Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9845–9875.
DOI: 10.3934/math.2026407
Received: 12 February 2026
Revised: 31 March 2026
Accepted: 08 April 2026
Published: 13 April 2026

Research article

$\mathfrak{B}$-statistical core and ideal core of double sequences in 2-normed spaces via RH-regular families

Hong-Zhen Yu[1], **Ömer Kişi**[2], **Mehmet Gürdal**[3] **and Qing-Bo Cai**[4,*]

[1] Business College, Jiangxi Institute of Fashion Technology, Jiangxi 330201, China
[2] Department of Mathematics, Bartın University, Bartın, Turkey
[3] Department of Mathematics, Suleyman Demirel University, Isparta 32260, Turkey
[4] School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China

* **Correspondence:** Email: qbcai@126.com.

Abstract: We investigated geometric limit sets of double sequences in finite dimensional 2-normed spaces when negligibility of index sets was measured by a density induced by a nonnegative Robison–Hamilton (RH)-regular family $\mathfrak{B}$ of four-dimensional matrices. First, using the $\mathfrak{B}$-density, we defined $\mathfrak{B}$-statistical limit superior and limit inferior for the scalar reductions generated by the seminorms $x \mapsto \|x, u\|$, and we studied the associated real cluster behavior. Next, we introduced $\mathfrak{B}$-statistical cluster points and the $\mathfrak{B}$-statistical core of a double sequence as the intersection of all closed convex sets that contained the sequence outside a $\mathfrak{B}$-density zero set. We obtained a disk-type representation of the core via intersections of sets of the form $\{x \in X : \|x - z, u\| \leq r\}$, where the radii were governed by $\mathfrak{B}$-statistical lim sup. We also proved a Knopp-type inclusion theorem: for a family of transforms satisfying a natural $\mathfrak{B}$-regularity condition, the Knopp core of each transform was contained in the $\mathfrak{B}$-statistical core of the original sequence. Finally, replacing $\mathfrak{B}$-density zero sets by a strongly admissible ideal $\mathcal{I}_2$ on $\mathbb{N}\times\mathbb{N}$, we defined ideal cores, established disk representations, compared the density-based and ideal cores, and identified an explicit condition under which the $\mathcal{I}_2$-core and the $\mathcal{I}_2^*$-core coincided.

Keywords: double sequences; statistical convergence; cluster points; core; ideal convergence; RH-regular matrices; 2 normed spaces
Mathematics Subject Classification: 40A35, 40C05, 46A45

1. Introduction

The concept of statistical convergence was introduced by Fast in 1951 [9], and it quickly became a useful alternative to ordinary convergence because it allows one to ignore a sparse exceptional set while keeping a quantitative control through density. After the pioneering definition, several matrix generated variants were developed by employing a regular nonnegative summability matrix, usually denoted by $\mathfrak{A}$, and these methods provided a natural bridge between summability theory and density-based convergence [4,8,17]. In a different direction, Steiglitz [32] proposed $\mathfrak{B}$-summability for families of matrices, and Kolk [17] used this family approach to formulate $\mathfrak{B}$-statistical convergence. The family setting is particularly convenient when one needs uniform control with respect to an auxiliary parameter, and it also allows one to treat many classical methods under a single umbrella.

The extension of statistical ideas to double sequences was initiated independently by Tripathy [33] and by Mursaleen and Edely [25]. The two-dimensional index set requires a careful choice of the underlying convergence notion and Pringsheim convergence plays the role of the reference point [27]. Regularity for four-dimensional matrix transformations was described by Hamilton [16] and Robison [28], and these criteria support the use of Robison–Hamilton (RH)-regular matrices as stable operators acting on double sequences. The matrix characterizations and further refinements for $\mathfrak{A}$-statistical methods in the double sequence setting were investigated in various directions, including the $\mathfrak{A}$-summability framework [8, 26, 29]. Many additional developments for generalized statistical convergence of double sequences, including ideal-based and weighted variants, can be found in [1, 18, 21]; see also [22–24] for approximation-theoretic applications.

A complementary viewpoint in this area is given by cluster point theory and core theory, where one replaces a single limit value by a geometric limit set that captures the asymptotic behavior of a bounded sequence. For matrix generated statistical methods, Demirci developed the theory of $\mathfrak{A}$-statistical cluster points and the $\mathfrak{A}$-statistical core for single sequences and obtained structural results that connect cluster sets with extremal limit values and convexity [6, 7]. Ideal-based refinements of cluster concepts were later considered by Gürdal and Savaş in the setting of A-cluster points via ideals [11]. These contributions indicate that the core and cluster point perspectives provide a robust geometric language that remains compatible with classical summability while capturing phenomena that are invisible to pointwise limits.

A second line of research arises from convergence defined through smallness with respect to an ideal or, equivalently, largeness with respect to the associated filter. Bernstein [2] introduced a filter based compactness notion that motivated later convergence concepts. Kostyrko, Šalát, and Wilczyński [19] developed the modern theory of $\mathcal{I}$-convergence and clarified its relation with summability methods, and Das et al. [5] extended ideal convergence to double sequences in metric spaces and studied its basic properties. Kumar [20] investigated $\mathcal{I}$ and $\mathcal{I}^*$ convergence for double sequences and related inclusion phenomena. This framework has been extended further in the presence of additional structures such as random 2-norms and random n-norms [13, 34] and it provides an effective language for separating negligible index sets from the dominant part of a sequence.

In normed type settings, statistical and ideal approaches interact naturally with geometric features. In particular, convergence notions in 2-normed spaces are governed by seminorms of the form $x \mapsto \|x, u\|$ for fixed nonzero directions u, and this makes it possible to express convergence, boundedness, and localization in a way that is sensitive to the underlying geometry. Ideal and statistical convergence phenomena in 2-normed spaces were studied in [35], while ideal characterizations for completions and related topics in n-normed spaces appear in [30]. Further developments connecting statistical ideas with operator theoretic contexts and localization can be found in [14, 15, 36], and statistical convergence has also been considered in alternative uncertainty type frameworks such as credibility spaces [31]. These studies suggest that the combination of matrix methods, density notions, and ideals yields a flexible yet stable toolkit for treating convergence beyond the classical regime.

In this paper we develop a core and cluster point theory for double sequences in finite dimensional 2-normed spaces under densities generated by a nonnegative RH-regular family $\mathfrak{B}$ of four-dimensional matrices. After introducing the associated notion of $\mathfrak{B}$-smallness on $\mathbb{N} \times \mathbb{N}$, we study $\mathfrak{B}$-statistical limit superior and limit inferior for the scalar reductions induced by the seminorms $x \mapsto \|x, u\|$, and then define the corresponding real and vector-valued cluster structures. This leads naturally to the $\mathfrak{B}$-statistical core, which we describe by a disk-type representation and relate to the convex hull of the cluster set.

We then pass to the ideal setting by replacing $\mathfrak{B}$-density zero exceptional sets with membership in a strongly admissible ideal on $\mathbb{N}\times\mathbb{N}$. In this way we obtain ideal versions of the core, compare them with the density-based construction, and establish an inclusion theorem of Knopp type for suitable matrix transforms.

The paper is organized as follows. In the next section we recall basic facts on 2-normed spaces, Pringsheim convergence, and RH-regular families, and we introduce the density generated by $\mathfrak{B}$. We then develop $\mathfrak{B}$-statistical limit superior and limit inferior and the corresponding cluster point theory. After that we introduce the $\mathfrak{B}$-statistical core, prove its fundamental properties and derive the disk-type representation. Finally, we study the ideal versions of the core and establish the main inclusion results and an explicit criterion for the coincidence of the $\mathcal{I}_2$-core and the $\mathcal{I}_2^*$-core.

2. Preliminaries

Let $\mathbb{N}$ be the set of positive integers. Throughout this paper, X denotes a real or complex vector space. A double sequence in X is a mapping $\rho\colon \mathbb{N} \times \mathbb{N} \to X$, and we write $\rho = (\rho_{k,l})_{k,l\in\mathbb{N}}$. The basic notion of convergence for double sequences is the Pringsheim convergence introduced by Pringsheim [27]. If $\mathfrak{h} \in X$, then ρ is said to converge to $\mathfrak{h}$ in the Pringsheim sense if for every neighborhood U of 0 there exists $N \in \mathbb{N}$ such that $\rho_{k,l} - \mathfrak{h} \in U$ whenever $k \geq N$ and $l \geq N$. We denote this limit by $P\lim_{k,l\to\infty} \rho_{k,l} = \mathfrak{h}$.

We work in the framework of 2-normed spaces. A 2-normed space is a pair $(X, \|\cdot, \cdot\|)$ with $\dim X \geq 2$ where $\|\cdot, \cdot\|\colon X \times X \to [0, \infty)$ satisfies the standard axioms. The value $\|x, y\|$ equals zero if, and only if, x and y are linearly dependent, symmetry holds, homogeneity holds with respect to scalars, and the triangle inequality holds in each argument in the form $\|x+y, z\| \leq \|x, z\|+\|y, z\|$. This structure has been used extensively in statistical and ideal convergence problems, including $\mathcal{I}$-statistical convergence in 2-normed spaces and related extensions [13, 35]. For each nonzero $u \in X$, we set $p_u(x) = \|x, u\|$. The family $(p_u)_{u\neq 0}$ defines a locally convex topology on X, and convergence in the 2-norm topology means convergence with respect to every seminorm p_u.

Matrix transformations of double sequences are generated by four-dimensional matrices. Let $A = (a_{m,n,k,l})$ be a four-dimensional matrix with scalar entries. Whenever the series converges we define

$$(A\rho)_{m,n} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l}\rho_{k,l}.$$

A four-dimensional matrix is called RH-regular if it preserves Pringsheim convergence of bounded scalar double sequences and preserves the corresponding limit. A Silverman–Toeplitz type characterization for this regularity was established by Hamilton [16] and Robison [28]. We use the following conditions, which are standard in this context [16, 28]. For each fixed (k, l), one has

$$P\lim_{m,n\to\infty} a_{m,n,k,l} = 0,$$

and one has

$$P\lim_{m,n\to\infty} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{m,n,k,l} = 1.$$

Moreover, for each fixed l, one has

$$P\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0,$$

and for each fixed k, one has

$$P\lim_{m,n\to\infty} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0.$$

Finally, the boundedness condition

$$\sup_{m,n} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |a_{m,n,k,l}| < \infty$$

is imposed, and together these assumptions guarantee RH-regularity in the usual sense [16, 28].

We also use matrix families in the sense of Steiglitz [32]. This viewpoint is useful when uniformity with respect to an auxiliary index is required and it has been employed by Kolk in the study of statistical convergence induced by a family of matrices [17]. Let J be a nonempty index set and let $\mathfrak{B} = (B_j)_{j\in J}$ be a family of four-dimensional matrices, where $B_j = (b_{m,n,k,l}(j))$. We assume that each B_j is nonnegative and RH-regular and that the constants appearing in the RH-regularity conditions can be chosen uniformly in j. This uniformity is the natural analogue of the family regularity conditions used for one-dimensional matrix families [17, 32].

The family $\mathfrak{B} = (B_j)_{j\in J}$ induces a density on $\mathbb{N}\times\mathbb{N}$ through characteristic functions, in the same spirit as matrix-generated densities in the classical statistical summability theory.

Definition 2.1 ($\mathfrak{B}$-density)**.** Let $E \subseteq \mathbb{N}\times\mathbb{N}$, and let χ_E denote its characteristic function. We say that E has $\mathfrak{B}$-density d and write

$$\delta_{\mathfrak{B}}(E) = d$$

if

$$P\lim_{m,n\to\infty} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{m,n,k,l}(j)\chi_E(k,l) = d$$

uniformly in $j \in J$. If the above limit does not exist, we define the upper and lower $\mathfrak{B}$-densities by

$$\overline{\delta}_{\mathfrak{B}}(E) = P\limsup_{m,n\to\infty} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{m,n,k,l}(j)\chi_E(k,l),$$

and

$$\underline{\delta}_{\mathfrak{B}}(E) = P\liminf_{m,n\to\infty} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{m,n,k,l}(j)\chi_E(k,l),$$

again uniformly in $j \in J$. We say that E has $\mathfrak{B}$-density zero if $\overline{\delta}_{\mathfrak{B}}(E) = 0$. A property is said to hold for $\mathfrak{B}$ almost all indices if it fails only on a set of $\mathfrak{B}$-density zero.

Definition 2.2 (Strongly admissible ideal)**.** An ideal $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ is called strongly admissible if

(i) $\mathcal{I}_2$ is nonempty and proper,
(ii) $\mathcal{I}_2$ is hereditary and closed under finite unions,
(iii) every singleton belongs to $\mathcal{I}_2$,
(iv) each row $\{i\}\times\mathbb{N}$ and each column $\mathbb{N}\times\{i\}$ belongs to $\mathcal{I}_2$ for every $i \in \mathbb{N}$.

The associated filter is

$$\mathcal{F}(\mathcal{I}_2) = \{F \subseteq \mathbb{N}\times\mathbb{N} : (\mathbb{N}\times\mathbb{N}) \setminus F \in \mathcal{I}_2\}.$$

Lemma 2.3. *Let $\mathfrak{B} = (B_j)_{j\in J}$ be a nonnegative RH-regular family of four-dimensional matrices. Then, every fixed row*

$$R_i = \{i\}\times\mathbb{N}$$

and every fixed column

$$C_i = \mathbb{N}\times\{i\}$$

has $\mathfrak{B}$-density zero. Consequently, every finite union of rows and columns has $\mathfrak{B}$-density zero.

Proof. Fix $i \in \mathbb{N}$. For the row R_i, one has

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{m,n,k,l}(j)\chi_{R_i}(k,l) = \sum_{l=1}^{\infty} b_{m,n,i,l}(j).$$

By the Hamilton–Robison conditions and the uniform RH-regularity assumption, the righthand side tends to 0 in the Pringsheim sense uniformly in j. Hence, $\overline{\delta}_{\mathfrak{B}}(R_i) = 0$. The column case is analogous. The final assertion follows from subadditivity of $\overline{\delta}_{\mathfrak{B}}$. □

Proposition 2.4. *For every set $A \subseteq \mathbb{N}\times\mathbb{N}$, one has*

$$\underline{\delta}_{\mathfrak{B}}(A^c) \geq 1 - \overline{\delta}_{\mathfrak{B}}(A).$$

In particular, if $\overline{\delta}_{\mathfrak{B}}(A) = 0$, then

$$\underline{\delta}_{\mathfrak{B}}(A^c) = 1.$$

Proof. Since $\chi_{A^c} = 1 - \chi_A$, we have

$$\sum_{k,l} b_{m,n,k,l}(j)\chi_{A^c}(k,l) = \sum_{k,l} b_{m,n,k,l}(j) - \sum_{k,l} b_{m,n,k,l}(j)\chi_A(k,l).$$

Taking lower Pringsheim limits on the left and using the RH-regularity relation

$$P\lim_{m,n\to\infty} \sum_{k,l} b_{m,n,k,l}(j) = 1$$

uniformly in j, we obtain

$$\underline{\delta}_{\mathfrak{B}}(A^c) \geq 1 - \overline{\delta}_{\mathfrak{B}}(A).$$

The last statement is immediate. □

Definition 2.5. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and let $\rho = (\rho_{k,l})$ be a double sequence in X. We say that ρ is $\mathfrak{B}$-statistically convergent to $\mathfrak{h} \in X$ if for every $\varepsilon > 0$ and every nonzero $u \in X$, the set

$$\Big\{(k,l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l} - \mathfrak{h}, u\| \geq \varepsilon\Big\}$$

has $\mathfrak{B}$-density zero.

We say that ρ is $\mathfrak{B}$-statistically bounded with respect to a nonzero $u \in X$ if there exists $M > 0$ such that

$$\overline{\delta}_{\mathfrak{B}}\Big(\{(k,l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l}, u\| > M\}\Big) = 0.$$

If this holds for every nonzero u, we call ρ $\mathfrak{B}$-statistically bounded. Boundedness assumptions of this type are standard in statistical and ideal approaches since they ensure that limit set constructions are meaningful and lead to compactness properties for cluster sets in the scalar reductions [6, 7].

We also use the standard P-ideal property in the ideal-theoretic part of the paper. Recall that a strongly admissible ideal $\mathcal{I}_2$ on $\mathbb{N} \times \mathbb{N}$ is called a P-ideal if for every sequence (E_r) in $\mathcal{I}_2$ there exists a set $E \in \mathcal{I}_2$ such that $E_r \setminus E$ is finite for each r. This condition is well known to be useful when comparing $\mathcal{I}$-type and $\mathcal{I}^*$-type convergence notions; see, for example, [5, 19, 20].

We finally introduce a compatibility condition for matrix transforms that will be used in the Knopp type inclusion results. In the one-dimensional setting, inclusion relations between matrix methods and statistical convergence, as well as stability requirements for strong summability and statistical methods, have been treated in several works, and these considerations motivate additional control on the mass of coefficients carried by exceptional index sets [4, 17, 18]. In the double sequence framework, RH-regularity is governed by the Hamilton and Robison conditions [16, 28], while the family viewpoint follows the line initiated by Steiglitz and employed by Kolk in the statistical convergence context [17, 32]. The next definition formulates a uniform version of this control for four-dimensional families.

Definition 2.6. Let $\mathfrak{B} = (B_j)_{j\in J}$ be a nonnegative RH-regular family of four-dimensional matrices. A family of four-dimensional matrices $\mathfrak{T} = (T_j)_{j\in J}$ is called $\mathfrak{B}$-regular if each $T_j = (t_{m,n,k,l}(j))$ is RH-regular and the RH-regularity bounds can be chosen uniformly in j. Moreover, for every set $E \subseteq \mathbb{N}\times\mathbb{N}$ satisfying $\overline{\delta}_{\mathfrak{B}}(E) = 0$, one has

$$P\lim_{m,n\to\infty} \sum_{(k,l)\in E} |t_{m,n,k,l}(j)| = 0$$

uniformly in j.

This condition expresses that the total mass of the coefficients of T_j assigned to an exceptional set of $\mathfrak{B}$-density zero becomes negligible uniformly in j. In particular, it is compatible with the use of $\mathfrak{B}$-density as the measure of smallness, and it allows one to compare geometric limit sets of ρ and of $\mathfrak{T}\rho$ without requiring pointwise control on the transforms.

3. $\mathfrak{B}$-statistical limit superior and limit inferior for double sequences

In this section we extend the classical notions of limit superior and limit inferior to the setting of double sequences by using the density induced by a nonnegative RH-regular family. The use of matrix generated densities is standard in matrix statistical convergence and summability theory, and it provides a stable way to measure the size of exceptional index sets beyond ordinary convergence [4, 8, 17]. In the double sequence framework, Pringsheim convergence serves as the reference notion [27], and RH-regularity of four-dimensional matrices is described by the Hamilton and Robison criteria [16,28]. The family viewpoint follows the line initiated by Steiglitz and used by Kolk in the context of $\mathfrak{B}$ statistical convergence [17, 32].

Throughout this section, $\rho = (\rho_{k,l})$ is a double sequence in a real or complex 2-normed space $(X, \|\cdot, \cdot\|)$, and $u \in X$ is a fixed nonzero vector.

We begin with the scalar setting.

Definition 3.1 ($\mathfrak{B}$-statistical lim sup and lim inf for real double sequences)**.** Let $y = (y_{k,l})$ be a real double sequence. We define

$$\mathfrak{B}\text{-}\limsup y = \inf\Big\{b \in \mathbb{R} : \overline{\delta}_{\mathfrak{B}}(\{(k,l) \in \mathbb{N}\times\mathbb{N} : y_{k,l} > b\}) = 0\Big\},$$

and

$$\mathfrak{B}\text{-}\liminf y = \sup\Big\{a \in \mathbb{R} : \overline{\delta}_{\mathfrak{B}}(\{(k,l) \in \mathbb{N}\times\mathbb{N} : y_{k,l} < a\}) = 0\Big\}.$$

If the defining set in the first formula is empty, we write $\mathfrak{B}\text{-}\limsup y = +\infty$. If the defining set in the second formula is empty, we write $\mathfrak{B}\text{-}\liminf y = -\infty$.

We now apply these notions to the scalar reduction generated by the seminorm $x \mapsto \|x, u\|$. Set

$$x^{(u)}_{k,l} = \|\rho_{k,l}, u\|, \qquad (k,l) \in \mathbb{N}\times\mathbb{N}.$$

Definition 3.2 ($\mathfrak{B}$-statistical $\limsup_u$ and $\liminf_u$)**.** For the double sequence ρ and the fixed nonzero vector $u \in X$, we define

$$\mathfrak{B}\text{-}\limsup_u \rho := \mathfrak{B}\text{-}\limsup(x^{(u)}_{k,l}), \qquad \mathfrak{B}\text{-}\liminf_u \rho := \mathfrak{B}\text{-}\liminf(x^{(u)}_{k,l}).$$

Equivalently,

$$\mathfrak{B}\text{-}\limsup_u \rho = \inf\Big\{b \in \mathbb{R} : \overline{\delta}_{\mathfrak{B}}(E_u(b)) = 0\Big\},$$

and

$$\mathfrak{B}\text{-}\liminf_u \rho = \sup\Big\{a \in \mathbb{R} : \overline{\delta}_{\mathfrak{B}}(F_u(a)) = 0\Big\},$$

where

$$E_u(b) = \{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\rho_{k,l}, u\| > b\}, \qquad F_u(a) = \{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\rho_{k,l}, u\| < a\}.$$

These definitions adapt the classical extremal description of lim sup and lim inf to the present density-based setting.

The sets $E_u(b)$ decrease as b increases and the sets $F_u(a)$ increase as a increases. This monotonicity yields the following basic inequality.

Proposition 3.3. *For every double sequence ρ and every nonzero $u \in X$, one has*

$$\mathfrak{B}\text{-}\liminf_{u} \rho \leq \mathfrak{B}\text{-}\limsup_{u} \rho.$$

Proof. Set

$$S = \{b \in \mathbb{R} \mid \overline{\delta}_{\mathfrak{B}}(E_u(b)) = 0\}, \qquad T = \{a \in \mathbb{R} \mid \overline{\delta}_{\mathfrak{B}}(F_u(a)) = 0\}.$$

It suffices to show that $a \leq b$ for every $a \in T$ and $b \in S$, because then $\sup T \leq \inf S$.

Fix $a \in T$ and $b \in S$. Suppose, toward a contradiction, that $a > b$. Then,

$$(\mathbb{N} \times \mathbb{N}) \setminus F_u(a) = \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l}, u\| \geq a\} \subseteq E_u(b),$$

since $\|\rho_{k,l}, u\| \geq a > b$ implies $\|\rho_{k,l}, u\| > b$. Because the matrices are nonnegative and satisfy

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{m,n,k,l}(j) = 1 \quad \text{(in the Pringsheim limit uniformly in } j\text{)},$$

Proposition 2.4 yields

$$\underline{\delta}_{\mathfrak{B}}(A^c) \geq 1 - \overline{\delta}_{\mathfrak{B}}(A).$$

Applying this to $A = F_u(a)$ and using $\overline{\delta}_{\mathfrak{B}}(F_u(a)) = 0$ gives $\underline{\delta}_{\mathfrak{B}}((\mathbb{N} \times \mathbb{N}) \setminus F_u(a)) = 1$. On the other hand, the inclusion $(\mathbb{N} \times \mathbb{N}) \setminus F_u(a) \subseteq E_u(b)$ yields

$$1 = \underline{\delta}_{\mathfrak{B}}((\mathbb{N} \times \mathbb{N}) \setminus F_u(a)) \leq \overline{\delta}_{\mathfrak{B}}(E_u(b)) = 0,$$

a contradiction. Hence, $a \leq b$, and, therefore, $\sup T \leq \inf S$, i.e., $\mathfrak{B}\text{-}\liminf_u \rho \leq \mathfrak{B}\text{-}\limsup_u \rho$. □

We say that ρ is $\mathfrak{B}$-statistically bounded with respect to u if there exists $M \in \mathbb{R}$ such that $\overline{\delta}_{\mathfrak{B}}(E_u(M)) = 0$. In this case, both $\mathfrak{B}\text{-}\limsup_u \rho$ and $\mathfrak{B}\text{-}\liminf_u \rho$ are finite. The next theorems provide the standard characterization of these extremal values in terms of $\mathfrak{B}$-density thresholds.

Theorem 3.4. *Let ρ be a double sequence in a 2-normed space and let $u \in X$ be nonzero. Assume that $\beta = \mathfrak{B}\text{-}\limsup_u \rho$ is finite. Then, for every $\varepsilon > 0$, one has*

$$\overline{\delta}_{\mathfrak{B}}(E_u(\beta + \varepsilon)) = 0 \quad \text{and} \quad \overline{\delta}_{\mathfrak{B}}(E_u(\beta - \varepsilon)) \neq 0.$$

Conversely, if $\beta \in \mathbb{R}$ satisfies these two conditions for every $\varepsilon > 0$, then $\beta = \mathfrak{B}\text{-}\limsup_u \rho$.

Proof. Let

$$S = \{b \in \mathbb{R} \mid \overline{\delta}_{\mathfrak{B}}(E_u(b)) = 0\}.$$

Let $\beta = \inf S$. Since β is finite, the set S is nonempty. Fix $\varepsilon > 0$. Since $\beta + \varepsilon > \beta = \inf S$, there exists $b \in S$ with $b < \beta + \varepsilon$. The monotonicity $E_u(\beta + \varepsilon) \subseteq E_u(b)$ yields

$$\overline{\delta}_{\mathfrak{B}}(E_u(\beta + \varepsilon)) \leq \overline{\delta}_{\mathfrak{B}}(E_u(b)) = 0,$$

hence the first claim. For the second claim, if $\overline{\delta}_{\mathfrak{B}}(E_u(\beta - \varepsilon)) = 0$ for some $\varepsilon > 0$, then $\beta - \varepsilon \in S$, which contradicts $\beta = \inf S$. Hence, $\overline{\delta}_{\mathfrak{B}}(E_u(\beta - \varepsilon)) \neq 0$ for every $\varepsilon > 0$.

Conversely, assume that $\beta \in \mathbb{R}$ satisfies the two stated relations for every $\varepsilon > 0$. Then, $\beta + \varepsilon \in S$ for all $\varepsilon > 0$, so $\inf S \leq \beta$. If $b < \beta$, choose $\varepsilon = \beta - b > 0$. Then, $\overline{\delta}_{\mathfrak{B}}(E_u(b)) \neq 0$, hence $b \notin S$. Therefore, β is a lower bound for S and, hence, $\beta = \inf S = \mathfrak{B}\text{-}\limsup_u \rho$. □

Theorem 3.5. *Let ρ be a double sequence in a 2-normed space and let $u \in X$ be nonzero. Assume that $\alpha = \mathfrak{B}\text{-}\liminf_u \rho$ is finite. Then, for every $\varepsilon > 0$, one has*

$$\overline{\delta}_{\mathfrak{B}}(F_u(\alpha - \varepsilon)) = 0 \quad \textit{and} \quad \overline{\delta}_{\mathfrak{B}}(F_u(\alpha + \varepsilon)) \neq 0.$$

Conversely, if $\alpha \in \mathbb{R}$ satisfies these two conditions for every $\varepsilon > 0$, then $\alpha = \mathfrak{B}\text{-}\liminf_u \rho$.

Proof. The proof follows the same pattern as that of Theorem 3.4, with $E_u(b)$ replaced by $F_u(a)$ and the infimum argument replaced by the corresponding supremum argument. We therefore omit the repetitive details. □

We next compare these $\mathfrak{B}$-statistical extremal values with the ordinary Pringsheim limit superior and limit inferior for the scalar double sequence $\|\rho_{k,l}, u\|$. For a real double sequence $(y_{k,l})$, we use the standard Pringsheim definitions

$$P\limsup_{k,l\to\infty} y_{k,l} = \inf_{N\in\mathbb{N}} \sup_{k\geq N,\ l\geq N} y_{k,l}, \qquad P\liminf_{k,l\to\infty} y_{k,l} = \sup_{N\in\mathbb{N}} \inf_{k\geq N,\ l\geq N} y_{k,l},$$

which are the natural two-dimensional analogues of the classical lim sup and lim inf [27].

Proposition 3.6. *For every double sequence ρ in X and every nonzero $u \in X$, one has*

$$P\liminf_{k,l\to\infty} \|\rho_{k,l}, u\| \leq \mathfrak{B}\text{-}\liminf_u \rho \leq \mathfrak{B}\text{-}\limsup_u \rho \leq P\limsup_{k,l\to\infty} \|\rho_{k,l}, u\|.$$

Proof. The middle inequality follows from Proposition 3.3. We prove the last inequality; the first inequality is analogous.

Let $\beta = P\limsup_{k,l\to\infty} \|\rho_{k,l}, u\|$. If $\beta = +\infty$, there is nothing to prove. Assume that $\beta < \infty$ and fix $\varepsilon > 0$. By the definition of $P\limsup$, there exists $N \in \mathbb{N}$ such that

$$\|\rho_{k,l}, u\| \leq \beta + \varepsilon \qquad \text{for all } k \geq N,\ l \geq N.$$

Hence, the set

$$G = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid \|\rho_{k,l}, u\| > \beta + \varepsilon\}$$

is contained in the union $([1,N)\times\mathbb{N}) \cup (\mathbb{N}\times[1,N))$ of finitely many rows and columns. For a nonnegative RH-regular family, each fixed row and each fixed column has $\mathfrak{B}$-density zero by the Hamilton–Robison conditions and uniformity, and finite unions preserve $\mathfrak{B}$-density zero. Therefore, $\overline{\delta}_{\mathfrak{B}}(G) = 0$. Thus, $\beta+\varepsilon$ belongs to the defining set in the infimum for $\mathfrak{B}\text{-}\limsup_u \rho$, and

$$\mathfrak{B}\text{-}\limsup_u \rho \leq \beta + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\mathfrak{B}\text{-}\limsup_u \rho \leq \beta$. □

The inequalities in Proposition 3.6 may be strict.

Example 3.7. Let $X = \mathbb{R}^2$ with the 2-norm $\|(x_1, x_2), (y_1, y_2)\| = |x_1 y_2 - x_2 y_1|$ and take $u = (0, 1)$. Define a double sequence $\rho = (\rho_{k,l})$ by

$$\rho_{k,l} = \begin{cases} (1,0) & \text{when } k = m^2 \text{ and } l = n^2 \text{ for some } m, n \in \mathbb{N}, \\ (0,0) & \text{otherwise.} \end{cases}$$

Then, $\|\rho_{k,l}, u\| = 1$ on the set of pairs of perfect squares, and it is 0 elsewhere. Since the set of square pairs meets every Pringsheim tail, one has $P\limsup_{k,l\to\infty} \|\rho_{k,l}, u\| = 1$. On the other hand, the set

$$Q = \{(k,l) \in \mathbb{N}\times\mathbb{N} : k = m^2,\ l = n^2 \text{ for some } m, n \in \mathbb{N}\}$$

has double natural density zero, because

$$\frac{1}{mn}\big|Q \cap (\{1,\dots,m\}\times\{1,\dots,n\})\big| = \frac{\lfloor\sqrt{m}\rfloor\lfloor\sqrt{n}\rfloor}{mn} \longrightarrow 0 \qquad (m, n \to \infty).$$

Hence, for the double Cesàro family, Q also has $\mathfrak{B}$-density zero. Consequently $\overline{\delta}_{\mathfrak{B}}(E_u(0)) = 0$, which yields $\mathfrak{B}\text{-}\limsup_u \rho = 0$. Hence, the rightmost inequality in Proposition 3.6 is strict.

The extremal quantities $\mathfrak{B}\text{-}\limsup_u \rho$ and $\mathfrak{B}\text{-}\liminf_u \rho$ will be used in Theorem 5.2 to derive the disk-type representation of the $\mathfrak{B}$-statistical core. We also record the corresponding real cluster description for the scalar reduction.

Definition 3.8 ($\mathfrak{B}$-statistical cluster points of a scalar reduction)**.** Let ρ be a double sequence in X and let $u \in X$ be nonzero. A real number γ is called a real $\mathfrak{B}$-statistical cluster point of ρ with respect to u if for every $\varepsilon > 0$ one has

$$\overline{\delta}_{\mathfrak{B}}(\{(k,l) \in \mathbb{N}\times\mathbb{N} : |\|\rho_{k,l}, u\| - \gamma| < \varepsilon\}) > 0.$$

The set of all such cluster points is denoted by $\Lambda_{\mathfrak{B},u}(\rho)$.

Theorem 3.9. *Assume that ρ is $\mathfrak{B}$-statistically bounded with respect to a nonzero $u \in X$. Then, $\Lambda_{\mathfrak{B},u}(\rho)$ is nonempty, closed, and bounded in $\mathbb{R}$, hence, it is compact.*

Proof. Let M be such that $\overline{\delta}_{\mathfrak{B}}(E_u(M)) = 0$. Then, $\|\rho_{k,l}, u\| \le M$ for $\mathfrak{B}$ almost all indices, hence, every $\mathfrak{B}$-statistical cluster point lies in $[0, M]$, and $\Lambda_{\mathfrak{B},u}(\rho)$ is bounded.

To show nonemptiness, partition $[0, M]$ into finitely many closed intervals $I_{q,1}, \dots, I_{q,N(q)}$ of length at most $1/q$. Let

$$A_{q,r} = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid \|\rho_{k,l}, u\| \in I_{q,r}\}.$$

Since $E_u(M)$ has upper $\mathfrak{B}$-density 0, the set $\bigcup_{r=1}^{N(q)} A_{q,r} = \{(k,l) \mid 0 \le \|\rho_{k,l}, u\| \le M\}$ has upper $\mathfrak{B}$-density 1. By subadditivity of $\overline{\delta}_{\mathfrak{B}}$ for nonnegative transforms, at least one $A_{q,r}$ must satisfy $\overline{\delta}_{\mathfrak{B}}(A_{q,r}) > 0$. Choose such an interval and denote it by J_q.

Choosing at each step an interval $J_{q+1} \subseteq J_q$ of length at most $1/(q+1)$ with the same property produces a nested sequence of nonempty closed intervals with diameters tending to zero. Let γ be the unique point in $\bigcap_{q=1}^{\infty} J_q$. Given $\varepsilon > 0$, choose q such that $1/q < \varepsilon$. Then, $J_q \subseteq (\gamma - \varepsilon, \gamma + \varepsilon)$ and the set $\{(k,l) \mid \|\rho_{k,l}, u\| \in J_q\}$ has positive upper $\mathfrak{B}$-density, so $\gamma \in \Lambda_{\mathfrak{B},u}(\rho)$.

To show closedness, let (γ_n) be a sequence in $\Lambda_{\mathfrak{B},u}(\rho)$ with $\gamma_n \to \gamma$. Fix $\varepsilon > 0$ and choose n such that $|\gamma_n - \gamma| < \varepsilon/2$. Then,

$$\{(k,l) \mid |\|\rho_{k,l}, u\| - \gamma| < \varepsilon\} \supseteq \{(k,l) \mid |\|\rho_{k,l}, u\| - \gamma_n| < \varepsilon/2\},$$

and the righthand set has positive upper $\mathfrak{B}$-density. Hence, $\gamma \in \Lambda_{\mathfrak{B},u}(\rho)$ and $\Lambda_{\mathfrak{B},u}(\rho)$ is closed. □

Theorem 3.10. *Assume that ρ is $\mathfrak{B}$-statistically bounded with respect to a nonzero $u \in X$. Then,*

$$\mathfrak{B}\text{-}\limsup_u \rho = \max \Lambda_{\mathfrak{B},u}(\rho), \qquad \mathfrak{B}\text{-}\liminf_u \rho = \min \Lambda_{\mathfrak{B},u}(\rho).$$

Proof. Let $\beta = \mathfrak{B}\text{-}\limsup_u \rho$ and fix $\varepsilon > 0$. By Theorem 3.4, one has $\overline{\delta}_{\mathfrak{B}}(E_u(\beta+\varepsilon)) = 0$ and $\overline{\delta}_{\mathfrak{B}}(E_u(\beta-\varepsilon)) \neq 0$. Consider

$$H_\varepsilon = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid \beta - \varepsilon < \|\rho_{k,l}, u\| \leq \beta + \varepsilon\}.$$

Since $E_u(\beta-\varepsilon) \subseteq E_u(\beta+\varepsilon) \cup H_\varepsilon$, subadditivity of $\overline{\delta}_{\mathfrak{B}}$ yields $\overline{\delta}_{\mathfrak{B}}(H_\varepsilon) > 0$. Hence, $\beta \in \Lambda_{\mathfrak{B},u}(\rho)$.

Now let $\gamma \in \Lambda_{\mathfrak{B},u}(\rho)$ and assume $\gamma > \beta$. Set $\varepsilon = (\gamma-\beta)/2$. Then,

$$\{(k,l) \mid |\|\rho_{k,l}, u\| - \gamma| < \varepsilon\} \subseteq E_u(\beta+\varepsilon).$$

However, $\overline{\delta}_{\mathfrak{B}}(E_u(\beta+\varepsilon)) = 0$, contradicting $\gamma \in \Lambda_{\mathfrak{B},u}(\rho)$. Therefore, no element of $\Lambda_{\mathfrak{B},u}(\rho)$ exceeds β, and since $\beta \in \Lambda_{\mathfrak{B},u}(\rho)$, we obtain $\beta = \max \Lambda_{\mathfrak{B},u}(\rho)$.

The proof for $\mathfrak{B}\text{-}\liminf_u \rho$ is analogous. Let $\alpha = \mathfrak{B}\text{-}\liminf_u \rho$. By Theorem 3.5, $\alpha \in \Lambda_{\mathfrak{B},u}(\rho)$, and no element of $\Lambda_{\mathfrak{B},u}(\rho)$ can be $< \alpha$. Hence, $\alpha = \min \Lambda_{\mathfrak{B},u}(\rho)$. □

Theorem 3.10 gives a useful extremal description of the real cluster behavior of the scalar reductions $\|\rho_{k,l}, u\|$ measured by $\mathfrak{B}$-density. This description will be used in Theorem 5.2 when we derive the disk representation of the $\mathfrak{B}$-statistical core.

4. $\mathfrak{B}$-statistical cluster points in 2-normed spaces

Cluster point theory describes accumulation behavior without requiring convergence. In the classical setting, a point is a cluster point if every neighborhood contains infinitely many terms. Statistical variants replace infinitude by largeness measured via density, and this approach has proved effective for matrix-generated statistical methods [6,7]. In this section we introduce $\mathfrak{B}$-statistical cluster points for double sequences in 2-normed spaces by using the $\mathfrak{B}$-density induced by a nonnegative RH-regular family. This is consistent with matrix statistical convergence [4, 8, 17] and with the family viewpoint in summability initiated by Steiglitz and developed further by Kolk [17,32]. The reference convergence notion for double sequences is Pringsheim convergence [27], while RH-regularity for four-dimensional matrices is governed by the Hamilton–Robison criteria [16,28].

Throughout this section, $(X, \|\cdot,\cdot\|)$ is a real or complex 2-normed space and $\rho = (\rho_{k,l})$ is a double sequence in X. Fix a nonzero direction $u \in X$.

Definition 4.1. A point $\gamma \in X$ is called a $\mathfrak{B}$-statistical cluster point of ρ with respect to u if for every $\varepsilon > 0$ the set

$$C_u(\gamma,\varepsilon) = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid \|\rho_{k,l} - \gamma, u\| < \varepsilon\}$$

satisfies $\overline{\delta}_{\mathfrak{B}}(C_u(\gamma,\varepsilon)) > 0$. The set of all such points is denoted by $\Gamma_{\mathfrak{B},u}(\rho)$.

The scalar reduction $\|\rho_{k,l}, u\|$ induces a corresponding real $\mathfrak{B}$-statistical cluster set in $\mathbb{R}$. We denote it by $\Lambda_{\mathfrak{B},u}(\rho)$ and refer to Definition 3.8 for the formal definition.

The next result shows that convergence along a set of positive upper $\mathfrak{B}$-density produces a $\mathfrak{B}$-statistical cluster point. Moreover, any $\mathfrak{B}$-statistical cluster point gives rise to an ordinary Pringsheim-convergent double subsequence.

Theorem 4.2. *Let $\gamma \in X$.*

(i) If there exists a set $E \subseteq \mathbb{N}\times\mathbb{N}$ with $\overline{\delta}_{\mathfrak{B}}(E) > 0$ such that $(\rho_{k,l})_{(k,l)\in E}$ converges to γ in the Pringsheim sense with respect to the seminorm $x \mapsto \|x, u\|$, then $\gamma \in \Gamma_{\mathfrak{B},u}(\rho)$.

(ii) If $\gamma \in \Gamma_{\mathfrak{B},u}(\rho)$, then there exists a double subsequence (ρ_{k_m,l_m}) with $k_m \to \infty$, $l_m \to \infty$ such that $\rho_{k_m,l_m} \to \gamma$ in the 2-norm topology with respect to u.

Proof. (i) Assume such a set E exists and fix $\varepsilon > 0$. Convergence along E means that there is $N \in \mathbb{N}$ such that

$$\|\rho_{k,l} - \gamma, u\| < \varepsilon \qquad \text{for all } (k,l) \in E \text{ with } k \geq N,\ l \geq N.$$

Hence,

$$E \setminus \Big(([1,N) \times \mathbb{N}) \cup (\mathbb{N} \times [1,N))\Big) \subseteq C_u(\gamma, \varepsilon).$$

For a nonnegative RH-regular family, every fixed row and every fixed column has $\mathfrak{B}$-density zero; therefore, the finite union $([1,N) \times \mathbb{N}) \cup (\mathbb{N} \times [1,N))$ has $\mathfrak{B}$-density zero. Using subadditivity of $\overline{\delta}_{\mathfrak{B}}$ gives

$$\overline{\delta}_{\mathfrak{B}}(E) \leq \overline{\delta}_{\mathfrak{B}}(C_u(\gamma,\varepsilon)) + \overline{\delta}_{\mathfrak{B}}\Big(([1,N) \times \mathbb{N}) \cup (\mathbb{N} \times [1,N))\Big) = \overline{\delta}_{\mathfrak{B}}(C_u(\gamma,\varepsilon)).$$

Since $\overline{\delta}_{\mathfrak{B}}(E) > 0$, we obtain $\overline{\delta}_{\mathfrak{B}}(C_u(\gamma,\varepsilon)) > 0$ for every $\varepsilon > 0$, hence $\gamma \in \Gamma_{\mathfrak{B},u}(\rho)$.

(ii) Assume $\gamma \in \Gamma_{\mathfrak{B},u}(\rho)$. Then, for each $m \in \mathbb{N}$ the set $C_u(\gamma, 1/m)$ has positive upper $\mathfrak{B}$-density, hence, it cannot be contained in finitely many rows or columns. Therefore, it meets arbitrarily far Pringsheim tails. Choose inductively $(k_m, l_m) \in C_u(\gamma, 1/m)$ such that

$$k_m \geq m, \quad l_m \geq m, \quad k_m > k_{m-1}, \quad l_m > l_{m-1} \qquad (m \geq 2).$$

Then, $k_m \to \infty$ and $l_m \to \infty$, and

$$\|\rho_{k_m,l_m} - \gamma, u\| < \frac{1}{m} \qquad (m \in \mathbb{N}),$$

which implies $\rho_{k_m,l_m} \to \gamma$ with respect to the seminorm $x \mapsto \|x, u\|$, i.e., in the induced 2-norm topology along this subsequence. □

We now relate the real cluster set $\Lambda_{\mathfrak{B},u}(\rho)$ to the $\mathfrak{B}$-statistical limit superior and limit inferior of the scalar reduction. This is consistent with the classical extremal viewpoint in the theory of infinite series [3].

Remark 4.3. For a fixed nonzero $u \in X$, the extremal description of the real cluster set $\Lambda_{\mathfrak{B},u}(\rho)$ has already been established in Theorem 3.10. We therefore use that result directly in what follows and do not restate it here.

Remark 4.4. The vector cluster set $\Gamma_{\mathfrak{B},u}(\rho)$ and the real cluster set $\Lambda_{\mathfrak{B},u}(\rho)$ encode different information. The set $\Lambda_{\mathfrak{B},u}(\rho)$ depends only on the scalar reduction $\|\rho_{k,l}, u\|$, while $\Gamma_{\mathfrak{B},u}(\rho)$ retains directional information through the condition $\|\rho_{k,l} - \gamma, u\| < \varepsilon$. For this reason, direct relations between $\Gamma_{\mathfrak{B},u}(\rho)$ and $\Lambda_{\mathfrak{B},u}(\rho)$ are generally indirect, and we mainly use the extremal description in Theorem 3.10.

Example 4.5. Let $X = \mathbb{R}^2$ with the 2-norm $\|(x_1, x_2), (y_1, y_2)\| = |x_1 y_2 - x_2 y_1|$ and fix $u = (0, 1)$. Define a double sequence $\rho = (\rho_{k,l})$ by

$$\rho_{k,l} = \begin{cases} (1,0) & \text{when } k = m^2 \text{ and } l = n^2 \text{ for some } m, n \in \mathbb{N}, \\ (0,0) & \text{otherwise.} \end{cases}$$

Let $\mathfrak{B}$ denote the double Cesàro method, viewed as a singleton nonnegative RH-regular family. Set

$$Q = \{(k,l) \in \mathbb{N} \times \mathbb{N} : k = m^2, \ l = n^2 \text{ for some } m, n \in \mathbb{N}\}.$$

Then, Q has double natural density zero, because

$$\frac{1}{mn}\big|Q \cap (\{1, \ldots, m\} \times \{1, \ldots, n\})\big| = \frac{\lfloor \sqrt{m} \rfloor \lfloor \sqrt{n} \rfloor}{mn} \longrightarrow 0 \qquad (m, n \to \infty).$$

Hence, Q has $\mathfrak{B}$-density zero as well. Therefore, for $0 < \varepsilon < 1$,

$$C_u((1,0), \varepsilon) = Q,$$

and, thus, $\overline{\delta}_{\mathfrak{B}}(C_u((1,0), \varepsilon)) = 0$, showing $(1,0) \notin \Gamma_{\mathfrak{B},u}(\rho)$. On the other hand, for $0 < \varepsilon < 1$ the set $C_u((0,0), \varepsilon)$ contains the complement of the square pairs and, hence, has $\mathfrak{B}$-density one. Thus,

$$\Gamma_{\mathfrak{B},u}(\rho) = \{(0,0)\}.$$

In particular, ρ is not Pringsheim convergent to $(0,0)$, but its $\mathfrak{B}$-statistical cluster behavior collapses to a single point.

Example 4.6. Let $X = \mathbb{R}^2$ with the same 2-norm and let $u = (0, 1)$. Define

$$\rho_{k,l} = \begin{cases} (0,1) & \text{when } k = l, \\ (0,0) & \text{when } k \neq l. \end{cases}$$

For the double Cesàro matrix $C = (c_{m,n,k,l})$, the diagonal set $\{(k,l) \mid k = l\}$ has double natural density zero and, therefore, C-density zero. Hence, with respect to C, one obtains $\Gamma_{C,u}(\rho) = \{(0,0)\}$.

On the other hand, the family viewpoint makes it possible to tune the induced density by choosing RH-regular families that place comparatively larger weight on selected regions, such as indices near the diagonal, while preserving nonnegativity and RH-regularity. For such a choice of $\mathfrak{B}$, the diagonal set need not be $\mathfrak{B}$-negligible, and this may enlarge the $\mathfrak{B}$-statistical cluster set by allowing $(0, 1)$ to appear together with $(0, 0)$.

This example is intended to emphasize that the choice of the family $\mathfrak{B}$ influences the notion of statistical largeness in $\mathbb{N} \times \mathbb{N}$ and therefore changes the resulting cluster sets. The family viewpoint is thus genuinely richer than the single-matrix case.

Remark 4.7. Example 4.6 highlights the role of the family parameter in the density concept. In applications, families often arise from an external process, and the uniformity requirement in the definition of $\mathfrak{B}$-density is precisely what makes the family approach stable.

5. $\mathfrak{B}$-statistical core for double sequences

Core theory provides a geometric description of limiting behavior by replacing a single limit value with a closed convex set determined by the asymptotic location of a sequence. For ordinary sequences, the classical Knopp core is obtained by intersecting closed convex sets that contain all but finitely many terms, and it is closely related to convexity and separation phenomena in summability theory. Statistical variants of the core were developed in the setting of matrix statistical convergence and cluster point theory, and the extremal descriptions via statistical limit superior and limit inferior play a central role in that approach [6, 7]. In the present setting we work with double sequences and $\mathfrak{B}$-density induced by a nonnegative RH-regular family, so admissibility is measured by $\mathfrak{B}$-density zero exceptional sets rather than finiteness. This choice is consistent with matrix generated statistical methods [4, 8, 17] and with the family viewpoint in summability [17, 32].

Let $(X, \|\cdot,\cdot\|)$ be a real or complex 2-normed space and let $\rho = (\rho_{k,l})$ be a double sequence in X.

A closed convex set $K \subseteq X$ is called $\mathfrak{B}$-admissible for ρ if the exceptional set

$$E(K) = \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid \rho_{k,l} \notin K\}$$

has $\mathfrak{B}$-density zero, meaning $\overline{\delta}_{\mathfrak{B}}(E(K)) = 0$. Admissible sets encode the idea that K contains $\rho_{k,l}$ for $\mathfrak{B}$ almost all indices. The next definition introduces the $\mathfrak{B}$-statistical core as the smallest closed convex set in this sense.

Definition 5.1. The $\mathfrak{B}$-statistical core of ρ, denoted by $\mathrm{core}_{\mathfrak{B}}(\rho)$, is the intersection of all $\mathfrak{B}$ admissible closed convex subsets of X. If there exists no $\mathfrak{B}$ admissible closed convex set, then $\mathrm{core}_{\mathfrak{B}}(\rho)$ is defined to be X.

By construction, $\mathrm{core}_{\mathfrak{B}}(\rho)$ is closed and convex as an intersection of closed convex sets. When ρ is $\mathfrak{B}$-statistically bounded, the core is contained in a bounded intersection of strips determined by the seminorms $\|\cdot, u\|$, and it is therefore bounded in finite dimensional spaces.

We now give an intrinsic description of the core in terms of the $\mathfrak{B}$-statistical limit superior of suitable scalar reductions.

From this point onward, we assume that ρ is $\mathfrak{B}$-statistically bounded. Then, for every $z \in X$ and every nonzero $u \in X$, the scalar double sequence

$$(\|\rho_{k,l} - z, u\|)$$

is also $\mathfrak{B}$-statistically bounded. Indeed, if

$$\overline{\delta}_{\mathfrak{B}}(\{(k,l) : \|\rho_{k,l}, u\| > M\}) = 0,$$

then by the triangle inequality,

$$\|\rho_{k,l} - z, u\| \leq \|\rho_{k,l}, u\| + \|z, u\|,$$

so

$$\overline{\delta}_{\mathfrak{B}}(\{(k,l) : \|\rho_{k,l} - z, u\| > M + \|z, u\|\}) = 0.$$

Hence, the quantity defined below is finite.

For $z \in X$, $u \neq 0$, and $r \geq 0$, define the closed u-disk

$$D_u(z,r) = \{\, w \in X : \ \|w - z, u\| \leq r \,\}.$$

The set $D_u(z,r)$ is closed and convex since the map $w \mapsto \|w - z, u\|$ is a continuous seminorm for the topology induced by the 2-norm.

For the scalar double sequence $(\|\rho_{k,l} - z, u\|)$, define

$$r_u(z) = \mathfrak{B}\text{-}\limsup(\|\rho_{k,l} - z, u\|).$$

Because of the boundedness assumption above, $r_u(z) < \infty$ for every $z \in X$ and every $u \neq 0$.

The next theorem shows that the $\mathfrak{B}$-statistical core of ρ can be recovered by intersecting these u-disks over all centers $z \in X$ and all directions $u \neq 0$.

Theorem 5.2. *Assume that ρ is $\mathfrak{B}$-statistically bounded. Then,*

$$\operatorname{core}_{\mathfrak{B}}(\rho) = \bigcap_{z \in X} \bigcap_{u \neq 0} D_u(z, r_u(z)).$$

Proof. Fix $z \in X$ and $u \neq 0$, and write

$$r = r_u(z) = \mathfrak{B}\text{-}\limsup(\|\rho_{k,l} - z, u\|).$$

We first show that $\operatorname{core}_{\mathfrak{B}}(\rho) \subseteq D_u(z,r)$. Let $\varepsilon > 0$. By Theorem 3.4, applied to the scalar double sequence $(\|\rho_{k,l} - z, u\|)$, the set

$$E_\varepsilon(z,u) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\rho_{k,l} - z, u\| > r + \varepsilon\}$$

has $\mathfrak{B}$-density zero. Hence, $D_u(z, r+\varepsilon)$ is a $\mathfrak{B}$-admissible closed convex set for ρ. By Definition 5.1,

$$\operatorname{core}_{\mathfrak{B}}(\rho) \subseteq D_u(z, r+\varepsilon) \qquad \text{for every } \varepsilon > 0.$$

Intersecting over $\varepsilon > 0$, we obtain

$$\operatorname{core}_{\mathfrak{B}}(\rho) \subseteq D_u(z,r).$$

Since z and u were arbitrary,

$$\operatorname{core}_{\mathfrak{B}}(\rho) \subseteq \bigcap_{z \in X} \bigcap_{u \neq 0} D_u(z, r_u(z)).$$

For the reverse inclusion, set

$$K = \bigcap_{z \in X} \bigcap_{u \neq 0} D_u(z, r_u(z)).$$

Let $L \subseteq X$ be any $\mathfrak{B}$-admissible closed convex set. We claim that $K \subseteq L$. Assume that $w \in K$ but $w \notin L$. Since L is closed and convex in the locally convex topology induced by the seminorms $p_u(x) = \|x, u\|$, a standard separation argument yields a continuous seminorm p and numbers $a < b$ such that

$$p(x) \leq a \quad \text{for all } x \in L, \qquad p(w) \geq b.$$

Because the topology is generated by $(p_u)_{u \neq 0}$, there exist $u \neq 0$ and $c > 0$ such that

$$p(\cdot) \leq c\, p_u(\cdot).$$

Consequently, for some $z \in X$ and some $r > 0$, one has

$$L \subseteq D_u(z,r) \qquad \text{but} \qquad w \notin D_u(z,r).$$

Since L is $\mathfrak{B}$-admissible and $L \subseteq D_u(z,r)$, the exceptional set

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\rho_{k,l} - z, u\| > r\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : \rho_{k,l} \notin L\}$$

has $\mathfrak{B}$-density zero. Hence, by Definition 3.1,

$$r_u(z) \leq r.$$

Therefore,

$$D_u(z, r_u(z)) \subseteq D_u(z,r).$$

Since $w \in K$, we have

$$w \in D_u(z, r_u(z)) \subseteq D_u(z,r),$$

which contradicts $w \notin D_u(z,r)$. Thus, $K \subseteq L$. Since L was arbitrary,

$$K \subseteq \text{core}_{\mathfrak{B}}(\rho).$$

This completes the proof. □

In the scalar case the core reduces to an interval determined by the $\mathfrak{B}$-statistical limit inferior and limit superior. This matches the classical extremal description in the one-dimensional statistical core theory and it is consistent with the role of liminf and limsup in summability [3, 6].

Corollary 5.3. *Let $\rho = (\rho_{k,l})$ be a real double sequence and let $\mathfrak{B}$ be as above. Then,*

$$\text{core}_{\mathfrak{B}}(\rho) = [\mathfrak{B}\text{-}\liminf \rho,\ \mathfrak{B}\text{-}\limsup \rho].$$

Proof. In the real case, the admissible closed convex sets are precisely closed intervals that contain $\rho_{k,l}$ for $\mathfrak{B}$ almost all indices. The left endpoint of the smallest such interval is determined by Theorem 3.5, while the right endpoint is determined by Theorem 3.4. Therefore, the intersection of all admissible intervals equals the stated interval. □

We record basic invariance properties of the $\mathfrak{B}$-statistical core.

Proposition 5.4. *Let ρ and σ be $\mathfrak{B}$-statistically bounded double sequences in X. If ρ is $\mathfrak{B}$-statistically convergent to $\mathfrak{h}$, then* $\text{core}_{\mathfrak{B}}(\rho) = \{\mathfrak{h}\}$. *If*

$$\overline{\delta}_{\mathfrak{B}}(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \rho_{k,l} \neq \sigma_{k,l}\}) = 0,$$

then

$$\text{core}_{\mathfrak{B}}(\rho) = \text{core}_{\mathfrak{B}}(\sigma).$$

If T is an affine isometry of X in the 2-norm topology, then

$$\text{core}_{\mathfrak{B}}(T\rho) = T(\text{core}_{\mathfrak{B}}(\rho)).$$

Proof. For the first statement, if ρ is $\mathfrak{B}$-statistically convergent to $\mathfrak{h}$, then every closed convex set containing $\mathfrak{h}$ contains $\rho_{k,l}$ for $\mathfrak{B}$ almost all indices, while any closed convex set not containing $\mathfrak{h}$ misses infinitely many terms on a set of positive upper density. Hence, the intersection of all admissible closed convex sets reduces to $\{\mathfrak{h}\}$.

For the second statement, a change on a $\mathfrak{B}$-density zero set does not alter admissibility because admissibility depends only on the exceptional set of indices. Therefore, the families of admissible sets for ρ and σ coincide and so do their intersections.

For the third statement, if K is admissible for ρ, then $T(K)$ is closed and convex, and the exceptional set for $T\rho$ with respect to $T(K)$ coincides with the exceptional set for ρ with respect to K. Therefore, admissibility is preserved under T and the equality follows by taking intersections. □

The core contains the convex hull of the $\mathfrak{B}$-statistical cluster set in every fixed direction. This inclusion is the natural analogue of the corresponding fact for matrix statistical cluster points in the single sequence setting [7].

Theorem 5.5. *Assume that ρ is $\mathfrak{B}$-statistically bounded. Then, for every nonzero $u \in X$, one has*

$$\mathrm{conv}(\Gamma_{\mathfrak{B},u}(\rho)) \subseteq \mathrm{core}_{\mathfrak{B}}(\rho).$$

Proof. Let K be a $\mathfrak{B}$ admissible closed convex set for ρ. Fix $\gamma \in \Gamma_{\mathfrak{B},u}(\rho)$. If $\gamma \notin K$, then since K is closed there exists $\varepsilon > 0$ such that $B_u(\gamma, \varepsilon) \cap K = \emptyset$, where $B_u(\gamma, \varepsilon) = \{x \in X \mid \|x - \gamma, u\| < \varepsilon\}$. The set of indices for which $\rho_{k,l} \in B_u(\gamma, \varepsilon)$ has positive upper $\mathfrak{B}$-density by the definition of cluster points, and this set is disjoint from the set of indices for which $\rho_{k,l} \in K$. Hence, $\rho_{k,l} \notin K$ on a set of positive upper density, which contradicts admissibility of K. Therefore, every $\mathfrak{B}$-statistical cluster point belongs to every admissible K, so $\Gamma_{\mathfrak{B},u}(\rho) \subseteq K$. Since K is convex, $\mathrm{conv}(\Gamma_{\mathfrak{B},u}(\rho)) \subseteq K$. Intersecting over all admissible K yields the inclusion. □

Remark 5.6. If one defines the classical Knopp core by requiring that the exceptional set is finite, then every Knopp admissible closed convex set is also $\mathfrak{B}$ admissible because every finite set has $\mathfrak{B}$-density zero for RH-regular families. Consequently, the $\mathfrak{B}$-statistical core is contained in the classical Knopp core whenever the latter is defined.

6. Knopp type inclusion theorems for $\mathfrak{B}$-statistical core

Knopp core theory provides a geometric invariant that links summability transformations with the limiting behavior of bounded sequences. In the one-dimensional setting, regular matrix methods yield the classical inclusion of the Knopp core of the transform into the Knopp core of the original sequence. Statistical versions of this principle were developed by replacing finiteness of exceptional sets with smallness measured by a density induced by a summability method, and the resulting A statistical core theory was systematically investigated in connection with A statistical cluster points and extremal limit notions [6, 7]. The purpose of the present section is to establish an analogue of this inclusion phenomenon for double sequences in 2 normed spaces when the density is generated by a nonnegative RH-regular family $\mathfrak{B}$.

For double sequences, Pringsheim convergence [27] is the natural reference notion, and the appropriate class of matrix transformations is given by four-dimensional RH-regular matrices in the sense of Hamilton and Robison [16,28]. We work with a family $\mathfrak{T} = (T_j)_{j\in J}$ of such matrices and view the transformed object as the collection of double sequences $(T_j\rho)_{j\in J}$. The main result gives conditions under which, for each fixed j, the Knopp core of $T_j\rho$ is contained in the $\mathfrak{B}$-statistical core of ρ. This viewpoint also fits the family framework in summability theory introduced for matrix families [32] and used to define density and statistical convergence associated with families of matrices [17]. The key assumption is a compatibility condition that forces the matrix coefficients of T_j to become negligible on sets of $\mathfrak{B}$-density zero, which is the two-dimensional family counterpart of the regularity requirement used in the A statistical core setting.

Let $\rho = (\rho_{k,l})$ be a bounded double sequence in a 2 normed space $(X, \|\cdot,\cdot\|)$. Boundedness is understood with respect to the seminorm family $p_u(x) = \|x, u\|$ in the sense that $\sup_{k,l} \|\rho_{k,l}, u\| < \infty$ for each fixed nonzero $u \in X$.

For any bounded double sequence $\eta = (\eta_{m,n})$ in X, we define its Knopp core by

$$\text{K-core}(\eta) = \bigcap_{N\in\mathbb{N}} \overline{\text{conv}}\{\eta_{m,n} \mid m \geq N,\ n \geq N\},$$

where the closure is taken in the topology induced by the 2 norm. The set K-core(η) is closed and convex, and it is nonempty when η is bounded.

Let $\mathfrak{T} = (T_j)_{j\in J}$ be a family of four-dimensional matrices with entries $t_{m,n,k,l}(j)$. For a double sequence ρ, we write

$$(T_j\rho)_{m,n} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{m,n,k,l}(j)\,\rho_{k,l},$$

whenever the series converges in X. We assume that the convergence is uniform in j for each fixed pair (m, n) and that the family is uniformly bounded in the sense that for each nonzero $u \in X$, the quantity

$$\sup_{m,n,j} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} t_{m,n,k,l}(j)\,\|\rho_{k,l}, u\|$$

is finite whenever ρ is bounded. We also assume that the matrices are nonnegative, meaning $t_{m,n,k,l}(j) \geq 0$ for all indices.

The $\mathfrak{B}$ regularity condition below is the natural compatibility requirement between the transformation $\mathfrak{T}$ and the $\mathfrak{B}$-density.

We recall the notion of a $\mathfrak{B}$-regular family of transforms from Definition 2.6. In the present section we additionally assume that the matrices T_j are nonnegative, so the absolute values in that definition may be dropped in the estimates.

The next lemma gives a seminorm representation of the Knopp core, which is the form that is compatible with the disk representation of the $\mathfrak{B}$-statistical core from the previous section.

Lemma 6.1. *Let $\eta = (\eta_{m,n})$ be a bounded double sequence in X. For $z \in X$ and $u \neq 0$, define*

$$L_u^{\mathrm{K}}(\eta, z) = P\limsup_{m,n\to\infty} \|\eta_{m,n} - z, u\|.$$

Then,

$$\text{K-core}(\eta) \subseteq \bigcap_{z\in X}\bigcap_{u\neq 0} D_u(z,\ L_u^{\mathrm{K}}(\eta, z)).$$

Proof. Fix $z \in X$ and $u \neq 0$, and write $p_u(x) = \|x, u\|$. For each $N \in \mathbb{N}$, set

$$S_N = \{\eta_{m,n} \mid m \geq N,\ n \geq N\}.$$

Since p_u is a continuous seminorm, it is convex, and for every convex combination $x = \sum_{r=1}^{q} \alpha_r x_r$ with $x_r \in S_N$, $\alpha_r \geq 0$, $\sum_{r=1}^{q} \alpha_r = 1$, we have

$$p_u(x - z) = p_u\Big(\sum_{r=1}^{q} \alpha_r (x_r - z)\Big) \leq \sum_{r=1}^{q} \alpha_r\, p_u(x_r - z) \leq \sup_{m \geq N,\ n \geq N} p_u(\eta_{m,n} - z).$$

By continuity, the same inequality holds for every $x \in \overline{\mathrm{conv}}(S_N)$ (closure taken in the 2-norm topology). Hence, for each N and every $x \in \overline{\mathrm{conv}}(S_N)$,

$$\|x - z, u\| \leq \sup_{m \geq N,\ n \geq N} \|\eta_{m,n} - z, u\|.$$

Now let $w \in \text{K-core}(\eta) = \bigcap_{N \in \mathbb{N}} \overline{\mathrm{conv}}(S_N)$. Then, the above estimate holds for all N, so

$$\|w - z, u\| \leq \inf_{N \in \mathbb{N}} \sup_{m \geq N,\ n \geq N} \|\eta_{m,n} - z, u\| = P\limsup_{m,n \to \infty} \|\eta_{m,n} - z, u\| = L_u^{\mathrm{K}}(\eta, z).$$

Therefore, $w \in D_u(z, L_u^{\mathrm{K}}(\eta, z))$ for every $z \in X$ and $u \neq 0$, and the desired inclusion follows. □

We now compare the Pringsheim limsup of the transformed sequence with the $\mathfrak{B}$-statistical limsup of the original sequence, and we do so in each seminorm direction.

Lemma 6.2. *Let ρ be a bounded double sequence in X and let $\mathfrak{T}$ be $\mathfrak{B}$ regular. Fix $z \in X$ and $u \neq 0$. Then, for every $j \in J$, one has*

$$P\limsup_{m,n \to \infty} \|(T_j \rho)_{m,n} - z, u\| \leq r_u(z),$$

where $r_u(z) = \mathfrak{B}\text{-}\limsup_u(\rho - z)$ is defined in Section 5.

Proof. Fix $j \in J$, $z \in X$, and $u \neq 0$. Set

$$r = r_u(z) = \mathfrak{B}\text{-}\limsup_{u}(\rho - z).$$

Let $\varepsilon > 0$ and consider the exceptional set

$$E = \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l} - z, u\| > r + \varepsilon\}.$$

By the definition of $\mathfrak{B}$-statistical limit superior, $\overline{\delta}_{\mathfrak{B}}(E) = 0$. Since $\mathfrak{T}$ is $\mathfrak{B}$ regular, one has

$$P\lim_{m,n \to \infty} \sum_{(k,l) \in E} t_{m,n,k,l}(j) = 0.$$

Boundedness of ρ with respect to u implies that

$$M = \sup_{k,l} \|\rho_{k,l} - z, u\|$$

is finite. Using nonnegativity of the coefficients and the 2 norm triangle inequality, we obtain

$$\|(T_j\rho)_{m,n} - z, u\| = \Big\|\sum_{k,l} t_{m,n,k,l}(j)\,(\rho_{k,l} - z) + \Big(\sum_{k,l} t_{m,n,k,l}(j) - 1\Big)z,\ u\Big\|$$
$$\leq \sum_{k,l} t_{m,n,k,l}(j)\,\|\rho_{k,l} - z, u\| + \Big|\sum_{k,l} t_{m,n,k,l}(j) - 1\Big|\,\|z, u\|.$$

Split the first sum into E and its complement. For $(k,l) \notin E$ one has $\|\rho_{k,l} - z, u\| \leq r + \varepsilon$, hence

$$\sum_{(k,l)\notin E} t_{m,n,k,l}(j)\,\|\rho_{k,l} - z, u\| \leq (r + \varepsilon)\sum_{k,l} t_{m,n,k,l}(j).$$

For the part over E, one has the estimate

$$\sum_{(k,l)\in E} t_{m,n,k,l}(j)\,\|\rho_{k,l} - z, u\| \leq M \sum_{(k,l)\in E} t_{m,n,k,l}(j).$$

RH-regularity yields

$$P\lim_{m,n\to\infty}\sum_{k,l} t_{m,n,k,l}(j) = 1 \quad \text{and} \quad P\lim_{m,n\to\infty}\Big|\sum_{k,l} t_{m,n,k,l}(j) - 1\Big| = 0,$$

and these limits are uniform in j by Definition 2.6. Therefore, taking $P\limsup_{m,n\to\infty}$ in the displayed inequality gives

$$P\limsup_{m,n\to\infty}\|(T_j\rho)_{m,n} - z, u\| \leq r + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude

$$P\limsup_{m,n\to\infty}\|(T_j\rho)_{m,n} - z, u\| \leq r = r_u(z).$$

□

We can now prove the Knopp type inclusion by combining the Knopp disk containment with the disk representation of the $\mathfrak{B}$-statistical core from Section 5.

Theorem 6.3. *Let ρ be a bounded double sequence in a 2 normed space X and let $\mathfrak{T}$ be $\mathfrak{B}$ regular. Then, for every $j \in J$, one has*

$$\text{K-core}(T_j\rho) \subseteq \text{core}_{\mathfrak{B}}(\rho).$$

Proof. Fix $j \in J$ and let $w \in \text{K-core}(T_j\rho)$. By Lemma 6.1, for every $z \in X$ and every $u \neq 0$, one has

$$w \in D_u\big(z,\ P\limsup_{m,n\to\infty}\|(T_j\rho)_{m,n} - z, u\|\big).$$

Lemma 6.2 yields

$$P\limsup_{m,n\to\infty}\|(T_j\rho)_{m,n} - z, u\| \leq r_u(z),$$

hence, $w \in D_u(z, r_u(z))$ for all z and u. Therefore,

$$w \in \bigcap_{z\in X}\bigcap_{u\neq 0} D_u(z, r_u(z)).$$

By Theorem 5.2 from Section 5, this intersection equals $\text{core}_{\mathfrak{B}}(\rho)$. Consequently, $w \in \text{core}_{\mathfrak{B}}(\rho)$ and the inclusion follows. □

A direct consequence is obtained by choosing $\mathfrak{T}$ to be the family $\mathfrak{B}$ itself.

Corollary 6.4. *Assume that $\mathfrak{B}$ is $\mathfrak{B}$ regular in the sense of Definition* 2.6. *Then, for every bounded double sequence ρ and every index j, one has*

$$\text{K-core}(B_j\rho) \subseteq \text{core}_{\mathfrak{B}}(\rho).$$

Proof. Apply Theorem 6.3 with $\mathfrak{T} = \mathfrak{B}$. □

The next corollary records the special case in which ρ is $\mathfrak{B}$-statistically convergent. In that situation, the $\mathfrak{B}$-statistical core collapses to a singleton by Proposition 5.4, so the inclusion is automatic but it is useful to state it explicitly.

Corollary 6.5. *Let ρ be $\mathfrak{B}$-statistically convergent to $\mathfrak{h} \in X$ and let $\mathfrak{T}$ be $\mathfrak{B}$ regular. Then, for every $j \in J$, one has*

$$\text{K-core}(T_j\rho) = \{\mathfrak{h}\} \subseteq \text{core}_{\mathfrak{B}}(\rho).$$

Proof. By Proposition 5.4, one has $\text{core}_{\mathfrak{B}}(\rho) = \{\mathfrak{h}\}$. Theorem 6.3 yields $\text{K-core}(T_j\rho) \subseteq \{\mathfrak{h}\}$. Since $\text{K-core}(T_j\rho)$ is nonempty, the equality follows. □

The $\mathfrak{B}$ regularity assumption cannot be omitted. A simple diagonal example shows that RH-regularity alone does not force the inclusion.

Example 6.6. Let $X = \mathbb{R}$ and let $\mathfrak{B}$ be the double Cesàro method. Define a family $\mathfrak{T} = (T_j)$ by

$$t_{m,n,k,l}(j) = \begin{cases} 1 & \text{when } k = m \text{ and } l = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $T_j\rho = \rho$ for every ρ, and each T_j is RH-regular. Let

$$\rho_{k,l} = \begin{cases} 1 & \text{when } k = l, \\ 0 & \text{when } k \neq l. \end{cases}$$

The diagonal set $\{(k,l) \mid k = l\}$ has $\mathfrak{B}$-density zero, so $\text{core}_{\mathfrak{B}}(\rho) = \{0\}$ by Corollary 5.3. On the other hand, the Knopp core of ρ is $[0,1]$ because both values 0 and 1 occur infinitely often in every Pringsheim tail. Therefore, $\text{K-core}(T_j\rho) = \text{K-core}(\rho) = [0,1]$ and the inclusion $\text{K-core}(T_j\rho) \subseteq \text{core}_{\mathfrak{B}}(\rho)$ fails.

The failure is explained by the second condition in Definition 2.6. For the diagonal set $E = \{(k,l) \mid k = l\}$, one has $\overline{\delta}_{\mathfrak{B}}(E) = 0$ but

$$\sum_{(k,l)\in E} t_{m,n,k,l}(j) = \begin{cases} 1 & \text{when } m = n, \\ 0 & \text{when } m \neq n, \end{cases}$$

so the Pringsheim limit required by $\mathfrak{B}$ regularity does not vanish.

Remark 6.7. For single sequences, inclusion of the Knopp core of a transform into a statistical core often forces a regularity condition on the transforming matrix under additional mild assumptions, and this is closely connected with the core theory developed for A statistical methods [6, 7]. In the present double sequence setting, a corresponding necessity statement can be formulated for families of four-dimensional matrices, but we restrict ourselves to the sufficient condition in Definition 2.6.

7. Ideal extensions for double cores

We now replace $\mathfrak{B}$-density smallness by smallness with respect to a strongly admissible ideal on $\mathbb{N}\times\mathbb{N}$. This leads to ideal analogues of the core constructions from Section 5 and allows us to compare density-based and ideal-based geometric limit sets within the same 2-normed framework.

Throughout this section, $\mathcal{I}_2$ denotes a strongly admissible ideal on $\mathbb{N}\times\mathbb{N}$, and $(X, \|\cdot,\cdot\|)$ is a real or complex 2-normed space.

7.1. $\mathcal{I}_2$ core associated with $\mathfrak{B}$

A closed convex set $K \subseteq X$ is called $\mathcal{I}_2$ admissible for ρ if the exceptional set

$$E(K) = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid \rho_{k,l} \notin K\}$$

belongs to $\mathcal{I}_2$. This condition means that $\rho_{k,l} \in K$ holds outside an $\mathcal{I}_2$ small set of indices.

Definition 7.1 ($\mathcal{I}_2$ core)**.** The $\mathcal{I}_2$ core of ρ, denoted by $\operatorname{core}_{\mathfrak{B},\mathcal{I}_2}(\rho)$, is the intersection of all $\mathcal{I}_2$ admissible closed convex sets $K \subseteq X$. If there exists no $\mathcal{I}_2$ admissible closed convex set, then $\operatorname{core}_{\mathfrak{B},\mathcal{I}_2}(\rho)$ is defined to be X.

The definition above depends only on the ideal smallness of exceptional sets. The symbol $\mathfrak{B}$ in the notation indicates that the section is part of the same framework as the $\mathfrak{B}$-statistical core, and the comparison results below clarify when the ideal core refines or enlarges the $\mathfrak{B}$-statistical core.

Definition 7.2 ($\mathcal{I}_2$-boundedness)**.** Let $\rho = (\rho_{k,l})$ be a double sequence in X. We say that ρ is $\mathcal{I}_2$-bounded with respect to a nonzero vector $u \in X$ if there exists $M > 0$ such that

$$\{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\rho_{k,l}, u\| > M\} \in \mathcal{I}_2.$$

If this holds for every nonzero $u \in X$, then ρ is called $\mathcal{I}_2$-bounded.

For $z \in X$ and $u \neq 0$, define

$$r_u^{\mathcal{I}_2}(z) = \inf\Big\{r \geq 0 \;\Big|\; \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid \|\rho_{k,l} - z, u\| > r\} \in \mathcal{I}_2\Big\}.$$

This quantity is the ideal analogue of the $\mathfrak{B}$-statistical limit superior of the scalar reduction, with ideal membership replacing $\mathfrak{B}$-density zero.

Theorem 7.3. *Assume that ρ is $\mathcal{I}_2$-bounded. Then,*

$$\operatorname{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) = \bigcap_{z\in X} \bigcap_{u\neq 0} D_u(z, r_u^{\mathcal{I}_2}(z)),$$

where

$$r_u^{\mathcal{I}_2}(z) = \inf\Big\{r \geq 0 : \{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\rho_{k,l} - z, u\| > r\} \in \mathcal{I}_2\Big\}.$$

Proof. Fix $z \in X$ and $u \neq 0$. Since ρ is $\mathcal{I}_2$-bounded, there exists $M > 0$ such that

$$\{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\rho_{k,l}, u\| > M\} \in \mathcal{I}_2.$$

By the triangle inequality,

$$\|\rho_{k,l} - z, u\| \leq \|\rho_{k,l}, u\| + \|z, u\|,$$

and, therefore,

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\rho_{k,l} - z, u\| > M + \|z, u\|\} \in \mathcal{I}_2.$$

Hence, $r_u^{\mathcal{I}_2}(z) < \infty$.

Now write

$$r = r_u^{\mathcal{I}_2}(z) = \inf\Big\{s \geq 0 : \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\rho_{k,l} - z, u\| > s\} \in \mathcal{I}_2\Big\}.$$

We first show that $\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) \subseteq D_u(z, r)$. Let $\varepsilon > 0$. By the definition of r as an infimum, there exists $s < r + \varepsilon$ such that

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l} - z, u\| > s\} \in \mathcal{I}_2.$$

Since $\{\|\rho_{k,l} - z, u\| > r + \varepsilon\} \subseteq \{\|\rho_{k,l} - z, u\| > s\}$ and $\mathcal{I}_2$ is closed under subsets, we have

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l} - z, u\| > r + \varepsilon\} \in \mathcal{I}_2.$$

Hence, $D_u(z, r+\varepsilon)$ is $\mathcal{I}_2$-admissible. By Definition 7.1, the $\mathcal{I}_2$-core is contained in every $\mathcal{I}_2$-admissible closed convex set, so $\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) \subseteq D_u(z, r+\varepsilon)$ for all $\varepsilon > 0$. Intersecting over $\varepsilon > 0$ yields $\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) \subseteq D_u(z, r)$. Since z and u were arbitrary,

$$\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) \subseteq \bigcap_{z \in X} \bigcap_{u \neq 0} D_u(z, r_u^{\mathcal{I}_2}(z)).$$

For the reverse inclusion, set

$$K = \bigcap_{z \in X} \bigcap_{u \neq 0} D_u(z, r_u^{\mathcal{I}_2}(z)).$$

Let $L \subseteq X$ be any $\mathcal{I}_2$-admissible closed convex set. We show that $K \subseteq L$. Assume that $w \in K$ but $w \notin L$. As in the proof of Theorem 5.2, a separation argument in the locally convex topology induced by $p_u(x) = \|x, u\|$ yields $z \in X$, $u \neq 0$, and $r_0 > 0$ such that

$$L \subseteq D_u(z, r_0) \qquad \text{but} \qquad w \notin D_u(z, r_0).$$

Since L is $\mathcal{I}_2$-admissible and $L \subseteq D_u(z, r_0)$, we have

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} \mid \|\rho_{k,l} - z, u\| > r_0\} \subseteq \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid \rho_{k,l} \notin L\} \in \mathcal{I}_2,$$

hence, $\{\|\rho_{k,l} - z, u\| > r_0\} \in \mathcal{I}_2$ and therefore $r_u^{\mathcal{I}_2}(z) \leq r_0$. Thus, $D_u(z, r_u^{\mathcal{I}_2}(z)) \subseteq D_u(z, r_0)$. Since $w \in K$, we have $w \in D_u(z, r_u^{\mathcal{I}_2}(z)) \subseteq D_u(z, r_0)$, contradicting $w \notin D_u(z, r_0)$. Therefore, $w \in L$ and $K \subseteq L$. As L was arbitrary, $K \subseteq \text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho)$. □

7.2. $\mathcal{I}_2^*$ core and its relation to $\mathcal{I}_2$ core

The ideal $\mathcal{I}_2^*$ principle is typically formulated by restricting attention to an index set from the filter $\mathcal{F}(\mathcal{I}_2)$. To avoid dependence on arbitrary values outside the restricting set, we encode this restriction at the level of admissibility rather than by redefining the sequence outside the set.

A closed convex set $K \subseteq X$ is called $\mathcal{I}_2^*$ admissible for ρ relative to $\mathfrak{B}$ if there exists a set $F \in \mathcal{F}(\mathcal{I}_2)$ such that the set

$$\{(k, l) \in F \mid \rho_{k,l} \notin K\}$$

has $\mathfrak{B}$-density zero.

Definition 7.4 ($\mathcal{I}_2^*$ core)**.** The $\mathcal{I}_2^*$ core of ρ relative to $\mathfrak{B}$, denoted by $\mathrm{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho)$, is the intersection of all $\mathcal{I}_2^*$ admissible closed convex sets $K \subseteq X$. If there exists no such set, then $\mathrm{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho)$ is defined to be X.

The next proposition gives the basic inclusions and clarifies when the two ideal cores coincide.

Proposition 7.5. *For every double sequence ρ, one has*

$$\mathrm{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho) \subseteq \mathrm{core}_{\mathfrak{B},\mathcal{I}_2}(\rho).$$

If the ideal of $\mathfrak{B}$-density zero sets is contained in $\mathcal{I}_2$, meaning that

$$\{E \subseteq \mathbb{N} \times \mathbb{N} \mid \overline{\delta}_{\mathfrak{B}}(E) = 0\} \subseteq \mathcal{I}_2,$$

then

$$\mathrm{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho) = \mathrm{core}_{\mathfrak{B},\mathcal{I}_2}(\rho).$$

Proof. Let K be $\mathcal{I}_2$ admissible. Take $F = (\mathbb{N} \times \mathbb{N}) \setminus E(K)$. Then, $F \in \mathcal{F}(\mathcal{I}_2)$ and $\{(k,l) \in F \mid \rho_{k,l} \notin K\}$ is empty, hence, it has $\mathfrak{B}$-density zero. Thus, every $\mathcal{I}_2$ admissible set is also $\mathcal{I}_2^*$ admissible, and since the family of $\mathcal{I}_2^*$ admissible sets is larger, the intersection defining $\mathrm{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho)$ is contained in the intersection defining $\mathrm{core}_{\mathfrak{B},\mathcal{I}_2}(\rho)$.

Assume now that every $\mathfrak{B}$-density zero set belongs to $\mathcal{I}_2$. Let K be $\mathcal{I}_2^*$ admissible, and choose $F \in \mathcal{F}(\mathcal{I}_2)$ such that the set

$$E_F(K) = \{(k,l) \in F \mid \rho_{k,l} \notin K\}$$

has $\mathfrak{B}$-density zero. Then, $E_F(K) \in \mathcal{I}_2$ by the assumption, and $(\mathbb{N} \times \mathbb{N}) \setminus F \in \mathcal{I}_2$ by definition of the filter. Since

$$E(K) \subseteq E_F(K) \cup ((\mathbb{N} \times \mathbb{N}) \setminus F),$$

it follows that $E(K) \in \mathcal{I}_2$. Hence, K is $\mathcal{I}_2$ admissible. Therefore, the $\mathcal{I}_2^*$ admissible and $\mathcal{I}_2$ admissible families coincide, and the two cores are equal. □

A disk representation can be stated for the $\mathcal{I}_2^*$ core as well. For $z \in X$ and $u \neq 0$, define

$$r_u^{\mathcal{I}_2^*}(z) = \inf\Big\{r \geq 0 \;\Big|\; \exists F \in \mathcal{F}(\mathcal{I}_2) \text{ such that } \overline{\delta}_{\mathfrak{B}}(\{(k,l) \in F \mid \|\rho_{k,l} - z, u\| > r\}) = 0\Big\}.$$

Theorem 7.6. *Assume that ρ is $\mathcal{I}_2$-bounded. Then,*

$$\mathrm{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho) = \bigcap_{z \in X} \bigcap_{u \neq 0} D_u(z, r_u^{\mathcal{I}_2^*}(z)),$$

where

$$r_u^{\mathcal{I}_2^*}(z) = \inf\Big\{r \geq 0 : \exists F \in \mathcal{F}(\mathcal{I}_2) \textit{ such that } \overline{\delta}_{\mathfrak{B}}(\{(k,l) \in F : \|\rho_{k,l} - z, u\| > r\}) = 0\Big\}.$$

Proof. Because ρ is $\mathcal{I}_2$-bounded, the quantities $r_u^{\mathcal{I}_2^*}(z)$ are finite for all $z \in X$ and all $u \neq 0$. The proof then follows the same pattern as that of Theorem 7.3. One uses $\mathcal{I}_2^*$-admissibility to obtain a filter set $F \in \mathcal{F}(\mathcal{I}_2)$ on which the exceptional set is $\mathfrak{B}$-density zero, and conversely one uses a separating neighborhood in the seminorm $p_u(x) = \|x, u\|$ to construct an $\mathcal{I}_2^*$-admissible disk excluding any point outside the intersection. □

Example 7.7. Let $X = \mathbb{R}$ and let $\mathfrak{B}$ be the double Cesàro method. Let $\mathcal{I}_2$ be the ideal consisting of all subsets of $\mathbb{N} \times \mathbb{N}$ contained in a finite union of rows and columns. This ideal is strongly admissible. Define

$$\rho_{k,l} = \begin{cases} 1 & \text{when } k = l, \\ 0 & \text{when } k \neq l. \end{cases}$$

The diagonal set $\{(k,l) \mid k = l\}$ has $\mathfrak{B}$-density zero, hence the $\mathfrak{B}$-statistical core satisfies $\text{core}_{\mathfrak{B}}(\rho) = \{0\}$ by the one-dimensional interval description in Section 5. On the other hand, the diagonal set does not belong to $\mathcal{I}_2$, so any closed convex set $K \subseteq \mathbb{R}$ that is $\mathcal{I}_2$ admissible must contain both 0 and 1. Consequently

$$\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) = [0, 1].$$

For the $\mathcal{I}_2^*$ core, take any $F \in \mathcal{F}(\mathcal{I}_2)$. The set F misses only finitely many rows and columns, hence $F \cap \{(k,l) \mid k = l\}$ is still a diagonal set and it has $\mathfrak{B}$-density zero. Therefore, the interval $(-\infty, \frac{1}{2}]$ is $\mathcal{I}_2^*$ admissible and, similarly, $(-\infty, \varepsilon]$ is $\mathcal{I}_2^*$ admissible for every $\varepsilon > 0$. This yields

$$\text{core}_{\mathfrak{B},\mathcal{I}_2^*}(\rho) = \{0\}.$$

Hence, the inclusion in Proposition 7.5 is strict for this choice of $\mathcal{I}_2$.

Finally, we record a stability property with respect to modifications on an ideal small set.

Proposition 7.8. *Let ρ and σ be double sequences in X. If*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} \mid \rho_{k,l} \neq \sigma_{k,l}\} \in \mathcal{I}_2,$$

then $\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) = \text{core}_{\mathfrak{B},\mathcal{I}_2}(\sigma)$.

Proof. A closed convex set K is $\mathcal{I}_2$ admissible for ρ if, and only if, it is $\mathcal{I}_2$ admissible for σ, because the exceptional sets differ by a subset of the index set where ρ and σ differ, and $\mathcal{I}_2$ is closed under finite unions and taking subsets. Therefore, the defining families coincide and so do their intersections. □

7.3. A Knopp type inclusion under ideal regularity

The Knopp type inclusion from Section 6 admits an ideal analogue when the compatibility requirement is formulated with respect to the ideal $\mathcal{I}_2$. This yields a direct bridge between the transformation theory and the ideal cores introduced above.

Definition 7.9. A family $\mathfrak{T} = (T_j)_{j \in J}$ of nonnegative four-dimensional matrices is called $\mathcal{I}_2$ regular if each T_j is RH-regular with uniform constants and for every set $E \in \mathcal{I}_2$ one has

$$P \lim_{m,n \to \infty} \sum_{(k,l) \in E} t_{m,n,k,l}(j) = 0$$

uniformly in j.

Corollary 7.10. *Let ρ be a bounded double sequence in a 2 normed space X and let $\mathfrak{T}$ be $\mathcal{I}_2$ regular. Then, for every $j \in J$, one has*

$$\text{K-core}(T_j \rho) \subseteq \text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho).$$

Proof. Fix $j \in J$, $z \in X$, and $u \neq 0$. Using nonnegativity and RH-regularity as in Lemma 6.2, one obtains the estimate

$$P\limsup_{m,n\to\infty} \|(T_j\rho)_{m,n} - z, u\| \leq r_u^{\mathcal{I}_2}(z),$$

where $r_u^{\mathcal{I}_2}(z)$ is defined before Theorem 7.3. The only change is that the exceptional set is now taken from $\mathcal{I}_2$ and the vanishing condition is provided by $\mathcal{I}_2$ regularity. Lemma 6.1 then yields

$$\text{K-core}(T_j\rho) \subseteq \bigcap_{z\in X} \bigcap_{u\neq 0} D_u(z, r_u^{\mathcal{I}_2}(z)).$$

Finally, Theorem 7.3 identifies the righthand side with $\text{core}_{\mathfrak{B},\mathcal{I}_2}(\rho)$. □

The corollary shows that the ideal cores are stable under the same type of matrix transformations considered in Section 6 once the negligible sets are measured by $\mathcal{I}_2$ rather than by $\mathfrak{B}$-density. In particular, when $\mathcal{I}_2$ is chosen as an ideal generated by $\mathfrak{B}$-density zero sets, the inclusion reduces to Corollary 6.4 and the ideal framework recovers the $\mathfrak{B}$-statistical theory as a special case.

8. Examples and comparisons

This section illustrates how the core and cluster objects react to three distinct choices. The first choice is sparsity relative to the selected RH-regular density. The second choice is the density itself, since different RH-regular families may assign different weights to the same index set. The third choice is the ideal, which may declare sets negligible even when they are not negligible for the density. The examples are kept minimal and each one is tied to the disk representation of the core and to the extremal characterization via $\mathfrak{B}$-statistical limsup and liminf.

8.1. Sparsity and collapse under a fixed density

Example 8.1. Let $X = \mathbb{R}^2$ with

$$\|(x_1, x_2), (y_1, y_2)\| = |x_1y_2 - x_2y_1|,$$

and let $u = (0, 1)$. Define

$$\rho_{k,l} = \begin{cases} (1,0) & \text{when } k = 2^m \text{ and } l = 3^n \text{ for some } m,n \in \mathbb{N}, \\ (0,0) & \text{otherwise.} \end{cases}$$

Let $\mathfrak{B}$ be the double Cesàro method. The sparse grid

$$G = \{(2^m, 3^n) : m, n \in \mathbb{N}\}$$

has double natural density zero, because

$$\frac{1}{mn}\Big|G \cap (\{1,\dots,m\} \times \{1,\dots,n\})\Big| = \frac{\lfloor \log_2 m\rfloor \lfloor \log_3 n\rfloor}{mn} \longrightarrow 0 \qquad (m,n\to\infty).$$

Hence, G has $\mathfrak{B}$-density zero as well. Therefore, ρ is $\mathfrak{B}$-statistically convergent to $(0, 0)$, and thus

$$\text{core}_{\mathfrak{B}}(\rho) = \{(0, 0)\}.$$

Example 8.1 represents the simplest collapse mechanism. The nonzero values are confined to a $\mathfrak{B}$ negligible set, so every seminorm direction sees the same limiting behavior and the core reduces to the singleton limit.

8.2. Change of the RH-regular density and change of the core

The next example shows that a change in the RH-regular family may change the core of the same sequence. The construction uses an RH-regular method that assigns a prescribed positive weight to the diagonal and distributes the remaining mass in the Cesàro manner.

Example 8.2. Let $X = \mathbb{R}^2$ with the same 2 norm and let $u = (0, 1)$. For $\alpha \in (0, 1)$, define a nonnegative four-dimensional matrix $B^{(\alpha)} = (b^{(\alpha)}_{m,n,k,l})$ by

$$b^{(\alpha)}_{m,n,k,l} = \frac{1-\alpha}{mn}\chi_{\{(k,l)\in\mathbb{N}^2:\ 1\le k\le m,\ 1\le l\le n\}}(k,l) + \frac{\alpha}{s}\chi_{\{(k,l)\in\mathbb{N}^2:\ 1\le k=l\le s\}}(k,l), \qquad s = \min\{m,n\}.$$

Consider the diagonal sequence

$$\rho_{k,l} = \begin{cases} (1,0) & \text{when } k = l, \\ (0,0) & \text{when } k \neq l. \end{cases}$$

Let $\mathfrak{B} = \{B^{(\alpha)}\}$ be the singleton family. For the diagonal set $D = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid k = l\}$, one has $\delta_{\mathfrak{B}}(D) = \alpha$. Hence, the scalar reduction $\|\rho_{k,l}, u\|$ takes the value 1 on a set of positive $\mathfrak{B}$-density and the value 0 on a set of $\mathfrak{B}$-density $1-\alpha$. Therefore,

$$\mathfrak{B}\text{-}\liminf_u \rho = 0, \qquad \mathfrak{B}\text{-}\limsup_u \rho = 1, \qquad \mathrm{core}_{\mathfrak{B}}(\rho) = \{(t,0) \in \mathbb{R}^2 \mid 0 \le t \le 1\}.$$

If instead $\mathfrak{B}$ is the double Cesàro method, then $\overline{\delta}_{\mathfrak{B}}(D) = 0$ and the same sequence satisfies $\mathrm{core}_{\mathfrak{B}}(\rho) = \{(0,0)\}$.

Example 8.2 isolates the density effect. The sequence is fixed and only the method changes. The disk representation from Section 5 explains the outcome since the radii are computed by the corresponding statistical limsup values, which depend on what the method considers negligible.

8.3. Change of the ideal and enlargement of the core

The ideal version can enlarge the core even when the density is fixed. The following example uses an ideal that regards finite unions of rows and columns as negligible.

Example 8.3. Let $X = \mathbb{R}^2$ and $u = (0, 1)$. Let $\mathfrak{B}$ be the double Cesàro method and let $\mathcal{I}_2$ consist of all subsets of $\mathbb{N}\times\mathbb{N}$ contained in finitely many rows and columns. Define

$$\rho_{k,l} = \begin{cases} (1,0) & \text{when } k = l, \\ (0,0) & \text{when } k \neq l. \end{cases}$$

Then, $\mathrm{core}_{\mathfrak{B}}(\rho) = \{(0,0)\}$ because the diagonal has Cesàro density zero. The diagonal does not belong to $\mathcal{I}_2$, hence no $\mathcal{I}_2$ admissible closed convex set can contain $(0,0)$ and exclude $(1,0)$. Consequently,

$$\mathrm{core}_{\mathfrak{B},\mathcal{I}_2}(\rho) = \{(t,0) \in \mathbb{R}^2 \mid 0 \le t \le 1\}.$$

Example 8.3 shows that the ideal and the density measure different notions of smallness. Even when the density core collapses, the ideal core may remain nontrivial, which is exactly what makes the ideal extension a genuine refinement rather than a restatement.

8.4. A compact comparison table

Table 1 summarizes the three mechanisms isolated above and the corresponding core outcomes.

Table 1. Comparison of the core behavior under sparsity, change of density, and change of ideal.

Example	Data of smallness	Resulting $\mathfrak{B}$-core	Resulting ideal core
8.1	a sparse dyadic-triadic grid has Cesàro density zero	$\{(0,0)\}$	not used
8.2	diagonal has density 0 or α	$\{(0,0)\}$ or segment	not used
8.3	$\mathcal{I}_2$ ignores finitely many rows and columns	$\{(0,0)\}$	segment 0 to 1

Remark 8.4. The regularity assumption in the Knopp type inclusion theorem cannot be dropped. The identity type family

$$t_{m,n,k,l}(j) = \begin{cases} 1 & \text{when } (k,l) = (m,n), \\ 0 & \text{otherwise} \end{cases}$$

is RH-regular but it fails $\mathfrak{B}$ regularity because it may concentrate full mass on a $\mathfrak{B}$-density zero set. This is the mechanism behind Example 6.6 from Section 6 and it explains why the additional compatibility requirement is needed in Theorem 6.3.

Remark 8.5. The sparse-collapse mechanism is not tied to any specific choice of sparse set. One may replace the support in Example 8.1 by other subsets of $\mathbb{N} \times \mathbb{N}$ having double natural density zero, such as pairs of powers of 2 or other logarithmically thin grids, and the same Cesàro-collapse phenomenon persists. One may also alter the values on such a sparse support so as to create additional ordinary cluster points while keeping the $\mathfrak{B}$-statistical core unchanged. This illustrates the distinction between topological accumulation and density-based accumulation.

The table and the three examples separate the three effects that drive the theory. Sparsity relative to the density collapses the core. A change of the RH-regular density changes the radii in the disk representation and may enlarge the core. A change of the ideal changes which exceptional sets are allowed and may enlarge the ideal core even when the density core collapses.

The three mechanisms isolated above also clarify the structure of the main results. The Knopp type inclusion theorem requires a compatibility condition because RH-regularity alone does not prevent concentration on negligible index sets. The dependence of the core on the chosen RH-regular family explains why the disk representation and the extremal descriptions are formulated method by method. Finally, the ideal example shows that $\mathcal{I}_2$ smallness may be genuinely different from $\mathfrak{B}$-density smallness, which justifies treating $\mathcal{I}_2$ core and $\mathcal{I}_2^*$ core as separate objects and motivates the additional hypotheses under which they coincide.

9. Conclusions

We developed a core and cluster point framework for double sequences in finite dimensional 2-normed spaces under densities generated by nonnegative RH-regular families. Using $\mathfrak{B}$-density, we introduced $\mathfrak{B}$-statistical lim sup and lim inf for scalar reductions, connected them to the corresponding real cluster behavior, and then defined $\mathfrak{B}$-statistical cluster points and the $\mathfrak{B}$-statistical core. The core admits an intrinsic disk-type representation in terms of the seminorm balls induced by $\|\cdot, u\|$, which provides a transparent geometric interpretation and yields stability under modifications on $\mathfrak{B}$-negligible index sets.

We also established a Knopp-type inclusion theorem for four-dimensional matrix transforms. Under a $\mathfrak{B}$-regularity condition ensuring that the transform does not concentrate mass on $\mathfrak{B}$-density-zero sets, the Knopp core of each transformed double sequence is contained in the $\mathfrak{B}$-statistical core of the original sequence. This clarifies why RH-regularity alone is insufficient and identifies the additional compatibility mechanism needed for core inclusions.

Finally, we incorporated strongly admissible ideals on $\mathbb{N} \times \mathbb{N}$ and introduced the $\mathcal{I}_2$-core and $\mathcal{I}_2^*$-core as ideal refinements of the density-based theory. We proved disk representations in the ideal setting and compared the resulting cores, including the equality statement in Proposition 7.5 under an explicit containment condition on negligible sets. These results are formulated in the setting of 2-normed spaces, but they also point toward a broader line of investigation in more general uniform-space frameworks.

Author contributions

Hong-Zhen Yu: Conceptualization, methodology, formal analysis, writing–original draft, writing–review and editing, funding acquisition; Ömer Kişi: Conceptualization, methodology, formal analysis, writing–original draft, writing–review and editing; Mehmet Gürdal: Conceptualization, methodology, formal analysis, writing–original draft, writing–review and editing; Qing-Bo Cai: Conceptualization, methodology, formal analysis, writing–original draft, writing–review and editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by Fujian Provincial Natural Science Foundation of China (Grant No. 2024J01792).

Conflict of interest

The authors declare no conflicts of interest.

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