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*Research article*

# Some properties of overpartitions into nonmultiples of two integers

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**Abstract:** We consider properties of overpartitions that are simultaneously $\ell$-regular and $\mu$-regular, where $\ell$ and $\mu$ are positive relatively prime integers. We prove a seven-way combinatorial identity related to these overpartitions. We also prove several congruence properties satisfied by this class of partitions (and a further related class) using both generating functions and modular forms with Radu's algorithm.

**Keywords:** overpartition; theta function; congruence; modular form; Radu's algorithm
**Mathematics Subject Classification:** 05A17, 11P81, 11P83

## 1. Introduction

A partition of a positive integer $n$ is a nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ that sum to $n$. The $\lambda_i$s are called parts of the partition. An overpartition of $n$ is a partition of $n$ where the first occurrence of each part size may be overlined. We denote the number of overpartitions of $n$ by $\overline{p}(n)$, with $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$, which enumerates the following overpartitions:

$$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (\overline{1},1,1).$$

The three overpartitions with no overlined parts are the ordinary partitions of 3. We will also use the alternative notation of a partition: $\lambda = (c_1^{u_1}, c_2^{u_2}, \dots, c_r^{u_r})$, where $c_1 > c_2 > \cdots > c_r > 0$, and the $u_i$s represent the multiplicities of the corresponding parts with $r \leq k$. For instance, the partition $(10, 10, 7, 7, 7, 5, 3, 2, 1, 1, 1)$ can be represented as $(10^2, 7^3, 5, 3, 2, 1^3)$.

Given a positive integer $\ell$, a partition $\lambda$ is called $\ell$-regular if no part of $\lambda$ is a multiple of $\ell$. Munagi and Sellers [10] studied combinatorial and arithmetic properties of overpartitions of $n$ with $\ell$-regular overlined parts, denoted by $A_\ell(n)$. Alanazi and Munagi [3] investigated the combinatorial identities for $\ell$-regular overpartitions of $n$, denoted by $\overline{R_\ell}(n)$. In addition, Alanazi et. al. [2] studied combinatorial and arithmetic properties of the overpartitions of $n$ with $\ell$-regular nonoverlined parts. We also note that Alanazi et. al. [4] and Shen [21] discussed various arithmetic properties of $\overline{R_\ell}(n)$. We also recall the generating function from [2],

$$\sum_{n\geq 0} \overline{R_\ell^*}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{n\ell})(1+q^n)}{1-q^n} = \frac{f_2 f_\ell}{f_1^2}, \tag{1.1}$$

where $\overline{R_\ell^*}(n)$ counts overpartitions of $n$ wherein nonoverlined parts are $\ell$-regular, and there are no restrictions on the overlined parts. Alanazi et. al. [2] proved several interesting results related to this function, some of which were recently extended by Sellers [20]. The second equality in (1.1) follows from the notation $f_n^k$ that is defined by $f_n^k := \prod_{i\geq 1}(1-q^{in})^k$ for all integers $n, k$ with $n > 0$.

In this paper, we extend the previous studies and consider properties of overpartitions that are simultaneously $\ell$-regular and $\mu$-regular, where $\ell$ and $\mu$ are positive relatively prime integers. Let $\overline{R_{\ell,\mu}}(n)$ be the number of overpartitions of $n$ with no parts divisible by $\ell$ or $\mu$, where $\gcd(\ell, \mu) = 1$. We will also call this function the number of $(\ell, \mu)$-regular overpartitions of $n$. It is easy to see that the generating function is given by

$$\sum_{n\geq 0} \overline{R_{\ell,\mu}}(n) q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)(1-q^{\ell n})(1-q^{\mu n})(1+q^{\ell\mu n})}{(1-q^n)(1+q^{\ell n})(1+q^{\mu n})(1-q^{\ell\mu n})} = \frac{f_2 f_\ell^2 f_\mu^2 f_{2\mu\ell}}{f_1^2 f_{2\ell} f_{2\mu} f_{\mu\ell}^2}. \tag{1.2}$$

This function was also recently studied by Nadji, Ahmia, and Ramirez [11], with follow-up work by Paudel, Sellers, and Wang [14], who generalized some of our results. Similar partition functions have also been studied by Ajeyakashi, Bharadwaj, and Chandankumar [1] and by Nadji, Saikia, and Sellers [12].

The paper is organized as follows. In Section 2, we prove a seven-way partition identity using generating functions and combinatorial techniques, thus establishing the equivalence of several sets of partitions. In Section 3, we prove some congruences modulo small powers of 2 satisfied by the functions $\overline{R_{\ell,\mu}}(n)$ and $\overline{R_\ell^*}(n)$. Then, in Section 4, we prove further congruences for $\overline{R_{\ell,\mu}}(n)$ and $\overline{R_\ell^*}(n)$ using modular forms, and we end the paper in Section 5 with concluding remarks.

## 2. A general partition theorem

Our first result is the following seven-way identity.

**Theorem 1.** *Assume the following notations:*

- $A_{\ell,\mu}(\ell n)$ *denotes the number of* $\mu$*-regular partitions of* $\ell n$*, where parts that are nonmultiples of* $\ell$ *appear* $\ell$ *times and parts that are multiples of* $\ell$ *appear fewer than* $\ell$ *times,*
- $B_{\ell,\mu}(2n)$ *denotes the number of* $(\ell, \mu)$*-regular partitions of* $2n$ *in which* $\ell$ *and* $\mu$ *are odd and odd parts occur with even multiplicities,*
- $C_{\ell,\mu}(2n)$ *denotes the number of* $\mu$*-regular partitions of* $2n$ *in which odd parts appear with multiplicities* $2, 4, \ldots, 2(\ell-2)$*, or* $2(\ell-1)$*, and even parts appear fewer than* $\ell$ *times, where* $\ell < \mu$ *and* $\ell$ *and* $\mu$ *are odd,*

- *$D_{\ell,\mu}(2n)$ denotes the number of $(\ell, 2\mu)$-regular partitions of $2n$ in which odd parts occur with even multiplicities and parts $\equiv \mu \pmod{2\mu}$ appear at most once, where $\mu$ is even,*
- *$E_{\ell,\mu}(2n)$ denotes the number of $(2\mu)$-regular partitions of $2n$ in which odd parts occur with multiplicities $2, 4, \ldots, 2(\ell-2)$, or $2(\ell-1)$, even parts appear fewer than $\ell$ times, and parts $\equiv \mu \pmod{2\mu}$ appear at most once, where $\ell < \mu$, and $\mu$ is even, and*
- *$F_{\ell,\mu}(\ell n)$ denotes the number of $(\ell^2, \mu)$-regular partitions of $\ell n$ in which parts that are nonmultiples of $\ell$ appear either $0$ or $\ell$ times.*

*Then,*

$$A_{\ell,\mu}(\ell n) = B_{\ell,\mu}(2n) = C_{\ell,\mu}(2n) = D_{\ell,\mu}(2n) = E_{\ell,\mu}(2n) = F_{\ell,\mu}(\ell n) = \overline{R_{\ell,\mu}}(n).$$

*Proof.* The generating function for $A_{\ell,\mu}(\ell n)$ consists of two parts as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} A_{\ell,\mu}(\ell n) q^{\ell n} &= \prod_{n=1}^{\infty} \frac{(1+q^{\ell(\ell n-1)})\cdots(1+q^{\ell(\ell n-(\ell-1))})}{(1+q^{\mu\ell(\ell n-1)})\ldots(1+q^{\mu\ell(\ell n-(\ell-1))})} \times \frac{(1+q^{\ell n}+\cdots+q^{(\ell-1)\ell n})}{(1+q^{\mu\ell n}+\cdots+q^{(\ell-1)\mu\ell n})} \\
&\quad \text{(where denominators cancel out terms with exponents divisible by } \mu) \qquad (2.1) \\
&= \prod_{n=1}^{\infty} \frac{(1+q^{\ell(\ell n-1)})\cdots(1+q^{\ell(\ell n-(\ell-1))})(1-q^{\ell^2 n})(1-q^{\mu\ell n})}{(1+q^{\mu\ell(\ell n-1)})\cdots(1+q^{\mu\ell(\ell n-(\ell-1))})(1-q^{\ell n})(1-q^{\mu\ell^2 n})} \\
&\quad \times \frac{(1+q^{\ell^2 n})(1+q^{\mu\ell^2 n})}{(1+q^{\ell^2 n})(1+q^{\mu\ell^2 n})} \\
&= \prod_{n=1}^{\infty} \frac{(1+q^{\ell n})(1-q^{\ell^2 n})(1-q^{\mu\ell n})(1+q^{\mu\ell^2 n})}{(1+q^{\mu\ell n})(1-q^{\ell n})(1-q^{\mu\ell^2 n})(1+q^{\ell^2 n})}, \qquad (2.2)
\end{aligned}$$

where the last equality follows from the complete residue-set property

$$\prod_{n=1}^{\infty}(1+q^{\ell(\ell n-1)})(1+q^{\ell(\ell n-2)})\cdots(1+q^{\ell(\ell n-(\ell-1))})(1+q^{\ell^2 n}) = \prod_{n=1}^{\infty}(1+q^{\ell n}).$$

Replacing $q^\ell$ by $q$ in (2.2) yields the generating function for $\overline{R_{\ell,\mu}}(n)$, that is, (1.2).

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{\ell,\mu}(2n) q^{2n} &= \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{2\mu n})(1-q^{2\ell(2n-1)})(1-q^{2\mu(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})(1-q^{2\ell\mu n})(1-q^{2\ell\mu(2n-1)})} \\
&= \prod_{n=1}^{\infty} \frac{(1+q^{2n})(1-q^{2\ell n})(1-q^{2\mu n})(1+q^{2\ell\mu n})}{(1-q^{2n})(1+q^{2\ell n})(1+q^{2\mu n})(1-q^{2\ell\mu n})}.
\end{aligned}$$

Then replacing $q^2$ by $q$ yields (1.2).

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{\ell,\mu}(2n) q^{2n} &= \prod_{n=1}^{\infty} \frac{(1+q^{2n}+\cdots+q^{(\ell-1)2n})(1+q^{2(2n-1)}+\cdots+q^{2(\ell-1)(2n-1)})}{(1+q^{2\mu n}+\cdots+q^{(\ell-1)2\mu n})(1+q^{2\mu(2n-1)}+\cdots+q^{2(\ell-1)\mu(2n-1)})} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{2\ell(2n-1)})(1-q^{2\mu n})(1-q^{2\mu(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})(1-q^{2\ell\mu n})(1-q^{2\ell\mu(2n-1)})}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} B_{\ell,\mu}(2n) q^{2n}. \tag{2.3}$$

The generating function of $2\mu$-regular partitions of $n$ in which odd parts occur with even multiplicities and each part $\equiv \mu \pmod{2\mu}$ appears at most once is

$$\prod_{n=1}^{\infty} \frac{1-q^{2\mu n}}{(1-q^{2n})(1+q^{2\mu n})(1-q^{2(2n-1)})}.$$

Thus, the generating function for $D_{\ell,\mu}(2n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} D_{\ell,\mu}(2n) q^{2n} &= \prod_{n=1}^{\infty} \frac{1-q^{2\mu n}}{(1-q^{2n})(1+q^{2\mu n})(1-q^{2(2n-1)})} \\ &\quad \times \frac{(1-q^{2\ell n})(1+q^{2\ell\mu n})(1-q^{2\ell(2n-1)})}{1-q^{2\ell\mu n}} \\ &= \sum_{n=0}^{\infty} B_{\ell,\mu}(2n) q^{2n}.\end{aligned}$$

Next, we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{\ell,\mu}(2n) q^{2n} &= \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})} \times \frac{(1-q^{2\mu n})}{(1-q^{2\ell\mu n})} \times \frac{1+q^{2\ell\mu n}}{1+q^{2\mu n}} \\ &= \prod_{n=1}^{\infty} \frac{(1+q^{2n})(1-q^{2\ell n})(1-q^{2\mu n})(1+q^{2\ell\mu n})}{(1-q^{2n})(1+q^{2\ell n})(1+q^{2\mu n})(1-q^{2\ell\mu n})}.\end{aligned}$$

Then replacing $q^2$ by $q$ gives (1.2).

Lastly,

$$\begin{aligned}\sum_{n=0}^{\infty} F_{\ell,\mu}(\ell n) q^{\ell n} &= \prod_{n=1}^{\infty} \frac{(1+q^{\ell(\ell n-1)})\ldots(1+q^{\ell(\ell n-(\ell-1))})(1-q^{\ell^2 n})(1-q^{\mu\ell n})}{(1+q^{\mu\ell(\ell n-1)})\ldots(1+q^{\mu\ell(\ell n-(\ell-1))})(1-q^{\ell n})(1-q^{\mu\ell^2 n})} \\ &= \prod_{n=1}^{\infty} \frac{(1+q^{\ell n})(1+q^{\mu\ell^2 n})(1-q^{\ell^2 n})(1-q^{\mu\ell n})}{(1+q^{\mu\ell n})(1+q^{\ell^2 n})(1-q^{\ell n})(1-q^{\mu\ell^2 n})}.\end{aligned}$$

Thus, replacing $q^\ell$ by $q$ yields (1.2) as desired. This completes the proof. □

## 3. Congruence properties via analytic & combinatorial techniques

Before stating and proving our results, we state the following lemmas, the proofs of which may be found in [10]. We recall the definition of Ramanujan's theta function

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1+q^{2n-1})^2(1-q^{2n}).$$

**Lemma 2.** *We have*

$$\varphi(-q^2)^2 = \varphi(q)\varphi(-q).$$

**Lemma 3.** *We have*

$$\frac{1}{\varphi(-q)} = \varphi(q)\varphi(q^2)^2\varphi(q^4)^4\cdots$$

We can rewrite the generating function of $\overline{R_{\ell,\mu}}(n)$ in terms of Ramanujan's theta function $\varphi(q)$ as follows:

$$\sum_{n\geq 0}\overline{R_{\ell,\mu}}(n)q^n = \frac{\varphi(-q^\ell)\varphi(-q^\mu)}{\varphi(-q)\varphi(-q^{\ell\mu})}. \tag{3.1}$$

Incorporating the results of Lemmas 2 and 3, we can rewrite

$$\sum_{n\geq 0}\overline{R_{\ell,\mu}}(n)q^n = \frac{\varphi(q)\varphi(q^2)^2\varphi(q^4)^4\cdots\varphi(q^{\ell\mu})\varphi(q^{2\ell\mu})^2\varphi(q^{4\ell\mu})^4\cdots}{\varphi(q^\ell)\varphi(q^{2\ell})^2\varphi(q^{4\ell})^4\cdots\varphi(q^\mu)\varphi(q^{2\mu})^2\varphi(q^{4\mu})^4\cdots}. \tag{3.2}$$

The easy corollary now follows.

**Corollary 4.** *For all $n \geq 1$, we have $\overline{R_{\ell,\mu}}(n) \equiv 0 \pmod 2$.*

*Proof.* Because $\varphi(q) = 1 + 2\sum_{n\geq 1} q^{n^2}$, we know that $\varphi(q) \equiv 1 \pmod 2$. So, (3.2) gives us

$$\sum_{n\geq 0}\overline{R_{\ell,\mu}}(n)q^n \equiv 1 \pmod 2.$$

This gives an analytic proof. □

We give a combinatorial proof as well.

*Combinatorial proof of Corollary 4.* We can obtain an overpartition by overlining the first occurrence of any distinct part of an ordinary partition $\lambda = (c_1^{u_1}, c_2^{u_2}, \ldots, c_r^{u_r})$ with $c_1 > c_2 > \cdots > c_r$. Because we may choose to either overline a part or not, the number of overpartitions obtainable from $\lambda$ is $\overline{p}(\lambda) := 2^r$. Thus, for all $n \geq 1$, $\overline{p}(n) \equiv 0 \pmod 2$. Analogously, if we consider only partitions $\lambda$ that are simultaneously $\ell$- and $\mu$-regular, we would obtain an even number of $(\ell,\mu)$-regular overpartitions. Thus, for all $n \geq 1$, we have $\overline{R_{\ell,\mu}}(n) \equiv 0 \pmod 2$. □

We now give a complete modulo 4 characterization of $\overline{R_{\ell,\mu}}(n)$.

**Theorem 5.** *For all $n \geq 1$,*

*(i) if $\ell$ is a square and $\mu$ is not, then*

$$\overline{R_{\ell,\mu}}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2 \text{ or } n = \mu k^2, \text{ where } \ell \text{ and } k \text{ are relatively prime;} \\ 0 \pmod 4 & \text{otherwise;} \end{cases}$$

*(ii) if $\ell$ and $\mu$ are both squares, then*

$$\overline{R_{\ell,\mu}}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2, \text{ where } k, \mu \text{ and } \ell \text{ are relatively prime;} \\ 0 \pmod 4 & \text{otherwise;} \end{cases}$$

*(iii) if neither $\ell$ nor $\mu$ is a square , then*

$$\overline{R_{\ell,\mu}}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2 \text{ , } n = \ell k^2 \text{ , } n = \mu k^2 \text{ or } n = \ell\mu k^2; \\ 0 \pmod 4 & \text{otherwise.} \end{cases}$$

*Analytic proof of Theorem 5.* Using (3.2), we have

$$\sum_{n\geq 0} \overline{R_{\ell,\mu}}(n)q^n \equiv \frac{\varphi(q)\varphi(q^{\ell\mu})}{\varphi(q^\ell)\varphi(q^\mu)} \pmod 4,$$

because $\varphi(q^i)^j \equiv 1 \pmod 4$ for any $j \geq 2$. Next, we deduce from Lemma 2, that

$$\varphi(q) = \frac{\varphi(-q^2)^2}{\varphi(-q)}.$$

Thus

$$\begin{aligned} \overline{R_{\ell,\mu}}(n)q^n &\equiv \frac{\varphi(q)\varphi(q^{\ell\mu})}{\varphi(q^\ell)\varphi(q^\mu)} \equiv \frac{\varphi(q)\varphi(q^\ell\mu)\varphi(-q^\ell)\varphi(-q^\mu)}{\varphi(-q^{2\ell})^2\varphi(-q^{2\mu})^2} \pmod 4 \\ &\equiv \varphi(q)\varphi(q^\ell\mu)\varphi(-q^\ell)\varphi(-q^\mu) \pmod 4, \end{aligned}$$

because $\varphi(-q^{2\ell})^2 \equiv 1 \pmod 4$.

Hence,

$$\begin{aligned} \sum_{n\geq 0} \overline{R_{\ell,\mu}}(n)q^n &\equiv \varphi(q)\varphi(q^\ell\mu)\varphi(-q^\ell)\varphi(-q^\mu) \pmod 4 \\ &= (1 + 2\sum_{n\geq 1} q^{n^2})(1 + 2\sum_{n\geq 1}(q^{\ell\mu})^{n^2})(1 + 2\sum_{n\geq 1}(-q^\ell)^{n^2})(1 + 2\sum_{n\geq 1}(-q^\mu)^{n^2}) \\ &\equiv 1 + 2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1}(q^{\ell\mu})^{n^2} + 2\sum_{n\geq 1}(-q^\ell)^{n^2} + 2\sum_{n\geq 1}(-q^\mu)^{n^2} \pmod 4 \\ &\equiv 1 + 2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1} q^{\ell\mu n^2} + 2\sum_{n\geq 1} q^{\ell n^2} + 2\sum_{n\geq 1} q^{\mu n^2} \pmod 4. \end{aligned}$$

A straightforward interpretation of the last congruence gives the three results as required. □

We give a combinatorial proof, as well.

*Combinatorial proof of Theorem 5.* Let $m(\ell|n)$ be the number of multiples $\ell$ dividing $n$ and define $\Delta(n, \ell, \mu) := \tau(n) - (m(\ell|n) + m(\mu|n) - m(\ell\mu|n))$, where $\tau(n)$ is the number of divisors of $n$. We claim that

$$\overline{R_{\ell,\mu}}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } \Delta(n, \ell, \mu) \text{ is odd;} \\ 0 \pmod 4 & \text{otherwise.} \end{cases} \tag{3.3}$$

We decompose $(\ell, \mu)$-regular overpartitions into two classes: those containing a unique part-size and those containing two or more different part-sizes. By the proof of Corollary 4, the latter class has cardinality of the form $m2^r, m > 0, r > 1$, which is divisible by 4. However, partitions with a single part-size arise from divisors of $n$. Each divisor $d$ of $n$ (excluding $\ell, \mu$, and $\ell\mu$) gives the ordinary partition $(d^{n/d})$, which in turn produces two $(\ell,\mu)$-regular overpartitions. Thus, $\Delta(n,\ell,\mu) \equiv 1 \pmod 2$ if and only if divisors of $n$ contribute an odd number of pairs of $(\ell,\mu)$-regular overpartitions. Hence, (3.3) follows.

We will use the easily proved relation $m(\ell|n) = \tau(n/\ell)$.

We consider part (i). In view (3.3), it will suffice to find the parity of $\Delta(n,\ell,\mu)$ under each constraint. If $n = k^2$, and $\ell \nmid n$, then $\tau(n)$ is odd, and $m(\ell|n) = 0$. Because both $m(\mu|n)$ and $m(\ell\mu|n)$ are even, it follows that $\Delta(n,\ell,\mu)$ is odd. Similarly, the case $\mu \nmid n$ implies that $\Delta(n,\ell,\mu)$ is odd. If $n = k^2$, and $\mu \mid n$, then $m(\mu|n)$ is even (because $\mu$ is not a square). Hence, $\Delta(n,\ell,\mu)$ is odd. However, if $n = \mu k^2$, then $\tau(n)$ is even, and $m(\mu|n) = \tau(k^2)$, which is odd. So, $\Delta(n,\ell,\mu)$ is odd. The proof of the first line of part (i) is complete.

For the second line, we consider the the following negations, given that $\ell$ is a square, and $\mu$ is not: (a) $n = k^2$, and $\ell \mid n$, and (b) $n \neq k^2$ and $n \neq \mu k^2$. In (a), we find that both $\tau(n)$ and $m(\ell|n)$ are odd. Because $m(\mu|n)$ and $m(\ell\mu|n)$ are clearly even, it follows that $\Delta(n,\ell,\mu)$ is even. In (b) it is clear that all the relevant functions are even. This completes the proof of part (i).

The other parts may be proved in a similar manner. For example, the first line of part (iii) may be established by noting that exactly one member of the set $\{\tau(n), m(\ell|n), m(\mu|n), m(\ell\mu|n)\}$ is odd at a time with all the others being even. □

We close this section by proving a general mod 8 congruence for the $\overline{R}^*_\ell(n)$ function. We will need the following lemma.

**Lemma 6.** *We have*

$$\sum_{n\geq 0} \overline{R_6^*}(3n+2)q^n \equiv 4f_6^3 \pmod 4.$$

*Proof.* We need the following identity [7, Theorem 1]:

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{3.4}$$

Using (3.4) in (1.1), we have the following:

$$\sum_{n\geq 0} \overline{R_6^*}(3n+2)q^n = 4\frac{f_2^3 f_6^3}{f_1^6} \equiv 4\frac{f_1^6 f_6^3}{f_1^6} \equiv 4f_6^3 \pmod 4.$$

□

**Theorem 7.** *Let $p \geq 5$ be a prime, and let $r$ with $1 \leq r \leq p-1$ be such that* $\mathrm{inv}(3,p)\cdot 4\cdot r + 1$ *is a quadratic nonresidue modulo $p$, where* $\mathrm{inv}(3,p)$ *is the inverse of* 3 *modulo $p$. Then, for all $n \geq 0$ we have* $\overline{R_6^*}(3(pn+r)+2) \equiv 0 \pmod 8$.

*Proof.* We need Jacobi's triple product identity

$$f_1^3 = \sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)/2}. \tag{3.5}$$

From Lemma 6 and (3.5), we have

$$\sum_{n\geq 0}\overline{R_6^*}(3n+2)q^n \equiv 4\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{3j(j+1)} \pmod{8}.$$

We are interested in values of the form $\overline{R_6^*}(3(pn+r)+2)$, and we want to know whether we can have $pn+r = 3j(j+1)$ for some non-negative integer $j$. If such a representation for $pn+r$ exists, then $r \equiv 3j(j+1) \pmod{p}$. Because $p \geq 3$, this is equivalent to $\mathrm{inv}(3,p)\cdot r \equiv j(j+1) \pmod{p}$, which in turn is equivalent to $\mathrm{inv}(3,p)\cdot 4\cdot r+1 \equiv (2j+1)^2 \pmod{p}$. From our assumption that $\mathrm{inv}(3,p)\cdot 4\cdot r+1$ is a quadratic nonresidue modulo $p$, the result now follows. □

## 4. Congruence properties via modular forms

In this section, we will use the theory of modular forms to find several congruences. The first result is given below.

**Theorem 8.** *For all $n \geq 0$, we have*

$$\overline{R_{2,3}}(9n+6) \equiv 0 \pmod{6}, \tag{4.1}$$
$$\overline{R_{4,3}}(6n+3) \equiv 0 \pmod{6}, \tag{4.2}$$
$$\overline{R_{4,3}}(6n+5) \equiv 0 \pmod{12}, \tag{4.3}$$
$$\overline{R_{4,3}}(9n+3) \equiv 0 \pmod{6}, \tag{4.4}$$
$$\overline{R_{4,3}}(12n+7) \equiv 0 \pmod{24}, \tag{4.5}$$
$$\overline{R_{4,3}}(12n+11) \equiv 0 \pmod{72}, \tag{4.6}$$
$$\overline{R_{4,9}}(8n+4) \equiv 0 \pmod{12}, \tag{4.7}$$
$$\overline{R_{4,9}}(12n+4) \equiv 0 \pmod{12}, \tag{4.8}$$
$$\overline{R_{4,9}}(12n+8) \equiv 0 \pmod{72}, \tag{4.9}$$
$$\overline{R_{4,9}}(16n+8) \equiv 0 \pmod{24}, \tag{4.10}$$
$$\overline{R_{4,9}}(18n+12) \equiv 0 \pmod{96}, \tag{4.11}$$
$$\overline{R_{4,9}}(24n+20) \equiv 0 \pmod{216}, \tag{4.12}$$
$$\overline{R_{4,9}}(18n+15) \equiv 0 \pmod{48}, \tag{4.13}$$
$$\overline{R_{4,9}}(96n+80) \equiv 0 \pmod{864}, \tag{4.14}$$
$$\overline{R_{8,27}}(36n+15) \equiv 0 \pmod{24}, \tag{4.15}$$
$$\overline{R_{8,27}}(36n+21) \equiv 0 \pmod{96}, \tag{4.16}$$
$$\overline{R_{8,27}}(36n+24) \equiv 0 \pmod{12}, \tag{4.17}$$
$$\overline{R_{8,27}}(36n+27) \equiv 0 \pmod{6}, \tag{4.18}$$
$$\overline{R_{16,81}}(36n+33) \equiv 0 \pmod{48}, \tag{4.19}$$
$$\overline{R_{16,81}}(72n+60) \equiv 0 \pmod{48}. \tag{4.20}$$

**Theorem 9.** *For all $n \geq 0$, we have*

$$\overline{R_3^*}(9n+4) \equiv 0 \pmod{12}, \tag{4.21}$$
$$\overline{R_3^*}(9n+7) \equiv 0 \pmod{48}, \tag{4.22}$$
$$\overline{R_6^*}(9n+5) \equiv 0 \pmod{24}, \tag{4.23}$$
$$\overline{R_6^*}(9n+8) \equiv 0 \pmod{96}. \tag{4.24}$$

**Remark 10.** *Alanazi et. al. [2] had obtained*

$$\overline{R_3^*}(9n+4) \equiv \overline{R_3^*}(9n+7) \equiv 0 \pmod{3}.$$

*Whearas Sellers [20] had very recently obtained*

$$\overline{R_3^*}(9n+4) \equiv \overline{R_3^*}(9n+7) \equiv 0 \pmod{4}.$$

*Our proof of the first congruence in Theorem 9 is independent of the techniques used in the proofs of the results of Alanazi et al. and Sellers.*

We can also get congruenes for finer arithmetic progressions; we list only two of them without proof here.

**Theorem 11.** *For all $n \geq 0$, we have*

$$\overline{R_6^*}(27n+11) \equiv 0 \pmod{64}, \tag{4.25}$$
$$\overline{R_6^*}(81n+47) \equiv 0 \pmod{24}. \tag{4.26}$$

Theorems 8 and 9 are proved in Section 4.4 using an algorithmic approach, due to Smoot [22].

The following two congruences will be proved using a manual implementation of an algorithm of Radu [16, 17], due to computational difficulties.

**Theorem 12.** *For all $n \geq 0$, we have*

$$\overline{R_{3,5}}(9n+3) \equiv 0 \pmod{6}, \tag{4.27}$$
$$\overline{R_{2,5}}(18n+9) \equiv 0 \pmod{6}. \tag{4.28}$$

We prove Theorem 12 in Section 4.5.

### *4.1. Preliminaries on the algorithmic approach*

In this section, we describe our methods for proving Theorems 8, 9, and 12. Theorems 8 and 9 are proved using a Mathematica implementation of an algorithm of Radu [16, 17] due to Smoot [22], and Theorem 12 is proved manually using Radu's algorithm. We first describe Radu's algorithm in Subsection 4.2 and then Smoot's implementation in Subsection 4.3.

Before proceeding further, we give a brief summary of the techniques here.

1. **Radu's Ramanujan-Kolberg algorithm:** The primary computational engine used in this section is an algorithm developed by Radu [16, 17]. The algorithm translates the problem of proving partition congruences into a finite, computationally verifiable number of checks. It does this by analyzing the generating functions of the partitions within the framework of the theory of modular forms, specifically by checking the order and bounds of modular functions at the cusps of congruence subgroups like $\Gamma_0(N)$.
2. **Smoot's Mathematica implementation:** Because Radu's algorithm requires heavy algebra and modular function manipulations, we use this in only one case, that of Theorem 12. For the other results, we use Smoot's Mathematica implementation [22] of Radu's algorithm (via the `RaduRK` Mathematica package).

### *4.2. Radu's algorithm*

For a positive integer $N$, we define the following matrix groups:

$$\Gamma := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \Gamma : n \in \mathbb{Z} \right\}.$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\},$$

and

$$[\Gamma : \Gamma_0(N)] := N \prod_{\ell | N} \left(1 + \frac{1}{\ell}\right),$$

where $\ell$ is a prime.

We need some preliminary results, which describe an algorithmic approach to proving partition concurrences, developed by Radu [16, 17]. For integers $M \geq 1$, suppose that $R(M)$ is the set of all the integer sequences

$$(r_\delta) := (r_{\delta_1}, r_{\delta_2}, r_{\delta_3}, \ldots, r_{\delta_k})$$

indexed by all the positive divisors $\delta$ of $M$, where $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$. For integers $m \geq 1$, $(r_\delta) \in R(M)$, and $t \in \{0, 1, 2, \ldots, m-1\}$, we define the set $P(t)$ as

$$P(t) := \Bigg\{ t' \in \{0, 1, 2, \ldots, m-1\} : t' \equiv ts + \frac{s-1}{24} \sum_{\delta | M} \delta r_\delta \pmod{m} \text{ for some } [s]_{24m} \in \mathbb{S}_{24m} \Bigg\}, \tag{4.29}$$

where $[x]_m$ denotes the residue class of $x$, $\mathbb{Z}_m^*$ denotes the set of the invertible elements of $\mathbb{Z}_m$, and $\mathbb{S}_m$ denotes the set of the squares of $\mathbb{Z}_m^*$.

For integers $N \geq 1$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(r_\delta) \in R(M)$, and $(r'_\delta) \in R(N)$, we also define

$$p(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \frac{\gcd(\delta(a + k\lambda c), mc)^2}{\delta m},$$

$$p'(\gamma) := \frac{1}{24}\sum_{\delta|N} r'_\delta \frac{\gcd(\delta,c)^2}{\delta}.$$

For integers $m \geq 1$, $M \geq 1$, $N \geq 1$, $t \in \{0, 1, 2, \ldots, m-1\}$, $k := \gcd\left(m^2-1, 24\right)$, and $(r_\delta) \in R(M)$, define $\Delta^*$ to be the set of all tuples $(m, M, N, t, (r_\delta))$ such that all of the following conditions are satisfied.

1. Prime divisors of $m$ are also prime divisors of $N$;
2. If $\delta \mid M$, then $\delta \mid mN$ for all $\delta \geq 1$ with $r_\delta \neq 0$;
3. $24 \mid kN \sum_{\delta|M} \frac{r_\delta mN}{\delta}$;
4. $8 \mid kN \sum_{\delta|M} r_\delta$;
5. $\frac{24m}{\left(-24kt - k\sum_{\delta|M}\delta r_\delta, 24m\right)} \mid N$;
6. If $2|m$ then either $4|kN$ and $8|\delta N$ or $2|s$ and $8|(1-j)N$, where $\prod_{\delta|M}\delta^{|r_\delta|} = 2^s \cdot j$.

We now state a result of Radu [17], which we use in completing the proof of Theorem 8.

**Lemma 13.** [17, Lemma 4.5] *Suppose that* $(m, M, N, t, (r_\delta)) \in \Delta^*$, $(r'_\delta) := (r'_\delta)_{\delta|N} \in R(N)$, $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ *is a complete set of representatives of the double cosets of* $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$*, and* $t_{\min} := \min_{t' \in P(t)} t'$,

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N}\delta r'_\delta - \frac{1}{m}\sum_{\delta|M}\delta r_\delta\right) - \frac{t_{min}}{m}, \tag{4.30}$$

$p(\gamma_j) + p'(\gamma_j) \geq 0$ *for all* $1 \leq j \leq n$*, and* $\sum_{n=0}^{\infty} A(n)q^n := \prod_{\delta|M} f_\delta^{r_\delta}$*. If for some integers* $u \geq 1$*, all* $t' \in P(t)$*, and* $0 \leq n \leq \lfloor \nu \rfloor$, $A(mn+t') \equiv 0 \pmod{u}$ *is true, then for integers* $n \geq 0$ *and all* $t' \in P(t)$*, we have* $A(mn+t') \equiv 0 \pmod{u}$.

The following lemma supports Lemma 13 in the proof of Theorem 8.

**Lemma 14.** [17, Lemma 2.6] *Let* $N$ *or* $N/2$ *be a square-free integer so that we have*

$$\bigcup_{\delta|N}\Gamma_0(N)\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\Gamma_\infty = \Gamma.$$

### *4.3. Smoot's implementation of Radu's algorithm*

We will also use Smoot's [22] implementation of Radu's algorithm [16, 17], which can be used to prove Ramanujan-type congruences. The algorithm takes as an input the generating function

$$\sum_{n=0}^{\infty} a_r(n)q^n = \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta},$$

and positive integers $m$ and $N$, where $M$ is another positive integer and $(r_\delta)_{\delta|M}$ is a sequence indexed by the positive divisors $\delta$ of $M$. With this input, Radu's algorithm tries to produce a set $P_{m,j}(j) \subseteq \{0, 1, \ldots, m-1\}$ which contains $j$ and is uniquely defined by $m, (r_\delta)_{\delta|M}$, and $j$. Then, it decides if there exists a sequence $(s_\delta)_{\delta|N}$ such that

$$q^\alpha \prod_{\delta|M} \prod_{n=1}^{\infty} (1-q^{\delta n})^{s_\delta} \cdot \prod_{j' \in P_{m,j}(j)} \sum_{n=0}^{\infty} a(mn+j')q^n$$

is a modular function with certain restrictions on its behavior on the boundary of $\mathbb{H}$.

Smoot [22] implemented this algorithm in Mathematica, and we use his `RaduRK` package, which requires the software package `4ti2`. Documentation on how to install and use these packages is available from Smoot [22]. We use this implemented `RaduRK` algorithm to prove Theorem 8 in the next section.

It is natural to guess that $N = m$ (which corresponds to the congruence subgroup $\Gamma_0(N)$), but this is not always the case, although they are usually closely related to one another. The determination of the correct value of $N$ is an important problem for the usage of `RaduRK`, and it depends on the $\Delta^*$ criterion described in the previous subsection. It is easy to check the minimum $N$, which satisfies this criterion, by running `minN[M, r, m, j]`.

### *4.4. Proofs of Theorems 8 and 9*

In this section, we prove Theorems 8 and 9 using Smoot's implementation described in Subsection 4.3.

*Proof of Theorem 8.* Because the proof of all congruences listed in Theorem 8 are similar, we only prove (4.1) here. The rest of the output can be obtained by visiting `https://manjilsaikia.in/publ/mathematica/smoot_lm.nb`. We use the procedure call

```
RK[12,12,{-2,3,2,-1,-3,1},9,6]
```

which gives a straight proof of (4.1). Here, we give the output of `RK`.

```
In[1] := RaduRK[12,12,{-2,3,2,-1,-3,1},9,6]
```

$$\prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n)\, q^n$$

$$\boxed{f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j')\, q^n = \sum_{g \in AB} g \cdot p_g(t)}$$

Modular Curve: $X_0(N)$

```
Out[2] =
```

| N: | 12 |
|---|---|
| {M,$(r_\delta)_{\delta\|M}$}: | {12, {−2, 3, 2, −1, −3, 1}} |
| m: | 9 |
| $P_{m,r}$(j): | {6} |
| $f_1$(q): | $\frac{(q;q)_\infty^5 \left(q^4;q^4\right)_\infty^2 \left(q^6;q^6\right)_\infty^9}{q^3 \left(q^2;q^2\right)_\infty \left(q^3;q^3\right)_\infty^3 \left(q^{12};q^{12}\right)_\infty^{12}}$ |
| t: | $\frac{\left(q^4;q^4\right)_\infty^4 \left(q^6;q^6\right)_\infty^2}{q \left(q^2;q^2\right)_\infty^2 \left(q^{12};q^{12}\right)_\infty^4}$ |
| AB: | {1} |
| $\left\{p_g(\text{t}):\ \text{g}\in\text{AB}\right\}$ | $\left\{6t^3+6t^2-6t-6\right\}$ |
| Common factor: | 6 |

The interpretation of this output is as follows.

The first entry in the procedure call `RK[12,12,{-2,3,2,-1,-3,1},9,6]` corresponds to specifying $N = 12$, which fixes the space of modular functions

$$M(\Gamma_0(N)) := \text{the algebra of modular functions for } \Gamma_0(N).$$

The second and third entries of the procedure call `RK[12,12,{-2,3,2,-1,-3,1},9,6]` give the assignment $\{M, (r_\delta)_{\delta|M}\} = \{12, (-2, 3, 2, -1, -3, 1)\}$, which corresponds to specifying $(r_\delta)_{\delta|M} = (r_1, r_2, r_4) = (-2, 3, 2, -1, -3, 1)$, so that

$$\sum_{n\geq 0} \overline{R}_{2,3}(n)q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \frac{f_2^3 f_3^2 f_{12}}{f_1^2 f_4 f_6^3}.$$

The last two entries of the procedure call `RK[12,12,{-2,3,2,-1,-3,1},9,6]` correspond to the assignment $m = 8$ and $j = 7$, which means that we want the generating function

$$\sum_{n\geq 0} \overline{R}_{2,3}(n)(mn + j)q^n = \sum_{n\geq 0} \overline{R}_{2,3}(n)(9n + 6)q^n.$$

So, $P_{m,r}(j) = P_{9,r}(6)$ with $r = (-2, 3, 2, -1, -3, 1)$.

The output $P_{m,r}(j) := P_{9,(-2,3,2,-1,-3,1)}(6) = \{6\}$ means that there exists an infinite product

$$f_1(q) = \frac{(q;q)_\infty^5 \left(q^4;q^4\right)_\infty^2 \left(q^6;q^6\right)_\infty^9}{q^3 \left(q^2;q^2\right)_\infty \left(q^3;q^3\right)_\infty^3 \left(q^{12};q^{12}\right)_\infty^{12}},$$

such that

$$f_1(q) \sum_{n\geq 0} \overline{R}_{2,3}(n)(9n + 6)q^n \in M(\Gamma_0(12)).$$

Finally, the output

$$t = \frac{\left(q^4;q^4\right)_\infty^4 \left(q^6;q^6\right)_\infty^2}{q \left(q^2;q^2\right)_\infty^2 \left(q^{12};q^{12}\right)_\infty^4}, \quad AB = \{1\}, \quad \text{and} \quad \{p_g(\text{t}):\ \text{g}\in AB\},$$

presents a solution to the question of finding a modular function $t \in M(\Gamma_0(12))$ and polynomials $p_g(t)$ such that

$$f_1(q)\sum_{n\geq 0}\overline{R}_{2,3}(9n+6)q^n = \sum_{g\in AB} p_g(t)\cdot g.$$

In this specific case, we see that the singleton entry in the set $\{p_g(\mathrm{t})\colon \mathrm{g}\in AB\}$ has the common factor 6, thus proving Eq (4.1). □

**Remark 15.** *The interested reader can refer to [15] or [5] for some more recent applications of the method.*

*Proof of Theorem 9.* Because the proof of Theorem 9 is similar to the proof of Theorem 8, we simply refer the reader to the output file available at `https://manjilsaikia.in/publ/mathematica/smoot_rl.nb`. □

### *4.5. Proof of Theorem 12*

We use the material in Subsection 4.2 without commentary. We give the full details for the proof of (4.27), and mention the parameters in Radu's algorithm for (4.28).

For the purposes of (4.27), it is enough to take

$$(m, M, N, t, (r_\delta)) = (9, 30, 30, 3, (-2, 1, 2, 2, -1, -1, -2, 1)).$$

It is routine to check that this choice satisfies the $\Delta^*$ conditions. By Eq (4.29), we see that $P(t) = \{3\}$. For the choice of $(r'_\delta) = (12, 0, 0, 0, 0, 0, 0, 0)$, we see that

$$p\left(\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\right) + p'\left(\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\right) \geq 0 \quad \text{for all } \delta \mid N,$$

and $\lfloor \nu \rfloor = 101$ for $t = 3$. We therefore only need to check the validity of (4.27) for all $n \leq \lfloor \nu \rfloor$, and then by Lemmas 13 and 14, we would have proved our result. This can be checked using Mathematica, and hence, the result follows.

For (4.28), we take

$$(m, M, N, t, (r_\delta)) = (18, 20, 60, 9, (-2, 3, -1, 2, -3, 1)),$$

which will give us

$$P(t) = \{9\}, \quad (r'_\delta) = (54, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad \text{and} \quad \lfloor \nu \rfloor = 1322.$$

We can then verify the congruence up to $n = 1322$ via Mathematica from which the result follows. □

## 5. Concluding remarks

1. It is possible to give a combinatorial proof of Theorem 1.
2. A modulo 8 characterization in the same vein as the modulo 4 characterization in Section 3 is also possible. We leave it as an open problem.
3. It is desirable to have elementary proofs of the results in Section 4.
4. A further study of congruences is also desired. To that end, we make the following conjecture.

   **Conjecture 16.** *For all $n \geq 0$, $\ell \geq 2$, and $1 \leq k \leq \ell$, we have*

   $$\overline{R_{4,9}}(4\ell n + 4k) \equiv 0 \pmod 6.$$

## Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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