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Research article

Topological morphisms and lifting correspondences of affine algebraic varieties

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Abstract: In general, the algebraic lifting correspondences of projective varieties consider a set of prime ideals with zero genus under morphisms. In this article, I propose the concept and formulation of topological double lifting correspondences of algebraic varieties considering topological based spaces and projective spaces preserving homeomorphism and separable fibrations. The finite sequence of lifting correspondences within a compact projective covering space are introduced. I show that double lifting correspondences admit several homeomorphisms between fundamental groups in the based topological spaces and in the projective covering spaces. The concept of topological endomorphism is extended to the real algebraic varieties in the distinguished Zariski open sets. It is illustrated that the symmetric automorphism in polynomial rings in a closed algebraic field generate equivalent classes of polynomials in subrings along with the formation of fixed points in projective varieties under certain conditions. It is shown that the homeomorphic topological lifting correspondences of projective varieties preserve the genus of algebraic varieties forming the fixed point under the covering functions, which is distinct compared to the Hopf fixed point. I introduce the topological concept of genus deformation morphism of a projective variety and show that there are no homeomorphic lifting correspondences in such cases. However, the covering functions are admissible in such cases considering that the fibers are discrete.

Keywords: topology; real algebraic variety; fiber; lifting; morphisms
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1. Introduction

In the domain of algebraic geometry, the concepts of Zariski topology play a central role, where the Zariski space is a not a Hausdorff topological space. This indicates that a Zariski topological space is not a separable space. On the other hand, in the domain of topology, the concepts of endomorphism and automorphism signify the continuous functions from a space to itself. In the case of endomorphism, the open sets map to open sets, and in the case of automorphism, a continuous function preserves homeomorphism. The interplays of real algebraic varieties, such as the Zariski topological spaces, associated sheaves, and the topological covering spaces, expose a rich set of properties. The involvements of projective varieties and projective spaces enrich it further. For example, it is shown by Chow that if $f \in R[x_1, x_2, \ldots\ldots, x_n]$ generates an algebraic variety then there is a normal projective subvariety generated by $g \in R[x_1, x_2, \ldots\ldots, x_n]$ in the respective projective space P_L^n, such that $h: Zr(f) \to Zr(g)$ is a topologically continuous function preserving the Zariski open and closed spaces [1]. Moreover, an Abelian (non singular) algebraic variety can be embedded in the projective space P_L^{2n}, where the self-intersection number of the subvariety in P_L^n is finite [2]. If we consider the complex field, then the Kodaira manifolds are projective algebraic if they are Hodge varieties and they can be embedded in the complex projective spaces [3]. In general, the real algebraic varieties are connected and smooth Zariski closed spaces over the closed real field R, which can be subjected to the morphisms within the topological spaces. Interestingly, the algebraic groups are considered algebraic varieties admitting covering spaces and projections [4]. For example, if $p: X \to Y$ is a covering map and the function $q: U \times D \to X$ over a discrete space D admits homeomorphism $(p \circ q): U \times D \to Y$, then there is another homeomorphism $u: U \to Y$ under the projection $v: U \times D \to U$. Note that the morphism of a real algebraic variety can be included within this formalism. In this context, the Randell theorem illustrates the existence of invariant moduli spaces under rearrangement (i.e., morphism), where manifolds can retain symmetry [5].

If we consider a linear space X and a projective space P_L^n, then there is an embedding $i_{emb}: X \to P_L^n$, which is locally complete [6]. On the other hand, the Berkovich algebraic curve X is a topologically separable, analytic, and compact curve admitting classically tame morphism $f: X \to Y$, where Y is another Berkovich curve [7]. The Berkovich algebraic curve X under the tame morphism requires connected covering spaces and the standard sheaf O_X with the stalks. However, the topological space generated by X can be further extended to form $G-$topological space X_G by enabling a set of non-analytic points in $X_G \backslash X$, and the sheaf O_X can be extended to the sheaf $O_{X(G)}$ accordingly [7]. Interestingly, from the topological point of view, the open cover of $A^o \subset X_G$ also covers $B^o \subset X$, indicating the extensions of the topological covering spaces. It is important to note that the algebraic lifting and the topological lifting employs a different set of formalisms because algebraic lifting involves sheaves and the topological lifting requires a homeomorphic covering map [8]. The interrelationships between the two formalisms are maintained by the lifting of associated line bundles. Interestingly, in the complex field, the vector line bundles can be employed to form a bundle of projective spaces, enabling the lifting of such projective bundles [9].

1.1. Motivations

In general, the topological homotopy lifting correspondences consider the retention of homeomorphism during the lift. It was mentioned that the algebraic lifting and the topological lifting correspondences employ different formalisms [8]. The generalized algebraic lifting of an affine real algebraic curve is not always straightforward, as pointed by Oort [10–12]. The topologically homeomorphic lifting is obstructed if the genus of an algebraic curves is large in the higher dimensions [11]. In other words, projection of an algebraic variety by a topological covering map may not always preserve the genus of the algebraic variety on the topological base space. In order to algebraically solve the problem, the local lifting of algebraic varieties is proposed based on automorphic group actions on the curves preserving the characteristic to zero [10]. However, this approach is difficult to generalize further in higher dimensions. In the case of complex algebraic curves, the Torelli morphism is employed forming the Torelli locus [11]. Both of these approaches assume that the algebraic variety is decomposable into multiple irreducible components, enabling morphism and local lifting of subvarieties. In other words, if an algebraic variety represents an irreducible Zariski topological space with a large genus, then it is difficult to formulate a homotopy lift. Thus, a generalized topological approach is required to gain deeper understandings by reducing additional conditions. On the other hand, the Grothendieck theorem of lifting algebraic variety states that if $(f_{lift(1)}, f_{lift(2)})$ is an equivalence class of liftings, then there is a bijective equivalence class of infinite sequence of liftings in R^n [13–15]. Note that such lifting may not always adhere to the principles of topological lifting if we do not consider whether a covering section is compact or non-compact. In the case of a complex manifold $M_{C(n)}$ of dimension n and the associated projective (n-1)-space under the holomorphic fibration $p: M_{C(n)} \to P_L^{n-1}$, the lifting of $\pi_1(M_{C(n)}, z_0)$ on the foliated projective space P_L^{n-1} requires monodromy of the locally flat (holomorphic) connections in the projective space [16]. Earlier, it was shown that if X is a smooth projective variety and P_L^n is a projective n-space, then there is an embedding $i_{emb}: X \to P_L^n$ and it enables pull back from the tangent space of X in the case of P_L^1 considering that the algebraic curve is a free rational variety [17]. This is essentially a contra-representation of lifting under the strict requirement of holomorphic foliated projective space as well as fibrations.

On the other hand, let us consider a set of algebraic curves $\{D_1, D_2\}$ of Berkovich variety (i.e., separated, quasi-smooth, and analytic real algebraic varieties). It is shown that such curves can be separately lifted to covering spaces [18]. However, the lifting considers several conditions such as, the existence of wild morphism $f: D_1 \to D_2$ with the availability of p −enhancements, and the curves are irreducible log-curves [18]. The effects of non-zero genus and homogeneity property of the projective varieties are not considered. Interestingly, the birational morphism of rational algebraic varieties within a projective space enables the pull back under the inclusion function [19]. Note that this formulation requires the (invertible) birational function of an algebraic variety D from the one-dimensional projective space, which is given as $v: P_L^1 \to D$. Thus, a topological generalization would be advantageous considering P_L^n enables deformations of the algebraic variety. The interesting questions are: (1) Can we topologically generalize these notions in the case of real algebraic varieties and the associated projective spaces considering symmetric automorphisms and endomorphisms in the polynomial rings? (2) can we avoid birational functions as a form of topological generalization admitting the homeomorphic covering maps? (3) What are the roles of distinguished Zariski open sets in this case if we consider multiple topologically based spaces admitting fundamental groups under homeomorphism? Moreover, an important question is: What are the interrelationships between the topological genus deformation function as a morphism of a projective algebraic variety and the corresponding homeomorphic lifting correspondence of such a variety in a projective space? I combine the elements of topology and algebraic geometry to address these questions.

1.2. Contributions

In general, the algebraic varieties are considered Zariski closed sets so that the complement spaces are the Zariski open distinguished spaces. The distinguished open spaces are important to form the boundary varieties and associated Nagy-Foiaş lifting [20]. Earlier, it was shown that automorphism in a polynomial ring can induce deformation in composite topological manifolds [21]. The interpretations of the Grothendieck theorem and the lifting in the domain of algebraic topology indicate that a closer relook into the lifting correspondences of a real projective variety within a covering section would expose interesting properties. In this article, I explore these aspects in detail considering a topological lifting correspondence and a sequence of topological lifting correspondences of an irreducible algebraic variety in a projective space. I formulate the endomorphism of real algebraic varieties under the real-valued evaluations in distinguished open sets, which is distinct compared to the endomorphism of compact topological manifolds $f: C^{\infty}(M) \to C^{\infty}(M)$. Moreover, I employ automorphisms in the closed algebraic field and in a polynomial ring to form two equivalence classes of polynomials, such as $\lambda_F -$equivalence class in the algebraic field and the $\beta_* -$ equivalence class in the polynomial subrings. The interplay of endomorphism and automorphisms includes the symmetry property. The $\beta -$ fixed points are formed within the projective varieties, which can be topologically lifted, preserving homeomorphism. The $\beta -$fixed points are distinct in the proposed formulation compared to the Hopf fixed points. This is because the Hopf fixed point depends on the extension of a continuous function $f: (X \subset R^n) \to (X \subset R^n)$, such that the homological trace of the extended function preserves the $tr(f_*) \neq 0$ condition [22]. Moreover, the Hopf fixed point does not consider projective varieties and its lifting correspondences. I propose a more generalized approach considering the homogeneous projective spaces and show two aspects of lifting correspondences such as: (1) Existence of double lifting correspondences in two topologically based spaces and, (2) a lift preserves the β *−fixed point* along the fiber. I illustrate that it is possible to admit a finite sequence of homeomorphic lifting correspondences of a real algebraic variety in the compact projective covering space. Recall that the topological lifting of an algebraic variety considers that the space is homogeneous, forming the corresponding projective variety. If the genus of an algebraic variety is not zero, then the topological lifting is not a straightforward approach. I introduce the concept of topological genus-deformation of a real algebraic variety and show that such an irreducible variety may not have homeomorphic lifting, although such a variety can have a covering map within a based topological space. The corresponding fiber at the base point is discrete within the projective space, enabling the genus-deformation function. Moreover, if the genus of the algebraic variety is more than one, then a choice of topological lifting is required, preserving the corresponding base point under the fibrations. Furthermore, the genus-deformation within the projective space can induce homeomorphism between fundamental groups. Note that the topological genus-deformation of an algebraic variety can be considered a special kind of morphism involving projective spaces; however, it is distinct compared to the surjective deformation over a complex field admitting a Galois group structure without altering its genus [23]. The surjective morphism of an n-dimensional complex algebraic variety (with divisors) into a projective n-space involves Galois embedding of the variety within N-dimensional ($N > n$) projective space and the linearization under function composition back to projective n-space [23]. However, the proposed genus-deformation map of an algebraic variety does not consider any linearization under projections and any composition of projections. Moreover, the distinction of the proposed formulation (in this article) is that I do not consider the shifting of base point of fundamental group homeomorphism under surjection and the connectedness of fiber products in the algebraic schemes for étale cover [24].

The rest of the article is organized as follows: The preliminary concepts are presented in Section 2. The definitions related to the proposed formulations are presented in Section 3. The topological and algebraic properties are detailed in Section 4. The applicational aspects and comparative analysis are presented in Section 5. Finally, in Section 6, I conclude the article.

2. Preliminaries

Let the zero-set $Zr(f)$ of $f \in R[x_1, x_2, \ldots\ldots, x_n]$ for $n \geq 3$ represent an algebraic curve with genus two, which is a Shimura variety. The lifting of such a Shimura variety is possible through the Hecke correspondences [13]. In the domain of commutative abstract algebra, it is shown that the lifting of the maps of modules between a set of finite graphs is possible, while retaining the homeomorphism under the path algebras [25]. Furukawa showed that if $Zr(f)$ is an irreducible algebraic variety with genus two equipped with divisor D on $Zr(f)^2$, then there is a lifting of $(Zr(f), D)$ with characteristic zero. However, it is considered that the divisor D induces endomorphism on the Jacobian of $Zr(f)$ and as a result, additional conditions about the differential of the algebraic variety are required [13]. It is shown that the symmetric lifting is not always admissible with varying dimensions, and the lifting of $Zr(f)$ to the field of reals R requires the sheaf of $Zr(f)$. The following proposition illustrates such conditions [13].

Proposition 2.1. *The lifting $f_{lift(1)}$ of $Zr(f)$ to R is possible if $f_{lift(1)}$ is a (proper) smooth scheme over the reals. Moreover, if $f_{lift(2)}$ is another lift, then $f_{lift(1)}$ must be a flat and closed sub-scheme of $f_{lift(1)} \times f_{lift(2)}$ over R.*

In the case of holomorphic complex manifold $M_{C(n)}$ of dimension n, a set of strict conditions is required to admit lifting of $\pi_1(M_{C(n)}, z_0)$ into the projective space P_L^{n-1}. Let us consider that P_L^{n-1} is a foliated holomorphic space admitting a holomorphic fiber bundle. The lifting of $\pi_1(M_{C(n)}, z_0)$ is defined as follows [16]:

Definition 2.1. *The representation of a locally flat as well as holomorphic connection in P_L^{n-1} is presented by the projection $p: \pi_1(M_{C(n)}, z_0) \to Aut(p^{-1}(z_0))$ admitting the lifting of $[f] \subset \pi_1(M_{C(n)}, z_0)$ as long as the lifting is within a holomorphic leaf and $p(.)$ is a monodromy.*

The characterizations of projective spaces, including the Fano varieties, are established based on the concept of free rational algebraic curves or morphisms. The definition of free rational morphism is presented as follows [17]:

Definition 2.2. *Let a projective 1-space be denoted by P_L^1 and X be a smooth projective variety. The morphism $f: P_L^1 \to X$ is defined as free if the pull back $f^*: T_X \to P_L^1$ from the tangent bundle T_X can be formed by the global sections of the corresponding rational curve.*

Interestingly, the projective varieties can be embedded into the projective n-space under certain additional conditions and it employs the concept of free rational morphism [17]. It can admit morphism into another variety, enabling fibration. These properties are presented in the following theorem [17]:

Theorem 2.1. *Let a projective n-space be denoted by P_L^n and X be a smooth projective variety admitting non-constant morphism $\pi: X \to Y$. There exists an embedding $i_{emb}: X \to P_L^n$ and a fiber at a smooth point $y \in Y$ given by $\pi^{-1}(y)$, which is isomorphic to a projective space.*

Theorem 2.1 asserts that if the morphism of an algebraic variety A from a topological space X

to space Y is not a constant then the injective embedding of A in a projective space retains fibration under fiber-projection from the base space to the projective space. The birational morphism of an algebraic variety is an important aspect of algebraic geometry because it involves projective spaces. It is shown that the projective morphism $h: D_1 \to D_2$ between the two algebraic varieties can be admitted in P_L^1. This observation is presented in the following theorem [19]:

Theorem 2.2. *If* $D_1, D_2 \subset P_L^n$ *are two non-degenerate rational varieties with* $deg > n$*, then there are birational morphisms* $f_{D1}: P_L^1 \to D_1$ *and* $f_{D2}: P_L^1 \to D_2$ *admitting inclusion* $\mu: P_L^1 \to P_L^1$ *such that it commutes as* $\mu \cong (f_{D2}^{-1} \circ (h \circ f_{D1}))$*, where* $h: D_1 \to D_2$ *is a projective transformation.*

This illustrates that the embedding of algebraic variety within a projective space under morphisms and the morphism of projective spaces in lower-dimensions are related concepts. However, the embedding of an algebraic variety within a projective space considers that the projective space is comparatively higher dimensional and the morphism of a projective space is generally formulated in lower dimension for simplicity. It is important to note that the endomorphism of real topological manifolds and complex surfaces behaves differently. In case of complex surfaces, the endomorphism of a blown-up surface at a point does not preserve the projective fibers [26]. Note that an algebraic variety may not be always stable (i.e., free from defect) in every dimension. The Kiepert trefoil is an algebraic curve (representing a Knot), which is relatively stable under topological deformations [27]. In general, the Segre, Veronese, and Grassmannian varieties are free from defects in relatively lower dimensions; however, it cannot be guaranteed in higher dimensions [28]. The determinations of existence of defects are analyzed by considering the projective algebraic varieties preserving isomorphism. Note that the morphisms, including fibration and co-fibration, can be formulated by employing the elements of Category theory [29]. This results in the concept of vertical fibration under functorial morphism. The mathematical objects such as algebraic varieties have applications in science and engineering. For example, the algebraic varieties are basic elements of *n*-dimensional Knot theory and they have applications in science and engineering considering 3-spaces [30].

3. Definitions

In this section, I present a set of definitions related to the proposed formulations. The distinguished Zariski open topological space of a $f \in F[x_1, x_2, \ldots\ldots, x_n]$ is denoted as D_f, and the disjoint union of two sets is denoted as $A \overset{\bullet}{\cup} B$. A point in a projective space is denoted as $(a_1, a_2, \ldots\ldots, a_n)_P$, and the projective n-space is denoted as $A_P^n(F)$, where $P_L^n \subset A_P^n(F)$ denotes a projective subspace. The genus of a real algebraic variety $Zr(f)$ is denoted as $\Gamma(Zr(f))$. I consider that the underlying real algebraic field is closed. First, I present the definition of the λ −endomorphism (here called endomorphism for simplicity) between two polynomial rings under the real valued finite evaluations.

Definition 3.1. *Let* F *be a closed algebraic field. A function* $\lambda: F[x_1, x_2, \ldots\ldots, x_n] \to F[y_1, y_2, \ldots\ldots, y_m]$ *is an endomorphism if* $\exists\{(a_1, a_2, \ldots\ldots, a_n), (b_1, b_2, \ldots\ldots, b_m)\} \subset (A^n(F) \overset{\bullet}{\cup} A^m(F))$ *, such that* $f((a_1, a_2, \ldots\ldots, a_n)) = g((b_1, b_2, \ldots\ldots, b_m))$ *for a* $f \in F[x_1, x_2, \ldots\ldots, x_n]$ *and* $g = (\lambda \circ f) \in F[y_1, y_2, \ldots\ldots, y_m]$*, where* $f((a_1, a_2, \ldots\ldots, a_n)) \in F$.

Example 3.1. *Let us consider two irreducible polynomials given by,* $f(x, y) = (x-1)^2 + (y-2)^3$ *and* $g(x, y, z) = (x-1)^2 + (y-2)^3 - (z-5)^4$ *such that* $g = (\lambda \circ f)$ *. This results in the conclusion that* $g(3,4,5) = f(3,4)$ *in the respective affine spaces.*

Note that λ −endomorphism indicates the existence of affine subspaces in the respective zero sets or within the respective distinguished affine spaces, where the polynomials in the corresponding polynomial rings can be evaluated with unique values. Interestingly, the endomorphism between two polynomial rings is not necessarily unique because there are many possible solutions satisfying the endomorphism condition within the affine topological spaces. Moreover, the endomorphism does not restrict to the condition that $f((a_1, a_2, \ldots\ldots, a_n)) = g((b_1, b_2, \ldots\ldots, b_m)) = 0 \in F$, indicating that endomorphism considers distinguished Zariski open sets with the zero-sets (i.e., the generalized form in the affine space over the closed algebraic field).

Example 3.2. *Let* $f(x, y) = 2xy + 1$ *and* $g(y) = 3y$ *be two polynomials, such that* $g = (\lambda \circ f)$. *If we choose* $\{(1,1), (0,1)\} \subset (D_f \cup D_g)$, *then we obtain* $f(x, y) = g(y)$. *Note that in this case* $n > m$ *and* $deg(f) > deg(g)$. *Similarly, if we select* $n = m$ *such that* $f(x) = 4x^2, g(y) = 2y$, *then we obtain* $f(1) = g(2)$ *in the algebraic field for the respective points in the distinguished Zariski open topological spaces. Moreover, if we consider* $f(x, y) = ax^2y^2 + by$ *and* $g(x, y) = cx + y^2$ *then* $\{(0,0), (x_i, y_i)\} \subset A^2(F)$ *can maintain* λ *−endomorphism if it preserves the conditions given by* $(x_i y_i - 1)(y_i^3 - by_i^2 - ay_i + c) = 0; x_i \in (0, +\infty)$ *for* $n = m$.

Remark 3.1. *It immediately results in the conclusion that* $(\lambda \circ f) \cong (\lambda^{-1} \circ g)$ *for* $n \neq m$ *in the algebraically closed field* F. *In other words, if we consider distinguished open* $D_f \subset A^n(F)$ *and* $D_g \subset A^m(F)$, *then the mutually disjoint spaces admit endomorphism for a set of polynomials* $\{f, g\}$ *of heterogeneous dimensions attaining converges in the algebraically closed field* F *when evaluated at non singular points.*

The following definition establishes a bridge between the automorphism within the algebraically closed field and the automorphism within two polynomial rings with the respective homogeneous dimensions, while preserving the endomorphism between two polynomial rings with heterogeneous dimensions.

Definition 3.2. *Let* $\lambda_F: F\backslash\{0\} \to F\backslash\{0\}$ *be an automorphism in the closed algebraic field and* $\beta: F[x_1, x_2, \ldots\ldots, x_n] \to F[x_1, x_2, \ldots\ldots, x_n]$ *be another automorphism within the polynomial ring of homogeneous dimensions. The set of polynomials* $\{f, g\}$ *is defined as* $\lambda_F - \mathtt{equivalent\ in}\ F$ *if* $f \in F[x_1, x_2, \ldots\ldots, x_n]$ *and* $g \cong (\lambda \circ f) \in F[y_1, y_2, \ldots\ldots, y_m]$ *preserve the following conditions:*

$$\exists h \in F[x_1, x_2, \ldots\ldots, x_n], h = (\beta^k \circ f),\ k \in [1, w] \subset Z^+, g \cong (\lambda \circ h),$$

$$f((a_1, a_2, \ldots\ldots, a_n)) = \lambda_F(g((b_1, b_2, \ldots\ldots, b_m))). \tag{1}$$

Example 3.3. *Let us consider a polynomial given by,* $f(x, y) = (x - 1)^2 + (y - 2)^3$ *and for* $k = 2$, $h(x, y) = (x - 1)^4 + (y - 2)^5$. *Suppose the polynomial* $g = (\lambda \circ h) = (x - 1)^4 + (y - 2)^5 - (z - 5)^4$. *This results in the conclusion that* $\lambda_F = 0.25$ *and* $\lambda_F . g(3,4,5) = f(3,4)$.

Note that we have considered $f((a_1, a_2, \ldots\ldots, a_n)) \neq g((b_1, b_2, \ldots\ldots, b_m))$ in the algebraically closed field in the absence of endomorphism. In addition, the automorphism does not generate any fixed-point, indicating that $\forall a, b \in [1, k], [a \neq b] \Rightarrow [Zr(\beta^a(f)) \cap Zr(\beta^b(f)) = \phi]$ and the zero-sets are considered dominant Zariski closed. The concept of $\lambda_F -$ equivalent class of polynomials is presented in the following commutative diagram (Figure 1):

$$\begin{tikzcd}
f \in F[x_1, x_2, \dots, x_n] \arrow[r, "\beta^k"] \arrow[d, dashed, "f((a_1, a_2, \dots, a_n))"'] & h \in F[x_1, x_2, \dots, x_n] \arrow[d, "\lambda"] \\
c \in F \setminus \{0\} & g \in F[y_1, y_2, \dots, y_m] \arrow[l, dashed, "(\lambda_F \circ g)((b_1, b_2, \dots, b_m))"]
\end{tikzcd}$$

Figure 1. Commutative diagram of $\lambda_F - equivalence$ in polynomial rings.

We can establish the interrelationship between the automorphisms within two polynomial subrings and the real-valued convergence in the algebraic field by the polynomials in the subrings under endomorphism. This results in the formations of equivalence classes of polynomials within the respective subrings as stated in the following definition:

Definition 3.3. *Let* $V \subset F[x_1, x_2, \dots\dots, x_n]$ *and* $W \subset F[y_1, y_2, \dots\dots, y_m]$ *be two subrings such that* $\forall f_i \in V, \exists g_i \in W$*, where* $g_i \cong (\lambda \circ f_i)$*. The set* $[f]_{\beta*} \subseteq V$ *and* $[g]_{\beta*} \subseteq W$ *are called* $\beta_* - equivalent$ *classes of polynomials in the respective subrings if* $\exists\{(a_1, a_2, \dots\dots, a_n), (b_1, b_2, \dots\dots, b_m)\} \subset (A^n(F) \dot{\cup} A^m(F))$ *such that* $(\beta_* \circ f_{i-1}) = (\beta_* \circ g_{i-1})$ *in the algebraic field* F *under evaluation for all* $\{f_{i-1}, f_i\} \subset [f]_{\beta*}$ *and* $\{g_{i-1}, g_i\} \subset [g]_{\beta*}$*, where* $\beta_*: V \dot{\cup} W \to V \dot{\cup} W$.

Remark 3.2. *Note that the polynomials in* $[f]_{\beta*}$ *maintain two conditions: (1)* $(\beta_* \circ f_{i-1}) = f_i; (\beta_* \circ g_{i-1}) = g_i$ *indicating that the automorphisms are uniform and symmetric in two subrings and (2)* $\lambda_F(r \in F) = 1$ *indicating that it is a finitely contractive identity preserving function within the closed field while forming the* $\beta_* - \mathbf{equivalent}$ *class of polynomials. Moreover, the polynomials in* $[f]_{\beta*}$ *and* $[g]_{\beta*}$ *preserve the convergence in algebraically closed field* F *when evaluated, respectively, at a set of points in the respective affine spaces of heterogeneous dimensions.*

The concept of $\beta_* - equivalent$ class of polynomials is presented in the commutative diagram below (Figure 2). In Figure 2, the Zariski open spaces $D_{[f]\beta*} = \cap_i(A^n(F) \setminus Zr(f_i \in [f]_{\beta*}))$ and $D_{[g]\beta*} = \cap_i(A^m(F) \setminus Zr(g_i \in [g]_{\beta*}))$ represent the respective distinguished open spaces for the corresponding $\beta_* - equivalent$ classes.

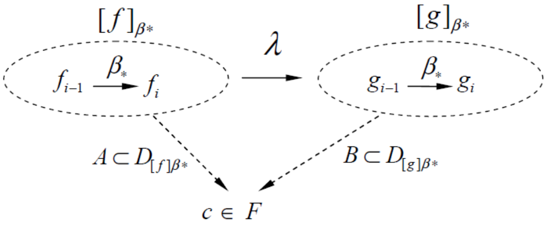

Figure 2. Commutative diagram of $\beta_* - equivalent$ class of polynomials.

If we relax the strict requirement of real-valued convergence of non-intersecting polynomials within a field F while maintaining the automorphism $\beta: F[x_1, x_2, \dots\dots, x_n] \to F[x_1, x_2, \dots\dots, x_n]$, then it admits the possibility of a complete intersection between two polynomials under automorphism. If we impose the condition that the automorphic polynomials are homogeneous, then it invites a set of interesting topological properties such as lifting correspondence between algebraic varieties in the projective spaces. The concept of generation of a fixed-point under the relaxed automorphism within a polynomial ring is presented in the following definition considering that the polynomials are homogeneous:

Definition 3.4. *If* $F[x_1, x_2, \ldots\ldots, x_n]$ *is a polynomial ring over the respective closed algebraic field and* $\{f, g\} \subset F[x_1, x_2, \ldots\ldots, x_n]$ *is a set of homogeneous polynomials, then* $(a_1, a_2, \ldots\ldots, a_n)_P \in A_P^n(F)$ *is a* β *–fixedpoint in an affine space if the following conditions are admitted.*

$$g = \beta(f), Zr(f) \cap Zr(g) = \{(a_1, a_2, \ldots\ldots, a_n)_P\}. \tag{2}$$

This immediately leads to the conclusion that the β *–fixedpoint* in an affine space preserves the properties of a complete intersection between two algebraic varieties and $(X - u)^2 | (fg)$, where $X = (x_1, x_2, \ldots\ldots, x_n)_P$ and $u = (a_1, a_2, \ldots\ldots, a_n)_P$ in the projective space. Note that there is a projective space P_L^n, which can admit the topological (isomorphic) projection given as $\pi_P : (Zr(f) \cup Zr(g)) \to P_L^n$. The topological projections can be suitably altered to admit the deformation of algebraic varieties, as presented in the following definition:

Definition 3.5. *Let the polynomial* $f \in F[x_1, x_2, \ldots\ldots, x_n]$ *be prime with* $\Gamma(Zr(f)) = 0$ *and* $g_{(a,b)} : Zr(f) \to X$ *be continuous in a connected topological space* X*, such that* $\Gamma(g_{(a,b)}) = 1$ *and* $X/(c{\sim}d)$ *is a quotient by identifying* $\{a, b\} \subset Zr(f)$ *under* $g_{(a,b)}(.)$*. The function* $g_{(a,b)}(.)$ *is defined as a topological genus-deformation map of the corresponding algebraic variety.*

Note that the genus-deformation map of an algebraic variety has resemblances to the concept of formation of discrete fibers in covering spaces under the topological projection to the base space.

Remark 3.3. *It can be observed that a genus-deformation map is a real algebraic variety in a topological space and there is a* $h \in R[x_1, x_2, \ldots\ldots, x_n]$ *such that* $h \in I(g_{(a,b)}(Zr(f)))$*, where* $\Gamma(Zr(h)) \geq \Gamma(g_{(a,b)}(Zr(f)))$*. If we consider that* $\Gamma(Zr(h)) = \Gamma(g_{(a,b)}(Zr(f)))$*, then* $h \in I(g_{(a,b)}(Zr(f)))$ *is a genus preserving ideal.*

4. Properties and lifting correspondences

In this section, I present a set of topological and algebraic properties related to the endomorphism and automorphism in the polynomial rings and real algebraic field. I present the topological lifting correspondences of the real algebraic varieties and the genus deformation morphism along with its consequences. I begin by presenting the topological and algebraic properties.

4.1. Topological and algebraic properties

The endomorphism within the polynomial rings of heterogeneous dimensions enforces convergence within the closed algebraic field under the morphism of algebraic varieties in heterogeneous affine spaces. The non-singular evaluation points in the respective distinguished open topological spaces are not necessarily unique. This property is presented in the following theorem:

Theorem 4.1. *If* $\lambda : F[x_1, x_2, \ldots\ldots, x_n] \to F[y_1, y_2, \ldots\ldots, y_m]$ *is an endomorphism such that* $\{f_i : i \in$

$[1,k]\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ *and* $\{g_i: i \in [1,k]\} \subset F[y_1,y_2,\ldots\ldots,y_m]$ *admit morphism* $\gamma: Zr(f_i) \to Zr(g_i)$ *then* $f_i((a_1,a_2,\ldots\ldots,a_n)) - g_i((b_1,b_2,\ldots\ldots,b_m)) = 0$, *where* $(a_1,a_2,\ldots\ldots,a_n) \in D_{f(i)}$ *and* $(b_1,b_2,\ldots\ldots,b_m) \in D_{g(i)}$.

Proof. Let $\lambda: F[x_1,x_2,\ldots\ldots,x_n] \to F[y_1,y_2,\ldots\ldots,y_m]$ be an endomorphism between the two polynomial rings over a closed real algebraic field F. Suppose we choose $\{f_i: i \in [1,k]\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ and $\{g_i: i \in [1,k]\} \subset F[y_1,y_2,\ldots\ldots,y_m]$ such that the continuous affine morphisms given by $\gamma: Zr(f_i) \to Zr(g_i)$ are admitted for every polynomials in the respective subrings. Thus, the respective set of ideals $I(Zr(\{f_i\}))$ and $I(Zr(\{g_i\}))$ admit the corresponding injective function $\gamma_*: I(Zr(\{g_i\})) \to I(Zr(\{f_i\}))$, where $Zr(\{f_i\}) \cap Zr(\{g_i\}) = \phi$ (i.e., mutually disjoint varieties). Let us choose $(a_1,a_2,\ldots\ldots,a_n) \in \bigcap_i D_{f(i)}$ and $(b_1,b_2,\ldots\ldots,b_m) \in \bigcap_i D_{g(i)}$ within the respective affine spaces. The endomorphism admits the condition that $f_i((a_1,a_2,\ldots\ldots,a_n)) = g_i((b_1,b_2,\ldots\ldots,b_m))$ in $F\backslash\{0\}$, where $g_i \cong (\lambda \circ f_i)$. Hence, we conclude that $f_i((a_1,a_2,\ldots\ldots,a_n)) - g_i((b_1,b_2,\ldots\ldots,b_m)) = 0$ and the non-singular evaluation points are not necessarily unique. □

If we enforce the condition that each $Zr(f_i) \subset A^n(F)$ is topologically separable for $f_i \in F[x_1,x_2,\ldots\ldots,x_n]$ then it results in the following lemma:

Lemma 4.1. *The* $\lambda_F - equivalent$ *class of polynomials in* $F[x_1,x_2,\ldots\ldots,x_n]$ *admits dominant Zariski closed sets and morphism, where each algebraic variety is topologically separable in the respective affine spaces.*

Proof. If we consider that $\{f,g\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ are in a $\lambda_F -$ equivalent class then they are dominant if they are irreducible algebraic varieties admitting the continuous affine morphism $\gamma: Zr(f) \to Zr(g)$. Moreover, there is no unique point $p = (a_1,a_2,\ldots\ldots,a_n) \in A^n(F)$ such that $(X-p)^2 | (fg)$, where $X = (x_1,x_2,\ldots\ldots,x_n)$. Hence, we conclude that there are $\overline{E} \subset A^n(F)$ and $\overline{G} \subset A^n(F)\backslash\overline{E}$ such that $Zr(f) \subset \overline{E}$ and $Zr(g) \subset \overline{G}$. □

Corollary 4.1. *These results indicate the following algebraic identities under* $\lambda_F - equivalence$ *class of polynomials in* $F[y_1,y_2,\ldots\ldots,y_m]$ *preserving the dominant algebraic varieties. Note that the algebraic identities indicate that* $\beta_n: F[x_1,x_2,\ldots\ldots,x_n] \to F[x_1,x_2,\ldots\ldots,x_n]$ *and* $\beta_m: F[y_1,y_2,\ldots\ldots,y_m] \to F[y_1,y_2,\ldots\ldots,y_m]$ *are bijective in the respective* $F[x_1,x_2,\ldots\ldots,x_n]$ *and* $F[y_1,y_2,\ldots\ldots,y_m]$ *of homogeneous dimensions, where real-valued evaluations are made in the distinguished open topological spaces.*

$$[(\beta_n \circ f_{i-1}) = f_i] \Rightarrow \left[\left(\lambda \circ (\beta_n \circ f_{i-1})\right) \cong (\lambda \circ f_i)\right], [(\lambda \circ f_{i-1})] \cong \left[\left(\beta_m^{-1} \circ (\lambda \circ f_i)\right)\right],$$

$$(\lambda_F \circ f_i) \cong (\lambda \circ f_{i-1}). \tag{3}$$

We can bridge between the bijective (automorphic) correspondences $\beta_n: F[x_1,x_2,\ldots\ldots,x_n] \to F[x_1,x_2,\ldots\ldots,x_n]$ and $\beta_m: F[y_1,y_2,\ldots\ldots,y_m] \to F[y_1,y_2,\ldots\ldots,y_m]$ as presented in the following theorem:

Theorem 4.2. *In the* $\lambda_F - equivalent$ *classes of polynomials in* $F[x_1,x_2,\ldots\ldots,x_n]$ *and* $F[y_1,y_2,\ldots\ldots,y_m]$, *the symmetry condition is maintained as,* $(\beta_n^{-k} \circ f_i) \cong (\beta_m^{-k} \circ g_i)$ *for every* $k \in [1,w]$.

Proof. Let us consider two $\lambda_F - equivalent$ classes of polynomial rings $F[x_1, x_2, \ldots\ldots, x_n]$ and $F[y_1, y_2, \ldots\ldots, y_m]$ associated to the endomorphism $\lambda: F[x_1, x_2, \ldots\ldots, x_n] \to F[y_1, y_2, \ldots\ldots, y_m]$. Suppose we consider $f_i \in F[x_1, x_2, \ldots\ldots, x_n]$ and $g_i \in F[y_1, y_2, \ldots\ldots, y_m]$ such that $g_i \cong (\lambda \circ f_i)$. As the functions $\beta_n: F[x_1, x_2, \ldots\ldots, x_n] \to F[x_1, x_2, \ldots\ldots, x_n]$ and $\beta_m: F[y_1, y_2, \ldots\ldots, y_m] \to F[y_1, y_2, \ldots\ldots, y_m]$ are bijective varieties inducing automorphisms in the respective polynomial rings, thus, $(\lambda \circ (\beta_n^{-k} \circ f_i)) \cong g_{i-k}$, where $(\beta_n \circ f_{i-1}) = f_i$. However, due to the endomorphism we obtain $(\lambda \circ f_{i-k}) \cong g_{i-k}$ and the $\lambda_F - equivalent$ class in $F[y_1, y_2, \ldots\ldots, y_m]$ admits the condition given by $(\beta_m^{-k} \circ g_i) = g_{i-k}$ for every $k \in [1, w]$. Hence, we conclude that $(\beta_n^{-k} \circ f_i) \cong (\beta_m^{-k} \circ g_i)$ for some $(a_1, a_2, \ldots\ldots, a_n) \in A^n(F)$ and $(b_1, b_2, \ldots\ldots, b_m) \in A^m(F)$ under the endomorphism. □

Corollary 4.2. The endomorphic $\lambda_F -$ equivalent classes of polynomial rings $F[x_1, x_2, \ldots\ldots, x_n]$ and $F[y_1, y_2, \ldots\ldots, y_m]$ preserve the following algebraic identities, where $\{(a_1, a_2, \ldots\ldots, a_n), (c_1, c_2, \ldots\ldots, c_n)\} \subset A^n(F)$.

$$(\lambda_F^k \circ f_i)((a_1, a_2, \ldots\ldots, a_n)) = (\beta_n^{-k} \circ f_i)((c_1, c_2, \ldots\ldots, c_n)),$$
$$(\lambda_F^k \circ (\lambda \circ f_i)) \cong (\beta_m^{-k} \circ g_i). \tag{4}$$

The following theorem illustrates the required condition to form the β −fixedpoints within the projective spaces considering different equivalent classes:

Theorem 4.3. *If homogeneous* $[f]_{\beta*}$ *and* $[g]_{\beta*}$ *do not admit respective* β *−fixedpoints then* $Zr(f_a), Zr(f_b)$ *and* $Zr(g_a), Zr(g_b)$ *are mutually disjoint in projective spaces* $A_P^n(F)$ *and* $A_P^m(F)$*, respectively, where* $\{f_a, f_b\} \subset [f]_{\beta*}$ *and* $\{g_a, g_b\} \subset [g]_{\beta*}$.

Proof. Let us consider two equivalent classes $[f]_{\beta*}$ and $[g]_{\beta*}$ in $F[x_1, x_2, \ldots\ldots, x_n]$ and $F[y_1, y_2, \ldots\ldots, y_m]$, respectively. Let us consider that the polynomials are homogeneous in $[f]_{\beta*}$ and in $[g]_{\beta*}$ preserving the condition $deg(f_a \in [f]_{\beta*}) \neq deg(g_a \in [g]_{\beta*})$. If we select $\{f_a, f_b\} \subset [f]_{\beta*}$ and $\{g_a, g_b\} \subset [g]_{\beta*}$ then there exist integers $u > 0$ and $v > 0$ such that $[b > a] \Rightarrow [f_b = (\beta_*^u \circ f_a)]$ and $[b > a] \Rightarrow [g_b = (\beta_*^v \circ g_a)]$. Note that polynomials in set $\{f_a, f_b, \lambda(f_a), \lambda(f_b)\} \subset ([f]_{\beta*} \cup [g]_{\beta*})$ converge at a point in $F\backslash\{0\}$ for some points in $A_P^n(F)$ and $A_P^m(F)$. However, if $[f]_{\beta*}$ and $[g]_{\beta*}$ do not admit respective $\beta -$ fixedpoints, then $Zr(f_b) \cap Zr(f_a) = \phi$ and $Zr(g_b) \cap Zr(g_a) = \phi$ in the projective spaces indicating that they are disjoint algebraic varieties in $A_P^n(F)$ and $A_P^m(F)$. □

Note that we can induce several classes of morphisms between $[f]_{\beta*}$ and $[g]_{\beta*}$ such as, $\gamma_{[f]}: [f]_{\beta*} \to [f]_{\beta*}$ indicating an endomorphism within an β_* −equivalent class and $\gamma_{[fg]}: [f]_{\beta*} \to [g]_{\beta*}$ indicating a standard morphism between the two β_* −equivalent classes. The endomorphism within an β_* −equivalent class is strictly projective.

4.2. Properties of lifting correspondences

Let us consider two mutually disjoint and based topological spaces (X, x_0) and (Y, y_0). Suppose an n-dimensional projective compact subspace is denoted as $P_L^n \subset A_P^n(F)$ such that $p_x: (E_X \subset P_L^n) \to (X, x_0)$ and $p_y: (E_Y \subset P_L^n) \to (Y, y_0)$ are two covering functions, where E_X, E_Y are mutually disjoint preserving $Hom(E_X, E_Y)$ property. This results in the topological double lifting correspondences of algebraic varieties in a projective space under injective topological embedding $i_{emb}(.)$ as presented in the following theorem:

Theorem 4.4. *If the function* $(i_{emb} \circ f): I \to (Y, y_0)$ *is continuous, then there are two topological lifting correspondences of a real algebraic variety in the projective* $P_L^n \subset A_P^n(F)$.

Proof. Let us consider the injective topological embedding $(i_{emb} \circ f): I \to (Y, y_0)$ and the corresponding $f_{l(X)}: f(I) \to E_X$ in the projective space $P_L^n \subset A_P^n(F)$ such that if $h \in F[x_1, x_2, \ldots\ldots, x_n]$ preserves $Isom(Zr(h), f(I))$ property, then $Zr(h)$ is a dominant as well as projective algebraic variety. If the function $p_y: (E_Y \subset P_L^n) \to (Y, y_0)$ is a covering map, then $f_{l(Y)}: Zr(h) \to (E_Y \subset P_L^n)$ is the first lift preserving the condition $(p_y \circ f_{l(Y)}) = (i_{emb} \circ f)$. Note that $(B = (i_{emb} \circ f)(I)) \subset Y$ is a connected topological subspace. If the continuous $g|_B: (Y, y_0) \to (X, x_0)$ is an injective variety, then the composition $g(B) = (p_x \circ f_{l(X)})(Zr(h))$ is the second lifting correspondence in P_L^n, where $p_x: (E_X \subset P_L^n) \to (X, x_0)$ is another covering map of the respective topological space. Hence, the projective n-space $P_L^n \subset A_P^n(F)$ admits double lifting correspondences of an affine real algebraic variety. □

Note that Theorem 4.4 illustrates that if there are two disjoint topological based spaces, then an algebraic variety can be topologically lifted into two subspaces of a projective space such that the corresponding covering maps can be established. Moreover, the homeomorphism can be preserved in the projective space between two subspaces containing liftings and between the two respective topological based spaces.

Remark 4.1. *The topological double lifting correspondences of an algebraic variety are illustrated in Figure 3. Moreover, note that the double lifting is not unique. This is because the continuous function* $v: (B \subset Y) \to E_X$ *is also a lifting correspondence under the respective covering function. Interestingly, in that case,* $Isom(v, f_{l(X)})$ *must be preserved.*

$$\begin{tikzcd}
 & E_Y \subset P_L^n \arrow[r, leftrightarrow, dashed, "Hom\,(E_X, E_Y)"] \arrow[d, "p_y"] & E_X \subset P_L^n \arrow[d, "p_x"] \\
Zr(h) \cong f(I) \arrow[ur, dashed, "f_{l(Y)}"] \arrow[r, "i_{emb}"'] \arrow[urr, dashed, bend right=70, "f_{l(X)}"'] & (Y, y_0) \arrow[r, leftrightarrow, dashed, "Hom\,(f, g)"] \arrow[r, bend right, "g|_B"'] & (X, x_0)
\end{tikzcd}$$

Figure 3. The topological double lifting in the projective n-space.

Lemma 4.2. *If* $P_L^n \subset A_P^n(F)$ *is a compact projective topological space then there is a finite sequence of lifting correspondences denoted by* $\langle v_{l(X(i))} \rangle_{i=1}^{k}$, *where every* $v_{l(X(i))}: Zr(h) \to (E_{X(i)} \subset P_L^n)$ *is a topological lifting of the respective algebraic variety under the corresponding covering function* $p_{x(i)}: E_{X(i)} \to (X_i, x_{i(0)})$.

Theorem 4.5. *If the two homogeneous polynomials* $\{f, g\} \subset F[x_1, x_2, \ldots\ldots, x_n]$ *have no common factor, then the topological lifting correspondences of respective algebraic varieties into a projective covering space are separable.*

Proof. Let us consider two homogeneous polynomials $\{f,g\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ such that they have no common factor and $(Zr(f) \cup Zr(g)) \subset B$, where B is a connected topological space. Suppose the continuous functions $v_{l(f)}: Zr(f) \to (E_X \subset P_L^n)$ and $v_{l(g)}: Zr(g) \to (E_X \subset P_L^n)$ are two topological lifting correspondences into a projective covering space such that $p_x: E_X \to (X,x_0)$ is the corresponding covering function. Let us assume that $v_{l(f)}(Zr(f)) \cap v_{l(g)}(Zr(g)) = \{b \in P_L^n\}$ in the projective covering space. Thus, we can find a covering function $p_x: E_X \to (X,x_0)$, such that $p_x(b) = x_0$. This indicates that there exists a unique injective embedding $i_{emb}: B \to (X,x_0)$, such that $(i_{emb}(B|_{Zr(f)})) \cap (i_{emb}(B|_{Zr(g)})) = \{p_x(b)\}$, preserving the homeomorphism property of the covering function. This results in the conclusion that $i_{emb}^{-1}(x_0)$ is an intersection point of $Zr(f)$ and $Zr(g)$ in B, which is a contradiction. Hence, the topological lifting correspondences $v_{l(f)}(Zr(f))$ and $v_{l(g)}(Zr(g))$ must be separated in $(E_X \subset P_L^n)$, such that $v_{l(f)}(Zr(f)) \subset U^o \subset E_X$ and $v_{l(g)}(Zr(g)) \subset V^o \subset (E_X \backslash \overline{U})$ in the projective covering space. □

Let us consider homogeneous $\{f,g\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ admitting a $\beta-$fixedpoint. This results in the following set of conclusions in terms of $\beta-$fixedpoints under the homeomorphic topological lifting correspondences in the projective spaces.

(1) Lifting of a genus 1 algebraic variety

If the algebraic variety $i_{emb}: Zr(f) \to (Y,y_0)$ has genus 1 such that $i_{emb}(Zr(f))\backslash\{y_0\}$ forms three topologically separated irreducible components of $Zr(f)$ in (Y,y_0), then there is a homeomorphic lifting correspondence $v_{l(f)}: (Zr(f) \subset B) \to (V^o \subset E_Y)$ such that $p_y(a_0) = y_0$, where $a_0 \in P_L^n$ is on the fiber of the lifting correspondence of a genus 1 algebraic variety within the projective covering space.

(2) Lifting of two genus 0 algebraic varieties with a $\beta-$fixedpoint

If $\{f,g\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ are homogeneous such that $u \in A^n(F)$ is a $\beta-$fixedpoint of $\{f,g\}$, then $\{u\}$ is preserved in the P_L^n and at the base $\{y_0\}$ of topological space (Y,y_0) under the homeomorphic lifting correspondences in the projective covering space P_L^n.

(3) Double lifting of two genus 0 algebraic varieties with a $\beta-$fixedpoint

The $\beta-$fixedpoint $u \in A^n(F)$ of homogeneous $\{f,g\} \subset F[x_1,x_2,\ldots\ldots,x_n]$ is preserved at $\{x_0,y_0\}$ under the homeomorphic double lifting correspondences in the projective covering sections in $(E_X \overset{\bullet}{\cup} E_Y) \subset P_L^n$ over the respective topological based spaces. Note that the fibers $\{p_x^{-1}(x_0), p_y^{-1}(y_0)\}$ are separated.

Corollary 4.3. *If the algebraic variety $Zr(f) \subset B$ has genus 1 in (Y,y_0) under the homeomorphic embedding and lifting correspondence preserving the fiber $p_y^{-1}(y_0)$ at $b_{y0} \in P_L^n$, then it forms a fundamental group $\pi_1(Y,y_0)$ and there are corresponding two group homeomorphisms given by, $h_{(YX)*}: \pi_1(Y,y_0) \to \pi_1(X,x_0)$ and $h_{(E)*}: \pi_1(E_Y,b_{y0}) \to \pi_1(E_X,b_{x0})$ under the lifting correspondences in the topological spaces.*

This property can be generalized further involving embedding within the covering sections in a projective space without lifting from the based topological spaces. The generalization under injective embeddings in P_L^n is illustrated in the commutative diagram below (Figure 4). Note that we have not considered lifting correspondences in this case. However, the homeomorphic double lifting correspondences admit this generalization within projective space if the projective space is a covering space.

$$\begin{tikzcd}
\pi_1(B,b_0) \arrow[r, hook, "i_{emb(EY)}"] \arrow[d, hook, "i_{emb(EX)}"'] & E_Y \subset P_L^n \arrow[dl, "h_{(YX)*}"] \\
E_X \subset P_L^n &
\end{tikzcd}$$

Figure 4. Fundamental group homeomorphism under embeddings in projective space.

In other words, it is illustrated in Figure 4 that the homeomorphism between fundamental groups can be formed in a projective space under the injective embeddings without involving the topological lifting correspondences within the based topological spaces. This indicates that the homeomorphism $h_{(YX)*}(.)$ can be considered the identity preserved under the morphism $\gamma_{(XY)}: (E_X \subset P_L^n) \to (E_Y \subset P_L^n)$ such that $(h_{(YX)*} \circ \gamma_{(XY)}) = Id_{EX}$. This indicates that the homeomorphisms of fundamental groups in based spaces and in a projective covering space can be treated separately. This further leads to an interesting property about the restrictions for homeomorphic lifting of a real algebraic variety under the genus-deformation if the variety is irreducible. A genus-deformation function cannot admit homeomorphic lifting although the path connecting (projective) covering spaces exist, enabling a covering map under the suitable embedding of a genus zero irreducible projective variety. This property is presented in the following theorem:

Theorem 4.6. *If* $X \subset V^o \subset P_L^n$ *is a connected smooth irreducible projective variety and the continuous* $g_{(a,b)}: X \to (Y, y_0)$ *is a genus-deformation function in a based topological space such that* $\{y_0\} = \{g(a) \sim g(b)\}$, *then there is no homeomorphic lifting* $f_{lift(g)}: (Y, y_0) \to V^o$ *for the covering map* $p_w: (\overline{V} \subset P_L^n) \to (W, w_0)$ *to the corresponding based topological space.*

Proof. Let us consider the projective irreducible and smooth real algebraic variety $Zr(f)$ for a homogeneous $f \in F[x_1, x_2, \ldots\ldots, x_n]$ such that the injective embedding $i_{emb(f)}: Zr(f) \to (V^o \subset P_L^n)$ preserves the $Hom(i_{emb}(Zr(f)), (X \subset \overline{V}))$ condition. Let us consider that $\Gamma(X) = 0$. If the continuous $g_{(a,b)}: X \to (Y, y_0)$ is a genus-deformation function such that $\Gamma(g_{(a,b)}(X)) = 1$ and $\{y_0\} = \{g(a) \sim g(b)\}$, then we can find another embedding $i_{emb(g)}: (Y, y_0) \to (W, w_0)$ into the corresponding based topological space (W, w_0). Note that it preserves the condition given by $i_{emb(g)}(y_0) = w_0$. Thus, we can find a covering map $p_w: (\overline{V} \subset P_L^n) \to (W, w_0)$ such that $p_w^{-1}(w_0) \subset X$ is a discrete fiber admitting the condition given by $(g_{(a,b)} \circ p_w^{-1})(w_0) = y_0$. Hence, a projective covering map exists. However, there is no homeomorphic lifting correspondence $f_{lift(g)}: (Y, y_0) \to V^o$ preserving $\Gamma(g_{(a,b)}(X)) > \Gamma(X)$ in the respective projective real algebraic varieties. □

Remark 4.2. *The representation of genus-deformation function admitting covering map without any homeomorphic lifting is illustrated in Figure 5.*

$$\begin{tikzcd}
Zr(f), \Gamma(Zr(f)) = 0 \arrow[d, hook, "i_{emb(f)}"] & \\
V^{o} \arrow[dr, "g_{(a,b)}"] \arrow[dd, "p_{w}"'] & \\
& (Y, y_0), \Gamma(g_{(a,b)}(X)) = 1 \arrow[dl, "i_{emb(g)}"] \\
(W, w_0) &
\end{tikzcd}$$

Figure 5. Genus-deformation function admitting covering spaces without homeomorphic lift.

Example 4.1. *Finally, we present an example of genus deformation under a covering map on a projective space. Let* $f \in R[x_1, x_2, \ldots\ldots, x_n]$ *be an irreducible polynomial generating a Zariski topological space by* $Zr(f) \subset R^3$ *such that* $\Gamma(f) = 0$*. If* $p: Zr(f) \to (B \subset R^2)$ *is a projection such that* $\exists a \in B$*, where* $p(Zr(f)) \backslash \{a\}$ *generates three topological components, then* $p: Zr(f) \to (B \subset R^2)$ *induces genus deformation, because, in this case,* $\Gamma(p(Zr(f))) = 1$*. Moreover, note that* $p(Zr(f))$ *is a reducible algebraic variety in* R^2.

5. Comparisons and applicational aspects

In general, the homotopy lift considers retention of homeomorphism under a covering map enabling fibration. In this article, the topological concepts of endomorphism and automorphism are extended to the elements in the intersection of the domains of algebraic geometry and algebraic topology. The formation of $\beta_* -$ equivalent class of polynomials in a polynomial ring is introduced in this article, admitting the non-constant morphism of corresponding real algebraic varieties. The double lifting correspondences preserving homeomorphism are introduced, retaining non-constant morphism of corresponding algebraic varieties, which is a generalization of homotopy lifting, which can be extended into a sequence of lifting correspondences involving projective spaces. In line with the existing concept of lifting of an algebraic variety, the sequence of lifting correspondences must not enable any constant morphism to preserve fibrations between projective spaces and the topological based spaces. The concept of symmetric automorphism of polynomial rings is introduced in this article and it results in the formation of fixed-points in the projective spaces under certain conditions. Note that the topological lifting correspondences preserve the fixed-point, which is a standard observation in the homotopy theory. However, the formation of fixed-point as presented in this article is distinct compared to the Hopf fixed-point and associated lifting. Moreover, it is shown in this article that there exists the genus deformation projection of algebraic varieties, which cannot have any homeomorphic lift. The Nagy-Foiaş lifting of an operator shows that a contractive and commutative operator can be projected to a commutative dilation operator in the vector spaces relating two diagonally opposite behaviors. The presented double lifting of algebraic variety within the projective spaces shows that it can admit non-constant morphism of an algebraic variety in the projective subspaces, retaining topological covering maps. Moreover, the morphism of an algebraic variety with genus one does not enforce shifting of base points of fundamental groups in topological based spaces under liftings in the respective projective spaces.

From an application point of view, the homotopy theory and lifting in the domain of algebraic topology have various applications in Computer Aided Design (CAD) and computer graphics, where continuous deformations of objects are studied [31,32]. Note that the computer simulation of object deformation in continuity is applicable in shape analysis in various engineering disciplines [31]. For example, the computable homotopy theory is applied to form algorithmic object classification in a computer vision system, where objects are binary images [33]. The homotopical image classification results in a rooted tree structure of classified binary images.

The homotopy lifting correspondences and double liftings presented in this article have potential applications in determining similarities between two objects and the pathways of deformations to convert an object into another. Moreover, the presented concept of homotopy lifting correspondences and genus deformations of algebraic varieties under projection may find applications in Computer Aided Manufacturing (CAM) involving the elements of computational geometry.

6. Conclusions

The formulations of algebraic lifting involving scheme or sheaf and the topological lifting correspondences involving covering spaces are distinct, although the concepts are almost equivalent. In this article, I introduce the endomorphism and automorphism of polynomials in a ring to form equivalence classes in the subrings under real-valued evaluations within an algebraically closed real field. The endomorphism and automorphisms within a polynomial ring and between two rings preserve the symmetry property under convergent real-valued evaluations in the real field. The formation of fixed points within the projective varieties and its topological lifting correspondence in a projective covering space are introduced. It is illustrated that if the projective covering space is compact, then there is a finite sequence of homeomorphic lifting correspondences of a projective variety, and there exist double lifting in topologically based spaces admitting a set of homeomorphisms of fundamental groups. The composition of homeomorphism of projective fundamental groups and its morphism in the projective space are identity preserving. This article shows that fibrations at fixed points in the projective varieties can preserve the projective lifting correspondences. However, if the topological morphism of a real projective variety alters the genus, then it cannot have any homeomorphic lifting correspondence, although the projective covering space admits discrete fibers. Note that the proposed topological deformation of a genus of a real projective variety and the fixed point preserving lifting correspondences are distinct compared to the Hopf fixed points and surjective Galois group deformation in a complex field involving an étale cover. This is because Hopf formulation requires the shifting of base points of the topological spaces.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest associated with this article.

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