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Research article

Spectral theory for transfer operators on non-standard symbolic spaces arising from transcendental dynamics

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Abstract: We study the shift map on a non-compact symbolic space X endowed with a metric ρ that is not compatible with the usual product topology. While abstract, this setting is motivated by symbolic codings arising from transcendental entire maps with Cantor bouquet Julia sets, where the natural metric differs from the standard product topology. We study a spectral theory of Ionescu-Tulcea and Marinescu type for transfer operators acting on Banach spaces of locally Hölder functions adapted to this non-standard metric. We prove quasi-compactness of the transfer operator and establish the existence of a spectral gap, showing that the spectral radius is isolated and corresponds to a simple leading eigenvalue equal to the topological pressure. We identify the associated eigenfunction, conformal measure, and Gibbs state and establish a variational principle. The arguments rely on general symbolic assumptions and do not rely on the product topology. As an application, we consider symbolic models arising from hyperbolic transcendental entire maps of finite order.

Keywords: transcendental entire maps; symbolic dynamics; spectral theory
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1. Introduction

The Ionescu-Tulcea and Marinescu theorem, established in their seminal paper [11], is a core result in functional analysis and has provided one of the key analytic tools for the spectral study of transfer operators acting on non-compact symbolic spaces with a countable alphabet (see [15, 20]). When an operator satisfies a Lasota-Yorke-type inequality, contracting with respect to a strong norm (typically Hölder regularity) and bounded with respect to a weaker norm (such as L^1), the theorem yields quasi-compactness of the operator. As a consequence, the spectral radius is isolated and corresponds to a finite-dimensional dominant eigenspace, while the remainder of the spectrum lies strictly inside a smaller disk. This spectral structure produces a positive eigenfunction and a conformal eigenmeasure, which together yield a Gibbs state.

We consider a symbolic model for the shift map acting on a non-compact space with an infinite alphabet, endowed with a metric ρ that is not compatible with the standard product topology generated by cylinders. While abstract in formulation, this setting is motivated by codings arising from transcendental entire maps with Cantor bouquet Julia sets, such as the exponential family $E_\omega(z) = \omega e^z$ for $0 < \omega < 1/e$ (see [6–8]). The endpoints of hairs admit symbolic encodings, but the metric reflecting their Euclidean geometry does not behave like the usual shift metric. This places our setting in a different geometric context from the classical countable Markov shift theory studied in [15, 19]; the latter serves as an important inspiration for our approach.

We define the shift map acting on a symbolic metric space (X, ρ) that satisfies topological mixing, local expansion, and approximation by compact subshifts of finite type Σ_N. These properties capture key dynamical features of transcendental entire maps with Cantor bouquet geometry. Under suitable mild assumptions on the metric and on the local Hölder regularity of the potential, the authors in [10] established the existence of a conformal measure on X, obtained as a weak limit of conformal measures on the compact approximations Σ_N, together with an invariant measure satisfying the Gibbs property on dynamically relevant subsets of X, constructed by averaging iterates of the conformal measure via a Banach limit.

The approximation in [10] does not provide a spectral description of the transfer operator on X, nor does it establish quasi-compactness adapted to the non-standard metric ρ. The present work provides this missing perspective and develops a thermodynamic formalism adapted to this non-standard symbolic setting. The novelty of this paper lies in adapting the quasi-compactness framework of the Ionescu-Tulcea and Marinescu theorem adapted to the metric ρ. Under additional spectral assumptions introduced here, we prove that the normalized transfer operator $\mathscr{L}_{\widehat{\phi}}$, acting on the Banach space H_α of locally Hölder continuous functions on (X, ρ), is quasi-compact and admits a spectral gap. As a consequence, the leading eigenvalue is simple and equal to the exponential of the pressure $P(\phi)$, and we obtain a strictly positive eigenfunction $h_\phi \in H_\alpha$. These objects together with the conformal measure m_ϕ constructed in [10] yield a σ-invariant measure $\mu_\phi = h_\phi m_\phi$, which is ergodic. Here, the pressure $P(\phi)$ is defined as the supremum of the pressures on the compact approximating subshifts Σ_N, with respect to the metric ρ.

In the present work, our approach is based on the spectral properties of the normalized transfer operator. Since its leading eigenvalue is simple, the associated invariant probability measure μ_ϕ is unique. Consequently, the invariant Gibbs measure constructed in [10] (see Remark 2.4) coincides with μ_ϕ provided that ϕ satisfies the hypotheses of both frameworks.

The structure of the paper is as follows: Section 2 introduces the setting, symbolic space, the metric, hypotheses, and remarks about assumptions. Section 3 states the results and Section 4 studies locally Hölder potentials, the transfer operator, and the pressure. Section 5 applies the Ionescu-Tulcea and Marinescu theorem to obtain the spectral results. Section 6 establishes the variational principle. Section 7 discusses the symbolic models arising from transcendental entire dynamics with Cantor bouquet geometry.

2. Symbolic framework and definitions

This section gathers the dynamical hypotheses of our non-standard symbolic framework together with the key definitions. These elements are needed to establish the main results.

2.1. The symbolic metric space

Let

$$\Sigma := \{\underline{a} = (a_0, a_1, \ldots) : a_j \in \mathbb{Z},\ j \geq 0\}$$

be the one-sided full shift over the countable alphabet $\mathbb{Z}$, endowed with the usual metric: For some $\theta \in (0, 1)$,

$$d_\theta(\underline{s}, \underline{t}) := \theta^{\inf\{k \geq 0 : s_k \neq t_k\}}, \qquad 0 < \theta < 1,$$

where, by convention, $\theta^\infty = 0$. The shift map $\sigma : \Sigma \to \Sigma$ is given by $\sigma(a_0 a_1 a_2 \cdots) = (a_1 a_2 a_3 \cdots)$.

For each integer $N \geq 1$, define the compact subshift

$$\Sigma_N := \{\underline{a} \in \Sigma : a_j \in \{-N, \ldots, N\} \text{ for all } j \geq 0\},$$

which represents the symbolic space restricted to a finite alphabet.

We now introduce a symbolic space $X \subset \Sigma$ that will serve as the central object of study. The space X satisfies the following properties:

- The space X is invariant under the shift, that is,

$$\sigma(X) = X = \sigma^{-1}(X).$$

  Hence, all iterates $\sigma^n|_X$ are well defined.
- The space X is endowed with a metric ρ that is not compatible with the standard metric d_θ.
- For every $N \geq 1$, the finite subshift $\Sigma_N \subset X$ is compact with respect to ρ.
- For any compact, forward-invariant subset $\Lambda \subset X$, there exists $N \geq 1$ such that $\Lambda \subseteq \Sigma_N$.

These assumptions ensure that X behaves as a locally compact symbolic space approximated by an increasing sequence of compact subshifts Σ_N.

A fundamental example of such a symbolic space arises in the dynamics of the *exponential map*

$$E_\omega(z) = \omega e^z, \qquad \omega \in (0, 1/e),$$

which is a prototypical transcendental entire function in the Eremenko-Lyubich class $\mathcal{B}$. The most interesting dynamics occur in the Julia set $J(E_\omega)$, which forms a Cantor bouquet. Its set of endpoints can be encoded by a symbolic space X of *allowable sequences*. Specifically, X consists of all sequences $\underline{a} = (a_0 a_1 \ldots) \in \Sigma$ corresponding to the itinerary of some endpoint $z \in \mathbb{C}$ under the dynamics of E_ω. Not all sequences appear as itineraries of endpoints, hence X is a proper, completely invariant subset of Σ encoding the dynamically relevant part of E_ω (see [6, 13]). See the simplified graph (Figure 1).

More generally, for transcendental entire functions f of finite order, with a bounded set of singular values and rapid derivative growth, the Julia set $J(f)$ is a Cantor bouquet. The endpoints of its dynamic rays are in one-to-one correspondence with symbolic sequences in $X \subset \Sigma$, and the dynamics of f restricted to the set of endpoints is conjugate to the shift map restricted to X.

To clarify this correspondence, we briefly recall the symbolic representation and the class of potentials satisfying the required properties in the general setting described in Section 7, as first established in [10].

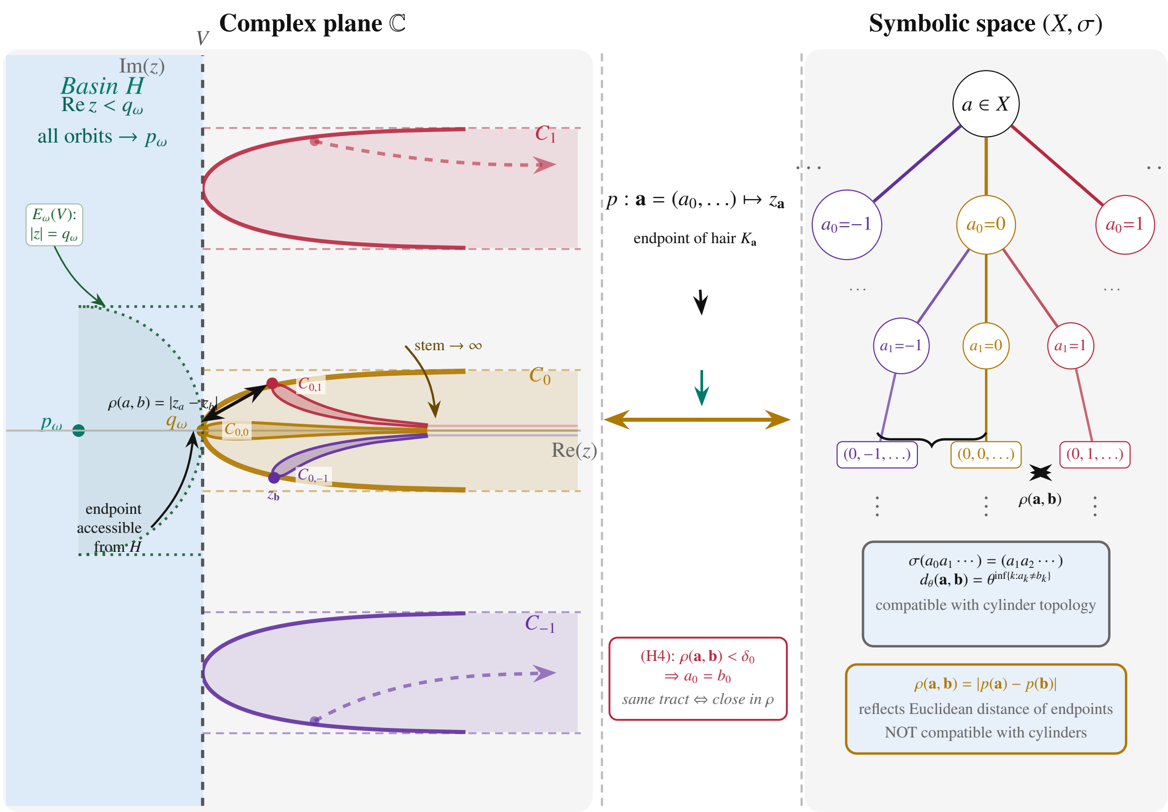

Figure 1. Exponential map: the basin of attraction H (left of q_ω) attracts all orbits to p_ω. "The fingers C_j lie in strips" $-\pi+2j\pi < \operatorname{Im} z < \pi+2j\pi$, and contain the hairs of the Cantor bouquet $J(E_\omega)$. Each hair carries a unique endpoint $z_{\mathbf{a}}$, coded by an allowable sequence $\mathbf{a} \in X$. The metric $\rho(\mathbf{a}, \mathbf{b}) = |z_{\mathbf{a}} - z_{\mathbf{b}}|$ reflects Euclidean geometry and is not compatible with the standard symbolic metric d_θ. See [2] and references therein.

2.2. *A metric defined on X and assumptions*

Let

$$\rho : X \times X \to [0, +\infty)$$

be a metric on X.

For $\underline{a} = (a_0 a_1 \ldots) \in \Sigma$ and a finite word $b^* = b_0 \ldots b_{n-1} \in \mathbb{Z}^n$, define the concatenation

$$b^*\underline{a} := (b_0, \ldots, b_{n-1}, a_0, a_1, \ldots), \quad \text{and for } A \subset \Sigma,\ b^*A := \{b^*\underline{a} : \underline{a} \in A\}.$$

For $\underline{a} \in X$ and $\delta > 0$, define

$$B(\underline{a}, \delta) := \{\underline{b} \in X : \rho(\underline{a}, \underline{b}) < \delta\}, \quad \mathbb{B}_0(\underline{a}, \delta) := \{\underline{b} \in B(\underline{a}, \delta) : b_0 = a_0\},$$

and the (n,δ)-dynamical ball

$$\mathbb{B}_n(\underline{a},\delta) := \left\{\underline{b} \in X : \sigma^j(\underline{b}) \in \mathbb{B}_0(\sigma^j(\underline{a}),\delta) \text{ for all } 0 \le j \le n-1\right\}.$$

Assume that there exists $\delta_0 = \delta_0(\rho) > 0$ such that ρ satisfies the following:

(H1) There exist $C > 0$ and $\lambda > 1$ such that for all $n \in \mathbb{N}$, any $\underline{a},\underline{b} \in X$ with $\rho(\underline{a},\underline{b}) < \delta_0$, and every $c^* \in \mathbb{Z}^n$,

$$\rho(c^*\underline{a}, c^*\underline{b}) \le C\lambda^{-n}\rho(\underline{a},\underline{b}).$$

(H2) For every $R > 0$, there exists $n \ge 1$ such that

$$\sigma^n(B(\underline{a},\delta_0)) \supset B(\underline{0},R) \quad \text{for all } \underline{a} \in B(\underline{0},R),$$

where $\underline{0} = (0,0,\ldots)$.

(H3) For every $\delta \in (0,\delta_0]$, setting $\delta' := \min\{\delta, \frac{\delta}{C}\}$, there exists $\ell = \ell(\delta) \ge 1$ such that for every $\underline{a} \in X$ and every $\underline{b},\underline{c} \in \mathbb{B}_0(\underline{a},\delta)$, there is a finite chain

$$\underline{b} = \underline{a}_0, \underline{a}_1, \ldots, \underline{a}_\ell = \underline{c}$$

satisfying $\rho(\underline{a}_j, \underline{a}_{j+1}) < \delta'$ for all $0 \le j \le \ell - 1$.

(H4) If $\rho(\underline{a},\underline{b}) < \delta_0$, then $a_0 = b_0$.

The symbolic coding for the exponential map $E_\omega(z) = \omega e^z$ with $\omega \in (0, 1/e)$ is given by the map

$$p\colon X \longrightarrow \mathbb{C}, \qquad \underline{a} \longmapsto z_{\underline{a}},$$

where $z_{\underline{a}}$ is the endpoint of the dynamic ray (hair) $K_{\underline{a}}$ associated with the allowable sequence $\underline{a} = (a_0, a_1, \ldots) \in X \subset \Sigma$. The non-standard metric on X is defined by

$$\rho(\underline{a},\underline{b}) = |p(\underline{a}) - p(\underline{b})| = |z_{\underline{a}} - z_{\underline{b}}|.$$

The metric ρ satisfies properties (H1)–(H4). See Subsection 7.4.

Note that the natural shift metric d_θ with $\theta \in (0,1)$, satisfies the condition (H1). Indeed, if we put $\delta_0 = 1$, $C = 1$ and $\lambda = 1/\theta$, then for every $\underline{a},\underline{b} \in \Sigma$, $u^* \in \mathbb{Z}^n$, we have $d_\theta(\underline{a},\underline{b}) \le 1$, $d(\underline{a},\underline{b}) = \theta^s$ with $s = \inf\{k : a_k \ne b_k\}$, then $d_\theta(u^*\underline{a}, u^*\underline{b}) \le \theta^{n+s} = \theta^n d(\underline{a},\underline{b})$.

Table 1 presents the main structural features of the classical shift metric d_θ and the non-standard metric ρ on X. The two columns describe distinct geometric situations; no comparison of strength or generality is intended.

Table 1. The metric d_θ and the non-standard metric ρ on X.

Standard shift metric d_θ	**Non-standard metric** ρ
Defined by $d_\theta(a,b) = \theta^{\inf\{k\geq 0: a_k\neq b_k\}}$ on the full shift Σ	A general metric defined on a non-compact subset X of Σ (e.g., $\rho(a,b) = \lvert p(a) - p(b)\rvert$ induced from the Euclidean geometry of the Julia set)
Compatible with the product topology generated by cylinders	Not assumed compatible with the cylinder topology; local structure is governed by hypotheses (H1)–(H4)
Contraction determined uniformly by the constant $\theta \in (0,1)$	Contraction along inverse branches controlled by hypothesis (H1), with constants $C > 0$ and $\lambda > 1$
Markov structure; a rich and complete thermodynamic formalism has been developed in this setting	Markov structure is not assumed; the framework is motivated by transcendental dynamics where this structure is not present, requiring adapted analytic tools
Purely symbolic metric	Metric reflects Euclidean geometry of dynamic rays (Cantor bouquet structure)

Remark 2.1. *Assumption (H2) ensures that the system (X,σ) is topologically mixing, and it ensures the existence of dense subsets in X (Lemma 4.3). Moreover, it guarantees the existence of controlled preimages, which is necessary for comparing the values of the transfer operator at different points and for constructing a conformal measure (see [10, Proposition 2.4]). Assumption (H3) guarantees that any two close points within the same 1-cylinder can be connected by a finite path of uniformly small steps (Lemma 4.4). This condition is crucial to extend Hölder regularity from small to large scales and to control the distortion of weakly Hölder continuous potentials and the transfer operator (see [10, Lemmas 2.9 and 2.10]).*

Remark 2.2. *Condition (H4) asserts that if two endpoints are sufficiently close in the Euclidean sense, then they belong to the same fundamental tract of the map, and therefore share the same first symbol. This property provides a minimal compatibility condition between the geometry of the bouquet and the symbolic partition, ensuring that local closeness in ρ corresponds to belonging to the same tract, without requiring global equivalence between the topologies induced by ρ and d_θ.*

2.3. Potentials, transfer operator definitions, and remarks

Definition 2.1. *Given $\delta_1 > 0$ and $\alpha \in (0,1]$, a function $\phi : X \to \mathbb{R}$ is said to be* uniformly δ_1-locally α-Hölder continuous *if there exists a constant $L \geq 0$ such that for all $\underline{a}, \underline{b}, \underline{c} \in X$ with $\underline{a}, \underline{b} \in \mathbb{B}_0(\underline{c}, \delta_1)$,*

$$|\phi(\underline{a}) - \phi(\underline{b})| \leq L\,(\rho(\underline{a}, \underline{b}))^{\alpha}. \tag{2.1}$$

We denote by H_{α,δ_1} the space of all bounded uniformly δ_1-locally α-Hölder potentials. This space is endowed with the norm

$$\|\phi\|_{\alpha,\delta_1} := |\phi|_{\alpha,\delta_1} + \|\phi\|_\infty,$$

where $\|\phi\|_\infty = \sup_{\underline{a}\in X} |\phi(\underline{a})|$ and

$$|\phi|_{\alpha,\delta_1} := \inf\left\{L \geq 0 : |\phi(\underline{a}) - \phi(\underline{b})| \leq L(\rho(\underline{a},\underline{b}))^\alpha \text{ whenever } \rho(\underline{a},\underline{b}) < \delta_1\right\}. \tag{2.2}$$

It follows from standard arguments that $(H_{\alpha,\delta_1}, \|\cdot\|_{\alpha,\delta_1})$ is a Banach space.

Let $\mathrm{CB}(X,\mathbb{R})$ denote the space of real-valued bounded continuous functions on X, equipped with the uniform norm $\|\cdot\|_\infty$. The transfer operator associated with a summable potential ϕ acts on $\mathrm{CB}(X,\mathbb{R})$ as,

$$\mathscr{L}_\phi\varphi(\underline{a}) = \sum_{\underline{b}:\sigma(\underline{b})=\underline{a}} e^{\phi(\underline{b})}\varphi(\underline{b}), \quad \text{for } \underline{a} \in X. \tag{2.3}$$

We denote by $\mathbb{1}$ the constant function equal to 1 on X. Then the transfer operator applied to $\mathbb{1}$ is

$$\mathscr{L}_\phi\mathbb{1}(\underline{a}) = \sum_{\underline{b}:\sigma(\underline{b})=\underline{a}} e^{\phi(\underline{b})}.$$

For $n \geq 0$, denote the Birkhoff sum: $S_n\phi := \sum_{j=0}^{n-1} \phi\circ\sigma^j$. For each $n \geq 1$ and $\underline{a} \in X$, the n-fold iterate of the transfer operator is given by

$$\mathscr{L}_\phi^n\varphi(\underline{a}) = \sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{a})}\varphi(c^*\underline{a}).$$

Let $\mathscr{L}_\phi^*$ denote the dual operator of $\mathscr{L}_\phi$, acting on finite signed Borel measures on X by

$$\int \varphi\, d(\mathscr{L}_\phi^* m) = \int \mathscr{L}_\phi\varphi\, dm.$$

Definition 2.2. *A potential $\phi : X \to \mathbb{R}$ is called* summable *if*

$$\sup_{a\in X}\left\{\mathscr{L}_\phi\mathbb{1}(\underline{a})\right\} < \infty.$$

A potential ϕ is rapidly decreasing *on X if*

$$\lim_{R\to\infty} \sup_{\underline{a}\in X\setminus\mathbb{B}(\underline{0},R)} \{\mathscr{L}_\phi\mathbb{1}(\underline{a})\} = 0. \tag{2.4}$$

Denote by $\mathcal{M}_\sigma(X)$ the space of σ-invariant probability measures on X. For $\nu \in \mathcal{M}_\sigma(X)$, we write $h_\nu(\sigma)$ for the measure-theoretic entropy of ν.

Definition 2.3. *The* topological pressure *of σ on X for a potential $\phi \in H_\alpha$ is defined as*

$$P(\phi) := \sup_{N\geq1} P(\Sigma_N, \phi|_{\Sigma_N}), \tag{2.5}$$

where

$$P(\Sigma_N, \phi|_{\Sigma_N}) := \sup\left\{h_\mu + \int \phi d\mu, \mu \text{ is inv. prob. measure on } \Sigma_N\right\} \tag{2.6}$$

is the variational principle for the pressure on Σ_N.

Definition 2.4. *A probability measure m on X is called $e^{P(\phi)-\phi}$-conformal if*

$$m(\sigma(A)) = \int_A e^{P(\phi)-\phi}\, dm,$$

for all measurable sets $A \subset X$ for which $\sigma : A \to \sigma(A)$ is a measurable bijection.

Remark 2.3. *In the context of symbolic codings arising from transcendental entire maps, the existence of a conformal measure was established in [10]. This conformal measure was constructed under a summability condition on potentials and suitable structural assumptions on the space, formulated as hypotheses (H1)–(H3). The construction builds upon the foundational theory of conformal measures for countable Markov shifts developed by Mauldin and Urbański [15].*

The key step is to approximate X by compact finite-state subsystems Σ_N, study the dual operator $\mathscr{L}_\phi^*$ restricted to Σ_N, and use Schauder-Tychonoff to obtain a conformal measure m_N supported on Σ_N. Under the summability hypothesis on ϕ, the tightness of $\{m_N\}$ holds (see [10, Section 3]). Hence by Prohorov's theorem [4], there exists a subsequence m_{N_k} converging weakly to a probability measure m on X. By continuity of the Radon-Nikodym derivatives and monotonicity $P_N(\phi) \nearrow P(\phi)$, taking limits in the conformal identity at level N_k yields

$$m(\sigma(A)) = \int_A e^{P(\phi)-\phi}\, dm, \tag{2.7}$$

that is, m is $\chi e^{-\phi}$-conformal, with $\chi := e^{P(\phi)}$.

Remark 2.4. *In [10], a probability measure η on X is called a Gibbs measure for a locally Hölder potential ϕ if it satisfies the Gibbs property: there is $C \geq 1$ such that for all $a \in X$, there exists $M = M(a)$ with the property that for all $n \geq 1$ and $c^* \in \mathbb{Z}^n$, we have*

$$C^{-1}M(a) \;\leq\; \frac{\eta(c^* B_0(a,\delta))}{\exp(S_n\phi(c^*a) - nP(\phi))} \;\leq\; C.$$

For more details about the existence of a conformal and σ-invariant measure, we refer the reader to the body of the paper [10].

For locally Hölder potential ϕ, we will denote such a measure by m_ϕ, which will be used to construct a σ-invariant absolutely continuous measure with respect to m_ϕ by means of the transfer operator and the Ionescu-Tulcea and Marinescu theorem, see Corollary 5.1.

3. Results

In this section, we first recall the Ionescu-Tulcea and Marinescu spectral theorem. It yields quasi-compactness under a Lasota-Yorke inequality and will be key in our setting.

Theorem 3.1 (Ionescu-Tulcea and Marinescu [11])**.** *Let $(F, |\cdot|)$ be a Banach space and let $E \subset F$ be a linear subspace endowed with a stronger norm $\|\cdot\|$ such that it satisfies the following:*

(E1) Every $\|\cdot\|$-bounded subset of E is relatively compact in $(F, |\cdot|)$.
(E2) If $(x_n) \subset E$ with $\sup_n \|x_n\| \leq K_1$ and $|x_n - x| \to 0$ in F, then $x \in E$ and $\|x\| \leq K_1$.

Let $T : F \to F$ be a bounded linear operator such that $T(E) \subset E$, the restriction $T|_E : E \to E$ is bounded with respect to $\|\cdot\|$, and which satisfies the following conditions:

(F1) There exists $K \geq 1$ for all $n \geq 1$, $|T^n x| \leq K\,|x|$ for all $x \in F$.
(F2) (Lasota-Yorke inequality) There exist $N \in \mathbb{N}$, $r \in (0,1)$ and $K_2 > 0$ such that

$$\|T^N x\| \;\leq\; r\,\|x\| \;+\; K_2\,|x|, \text{ for all } x \in E.$$

Then $T|_E$ is quasi-compact. *In particular:*

(1) There are at most finitely many eigenvalues of T on the unit circle $\{|\lambda| = 1\}$, say, $\gamma_1, \ldots, \gamma_p$.
(2) For each i, the eigenspace $F_i := \{x \in F : Tx = \gamma_i x\}$ satisfies $F_i \subset E$ and $\dim F_i < \infty$.
(3) There exist bounded projections $T_i : E \to F_i$ and a bounded operator $S : E \to E$ such that

$$T \;=\; \sum_{i=1}^{p} \gamma_i T_i \;+\; S, \quad T_i^2 = T_i, \;\; T_i T_j = 0\ (i \neq j), \;\; T_i S = S T_i = 0,$$

with $\sup_{n\geq 1} |S^n| < \infty$ and $\|S^n\| \leq M\,\delta^n$ for some $M > 0$ and $\delta \in (0,1)$.

3.1. Main results

Throughout the remainder of the paper, we assume that hypotheses (H1)–(H4) are satisfied. Let $\delta_0 > 0$, $C > 0$, and $\lambda > 1$ be the constants from (H1).

Recall that $H_{\alpha,\delta}$ denotes the space of bounded potentials that are uniformly δ-locally α-Hölder continuous. For simplicity, whenever no confusion arises, for a fixed δ, we write $H_{\alpha,\delta} = H_\alpha$, and we denote the seminorm $|g|_{\alpha,\delta}$ defined in (2.2) simply by $|g|_\alpha$.

Set $\chi := e^{P(\phi)}$ and $\widehat{\phi} := \phi - P(\phi)$, so that $\mathscr{L}_{\widehat{\phi}} = \chi^{-1}\mathscr{L}_\phi$.

Theorem 3.2 (Spectral decomposition for the normalized operator)**.** *Let $\phi \in H_\alpha$ be summable and rapidly decreasing, and satisfy $\sup_{n\geq 1} \left\|\mathscr{L}_\phi^n \mathbb{1}\right\|_\infty < \infty$. Then there exist finitely many eigenvalues $\gamma_1, \ldots, \gamma_p$ of modulus 1, finite rank projections $Q_i : H_\alpha \to H_\alpha$, and a bounded operator $S : H_\alpha \to H_\alpha$ such that*

$$\mathscr{L}_{\widehat{\phi}}^n = \sum_{i=1}^{p} \gamma_i^n Q_i + S^n \quad \text{for all } n \geq 1,$$

with $Q_i \cdot Q_j = 0$ for $i \neq j$, $Q_i \cdot S = S \cdot Q_i = 0$, and $\|S^n\|_{H_\alpha \to H_\alpha} \leq C\,\xi^n$ for some $C > 0$ and $\xi \in (0,1)$. In particular, $\mathscr{L}_{\widehat{\phi}}$ is quasi-compact on H_α. Moreover, 1 is an eigenvalue of maximal modulus, as established in Corollary 3.1.

Proof. The proof is based on several technical estimates that will be established in Section 5. Lemma 5.1 gives a Lasota-Yorke inequality for $\mathscr{L}_\phi^n$, and after rescaling by χ, the same holds for $\mathscr{L}_{\widehat{\phi}}^n$. Lemma 5.2 provides the compact embedding required for condition $(E1)$ in Theorem 3.1, while $(E2)$ is standard. We may therefore apply Theorem 3.1. □

Corollary 3.1. *The number 1 is an isolated and simple eigenvalue of $\mathscr{L}_{\widehat{\phi}} : H_\alpha \to H_\alpha$. Its eigenspace is generated by a strictly positive function $h \in H_\alpha$ with $\int h\, dm_\phi = 1$, and*

$$\lim_{R\to\infty} \sup_{\underline{a} \in X \setminus \mathbb{B}(\underline{0},R)} h(\underline{a}) = 0.$$

Proof. The proof is given in Section 5.1. □

Corollary 3.2. *Let $F = CB(X, \mathbb{R})$ with the norm $|g| = \|g\|_\infty$ and let $E = H_\alpha$ with the norm $\|g\| = \|g\|_\infty + |g|_\alpha$. Assume that the metric dynamical system (X, σ, ρ) satisfies (H1)–(H4) and let $\phi \in H_\alpha$ be summable. Then $\mathscr{L}_\phi : E \to E$ is quasi-compact. Moreover,*

(1) The spectral radius of $\mathscr{L}_\phi$ is equal to the pressure $e^{P(\phi)}$.
(2) There exists a simple maximal eigenvalue $e^{P(\phi)}$ with a positive eigenfunction $h_\phi \in H_\alpha$ and a Borel probability measure m_ϕ such that

$$\mathscr{L}_\phi h_\phi = e^{P(\phi)} h_\phi, \qquad \mathscr{L}_\phi^* m_\phi = e^{P(\phi)} m_\phi.$$

(3) The rest of the spectrum of $\mathscr{L}_\phi$ is contained in a disk of radius strictly smaller than $e^{P(\phi)}$.

Proof. Set $\hat{\phi} := \phi - P(\phi)$, so that $\mathscr{L}_{\hat{\phi}} = e^{-P(\phi)} \mathscr{L}_\phi$. Since $\phi \in H_\alpha$ is summable, $\hat{\phi}$ satisfies the same hypotheses, and Theorem 3.2 applies to $\mathscr{L}_{\hat{\phi}}$. Items (1) and (3) follow from rescaling the spectrum of $\mathscr{L}_{\hat{\phi}}$ by $e^{P(\phi)}$, using that its spectral radius equals 1 and its leading eigenvalue is simple and isolated by Corollary 3.1. Item (2) follows directly from Corollary 3.1 and Remark 5.1. □

4. Preliminary lemmas on (X, ρ) and locally Hölder potentials

In the following lemma, we give some basic properties, which follow directly from the definitions and will be used several times.

Lemma 4.1. *For all $\delta > 0$, $\underline{a} \in X$, $n, m \geq 0$, and $c^* \in \mathbb{Z}^n$, we have*

$$\sigma^n(\mathbb{B}_{m+n}(\underline{a}, \delta)) \subseteq \mathbb{B}_m(\sigma^n \underline{a}, \delta), \tag{4.1}$$

and

$$\mathbb{B}_{m+n}(c^* \underline{a}, \delta) \subseteq c^* \mathbb{B}_m(\underline{a}, \delta). \tag{4.2}$$

The following lemmas hold, each relying on at least one of the hypotheses (H1)–(H4), and their proofs were given in [10].

Lemma 4.2. *Fix $n_0 \geq 0$ such that $C\lambda^{-n_0} \leq \min\{1, 1/C\}$. Then for all $\delta \in (0, \delta_0]$, $n, m \geq 0$, $c^* \in \mathbb{Z}^n$, and $\underline{a} \in X$, we have*

(i) $\mathbb{B}_{m+n}(c^* \underline{a}, \delta) \subseteq c^* \mathbb{B}_m(\underline{a}, \delta) \subseteq \mathbb{B}_m(c^* \underline{a}, C\lambda^{-n}\delta)$. In particular, taking $n = n_0$,

$$\mathbb{B}_{m+n_0}(c^* \underline{a}, \delta) \subseteq \mathbb{B}_m(c^* \underline{a}, \min\{\delta, \delta/C\}).$$

(ii) $\mathbb{B}_{m+n}(c^* \underline{a}, \min\{\delta, \delta/C\}) \subseteq c^* \mathbb{B}_m(\underline{a}, \min\{\delta, \delta/C\}) \subseteq \mathbb{B}_{m+n}(c^* \underline{a}, \delta)$.

Lemma 4.3. *The following hold:*

(1). For all $\underline{a} \in X$ and $r > 0$, there exists $n > 0$ such that $B(\sigma^n \underline{a}, \delta_0) \subseteq \sigma^n B(\underline{a}, r)$.
(2). The set $\bigcup_{N \geq 1} \Sigma_N$ is dense in X.
(3). The system (X, σ) is topologically mixing.

For $\underline{a}, \underline{b} \in X$ and $k \geq 1$, define the metric

$$\rho_k(\underline{a}, \underline{b}) := \max_{0 \leq j \leq k} \rho(\sigma^j \underline{a}, \sigma^j \underline{b}).$$

Lemma 4.4. *Let $\delta \in (0, \delta_0]$ and set $\delta' = \min\{\delta, \delta/C\}$. Then for all $k \geq 1$, $\underline{d} \in X$, and $\underline{a}, \underline{b} \in \mathbb{B}_k(\underline{d}, \delta)$, there exists a finite chain*

$$\underline{a} = \underline{c}_0, \underline{c}_1, \ldots, \underline{c}_{\ell^k} = \underline{b},$$

such that $\rho_k(\underline{c}_j, \underline{c}_{j+1}) < \delta'$ for every $0 \leq j < \ell^k$.

Proof. This follows directly from assumption ($H3$), applied successively to the iterates $\sigma^j(\underline{d})$ for $0 \leq j \leq k$. For details, see [10] □

Lemma 4.5. *For every sufficiently small $\delta > 0$ and any $\underline{a}, \underline{b} \in X$, if $\underline{b} \in \mathbb{B}_n(\underline{a}, \delta)$, then, $a_j = b_j$, for all $0 \leq j \leq n - 1$.*

Proof. Immediate from assumption ($H4$). □

Remark 4.1. *Let $\delta \in (0, \delta_0]$ and set $\delta' = \min\{\delta, \delta/C\}$. Let $\phi \in H_\alpha$, satisfying the summable condition, then, there exists $K > 0$, such that*

$$|e^{S_n\phi(c^*\underline{a})} - e^{S_n\phi(c^*\underline{b})}| \leq K\, e^{S_n\phi(c^*\underline{b})}\, \rho(\underline{a}, \underline{b})^\alpha,$$

for all $\underline{a}, \underline{b}$ satisfying $\rho(\underline{a}, \underline{b}) < \delta'$. Potentials satisfying this bound will be called dynamically α-Hölder.

Let $\delta_0 > 0$ be the constant associated with the metric assumptions. Given $\delta \in (0, \delta_0]$ and $n \geq 0$, we define the (n, δ)-*variation* of the potential $\phi : X \to \mathbb{R}$ by

$$\mathrm{Var}_n(\phi) := \sup_{\underline{a} \in X} \sup_{\underline{b}, \underline{c} \in \mathbb{B}_n(\underline{a}, \delta)} |\phi(\underline{b}) - \phi(\underline{c})|.$$

Definition 4.1. *Let $\delta \in (0, \delta_0]$. We say that a continuous potential ϕ has weak Hölder regularity if there exist $\tilde{C} > 0$ and $0 < r < 1$ such that, for all $n \geq 0$, we have*

$$\mathrm{Var}_n(\phi) \leq \tilde{C} r^n. \tag{4.3}$$

Notice that for $\delta \in (0, \delta_0]$ and $\alpha \in (0, 1]$, every uniformly δ-locally α-Hölder continuous potential is weakly Hölder continuous with constants $r = \lambda^{-\alpha}$ and $\tilde{C} = LC^\alpha \delta^\alpha$ (compare [10, Lemma 2.8]).

Lemma 4.6. *Let $\delta \in (0, \delta_0]$ and set $\delta' = \min\{\delta, \frac{\delta}{C}\}$. For $m \geq n - 1$, $c^* \in \mathbb{Z}^m$, and $\underline{a}, \underline{b} \in X$, such that $\rho(\underline{a}, \underline{b}) < \delta'$, we have*

$$\left| S_n\phi(c^*\underline{a}) - S_n\phi(c^*\underline{b}) \right| \leq \lambda^{-\alpha(m-n+1)} \frac{L\, C^\alpha}{1 - \lambda^{-\alpha}}\, \rho(\underline{a}, \underline{b})^\alpha.$$

Proof. By the local Hölder regularity of ϕ with respect to ρ and Lemma 4.2 part (ii),

$$\left| S_n\phi(c^*\underline{a}) - S_n\phi(c^*\underline{b}) \right| \leq \sum_{j=0}^{n-1} \left| \phi(\sigma^j(c^*\underline{a})) - \phi(\sigma^j(c^*\underline{b})) \right| \leq L \sum_{j=0}^{n-1} \rho(\sigma^j(c^*\underline{a}), \sigma^j(c^*\underline{b}))^\alpha$$

$$\leq L \sum_{j=0}^{n-1} (C\,\lambda^{-(m-j)}\,\rho(\underline{a},\underline{b}))^{\alpha} = L C^{\alpha} \sum_{j=0}^{n-1} \lambda^{-\alpha(m-j)} \rho(\underline{a},\underline{b})^{\alpha}$$

$$\leq L C^{\alpha} \frac{\lambda^{-\alpha(m-n+1)}}{1-\lambda^{-\alpha}} \rho(\underline{a},\underline{b})^{\alpha}.$$

□

Lemma 4.7. *Let $\delta \in (0,\delta_0]$ and set $\delta' = \min\{\delta, \frac{\delta}{C}\}$. For $m \geq n-1$, $c^* \in \mathbb{Z}^m$, and $\underline{a},\underline{b}$, such that $\rho(\underline{a},\underline{b}) < \delta'$, we have*

$$\left|e^{S_n\phi(c^*\underline{a})} - e^{S_n\phi(c^*\underline{b})}\right| \leq \frac{2\,L\,C^{\alpha}}{1-\lambda^{-\alpha}}\,\lambda^{-\alpha(m-n+1)}\,e^{S_n\phi(c^*\underline{b})}\,\rho(\underline{a},\underline{b})^{\alpha}.$$

Proof. Set $K_{\alpha} := \dfrac{L\,C^{\alpha}}{1-\lambda^{-\alpha}}\,\lambda^{-\alpha(m-n+1)}$. By Lemma 4.6,

$$\left|S_n\phi(c^*\underline{a}) - S_n\phi(c^*\underline{b})\right| \leq K_{\alpha}\,\rho(\underline{a},\underline{b})^{\alpha}.$$

Then,

$$\frac{\left|e^{S_n\phi(c^*\underline{a})} - e^{S_n\phi(c^*\underline{b})}\right|}{e^{S_n\phi(c^*\underline{b})}} = \left|e^{S_n\phi(c^*\underline{a}) - S_n\phi(c^*\underline{b})} - 1\right| \leq e^{K_{\alpha}\rho(\underline{a},\underline{b})^{\alpha}} - 1.$$

Taking $\delta > 0$ sufficiently small such that $K_{\alpha}\rho(\underline{a},\underline{b})^{\alpha} \leq 1$, we use $e^t - 1 \leq 2t$ for $0 \leq t \leq 1$ to conclude

$$\left|e^{S_n\phi(c^*\underline{a})} - e^{S_n\phi(c^*\underline{b})}\right| \leq 2K_{\alpha}\,e^{S_n\phi(c^*\underline{b})}\,\rho(\underline{a},\underline{b})^{\alpha}.$$

□

For a potential $\phi \in H_{\alpha}$, we define

$$A_n = A_n(\phi) := \exp\left(\sum_{k>n} \mathrm{Var}_k(\phi)\right).$$

Then $A_0 \geq A_1 \geq A_2 \geq \cdots$.

For $n \geq 0$, $\underline{a} \in X$, $\delta > 0$, and $n, m \geq 0$, we define

$$\|S_n\phi\|_{\mathbb{B}_m(\underline{a},\delta)} := \sup_{\underline{b}\in\mathbb{B}_m(\underline{a},\delta)} S_n\phi(\underline{b}).$$

Lemma 4.8. *Let $\phi \in H_{\alpha}$ and let $m, n \geq 0$ with $m \geq n$. Then for every $\underline{a} \in X$, the following statements hold.*

(i) $\mathrm{Var}_m(S_n\phi) \leq \log A_{m-n}$.

(ii) For all $\underline{b},\underline{c} \in \mathbb{B}_m(\underline{a},\delta)$,

$$S_n\phi(\underline{c}) - \log A_{m-n} \leq S_n\phi(\underline{b}) \leq S_n\phi(\underline{c}) + \log A_{m-n}.$$

(iii) For every $a \in X$,

$$
\begin{aligned}
&A_1^{-2} \sum_{(b_0,\dots,b_{n-1},\,c_0,\dots,c_{m-1})} e^{\|S_{n+m}\phi\|_{\mathbb{B}_{n+m+1}(b_0,\dots,b_{n-1},c_0,\dots,c_{m-1},a,\delta)}} \\
\le &\left(\sum_{(b_0,\dots,b_{n-1})} e^{\|S_n\phi\|_{\mathbb{B}_{n+1}(b_0,\dots,b_{n-1},a,\delta)}} \right)\left(\sum_{(c_0,\dots,c_{m-1})} e^{\|S_m\phi\|_{\mathbb{B}_{m+1}(c_0,\dots,c_{m-1},a,\delta)}} \right) \\
\le &A_1^{2} \sum_{(b_0,\dots,b_{n-1},\,c_0,\dots,c_{m-1})} e^{\|S_{n+m}\phi\|_{\mathbb{B}_{n+m+1}(b_0,\dots,b_{n-1},c_0,\dots,c_{m-1},a,\delta)}}.
\end{aligned}
$$

Proof. (i) Let $\underline{b}, \underline{c} \in \mathbb{B}_m(\underline{a}, \delta)$. By property (4.1), we have $\sigma(\mathbb{B}_m(\underline{a}, \delta)) \subset \mathbb{B}_{m-1}(\sigma(\underline{a}), \delta)$, and therefore for every $0 \le j \le m$, $\sigma^j(\underline{b}), \sigma^j(\underline{c}) \in \mathbb{B}_{m-j}(\sigma^j(\underline{a}), \delta)$. Hence,

$$|S_n\phi(\underline{b}) - S_n\phi(\underline{c})| = \Big| \sum_{j=0}^{n-1} (\phi(\sigma^j\underline{b}) - \phi(\sigma^j\underline{c})) \Big| \le \sum_{j=0}^{n-1} \mathrm{Var}_{m-j}(\phi) = \sum_{k>m-n} \mathrm{Var}_k(\phi) = \log A_{m-n}.$$

(ii) For $\underline{b}, \underline{c} \in \mathbb{B}_m(\underline{a}, \delta)$, by part (i),

$$|S_n\phi(\underline{b}) - S_n\phi(\underline{c})| \le \mathrm{Var}_m(S_n\phi) \le \log A_{m-n},$$

which yields the desired two-sided bound.
(iii) Let $\underline{d} \in \mathbb{B}_{n+m+1}(b_0, \dots, b_{n-1}, c_0, \dots, c_{m-1}, a, \delta)$. Then

$$S_{n+m}\phi(\underline{d}) = S_n\phi(\underline{d}) + S_m\phi(\sigma^n\underline{d}).$$

By Lemma 4.5 and part (ii) with $m = 1$, we obtain

$$S_n\phi(\underline{d}) + \|S_m\phi\|_{\mathbb{B}_{m+1}(c_0,\dots,c_{m-1},a,\delta)} - 2\log A_1 \le S_{n+m}\phi(\underline{d}) \le S_n\phi(\underline{d}) + \|S_m\phi\|_{\mathbb{B}_{m+1}(c_0,\dots,c_{m-1},a,\delta)} + 2\log A_1.$$

Taking the supremum over $\underline{d}$ in the corresponding $(n + m + 1)$-ball, then exponentiating and summing over all admissible words, yields the desired inequalities. □

For $\underline{a} \in X$, define the *partition function*

$$\widehat{Z}_n(\phi, \underline{a}) := \sum_{\underline{d}:\,\sigma^n(\underline{d})=\underline{a}} e^{S_n\phi(\underline{d})} = \sum_{(d_0,\dots,d_{n-1})} e^{S_n\phi(d_0d_1\dots d_{n-1}\underline{a})}.$$

Lemma 4.9. *Let $\phi \in H_\alpha$ be a summable potential. Then for all $n \ge 1$ and $\underline{a} \in X$,*

$$\widehat{Z}_n(\phi, \underline{a}) \le \|\mathscr{L}_\phi \mathbb{1}\|_\infty^n.$$

Proof. It is immediate from the definition. For completeness, we provide a proof by induction on n. For $n = 1$, we have

$$\widehat{Z}_1(\phi, \underline{a}) = \sum_{\sigma(\underline{b})=\underline{a}} e^{\phi(\underline{b})} = (\mathscr{L}_\phi \mathbb{1})(\underline{a}) \le \|\mathscr{L}_\phi \mathbb{1}\|_\infty.$$

Assume that the statement holds for $n \le k - 1$. Then, using the decomposition of k-preimages through one-step extensions,

$$\widehat{Z}_k(\phi, \underline{a}) = \sum_{\sigma^k(\underline{b})=\underline{a}} e^{S_k\phi(\underline{b})} = \sum_{\sigma(\underline{d})=\underline{a}} e^{\phi(\underline{d})} \sum_{\sigma^{k-1}(\underline{b})=\underline{d}} e^{S_{k-1}\phi(\underline{b})} \le \|\mathscr{L}_\phi \mathbb{1}\|_\infty \|\mathscr{L}_\phi \mathbb{1}\|_\infty^{k-1} = \|\mathscr{L}_\phi \mathbb{1}\|_\infty^k.$$

□

Let $Y \subset X \subset \Sigma$ be a shift-invariant subset, that is, $\sigma^{-1}(Y) = Y$. Define the restricted partition function

$$\widehat{Z}_n(Y, \phi|_Y, \underline{a}) := \sum_{\sigma^n(\underline{b})=\underline{a},\, \underline{b}\in Y} e^{S_n\phi(\underline{b})}.$$

Lemma 4.10. *Let $\phi \in H_\alpha$. Then the following properties hold:*

(i) For every $\underline{a}, \underline{b} \in X$, there exist constants $C > 0$ and $\ell \geq 1$ such that

$$\widehat{Z}_n(\phi, \underline{a}) \leq A_0\, \widehat{Z}_{n+\ell}(\phi, \underline{b}).$$

(ii) For all $\underline{a} \in X$ and $m, n \geq 1$,

$$\widehat{Z}_n(\phi, \underline{a})\, \widehat{Z}_m(\phi, \underline{a}) \leq A_1^2\, \widehat{Z}_{n+m}(\phi, \underline{a}).$$

Proof. (i) By assumption ($H3$), there exist $\ell \geq 1$ and a finite word $d^* = d_0 \dots d_{\ell-1}$ such that $d^*\underline{b} \in \mathbb{B}(\underline{a}, \delta')$. Moreover, for each $n \geq 1$ and $c^* \in \mathbb{Z}^n$, we have $c^*d^*\underline{b} \in \mathbb{B}_n(c^*\underline{a}, \delta)$. Then, by Lemma 4.8 part (ii),

$$\sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{a})} \leq A_0 \sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*d^*\underline{b})} \leq A_0\, \widehat{Z}_{n+\ell}(\phi, \underline{b}).$$

(ii) By Lemma 4.8 part (iii),

$$\Big(\sum_{b_0,\dots,b_{n-1}} e^{S_n\phi(b_0\dots b_{n-1}\underline{a})}\Big)\Big(\sum_{c_0,\dots,c_{m-1}} e^{S_m\phi(c_0\dots c_{m-1}\underline{a})}\Big) \leq A_1^2 \sum_{b_0,\dots,b_{n-1},c_0,\dots,c_{m-1}} e^{S_{n+m}\phi(b_0\dots b_{n-1}c_0\dots c_{m-1}\underline{a})} = A_1^2\, \widehat{Z}_{n+m}(\phi, \underline{a}).$$

□

Proposition 4.1. *Let $\phi \in H_\alpha$. Then the limit $\lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\phi, \underline{a})$ exists, is finite, and does not depend on $\underline{a}$. Moreover, it is never $-\infty$. If in addition $\|\mathscr{L}_\phi \mathbb{1}\|_\infty < \infty$, then this limit is finite.*

Proof. Fix $\underline{a} \in X$ and set $a_n = \log \widehat{Z}_n(\phi, \underline{a})$. By Lemma 4.10 part (ii), the sequence $\{a_n\}_{n\geq1}$ satisfies

$$a_n + a_m \leq a_{n+m} + \log A_1.$$

Hence, $\{a_n\}$ is almost subadditive. By the generalized Fekete lemma*, the limit $\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\phi, \underline{a})$ exists.

Lemma 4.10 part (i) implies that the limit does not depend on $\underline{a}$. To see that it is never $-\infty$, note that we can take $\underline{a} \in \Sigma_N \subset X$, so that

$$\lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\phi, \underline{a}) \geq \lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\Sigma_N, \phi, \underline{a}) > -\infty.$$

Finally, if $\|\mathscr{L}_\phi \mathbb{1}\|_\infty < \infty$, then by Lemma 4.9, $\widehat{Z}_n(\phi, \underline{a}) \leq \|\mathscr{L}_\phi \mathbb{1}\|_\infty^n$, which ensures that the limit is finite. □

*Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that $a_n + a_m \leq a_{n+m}$ for all $n, m \geq 1$, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\sup_{n\geq1} \frac{a_n}{n}$.
Let c be a fixed constant such that for all $n, m \geq 1$, $a_n + a_m \leq a_{n+m} + c$. Then $b = \lim_{n\to\infty} \frac{a_n}{n}$ exists and $\frac{a_n}{n} \leq b + \frac{c}{n}$.

For each integer $N \geq 1$, consider the restricted pressure on Σ_N, defined as follows:

$$P(\Sigma_N, \phi|_{\Sigma_N}) := \lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\Sigma_N, \phi, \underline{a}),$$

which exists by the almost-subadditivity established in Lemma 4.10 part (ii), restricted to Σ_N and satisfies the variational principle in thermodynamics; see Eq (2.6).

Proposition 4.2. *If $\phi \in H_\alpha$, then* $\lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\phi, \underline{a}) = \sup_{N\geq 1} P(\Sigma_N, \phi|_{\Sigma_N}) = P(\phi)$.

Proof. Set $\mathscr{Z}(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\phi, \underline{a})$. To prove the upper bound, observe that for every $N \geq 1$, $\widehat{Z}_n(\Sigma_N, \phi, \underline{a}) \leq \widehat{Z}_n(\phi, \underline{a})$, hence

$$P(\Sigma_N, \phi|_{\Sigma_N}) = \lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\Sigma_N, \phi, \underline{a}) \leq \lim_{n\to\infty} \frac{1}{n} \log \widehat{Z}_n(\phi, \underline{a}) = \mathscr{Z}(\phi).$$

Taking the supremum, it yields $\sup_{N\geq 1} P(\Sigma_N, \phi|_{\Sigma_N}) \leq \mathscr{Z}(\phi)$.

To prove the lower bound, we split the argument into two cases:

Case 1. $\mathscr{Z}(\phi) < \infty$. Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ large so that

$$\mathscr{Z}(\phi) < \frac{1}{m} \log \widehat{Z}_m(\phi, \underline{a}) + \varepsilon \quad \text{and} \quad m > \frac{2\log A_1}{\varepsilon}. \tag{4.4}$$

Since $\Sigma_N \nearrow X$ and the sums are over words of fixed length m, there exists M such that

$$\frac{1}{m} \log \widehat{Z}_m(\phi, \underline{a}) < \frac{1}{m} \log \widehat{Z}_m(\Sigma_M, \phi, \underline{a}) + \varepsilon. \tag{4.5}$$

Let $a_n := \log \widehat{Z}_n(\Sigma_M, \phi, \underline{a})$. By Lemma 4.10 part (ii) (applied on Σ_M),

$$a_n + a_m \leq a_{n+m} + 2\log A_1.$$

Thus, by the almost-subadditive Fekete lemma,

$$\frac{a_m}{m} \leq \lim_{n\to\infty} \frac{a_n}{n} + \frac{2\log A_1}{m} = P(\Sigma_M, \phi|_{\Sigma_M}) + \frac{2\log A_1}{m} \leq P(\Sigma_M, \phi|_{\Sigma_M}) + \varepsilon,$$

where we use (4.4) for the last inequality. Equivalently,

$$\frac{1}{m} \log \widehat{Z}_m(\Sigma_M, \phi, \underline{a}) \leq P(\Sigma_M, \phi|_{\Sigma_M}) + \varepsilon. \tag{4.6}$$

Combining (4.4)–(4.6), we obtain

$$\mathscr{Z}(\phi) < \frac{1}{m} \log \widehat{Z}_m(\phi, \underline{a}) + \varepsilon < \frac{1}{m} \log \widehat{Z}_m(\Sigma_M, \phi, \underline{a}) + 2\varepsilon \leq P(\Sigma_M, \phi|_{\Sigma_M}) + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\mathscr{Z}(\phi) \leq \sup_N P(\Sigma_N, \phi|_{\Sigma_N}).$$

Case 2. $\mathscr{Z}(\phi) = \infty$. Then, for every $R > 0$, there exists m such that

$$\frac{1}{m}\log \widehat{Z}_m(\phi, \underline{a}) > R.$$

Since $\Sigma_N \subset \Sigma_{N+1} \subset X$, the partition functions satisfy $\hat{Z}_m(\Sigma_N, \phi, a) \nearrow \hat{Z}_m(\phi, a)$ as $N \to \infty$. Therefore, there exists $M \geq 1$ such that

$$\frac{1}{m}\log \hat{Z}_m(\Sigma_M, \phi, a) > \frac{1}{m}\log \hat{Z}_m(\phi, a) - \varepsilon > R - \varepsilon.$$

As in the previous case, by almost subadditivity,

$$\frac{1}{m}\log \widehat{Z}_m(\Sigma_M, \phi, \underline{a}) \leq P(\Sigma_M, \phi|_{\Sigma_M}) + \varepsilon,$$

and hence $P(\Sigma_M, \phi|_{\Sigma_M}) \geq R - 2\varepsilon$. Since $R > 0$ is arbitrary, we conclude that

$$\sup_N P(\Sigma_N, \phi|_{\Sigma_N}) = \infty = \mathscr{Z}(\phi).$$

In both cases, we have shown that $\mathscr{Z}(\phi) \leq \sup_N P(\Sigma_N, \phi|_{\Sigma_N})$, which completes the proof.

□

5. Lasota-Yorke estimates and pre-compactness

This section is devoted to proving Theorem 3.2. We verify the hypotheses of Theorem 3.1 in the following lemmas. In particular, we prove a type of Lasota-Yorke inequality for the transfer operator.

Lemma 5.1. *Let $\phi \in H_\alpha$ be summable, and dynamically α-Hölder. Then,*

(1) For all $g \in H_\alpha$ and all $n \geq 1$,

$$\|\mathscr{L}_\phi^n g\|_\infty \leq \|\mathscr{L}_\phi \mathbb{1}\|_\infty^n \|g\|_\infty.$$

(2) $\mathscr{L}_\phi(H_\alpha) \subset H_\alpha$. Moreover, if

$$Q_\phi := \sup_{n\geq 1} \|\mathscr{L}_\phi^n \mathbb{1}\|_\infty < \infty,$$

then there exists $c_1 > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and all $g \in H_\alpha$, we have, the Lasota-Yorke estimate,

$$\|\mathscr{L}_\phi^n g\|_\alpha \leq \frac{1}{2}\|g\|_\alpha + c_1\|g\|_\infty. \tag{5.1}$$

Proof. Item (1) follows immediately from the definitions, since

$$\sum_{\sigma^n(\underline{b})=\underline{a}} e^{S_n\phi(\underline{b})} = \sum_{\sigma^{n-1}(\underline{d})=\underline{a}} e^{S_n\phi(\underline{d})} \cdot \sum_{\sigma(\underline{b})=\underline{d}} e^{\phi(\underline{b})}.$$

To prove part (2), set

$$K = K_\alpha := \frac{LC^\alpha}{1-\lambda^{-\alpha}}\,\lambda^{-\alpha(m-n+1)}.$$

Then, we obtain

$$
\begin{aligned}
&|\mathscr{L}_\phi^n g(\underline{b}) - \mathscr{L}_\phi^n g(\underline{a})| = \left|\sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{b})} g(c^*\underline{b}) - \sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{a})} g(c^*\underline{a})\right| \\
&\le \sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{a})} \left|g(c^*\underline{b}) - g(c^*\underline{a})\right| + \sum_{c^*\in\mathbb{Z}^n} \left|e^{S_n\phi(c^*\underline{b})} - e^{S_n\phi(c^*\underline{a})}\right| \left|g(c^*\underline{b})\right| \\
&\le \sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{a})} |g|_\alpha \rho(c^*\underline{b}, c^*\underline{a})^\alpha + \sum_{c^*\in\mathbb{Z}^n} \|g\|_\infty \left|e^{S_n\phi(c^*\underline{b})}\right| K\rho(\underline{a},\underline{b})^\alpha \\
&\le \sum_{c^*\in\mathbb{Z}^n} e^{S_n\phi(c^*\underline{a})} |g|_\alpha C^\alpha \lambda^{-\alpha n} \rho(\underline{b},\underline{a})^\alpha + \sum_{c^*\in\mathbb{Z}^n} \|g\|_\infty e^{S_n\phi(c^*\underline{b})} K\rho(\underline{a},\underline{b})^\alpha \\
&\le |g|_\alpha \mathscr{L}_\phi^n \mathbb{1}(\underline{a}) C^\alpha \lambda^{-\alpha n} \rho(\underline{b},\underline{a})^\alpha + K\|g\|_\infty \mathscr{L}_\phi^n \mathbb{1}(\underline{b}) \rho(\underline{a},\underline{b})^\alpha \\
&\le \|\mathscr{L}_\phi^n \mathbb{1}\| (|g|_\alpha C^\alpha \lambda^{-\alpha n} + K\|g\|_\infty) \rho(\underline{a},\underline{b})^\alpha.
\end{aligned}
$$

So, we have that,

$$
|\mathscr{L}_\phi^n g|_\alpha \le \mathscr{L}_\phi^n \mathbb{1}(|g|_\alpha C^\alpha \lambda^{-\alpha n} + K\|g\|_\infty) \le \mathscr{L}_\phi^n \mathbb{1}(C^\alpha \lambda^{-\alpha n}\|g\|_\alpha + K\|g\|_\infty) < \infty, \tag{5.2}
$$

and we also have $\mathscr{L}_\phi^n g \in H_\alpha$.

Suppose that $Q_\phi := \sup_{n\ge1}\{\|\mathscr{L}_\phi^n \mathbb{1}\|_\infty\} < \infty$. Then by Eq (5.2), we have

$$
\begin{aligned}
\|\mathscr{L}_\phi^n g\|_\alpha &= |\mathscr{L}_\phi^n g|_\alpha + \|\mathscr{L}_\phi^n g\|_\infty \le \mathscr{L}_\phi^n \mathbb{1}(C^\alpha \lambda^{-\alpha n}\|g\|_\alpha + K\|g\|_\infty) + \|\mathscr{L}_\phi^n g\|_\infty \\
&\le C^\alpha \lambda^{-\alpha n} Q_\phi \|g\|_\alpha + KQ_\phi\|g\|_\infty + \|\mathscr{L}_\phi^n g\|_\infty \le C^\alpha \lambda^{-\alpha n} Q_\phi \|g\|_\alpha + Q_\phi(K+1)\|g\|_\infty.
\end{aligned}
$$

Taking $n \ge 1$ so large that $C^\alpha \lambda^{-\alpha n} Q_\phi \le 1/2$, we obtain

$$
\|\mathscr{L}_\phi^n g\|_\alpha \le \frac{1}{2}\|g\|_\alpha + c_1\|g\|_\infty.
$$

□

Lemma 5.2. *Assume that $\phi \in H_\alpha$ is summable and dynamically α-Hölder, satisfies the rapidly decreasing property* (2.4)*, and*

$$
Q_\phi = \sup_{n\ge1} \left\|\mathscr{L}_\phi^n \mathbb{1}\right\|_\infty < \infty. \tag{5.3}
$$

Let $B \subset H_\alpha$ be bounded in the $\|\cdot\|_\infty$-norm. Then $\mathscr{L}_\phi(B)$ is relatively compact in $CB(X,\mathbb{R})$ with the $\|\cdot\|_\infty$-norm (hence, precompact in H_α with respect to $\|\cdot\|_\infty$).

Proof. Let $(g_n)_{n\ge1} \subset B$. By the dynamically Hölder distortion (Lemma 4.7), there is $K = K_\alpha > 0$ such that for all $\underline{a}, \underline{b}$ with $\rho(\underline{a},\underline{b}) < \delta'$,

$$
\left|e^{\phi(c^*\underline{a})} - e^{\phi(c^*\underline{b})}\right| \le K\, e^{\phi(c^*\underline{b})}\, \rho(\underline{a},\underline{b})^\alpha.
$$

Using (H1) and summability, for every n, we obtain the equicontinuity estimate

$$
\left|\mathscr{L}_\phi(g_n)(\underline{b}) - \mathscr{L}_\phi(g_n)(\underline{a})\right| \le \|\mathscr{L}_\phi \mathbb{1}\|_\infty \Big(C^\alpha \lambda^{-\alpha}|g_n|_\alpha + K\|g_n\|_\infty\Big) \rho(\underline{a},\underline{b})^\alpha,
$$

so $\{\mathscr{L}_\phi(g_n)\}$ is equicontinuous on X. Boundedness follows from $\|\mathscr{L}_\phi(g_n)\|_\infty \le \|\mathscr{L}_\phi\mathbb{1}\|_\infty\|g_n\|_\infty$. Fix $\varepsilon > 0$. By (2.4), choose R with $\sup_{x\notin\mathbb{B}(\underline{0},R)} \mathscr{L}_\phi\mathbb{1}(x) \le \varepsilon/(2M)$, where $M := \sup_n \|g_n\|_\infty < \infty$. Then $\sup_{x\notin\mathbb{B}(\underline{0},R)} |\mathscr{L}_\phi(g_n)(x)| \le \varepsilon/2$ for all n.

By Arzelà-Ascoli on the compact $\overline{\mathbb{B}}(\underline{0}, R)$, extract a subsequence $\mathscr{L}_\phi(g_{n_j})$ converging uniformly on $\overline{\mathbb{B}}(\underline{0}, R)$ to some ψ. The tail control gives uniform convergence on X, hence $\|\mathscr{L}_\phi(g_{n_j}) - \psi\|_\infty \to 0$. Since $\sup_j |\mathscr{L}_\phi(g_{n_j})|_\alpha < \infty$ (by the same distortion bound), we have $\psi \in H_\alpha$. Thus, $\mathscr{L}_\phi(B)$ is relatively compact in $\|\cdot\|_\infty$. □

Remark 5.1 (Conformal measure)**.** *Let m_ϕ be a $\chi e^{-\phi}$-conformal probability measure for σ, stated in [10], then m_ϕ is a fixed point of the dual of $\mathscr{L}_{\widehat{\phi}}$, for $\widehat{\phi} := \phi - P(\phi)$. That is,*

$$\mathscr{L}^*_{\widehat{\phi}} m_\phi = m_\phi.$$

Lemma 5.3. *Assume ϕ is summable and rapidly decreasing, and satisfies* (5.3)*. Then for every $\varepsilon > 0$, there exists $R > 0$ such that*

$$\inf_{n\ge 0} \sup_{\underline{a}\in\overline{\mathbb{B}}(\underline{0},R)} \{\mathscr{L}^n_{\widehat{\phi}}\mathbb{1}(\underline{a})\} \ \ge\ 1 - \varepsilon.$$

Proof. Suppose by contradiction that, for some $\varepsilon > 0$ and every R,

$$\inf_{n\ge 0} \sup_{\underline{a}\in\overline{\mathbb{B}}(\underline{0},R)} \mathscr{L}^n_{\widehat{\phi}}\mathbb{1}(\underline{a}) < 1 - \varepsilon.$$

Let $m = m_\phi$ be the $e^{P(\phi)-\phi}$-conformal probability measure whose existence was established in [10], as mentioned in Remark 5.1. Use (5.3) and choose R large so that $m(X \setminus \overline{\mathbb{B}}(\underline{0}, R)) < \varepsilon/(2Q_\phi)$. Then, for all n,

$$\begin{aligned} 1 = \int \mathscr{L}^n_{\widehat{\phi}}\mathbb{1} dm &= \int_{\overline{\mathbb{B}}(0,R)} \mathscr{L}^n_{\widehat{\phi}}\mathbb{1} dm + \int_{X\setminus\overline{\mathbb{B}}(0,R)} \mathscr{L}^n_{\widehat{\phi}}\mathbb{1} dm \\ &\le (1-\varepsilon)\, m(\overline{\mathbb{B}}(\underline{0}, R)) + Q_{\widehat{\phi}}\, m(X \setminus \overline{\mathbb{B}}(\underline{0}, R)) < 1 - \varepsilon + Q_\phi \tfrac{\varepsilon}{2Q_\phi}, \end{aligned}$$

which is a contradiction.

For the uniform comparability on $\overline{\mathbb{B}}(\underline{0}, R)$, apply Lemma 4.6 to obtain $K_R > 0$ with $|S_n\widehat{\phi}(c^*\underline{a}) - S_n\widehat{\phi}(c^*\underline{b})| \le K_R$ when $\underline{a}, \underline{b} \in \overline{\mathbb{B}}(\underline{0}, R)$. Hence,

$$\exp(-K_R) \le \frac{\mathscr{L}^n_{\widehat{\phi}}\mathbb{1}(\underline{a})}{\mathscr{L}^n_{\widehat{\phi}}\mathbb{1}(\underline{b})} \le \exp(K_R),$$

which yields the stated lower bound. □

5.1. Proof of Corollary 3.1

Proof. From Lemma 5.1, we have that, for some constant $C_1 > 0$ and for all $n \ge 1$,

$$\|\mathscr{L}^n_{\widehat{\phi}}\mathbb{1}\|_\alpha \le C_1.$$

Thus, we have

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \mathscr{L}_{\widehat{\phi}}^{j} \mathbb{1} \right\|_{\alpha} = \left\| \mathscr{L}_{\widehat{\phi}} \Big(\frac{1}{n} \sum_{j=0}^{n-1} \mathscr{L}_{\widehat{\phi}}^{j} \mathbb{1} \Big) \right\|_{\alpha} \leq C_1, \tag{5.4}$$

for every $n \geq 1$. Then from Lemma 5.2, there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\geq 1}$ and a function $h \in H_\alpha$ such that for every $\underline{a}$,

$$\frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j} \mathbb{1}(\underline{a}) \longrightarrow h(\underline{a}), \text{ as } k \to \infty,$$

where the convergence holds in $CB(X, \mathbb{R})$, and by (5.4), we have $\|h\|_\alpha \leq C_1$, thus $h \in H_\alpha$. Moreover, let $m = m_\phi$ denote the $\chi e^{-\phi}$-conformal measure as Remark 5.1. Since m is a fixed point of the operator conjugate to $\mathscr{L}_{\widehat{\phi}}$, for every $j \geq 0$,

$$\int \mathscr{L}_{\widehat{\phi}}^{j} dm = 1,$$

then,

$$\int \frac{1}{n} \sum_{j=0}^{n-1} \mathscr{L}_{\widehat{\phi}}^{j} dm = 1, \text{ for every } n \geq 1.$$

Now, applying Lebesgue's dominated convergence theorem and the fact that ϕ has the property (5.3), we obtain $\int h\, dm = 1$. Since ϕ is summable and $\mathscr{L}_{\widehat{\phi}}^{j} \mathbb{1}$ is uniformly bounded by $Q_\phi < \infty$, dominated convergence also allows us to interchange the limit with the sum over $\sigma^{-1}(a)$. Then,

$$\begin{aligned}
\mathscr{L}_{\widehat{\phi}} h(\underline{a}) &= \chi^{-1} \sum_{\underline{b} \in \sigma^{-1}(\underline{a})} e^{\phi(b)} h(b) \leq \chi^{-1} \sum_{\underline{b} \in \sigma^{-1}(\underline{a})} e^{\phi(b)} \left(\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j} (\mathbb{1})(\underline{b}) \right) \\
&= \lim_{k\to\infty} \chi^{-1} \sum_{\underline{b} \in \sigma^{-1}(\underline{a})} e^{\phi(b)} . \frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j} (\mathbb{1})(\underline{b}) = \lim_{k\to\infty} \mathscr{L}_{\widehat{\phi}} \left(\frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j} (\mathbb{1}) \right) (\underline{a}) \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j+1} (\mathbb{1})(\underline{a}).
\end{aligned}$$

Since ϕ is summable, Lemma 4.9 ensures that there is C_1 such that $\|\mathscr{L}_{\widehat{\phi}} \mathbb{1}\|_\infty \leq C_1$ for all $n \geq 0$. Therefore, for every $k \geq 1$,

$$\left| \frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j} \mathbb{1}(\underline{a}) - \frac{1}{n_k} \sum_{j=1}^{n_k} \mathscr{L}_{\widehat{\phi}}^{j+1} \mathbb{1}(\underline{a}) \right| = \frac{1}{n_k} |\mathscr{L}_{\widehat{\phi}} \mathbb{1}(\underline{a}) - \mathscr{L}_{\widehat{\phi}}^{n_k+1} \mathbb{1}(\underline{a})| \leq \frac{2C_1}{n_k},$$

which implies $\mathscr{L}_{\widehat{\phi}} h(\underline{a}) \leq h(\underline{a})$.

Since $\mathscr{L}_{\widehat{\phi}}^{*} m = m$, integrating both sides yields $\int \left(h - \mathscr{L}_{\widehat{\phi}} h \right) dm = 0$. Given that $h - \mathscr{L}_{\widehat{\phi}} h \geq 0$ and m has full support on X, we conclude that $h = \mathscr{L}_{\widehat{\phi}} h$ holds m-almost everywhere. By continuity of both functions in H_α, this equality in fact holds everywhere on X, that is, $\mathscr{L}_{\widehat{\phi}} h = h$ on X.

Moreover, by Lemma 5.3, $h(\underline{a}) > 0$, for all $\underline{a} \in X$. Since $h = \mathscr{L}_{\widehat{\phi}} h$, we have that for all $\underline{a}$,

$$\mathscr{L}_{\widehat{\phi}} h(\underline{a}) \leq \|\mathscr{L}_{\widehat{\phi}} h\|_\infty \leq \|\mathscr{L}_{\widehat{\phi}}(\mathbb{1})\|_\infty \|h\|_\infty,$$

and

$$\lim_{R\to\infty} \sup_{\underline{a}\in X\setminus\mathbb{B}(\underline{0},R)} \{\mathscr{L}_{\widehat{\phi}}\mathbb{1}(\underline{a})\} = 0.$$

Then

$$\lim_{R\to\infty} \sup_{\underline{a}\in X\setminus\mathbb{B}(0,R)} \{h(\underline{a})\} = 0.$$

Now, applying Theorem 3.2, we find that 1 is an isolated eigenvalue of $\mathscr{L}_{\widehat{\phi}}$. □

We construct a σ-invariant absolutely continuous measure with respect to the conformal measure m_ϕ, which follows from the Ionescu-Tulcea and Marinescu theorem for the Perron-Frobenius operator.

Corollary 5.1. *The measure $\mu_\phi = hm_\phi$ is σ-invariant; that is, $\mu_\phi \circ \sigma^{-1} = \mu_\phi$, which is absolutely continuous with respect to the $\chi e^{-\phi}$-conformal measure m_ϕ. It is ergodic with respect to σ. In particular, $\mu_\phi(X) = 1$.*

Proof. Let $\widehat{\phi} = \phi - P(\phi)$ and let m_ϕ be the $e^{P(\phi)-\phi}$-conformal measure. By Theorems 3.1 and 3.2, the normalized operator $\mathscr{L}_{\widehat{\phi}}$ is quasi-compact, 1 is a simple isolated eigenvalue, and there exists $h \in H_\alpha$, $h > 0$, such that $\mathscr{L}_{\widehat{\phi}}h = h$ and $\int h\, dm_\phi = 1$.

Define μ_ϕ by $d\mu_\phi = h\, dm_\phi$. Then $\mu_\phi(X) = 1$, hence μ_ϕ is a probability and clearly $\mu_\phi \ll m_\phi$. For every $g \in CB(X,\mathbb{R})$,

$$\int g\circ\sigma\, d\mu_\phi = \int g(\sigma(x))\, h(x)\, dm_\phi(x) = \int \mathscr{L}_{\widehat{\phi}}(gh)\, dm_\phi.$$

Since $\mathscr{L}^*_{\widehat{\phi}}m_\phi = m_\phi$ and $\mathscr{L}_{\widehat{\phi}}h = h$, we obtain $\int g\circ\sigma\, d\mu_\phi = \int g\, d\mu_\phi$. Therefore, μ_ϕ is σ-invariant.

Let $k \geq 1$ and let A be a σ^k-invariant measurable set. Set $u := h\,\mathbb{1}_A \in L^1(m_\phi)$. Then for all $g \in CB(X,\mathbb{R})$,

$$\int g\, u\, dm_\phi = \int g\circ\sigma^k\, u\, dm_\phi = \int \mathscr{L}^k_{\widehat{\phi}}(gu)\, dm_\phi.$$

Hence, $\mathscr{L}^k_{\widehat{\phi}}u = u$ in $L^1(m_\phi)$. By quasi-compactness and simplicity of the peripheral eigenvalue 1, the eigenspace for 1 is span$\{h\}$. Therefore, $u = ch\ m_\phi$-a.e., and thus $\mathbb{1}_A = c\ \mu_\phi$-a.e., implying $\mu_\phi(A) \in \{0,1\}$. Hence, μ_ϕ is ergodic.

Since the leading eigenvalue of the normalized transfer operator is simple, the corresponding invariant probability measure is unique. Consequently, the invariant Gibbs measure constructed in [10] coincides with the measure μ_ϕ obtained here whenever both constructions apply. □

6. Variational principle

For each $\mu \in \mathcal{M}_\sigma(X)$, we write $h_\mu(\sigma)$ for the measure-theoretic entropy of σ with respect to μ.

Write $\mathbb{N} = \{1,2,\cdots\}$. Let $\mathcal{P} = \{\xi_n\}_{n\in\mathbb{N}}$ be a countable partition of X such that

(1) $\mathrm{diam}(\xi_n) < \delta_0$ for all n;

(2) for every N, there exists $M(N)$ such that $\bigcup_{n=0}^{M(N)} \xi_n = \overline{B(0,N)}$;

(3) $\overline{\mathrm{Int}(\xi_n)} \supset \xi_n$.

Let $\mathcal{P}_N = \{\xi_n : n < M(N)\} \cup \{\bigcup_{n \geq M(N)} \xi_n\}$. We denote $\{\bigcup_{n \geq M(N)} \xi_n\}$ by $\xi_{\geq M(n)}$.

Then $\{\mathcal{P}_N\}$ is an increasing sequence of partitions, each of finite entropy, whose union $\bigcup_k \mathcal{P}_{N_k}$ generates the Borel σ-algebra of X. Moreover, $\mathcal{P}_{N_1} \subset \mathcal{P}_{N_2} \subset \cdots$ and $\mathcal{H}_\mu(\mathcal{P}_{N_k}) < \infty$ for every $\mu \in \mathcal{M}_\sigma(X)$. Then,

$$h_\mu(\sigma) = \lim_{N\to\infty} h_\mu(\sigma, \mathcal{P}_N), \text{ with } h_\mu(\sigma, \mathcal{P}_N) = \lim_{n\to\infty} \frac{1}{n}\mathcal{H}_\mu(\mathcal{R}_0^n). \tag{6.1}$$

Now, we fix N, and by $\mathcal{R}$, we denote $\mathcal{P}_N$. Let

$$\mathcal{R}_0^n := \mathcal{R} \vee \sigma^{-1}(\mathcal{R}) \vee \ldots \vee \sigma^{-n}(\mathcal{R}).$$

By α^*, we denote finite sequences $\alpha_0 \ldots \alpha_n$ of elements from $\mathcal{R}$. Then

$$[\alpha^*] := [\alpha_0, \ldots, \alpha_n] = \bigcap_{j=0}^{n} \sigma^{-j}(\alpha_j).$$

Let

$$P_n(\beta_0, \beta_n) := \frac{1}{n} \log \sum_{\substack{\alpha^* = \beta_0 \cdots \beta_n \\ [\alpha^*] \in \mathcal{R}_0^n}} e^{\|S_n\phi\|_{[\alpha^*]}}.$$

Lemma 6.1. *Let* $\|\mathscr{L}_\phi \mathbb{1}\|_\infty < +\infty$.

(1) $\sup_{\underline{c} \in X} \phi(\underline{c}) < \infty$.
(2) If $\beta_n \neq \xi_{\geq M(N)}$, *then* $\limsup_{n\to\infty} P_n(\beta_0, \beta_n) \leq P(\phi)$.
(3) If $\beta_n = \xi_{\geq M(N)}$, *then* $\limsup_{n\to\infty} P_n(\beta_0, \beta_n) \leq \sup_X \phi + \log \|\mathscr{L}_\phi \mathbb{1}\|_\infty$.

Proof. (1) Since $\|\mathscr{L}_\phi \mathbb{1}\|_\infty < +\infty$, there exists $\tilde{C}$ such that

$$\mathscr{L}_\phi \mathbb{1}(\underline{a}) = \sum_{\sigma(\underline{b}) = \underline{a}} e^{\phi(\underline{b})} < \tilde{C}.$$

Let $\underline{c} \in X, \sigma(\underline{c}) = \underline{a}$, then $e^{\phi(\underline{c})} \leq \sum_{\sigma(\underline{b}) = \underline{a}} e^{\phi(\underline{b})} < \tilde{C}$.

Therefore, $\phi(\underline{c}) < \tilde{C}'$.

(2) We assume that $\beta_n \neq \xi_{\geq M(N)}$. Fix $\underline{a} \in \beta_n$. Since, by Lemma 4.8(ii),

$$\sum_{\substack{\alpha^* = \beta_0 \ldots \beta_n \\ [\alpha^*] \in \mathcal{R}_0^n}} e^{\|S_n\phi\|_{[\alpha^*]}} \leq A_1 \sum_{\substack{\sigma^n(\underline{b}) = \underline{a} \\ \underline{b} \in \beta_0}} e^{S_n\phi(\underline{b})},$$

then,

$$P_n(\beta_0, \beta_n) \leq \frac{1}{n} \log A_1 + \frac{1}{n} \log(\mathscr{L}_\phi^n \mathbb{1}_{\beta_0})(\underline{a}).$$

Therefore,

$$\limsup_{n\to\infty} P_n(\beta_0, \beta_n) \leq P(\phi).$$

(3) Suppose that $\beta_n = \xi_{\geq M(N)}$. For every $[\alpha^*] \in \mathcal{R}_0^n$, we choose a point $x_{\alpha^*} \in [\alpha^*]$ such that

$$\|S_n\phi\|_{[\alpha^*]} < S_n\phi(x_{\alpha^*}) + \log 2. \tag{6.2}$$

For $k > 0$, we define

$$\mathcal{A}_k := \{[\alpha^*] \in \mathcal{R}_0^n : \alpha_0 = \beta_0, \alpha_k \neq \beta_n \text{ and } \alpha_j = \beta_n, \forall j > k\},$$
$$\mathcal{A}_0 := \{[\alpha^*] \in \mathcal{R}_0^n : \alpha_0 = \beta_0 \text{ and } \alpha_j = \beta_n, \forall j > 0\},$$
$$S_k := \sum_{\alpha^*:[\alpha^*]\in\mathcal{A}_k} e^{S_n\phi(x_{\alpha^*})},$$

and by k_n, we denote the index k such that $S_k = \max_{0\leq\ell<n} S_\ell$. Note that

$$S_{k_n} = \sum_{\alpha^*:[\alpha^*]\in\mathcal{A}_{k_n}} e^{S_{k_n}\phi(x_{\alpha^*})+S_{n-k_n}\phi(\sigma^{k_n}(x_{\alpha^*}))} \leq \exp\left((n-k_n)\sup_{\underline{c}\in X}\phi(\underline{c})\right) \sum_{\alpha^*:[\alpha^*]\in\mathcal{A}_{k_n}} e^{S_{k_n}\phi(x_{\alpha^*})}.$$

Since $k_n \leq n-1$ by definition of $\mathcal{A}_{k_n}$, we have $A_{n-k_n} \leq A_1$. Now, we choose arbitrary $y_{\xi_i} \in \xi_i$. Then, by Lemma 4.8 (ii),

$$\begin{aligned} S_{k_n} &\leq \exp\left((n-k_n)\sup_{\underline{c}\in X}\phi(\underline{c})\right)A_{n-k_n}\sum_{i=1}^{M(N)}\sum_{\substack{\sigma^n(\underline{b})=y_{\xi_i}\\ \underline{b}\in\beta_0}} e^{S_{k_n}\phi(\underline{b})} \\ &\leq A_1\exp\left((n-k_n)\sup_{\underline{c}\in X}\phi(\underline{c})\right)M(N)\|\mathscr{L}_\phi\mathbb{1}_{\beta_0}\|_\infty^{k_n} \\ &\leq A_1 M(N)\exp\left(n\sup_{\underline{c}\in X}\phi(\underline{c})\right)\|\mathscr{L}_\phi\mathbb{1}_{\beta_0}\|_\infty^{n}. \end{aligned}$$

Hence, using (6.2), we get

$$P_n(\beta_0,\beta_n) \leq \frac{1}{n}\log\sum_{k=0}^{n-1}\sum_{\alpha^*:[\alpha^*]\in\mathcal{A}_k} e^{S_n\phi(x_{\alpha^*})+\log 2} \leq \frac{1}{n}\log\sum_{k=0}^{n-1} 2S_k \leq \sup_{\underline{c}\in X}\phi(\underline{c}) + \log\|\mathscr{L}_\phi\mathbb{1}_{\beta_0}\|_\infty + \frac{\log(nA_1M(N))}{n}.$$

□

Theorem 6.1. *Let $\phi \in H_\alpha$ such that $\|\mathscr{L}_\phi\mathbb{1}\|_\infty < +\infty$. Then,*

$$P(\phi) = \sup\left\{h_\mu(\sigma) + \int\phi\, d\mu : \mu \in \mathcal{M}_\sigma(X)\,; -\int\phi < \infty\right\} < \infty.$$

Proof. Let $\mathcal{P} = \{\xi_n\}_{n\in\mathbb{N}}$ be the countable partition given in the setup. Recall the block version of the pressure

$$P_n(\beta_0,\beta_n) := \frac{1}{n}\log\sum_{\substack{\alpha^*=\beta_0\cdots\beta_n\\ [\alpha^*]\in\mathcal{R}_0^n}} e^{\|S_n\phi\|_{[\alpha^*]}}.$$

Then,

$$\frac{1}{n}\mathcal{H}_\mu(\mathcal{R}_0^n) + \int\phi d\mu = \frac{1}{n}\Big(\mathcal{H}_\mu(\mathcal{R}_0^n) + \int S_n\phi d\mu\Big) \leq \frac{1}{n}\sum_{[\alpha^*]\in\mathcal{R}_0^n}\mu([\alpha^*])\log\frac{e^{\|S_n\phi\|_{[\alpha^*]}}}{\mu([\alpha^*])}$$

$$= \frac{1}{n} \sum_{\beta_0, \beta_n \in \mathcal{R}} \mu(\beta_0 \cap \sigma^{-n}(\beta_n)) \sum_{\substack{\alpha^* = \beta_0 \cdots \beta_n \\ [\alpha^*] \in \mathcal{R}_0^n}} \frac{\mu([\alpha^*])}{\mu(\beta_0 \cap \sigma^{-n}(\beta_n))} \log \frac{e^{\|S_n \phi\|_{[\alpha^*]}}}{\mu([\alpha^*])}.$$

By the standard convexity argument (Jensen's inequality for log) applied to the conditional distribution of α^* given the endpoints β_0, β_n, we obtain

$$\begin{aligned}
&\frac{1}{n}\mathcal{H}_\mu(\mathcal{R}_0^n) + \int \phi d\mu \\
\leq &\frac{1}{n} \sum_{\beta_0, \beta_n \in \mathcal{R}} \mu(\beta_0 \cap \sigma^{-n}(\beta_n)) \log \sum_{\substack{\alpha^* = \beta_0 \cdots \beta_n \\ [\alpha^*] \in \mathcal{R}_0^n}} e^{\|S_n \phi\|_{[\alpha^*]}} + \frac{1}{n} \sum_{\beta_0, \beta_n \in \mathcal{R}} -\mu(\beta_0 \cap \sigma^{-n}(\beta_n)) \log \mu(\beta_0 \cap \sigma^{-n}(\beta_n)) \\
= &\sum_{\beta_0, \beta_n \in \mathcal{R}} \mu(\beta_0 \cap \sigma^{-n}(\beta_n)) P_n(\beta_0, \beta_n) + \frac{1}{n}\mathcal{H}_\mu(\mathcal{R} \vee \sigma^{-n}\mathcal{R}).
\end{aligned}$$

Then, since μ is σ-invariant and $\mathcal{R}$ is a finite partition with $\mathcal{H}_\mu(\mathcal{R}) < \infty$, we have $\mathcal{H}_\mu(\sigma^{-n}\mathcal{R}) = H_\mu(\mathcal{R})$, and hence by subadditivity of entropy,

$$H_\mu(\mathcal{R} \vee \sigma^{-n}\mathcal{R}) \leq H_\mu(\mathcal{R}) + H_\mu(\sigma^{-n}\mathcal{R}) = 2H_\mu(\mathcal{R}) < \infty,$$

and then

$$\frac{1}{n}\mathcal{H}_\mu(\mathcal{R}_0^n) + \int \phi d\mu \leq \frac{1}{n} \sum_{\beta_0, \beta_n \in \mathcal{R}} \mu(\beta_0 \cap \sigma^{-n}(\beta_n)) P_n(\beta_0, \beta_n) + \frac{2\mathcal{H}_\mu(\mathcal{R})}{n}.$$

Fix $\varepsilon > 0$. By Lemma 6.1 and finiteness of $\mathcal{R}$, there exists $n_0 = n_0(N, \varepsilon)$ such that, for all $n \geq n_0$ and for all $\beta_0, \beta_n \in \mathcal{R}$,

$$\beta_n \neq \xi_{\geq M(N)} \Rightarrow P_n(\beta_0, \beta_n) \leq P(\phi) + \varepsilon, \qquad \beta_n = \xi_{\geq M(N)} \Rightarrow P_n(\beta_0, \beta_n) \leq C_\phi + \varepsilon,$$

where $C_\phi := \sup_X \phi + \log \|\mathscr{L}_\phi \mathbb{1}\|_\infty < \infty$ (Lemma 6.1 part (1)). Hence, for $n \geq n_0$,

$$\begin{aligned}
\sum_{\beta_0, \beta_n} \mu(\beta_0 \cap \sigma^{-n}\beta_n) P_n(\beta_0, \beta_n) &\leq \mu(\xi_{\geq M(N)}^{\mathsf{c}} \cap \sigma^{-n}\xi_{\geq M(N)}^{\mathsf{c}}) (P(\phi) + \varepsilon) + \mu(\xi_{\geq M(N)} \cup \sigma^{-n}\xi_{\geq M(N)}) (C_\phi + \varepsilon) \\
&\leq (1 - \mu(\xi_{\geq M(N)})) P(\phi) + \mu(\xi_{\geq M(N)}) C_\phi + \varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\beta_0, \beta_n \in \mathcal{R}} \mu(\beta_0 \cap \sigma^{-n}(\beta_n)) P_n(\beta_0, \beta_n) \\
\leq &\mu(\xi_{\geq M(N)}^{C} \cap \sigma^{-n}\xi_{\geq M(N)}^{C})(P(\phi) + \varepsilon) + \mu(\xi_{\geq M(N)} \cup \sigma^{-n}\xi_{\geq M(N)})(\sup_X \phi + \log \|\mathscr{L}_\phi \mathbb{1}\|_\infty + \varepsilon) \\
\leq &(1 - \mu(\xi_{\geq M(N)}))(P(\phi) + \varepsilon) + \mu(\xi_{\geq M(N)})(C_\phi + \varepsilon) \\
= &(1 - \mu(\xi_{\geq M(N)}))P(\phi) + (\sup_X \phi + \log \|\mathscr{L}_\phi \mathbb{1}\|_\infty)\mu(\xi_{\geq M(N)}) + \varepsilon.
\end{aligned}$$

Therefore, for all $n \geq n_0$,

$$\frac{1}{n}\mathcal{H}_\mu(\mathcal{R}_0^n) + \int \phi \, d\mu \leq (1 - \mu(\xi_{\geq M(N)})) P(\phi) + \mu(\xi_{\geq M(N)}) C_\phi + \varepsilon + \frac{2\mathcal{H}_\mu(\mathcal{R})}{n},$$

passing to the limit as $n \to \infty$ (for fixed N), gives

$$h_\mu(\sigma, \mathcal{R}) + \int \phi \, d\mu \le (1 - \mu(\xi_{\ge M(N)}))\, P(\phi) + \mu(\xi_{\ge M(N)})\, C_\phi + \varepsilon.$$

Now, let $N \to \infty$. Since $\{\mathcal{P}_N\}$ exhausts X and $\xi_{\ge M(N)} \downarrow \emptyset$, we have $\mu(\xi_{\ge M(N)}) \to 0$. Using $h_\mu(\sigma) = \lim_{N\to\infty} h_\mu(\sigma, \mathcal{P}_N)$, we obtain

$$h_\mu(\sigma) + \int \phi \, d\mu \le P(\phi) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $h_\mu(\sigma) + \int \phi \, d\mu \le P(\phi)$ for all invariant μ.

On the other hand, for each fixed N, the subshift Σ_N is compact with respect to the metric ρ. The classical Ruelle variational principle on compact shifts yields a Gibbs state $\mu_N \in \mathcal{M}_\sigma(\Sigma_N)$ such that $P(\Sigma_N, \phi|_{\Sigma_N}) = h_{\mu_N}(\sigma) + \int \phi \, d\mu_N$. Taking the supremum over N, we get

$$P(\phi) = \sup_N P(\Sigma_N, \phi|_{\Sigma_N}) \le \sup_{\mu \in \mathcal{M}_\sigma(X)} \Big\{ h_\mu(\sigma) + \int \phi \, d\mu \Big\}.$$

□

7. Symbolic dynamics for transcendental entire maps

This section collects the main structural and dynamical ingredients concerning symbolic codings induced by transcendental entire maps that are needed for the applications developed in this paper. Part of the material was established in [10]; we include it here in a self-contained form for the convenience of the reader, together with the verification of hypotheses (H1)–(H4) for the exponential map E_ω.

Denote by $\mathrm{Sing}(f^{-1})$ the set of finite singularities of the inverse function f^{-1}, which is the set of critical values (images of critical points) and asymptotic values of f together with their finite limit points. The *post-singular* set $PS(f)$ of f is defined as

$$PS(f) := \bigcup_{n=0}^{\infty} f^n \left(\mathrm{Sing}\left(f^{-1}\right)\right).$$

Definition 7.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire map.*

(1) We say that f is of finite order *if $\rho_f := \limsup_{z\to\infty} \dfrac{\log\log|f(z)|}{\log|z|}$ is finite.*

(2) f satisfies the rapid derivative growth condition*: There are $\alpha_2 > 0$, $\alpha_1 > \alpha_2$, and $\kappa > 0$ such that for every $z \in J(f)$, we have*

$$|f'(z)| \ge \kappa^{-1} |z|^{\alpha_1} |f(z)|^{\alpha_2}.$$

(3) The set $\mathrm{Sing}(f^{-1})$ *is contained in a compact subset of the immediate basin $B = B(z_0)$ of an attracting fixed point $z_0 \in \mathbb{C}$.*

Denote by $\mathcal{F}$ the class of transcendental entire functions f satisfying (1)–(3).

Remark 7.1. *Each $f \in \mathscr{F}$ belongs to the* Eremenko-Lyubich *class*

$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} : \mathrm{Sing}(f^{-1}) \text{ is bounded}\}.$$

It was proved in [9] that for $f \in \mathcal{B}$, all the Fatou components of f are simply connected. Hence, the immediate basin B is simply connected.

Remark 7.2. *Each $f \in \mathscr{F}$ is hyperbolic in the sense that the closure $\overline{PS(f)}$ is disjoint from the Julia set and $\overline{PS(f)}$ is compact. We have that f has no wandering and no Baker domains, so B is the only Fatou component of f; see [8,9,12]. Examples in the class $\mathscr{F}$ include the family $\lambda \exp(z)$ for $\lambda \in (0, 1/e)$, and the family of maps $\lambda \sin(z)$ for $\lambda \in (0, 1)$; other examples are the expanding entire maps $\sum_{j=0}^{p+q} a_j e^{(j-p)z}$, $p, q > 0$, $a_j \in \mathbb{C}$, studied early in [5].*

Remark 7.3. *Note that, if f satisfies (3), then f is in the class of transcendental entire maps of* disjoint-type*, which is a class of maps satisfying a strong form of hyperbolicity. This was studied in several papers including [1, 18]. For more details, see [17, Section 3.3].*

7.1. Symbolic representation

Fix $f \in \mathscr{F}$. Since the immediate attraction basin $B = B(z_0)$ of an attracting fixed point z_0 is simply connected, there exists a bounded simply connected domain $D \subset \mathbb{C}$, such that its closure $\overline{D} \subset B$, and its boundary ∂D is an analytic Jordan curve. Moreover, $\overline{\text{Sing}(f^{-1})} \subset D$ and $\overline{f(D)} \subset D$; for more details, see [2, Lemma 3.1]. Following [2], the pre-images of $\mathbb{C} \backslash \overline{D}$ by f consist of countably many unbounded connected components called *tracts* of f. We denote the collection of all these tracts by $\mathscr{R}$.

Since the closure of each tract is simply connected, there exists an open simple arc $\alpha : (0, \infty) \to \mathbb{C} \backslash \overline{D}$, which is disjoint from the union of the closures of all tracts and such that $\alpha(t)$ tends to a point of ∂D as t tends to 0^+, and $\alpha(t)$ tends to ∞ as t tends to $+\infty$. We use this curve to define the fundamental domains on each tract as follows: Since for every $T \in \mathscr{R}$, the map $f|_T$ is a cover of $\mathbb{C} \backslash \overline{D}$, and $T \backslash f^{-1}(\alpha)$ is the union of infinitely many disjoint simply connected domains S, such that the function

$$f|_S : S \to \mathbb{C} \backslash (\overline{D} \cup \alpha)$$

is bijective. Given $T \in \mathscr{R}$, we denote by $\mathcal{S}_T$ the collection of connected components of $T \backslash f^{-1}(\alpha)$. The elements of

$$\mathcal{S} := \bigcup_{T \in \mathscr{R}} \mathcal{S}_T \tag{7.1}$$

are called fundamental domains.

For each $S \in \mathcal{S}$, we have that the restriction $f|_S$ is univalent, so we denote its inverse branch by $g_S := (f|_S)^{-1} : \mathbb{C} \backslash (\overline{D} \cup \alpha) \to S$. For $n \geq 1$ and each $j \in \{0, 1, \cdots, n\}$, denote by S_j an element of $\mathcal{S}$ and put $g_{S_0 S_1 \cdots S_n} = g_{S_0} \circ \cdots \circ g_{S_n}$. Then,

$$g_{S_0 \cdots S_n}(\mathbb{C} \backslash (\overline{D} \cup \alpha)) = \left\{ z \in \mathbb{C} : f^j(z) \in S_j, \text{ for every } j = 0, \cdots, n \right\}. \tag{7.2}$$

For each sequence $\underline{S} = (S_0 S_1 \cdots) \in \mathcal{S}^{\mathbb{N}}$, let $K_{\underline{S}} := \bigcap_{n=0}^{\infty} g_{S_0 S_1 \cdots S_n}(\mathbb{C} \backslash (\overline{D} \cup \alpha))$. Then, the Julia set of f is given by the disjoint union of $K_{\underline{S}}$, that is,

$$J(f) = \bigsqcup_{\underline{S} \in \mathcal{S}^{\mathbb{N}}} K_{\underline{S}}.$$

Since f satisfies conditions (1) and (3), it follows from [2, 18], that the Julia set $J(f)$ is a Cantor bouquet, that is, a union of uncountably many pairwise disjoint curves called hairs that tend to infinity.

Each hair is attached to a unique point in $J(f)$ that is accessible from the immediate basin B, referred to as the endpoint of the hair. More precisely, for each symbolic sequence $\underline{S}$, the associated set $K_{\underline{S}}$ is either empty, or there exists a homeomorphism

$$h_{\underline{S}} : [0, +\infty) \to K_{\underline{S}},$$

such that $\lim_{t\to+\infty} h_{\underline{S}}(t) = \infty$, and for every $t > 0$, $\lim_{n\to+\infty} f^n(h_{\underline{S}}(t)) = \infty$. In the latter case, the point $z_{\underline{S}} := h_{\underline{S}}(0)$ is the unique point of $K_{\underline{S}}$ that is accessible from the immediate basin B, in the sense that there exists a continuous curve $v : [0, \infty) \to B$, such that

$$\lim_{t\to\infty} v(t) = z_{\underline{S}}.$$

See [1] for this construction, which generalizes earlier results for the exponential map with an attracting fixed point in [6].

Let $\Sigma := \{\underline{s} = (s_0 s_1 \ldots) : s_j \in \mathbb{Z},\ \text{for all } j \geq 0\}$ be the full shift space, and the shift metric for some $\theta \in (0, 1)$ is given by

$$d(\underline{s}, \underline{t}) = \theta^{\inf\{k: s_k \neq t_k\}\cup\{\infty\}}. \tag{7.3}$$

For every $n \geq 1$, we denote a finite word $s_0 \cdots s_{n-1}$ in $\mathbb{Z}^n$ simply by s^*, so we use the notation for cylinders:

$$[s^*] = \{\underline{w} \in \Sigma : w_i = s_i, 0 \leq i \leq n-1\},$$

and for $s \in \mathbb{Z}$, we simply denote $[s] = \{\underline{w} \in \Sigma : w_0 = s\}$. Let $\sigma : \Sigma \longrightarrow \Sigma$ be the left-sided shift map, given by $\sigma(s_0 s_1 \cdots) = (s_1 s_2 \cdots)$.

Observe that by definition, the set $\mathcal{S}$ given in (7.1) is countably infinite, so we identify $\mathcal{S}$ with $\mathbb{Z}$. Put

$$X := \{\underline{S} \in \mathcal{S}^{\mathbb{N}} : K_{\underline{S}} \neq \emptyset\} \subset \Sigma. \tag{7.4}$$

Let

$$Z = \bigcup_{\underline{S}\in X} K_{\underline{S}}.$$

From (7.2), we have, for each $\underline{S} \in \mathcal{S}^{\mathbb{N}}$, $f(K_{\underline{S}}) = K_{\sigma(\underline{S})}$, then the function f on the Julia set $J(f)$ is semi-conjugate to σ on X; however, f on the set

$$\mathcal{EP} := \{z_{\underline{S}} = h_{\underline{S}}(0) : \underline{S} \in X\}$$

is conjugate to $\sigma|_X$. Hence, the set X is completely σ-invariant.

The set $\mathcal{EP}$ defined above, is the set of *endpoints* of hairs $K_{\underline{S}}$ and it satisfies the following properties: It is the set of accessible points from the immediate attraction basin B. It is totally disconnected, however, $\mathcal{EP} \cup \{\infty\}$ is connected, see [3]. Moreover, the Hausdorff dimension of this set is equal to 2 (see [2]), generalizing previous results of Karpińska [13] for the exponential map $E_\omega(z) = \omega e^z$ with parameters $\omega \in (0, 1/e)$. This exponential map is probably the best known example in the family $\mathscr{F}$, its Julia set is a Cantor bouquet, and the set of endpoints is modeled by the symbolic space of all allowable sequences; see [6, 14]. †

† A sequence $\underline{S} = (s_0 s_1 \ldots) \in \Sigma$ is *allowable* if there exists a unique dynamic ray $K_{\underline{S}}$ of f such that, for each $n \geq 0$, the point $f^n(z)$ lies in the fundamental domain indexed by s_n for every $z \in K_{\underline{S}}$.

7.2. Induced metric on X

Let X be a symbolic space encoding the itineraries of endpoints of $J(f)$. We define a metric ρ on X, which does not necessarily generate the topology induced by the cylinder sets

$$\rho(\underline{s}, \underline{w}) := |h_{\underline{s}}(0) - h_{\underline{w}}(0)|.$$

7.3. A different class of potentials

Let ρ_f be the order of f, and let $\alpha_1, \alpha_2 > 0$ be the constants given by the rapid derivative growth condition of f. Fix $\tau \in (0, \alpha_2)$ and define $\gamma : \mathbb{C}\setminus\{0\} \to \mathbb{R}\cup\{\infty\}$ by $\gamma(z) := |z|^{-\tau}$. Let θ be the Riemannian metric on $\mathbb{C} \setminus \{0\}$ defined by

$$d\theta(z) = \gamma(z)\,|dz|.$$

We differentiate f with respect to θ instead of the Euclidean metric; for each $z \in \mathbb{C} \setminus \{0\}$, this gives

$$|f'(z)|_\theta \;=\; |f'(z)|\,\frac{\gamma(f(z))}{\gamma(z)} \;=\; |f'(z)|\,\frac{|z|^\tau}{|f(z)|^\tau}. \tag{7.5}$$

Definition 7.2. *We denote by $\mathscr{C}$ the set of functions $\psi : \bigcup_{S\in\mathcal{S}} S \to \mathbb{R}^+$ that are bounded from above and constant on each $S \in \mathcal{S}$:*

$$\mathscr{C} \;:=\; \left\{\psi : \bigcup_{S\in\mathcal{S}} S \;\longrightarrow\; \mathbb{R}^+ \;\middle|\; \psi \textit{ is bounded from above and constant on each } S \in \mathcal{S}\right\}.$$

The *class of potentials* associated with $f \in \mathscr{F}$ is

$$\mathscr{P}_f \;:=\; \left\{\phi_{\psi,t}(z) = \log\psi(z) - t\log|f'(z)|_\theta \;\middle|\; \psi \in \mathscr{C}, \quad t > \frac{\rho_f}{\alpha_1+\tau}\right\}.$$

Remark 7.4. *The class $\mathscr{P}_f$ contains the potentials $-t\log|f'|_\theta$, which by* (7.5) *are cohomologous to* $-t\log|f'|$.

Let $C_b(J(f),\mathbb{R})$ denote the Banach space of bounded continuous functions on $J(f)$, equipped with the supremum norm. For each potential $\phi \in \mathscr{P}_f$, the associated transfer operator $\mathscr{L}_\phi$ acts continuously on $C_b(J(f),\mathbb{R})$ by

$$\mathscr{L}_\phi\psi(z) \;=\; \sum_{f(w)=z} \psi(w)\,e^{\phi(w)}, \qquad \psi \in C_b(J(f),\mathbb{R}).$$

The following estimate is central to the summability properties of $\mathscr{P}_f$. Let $f \in \mathscr{F}$ and let $\phi_{c,t} = \log c - t\log|f'|_\theta \in \mathscr{P}_f$. Using the conjugacy $H : X \to \mathcal{EP}$, a direct computation yields

$$\begin{aligned}\mathscr{L}_{\phi_{c,t}\circ H}\,\psi\circ H(\mathbf{w}) &= \sum_{\sigma(\mathbf{s})=\mathbf{w}} \psi(z_{\mathbf{s}})\,c(z_{\mathbf{s}})\,|f'(z_{\mathbf{s}})|_\theta^{-t} = \sum_{\sigma(\mathbf{s})=\mathbf{w}} \psi(z_{\mathbf{s}})\,c(z_{\mathbf{s}})\,|f'(z_{\mathbf{s}})|^{-t}\,|z_{\mathbf{s}}|^{-\tau t}\,|f(z_{\mathbf{s}})|^{\tau t}\\ &= |z_{\mathbf{w}}|^{\tau t}\sum_{\sigma(\mathbf{s})=\mathbf{w}} \psi(z_{\mathbf{s}})\,c(z_{\mathbf{s}})\,|f'(z_{\mathbf{s}})|^{-t}\,|z_{\mathbf{s}}|^{-\tau t}.\end{aligned}$$

Applying the rapid derivative growth condition gives

$$\mathscr{L}_{\phi_{c,t}\circ H}(\mathbb{1})(\mathbf{w}) \le \kappa^t\,|z_{\mathbf{w}}|^{\tau t}\sum_{\sigma(\mathbf{s})=\mathbf{w}} c(z_{\mathbf{s}})\,|z_{\mathbf{s}}|^{-\alpha_1 t}\,|f(z_{\mathbf{s}})|^{-\alpha_2 t}\,|z_{\mathbf{s}}|^{-\tau t} = \kappa^t\,|z_{\mathbf{w}}|^{\tau t}\sum_{\sigma(\mathbf{s})=\mathbf{w}} c(z_{\mathbf{s}})\,|z_{\mathbf{s}}|^{-(\alpha_1+\tau)t}\,|z_{\mathbf{w}}|^{-\alpha_2 t}$$

$$\leq \frac{\kappa^t}{|z_{\mathbf{w}}|^{t(\alpha_2-\tau)}} \sup_{\mathbf{s}\in\mathcal{S}^{\mathbb{N}}} c(z_{\mathbf{s}}) \sum_{\sigma(\mathbf{s})=\mathbf{w}} |z_{\mathbf{s}}|^{-(\tau+\alpha_1)t}.$$

Since f is a transcendental entire function of finite order ρ_f and $t > \rho_f/(\tau+\alpha_1)$, the Borel-Picard Theorem [16, Theorem 3.5] implies that the last series has an exponent of convergence equal to ρ_f, and hence converges. By [16, Proposition 3.6], there exists $\mathcal{M}_t > 0$ such that, for all $\mathbf{w} \in X$,

$$\mathcal{L}_{\phi_{c,t}\circ H}(\mathbb{1})(\mathbf{w}) \;\leq\; \frac{\mathcal{M}_t}{|z_{\mathbf{w}}|^{t(\alpha_2-\tau)}} \sup_{\mathbf{s}\in\mathcal{S}^{\mathbb{N}}} c(z_{\mathbf{s}}). \tag{7.6}$$

Consequently, (7.6) implies both

$$\lim_{R\to\infty} \sum_{s\in\mathbb{Z}} \exp\left(\sup_{\mathbf{w}\in[s]\cap(X\setminus B(\mathbf{0},R))} \phi_{c,t}\circ H(\mathbf{w}) \right) = 0, \qquad \text{and} \qquad \lim_{z_{\mathbf{w}}\to\infty} \mathcal{L}_{\phi_{c,t}\circ H}\, \mathbb{1}(\mathbf{w}) = 0,$$

which verify the rapidly decreasing condition for potentials in $\mathscr{P}_f$.

7.4. *The symbolic space X and properties* $(H1)$–$(H4)$ *from the exponential map*

The most representative example in the class $\mathscr{F}$ is the *exponential map*

$$E_\omega(z) = \omega e^z, \qquad \omega \in (0, 1/e).$$

The map E_ω belongs to $\mathscr{F}$, and since it is an entire function of order 1, it satisfies the rapid derivative growth condition with $\alpha_1 = 0$ and $\alpha_2 = 1$, and has a unique singular value at 0. Since this singular value lies in the attracting basin of the fixed point, the map is hyperbolic. Moreover, the classical potentials

$$-t\log|z| = -t\log|E'_\omega(z)| + \log\gamma_1 - \log\gamma_1 \circ E_\omega,$$

where

$$\gamma_1 = |z|^{-t},$$

are tame and belong to the class $\mathscr{P}_{E_\omega}$.

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Then

$$E_\omega(\{z : \operatorname{Re} z < \ln(1/\omega)\}) = \mathbb{D} \setminus \{0\}.$$

Since $\mathbb{D} \subset \{z : \operatorname{Re} z < 1\} \subset \{z : \operatorname{Re} z < \ln(1/\omega)\}$, where the last inclusion holds because $\ln(1/\omega) > 1$ for $\omega \in (0, 1/e)$, it follows that $\overline{E_\omega(\mathbb{D})} \subset \mathbb{D}$. Since the immediate basin B of the attracting fixed point is the unique Fatou component of E_ω, we also have $\overline{\mathbb{D}} \subset B$. Furthermore,

$$E_\omega^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}) = \{z : \operatorname{Re} z > \ln(1/\omega)\} = \{z : \operatorname{Re} z > q_\omega\},$$

where $q_\omega = \ln(1/\omega)$ is the repelling fixed point on the real axis. So, the unique tract of E_ω is the right half-plane $T = \{z : \operatorname{Re} z > \ln(1/\omega)\}$.

To define the fundamental domains, consider the ray $\alpha : (0,\infty) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ defined by $\alpha(t) = -(1+t)$. Then,

$$E_\omega^{-1}(\alpha(0,\infty)) = \bigcup_{k\in\mathbb{Z}} \{x + (2k-1)\pi i : x > \ln(1/\omega)\}.$$

For each $k \in \mathbb{Z}$, the corresponding fundamental domain is

$$S_k = \{z : \operatorname{Re} z > \ln(1/\omega),\ (2k-1)\pi < \operatorname{Im} z < (2k+1)\pi\},$$

so $T \setminus E_\omega^{-1}(\alpha(0,\infty))$ is the disjoint union of the domains $\{S_k\}_{k\in\mathbb{Z}}$.

For a finite word $c^* = (c_0 \dots c_{n-1}) \in \mathbb{Z}^n$, the corresponding inverse branch is

$$g_{c^*} = g_{c_0} \circ g_{c_1} \circ \cdots \circ g_{c_{n-1}},$$

where $g_k : \mathbb{C} \setminus \overline{D} \to S_k$ is the univalent inverse branch of E_ω over the strip S_k.

Note that, the inverse branch g_k is a branch of the complex logarithm, and its derivative satisfies

$$|g_k'(z)| = \frac{1}{|E_\omega'(g_k(z))|} = \frac{1}{|E_\omega(g_k(z))|} = \frac{1}{|z|}.$$

Since $|z| \to \infty$ as $\operatorname{Re} z \to \infty$, each branch g_k is a contraction. Applying the chain rule to $g_{c^*} = g_{c_0} \circ \cdots \circ g_{c_{n-1}}$ and using the rapid derivative growth condition

$$|E_\omega'(z)| = |E_\omega(z)| = \omega e^{\operatorname{Re} z},$$

which grows exponentially in $\operatorname{Re} z$, one obtains

$$|p(c^*\underline{a}) - p(c^*\underline{b})| = |g_{c^*}(p(\sigma^n\underline{a})) - g_{c^*}(p(\sigma^n\underline{b}))| \le C\, e^{-\alpha n}\, |p(\underline{a}) - p(\underline{b})|,$$

for some $\alpha > 0$ coming from the expansion rate in the real direction. Setting $\lambda = e^\alpha > 1$ verifies hypothesis (H1).

On the other hand, geometrically, the expansion $|E_\omega'(z)| = \omega e^{\operatorname{Re} z} \to \infty$ as $\operatorname{Re} z \to \infty$ implies that real parts of iterated preimages tend to $+\infty$, while the $2\pi i$-periodicity of E_ω distributes them across all strips S_k. Together, these two facts imply that, for any $R > 0$, there exists $n \ge 1$ such that E_ω^n maps a neighbourhood of $z_\mathbf{0}$ onto a region whose intersection with every strip S_k with $|k| \le R$ is non-empty, which is precisely (H2) in symbolic terms.

The condition (H3) is more intuitive. Note that all sequences in a ball $B_0(\underline{a}, \delta)$ share the first symbol $a_0 = k$, so their endpoints lie in the convex strip S_k. Then, any two such endpoints in S_k can therefore be joined by a straight segment inside S_k; subdividing it into steps of length less than δ' yields the required finite chain.

Finally, condition (H4) is the most direct and follows immediately from the geometry of the fundamental domains, because consecutive strips S_k and S_{k+1} are separated by the line $\operatorname{Im} z = (2k+1)\pi$, so endpoints in different strips satisfy $|z_{\underline{a}} - z_{\underline{b}}| \ge 2\pi$. Taking $\delta_0 < 2\pi$ gives the result.

A non-tame potential. Following [5], let $c : J(E_\omega) \to \mathbb{R}^+$ be a function that is constant on $J(E_\omega) \cap (S_{-k} \cup S_k)$, with value c_k, and assume that the sequence $(c_k)_{k\in\mathbb{Z}}$ of positive numbers satisfies

$$\lim_{k\to\infty} \frac{\log c_k}{\log k} = -\infty.$$

For $t > 0$, define the potential

$$\phi(z) = \log\bigl(c(z)\, |z|^{-t}\bigr), \qquad c(z) = c_k \text{ if } z \in S_{-k} \cup S_k.$$

Since $c_k \to 0$ as $k \to \infty$, the potential ϕ is *not tame*, that is, does not belong to the class introduced in [16]:

$$\mathscr{T}_{E_\omega} := \left\{ \phi = h - t \log |E'_\omega|_\theta \;\middle|\; h \text{ is a bounded weakly Hölder function, } t > \frac{\rho'}{\alpha_1 + \alpha_2} \right\}.$$

Nevertheless, it belongs to $\mathscr{P}_{E_\omega}$ because c is bounded on each fundamental domain S_k and $|E'_\omega(z)|_\theta = |z|$.

Although the class $\mathscr{F}$ does not include most of the functions considered in [16], the exponential map E_ω satisfies $\mathscr{P}_{E_\omega} \cap \mathscr{T}_{E_\omega} \neq \emptyset$ and $\mathscr{P}_{E_\omega} \setminus \mathscr{T}_{E_\omega} \neq \emptyset$.

8. Conclusions

This abstract symbolic setting is motivated by symbolic codings of transcendental entire maps whose Julia sets have Cantor bouquet structure. Our main contribution is the proof of quasi-compactness and the existence of a spectral gap for the normalized transfer operator acting on spaces of locally Hölder functions adapted to the non-standard metric ρ. As a consequence, we obtain a simple maximal eigenvalue equal to the topological pressure, a strictly positive eigenfunction, which, together with the associated conformal measure, yields a unique σ-invariant Gibbs state in this non-classical symbolic setting.

These results complement the approach developed in [10], where the existence of a conformal and an invariant Gibbs measure was established without a spectral description. The present work provides the missing spectral structure and opens the way to the study of statistical properties in this framework, without relying on compatibility with the cylinder topology.

Beyond the exponential family considered as the primary application, the framework introduced here may be extended to other classes of transcendental entire maps exhibiting Cantor bouquet geometry.

Author contributions

Irene Inoquio-Renteria and Rodolfo Viera: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. Both authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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