\}$.

Consider the complementary set Δ^c and the corresponding linear code C_{Δ^c} over $\mathbb{F}_3$. A direct computation shows that this code has parameters

$$[6527,\ 8,\ 4344].$$

In particular, the relative minimum distance satisfies

$$\frac{4344}{6527} \approx 0.6655,$$

which is very close to the asymptotic value $(q-1)/q = 2/3$ predicted by the previous theorem.

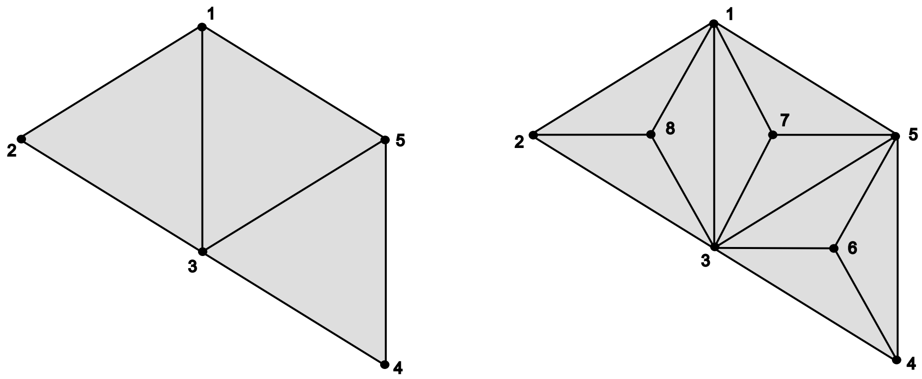

Figure 2. Subdivision of Δ_0 presented in Example 4.10.

Example 4.11. Continuing with the sets from the previous example, the associated anticode to Δ_0 over $\mathbb{F}_2$ has parameters

$$[16, 5, 6],$$

giving a distance-to-length ratio of $6/16 = 0.375$.

The anticode over $\mathbb{F}_2$ associated with Δ, on the other hand, has parameters

$$[222, 8, 107],$$

with a distance-to-length ratio of approximately $107/222 \approx 0.482$, which is a significant improvement over the original complex.

5. Construction of codes over $\mathbb{F}_2$ using geometric properties

In this section, we focus on the construction of linear codes over the field $\mathbb{F}_2$ using the geometric properties of simplicial complexes developed in the previous sections. In particular, our aim is to introduce a geometric framework that allows one to construct and interpret such codes. While some of the parameters obtained here may also be achievable via existing constructions, our approach provides a unified geometric perspective that clarifies the underlying structure of these codes and their properties.

There are several reasons for choosing $\mathbb{F}_2$ as the base field. First, as observed earlier, for this class of codes, the minimum distance is independent of the size of the field. Moreover, working over $\mathbb{F}_2$ allows us to interpret the weight of a codeword from a geometric point of view, i.e., in terms of intersections with simplices rather than sums modulo q.

Finally, from a computational point of view, constructions over $\mathbb{F}_2$ are significantly less expensive and, therefore, more amenable to explicit calculations and experimentation.

Let Δ be a simplicial complex on the vertex set $[N{+}1]$. For any vector $u \in \mathbb{F}_2^k$, the weight of the corresponding codeword admits a geometric interpretation as the number of simplices of Δ whose intersection with the support of u has odd cardinality. That is,

$$w(c_{\Delta^*}(u)) = \#\{\sigma \in \Delta : |\,\mathrm{supp}(u) \cap \sigma| \equiv 1 \pmod 2\}.$$

We begin with the simplest possible example. Let Δ be a simplicial complex consisting of a single maximal face A_1 with $|A_1| = N + 1$. Equivalently, Δ is the standard N–simplex.

From the results established above in Corollary 3.2, it follows directly that the associated code C_{Δ^*} has parameters

$$[2^{N+1} - 1,\ N{+}1,\ 2^N].$$

In particular, all codes arising from simplicial complexes of this form are optimal.

Proposition 5.1. *The linear code associated with the $(N-1)$–skeleton of an N–simplex has parameters*

$$[2^{N+1} - 2,\ N + 1,\ 2^N - 1].$$

In particular, these codes are length-optimal.

Proof. As shown in the previous section, passing to the boundary of a simplicial complex decreases the length of the associated code by the number of maximal faces of the original complex, and decreases the minimum distance by at most one per maximal face removed. Since an N–simplex has exactly one maximal face, both the length and the minimum distance decrease by exactly one.

Therefore, the resulting code has parameters

$$[2^{N+1} - 2,\ N + 1,\ 2^N - 1].$$

It remains to verify that these parameters attain the Griesmer bound.

$$\begin{aligned}\sum_{n=0}^{N}\left\lceil \frac{2^N-1}{2^n}\right\rceil &= 2^N - 1 + \sum_{n=1}^{N}\left\lceil \frac{2^N-1}{2^n}\right\rceil = 2^N - 1 + \sum_{n=1}^{N}\left\lceil \frac{2^N}{2^n} - \frac{1}{2^n}\right\rceil \\ &= 2^N - 1 + \sum_{n=1}^{N}\frac{2^N}{2^n} = 2^N - 1 + 2^N - 1 = 2^{N+1} - 2.\end{aligned}$$

This shows that the code meets the Griesmer bound and is therefore optimal. □

Theorem 5.2. *An infinite family of distance-optimal linear codes can be constructed by taking the cone over the $(N-1)$–skeleton of an N–simplex. These codes have parameters*

$$[\,2(2^{N+1}-2)+1,\ N+2,\ 2(2^{N}-1)].$$

Proof. From the previous result, the linear code associated with the $(N-1)$–skeleton of an N–simplex has parameters

$$[\,2^{N+1}-2,\ N+1,\ 2^{N}-1\,].$$

Taking the cone over this simplicial complex modifies the parameters in a controlled way, according to the geometric results established in Proposition 4.4. In particular, the length is doubled and increased by one, the dimension increases by one, and the distance is doubled, yielding a new code with parameters

$$[\,2(2^{N+1}-2)+1,\ N+2,\ 2(2^{N}-1)\,].$$

We now analyze the Griesmer bound. A direct computation shows that

$$\begin{aligned}\sum_{n=0}^{N+1}\left\lceil\frac{2(2^{N}-1)}{2^{n}}\right\rceil &= 2(2^{N}-1)+(2^{N}-1)+\sum_{n=1}^{N}\left\lceil\frac{2^{N}-1}{2^{n}}\right\rceil\\ &= 2(2^{N}-1)+(2^{N}-1)+\sum_{n=1}^{N}\frac{2^{N}}{2^{n}}\\ &= 2(2^{N}-1)+(2^{N}-1)+(2^{N}-1)\\ &= 2(2^{N+1}-2).\end{aligned}$$

Therefore, the length of the code exceeds the Griesmer bound by exactly one unit.

However, no code with the same length and dimension can have strictly larger minimum distance. Indeed, assume that there exists a code with parameters $[\,2(2^{N+1}-2)+1,\ N+2,\ 2(2^{N}-1)+1\,]$. Then

$$\begin{aligned}\sum_{n=0}^{N+1}\left\lceil\frac{2(2^{N}-1)+1}{2^{n}}\right\rceil &= 2^{N+1}-1+\sum_{n=1}^{N+1}\left\lceil\frac{2^{N+1}-1}{2^{n}}\right\rceil\\ &= 2^{N+1}-1+\sum_{n=1}^{N+1}2^{N+1-n}\\ &= 2^{N+2}-2\\ &> 2(2^{N+1}-2)+1,\end{aligned}$$

which contradicts the Griesmer bound. Hence, no code with these parameters can have larger minimum distance, and the constructed code is distance-optimal. □

Example 5.3. Consider the triangulation of the torus shown in Figure 3. Let Δ be the associated simplicial complex. Then the length of the code C_{Δ^*} is 42, corresponding to the number of elements of Δ^* (i.e., $|\Delta|-1$). Its dimension is 7, given by the number of vertices.

Moreover, the minimum distance of the code is 13, which coincides with the number of elements in the link of each vertex. Indeed, each vertex is contained in 6 two-dimensional faces and 6 edges, together with the vertex itself.

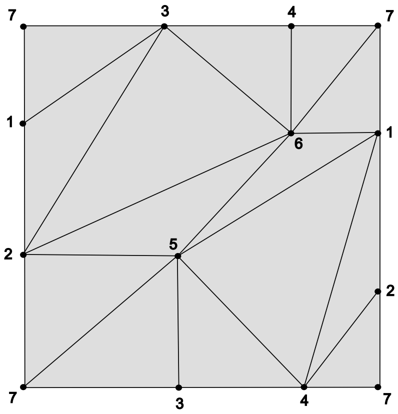

Figure 3. Torus triangulation.

Example 5.4. In the following example, we use the results proved throughout the article to calculate the parameters of the linear code associated with the triangulation of the pseudomanifold (see Figure 4)

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^4 + y^4 = xyz, z^2 < 1\}.$$

Let Δ be the simplicial complex defined as

$$\Delta = \langle \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\} \rangle.$$

The associated code has parameters $[13, 4, 6]$, where the length corresponds to the number of elements of Δ^*, the dimension is given by the number of vertices, and the minimum distance is determined by the size of the smallest vertex link. For example, the link of vertex 1 is $\text{link}(\{1\}) = \{\emptyset, \{0\}, \{2\}, \{3\}, \{0, 2\}, \{0, 3\}\}$.

Now consider two disjoint copies of Δ. The resulting code associated to the new simplicial complex has parameters $[26, 8, 6]$. If we identify the edges $\{0, 1\} \sim \{0', 1'\}$, then the number of vertices decreases by two and the number of faces is reduced accordingly, yielding a new code with parameters $[23, 6, 6]$. The minimum distance remains unchanged since there are still vertices whose links contain 6 elements.

Repeating this construction, by taking two disjoint copies of the resulting complex and identifying the vertices $\{0\}$ and $\{0''\}$ in each copy, we obtain a simplicial complex that provides a triangulation of the pseudomanifold. The associated code has parameters $[45, 11, 6]$.

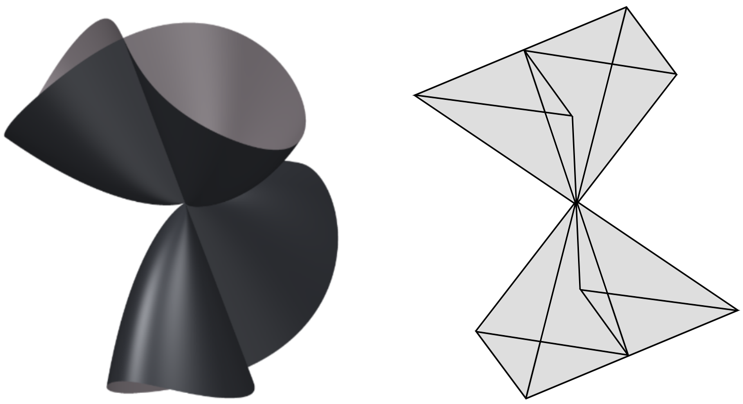

Figure 4. Pseudomanifold of Example 5.4. and its triangulation.

6. Conclusions

In this work, a geometric approach to the study of linear codes associated with simplicial complexes has been developed. The main result, Theorem 3.1, allows one to dispense with the weight distribution function and provides a geometric interpretation of the distance.

Furthermore, we analyze how certain topological operations on simplicial complexes affect the resulting code, which in turn makes it possible to understand how to modify its parameters. In addition, Theorem 4.9 is introduced, which studies the parameters of the anticode in terms of the structure of the complex and provides the conditions for the construction of asymptotically optimal codes.

From the geometric perspective developed in this work, some known families of optimal codes are recovered. Moreover, examples of codes and their parameters arising from triangulations of manifolds are presented, illustrating the applicability of the results obtained.

Author contributions

Antonio Jesús Lorite López, Daniel Camazón Portela and Juan Antonio López Ramos: Conceptualization, methodology, investigation, analysis, writing-original draft, writing-review and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Juan Antonio López Ramos and Dr. Daniel Camazon Portela are the Guest Editors of special issue "Interactions Between Ring Theory and Coding Theory" for AIMS Mathematics. They were not involved in the editorial review and the decision to publish this article. All authors declare no conflicts of interest in this paper.

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