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Research article

Automorphisms of the totally isotropic subspace inclusion graph of unitary spaces

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Abstract: Let F_{q^2} be a finite field of q^2 elements, where q is a power of a prime and $\mathbb{U}_n$ an n-dimensional unitary space over F_{q^2}. The inclusion graph of the totally isotropic subspace of $\mathbb{U}_n$, written as $\mathcal{I}n(\mathbb{U}_n)$, is a graph which has all totally isotropic subspaces of $\mathbb{U}_n$ as its vertices and two distinct, totally isotropic subspaces U_1, U_2 of $\mathbb{U}_n$ which are adjacent if and only if $U_1 \subset U_2$ or $U_2 \subset U_1$. In this paper, all automorphisms of $\mathcal{I}n(\mathbb{U}_n)$ are determined definitely.

Keywords: automorphisms; totally isotropic subspace; unitary space; inclusion graph
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1. Introduction

Let F_{q^2} be a finite field of any characteristic and $n \geq 2$ an integer. Then, F_{q^2} has an involutive automorphism

$$- : F_{q^2} \to F_{q^2},$$
$$a \mapsto \overline{a} = a^q,$$

and the fixed field of this automorphism is F_q. Let

$$A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$$

be an $m \times n$ matrix over F_{q^2}. We use $\overline{A}$ to denote the matrix obtained from A by applying the automorphism $a \mapsto \overline{a}$ to all the mn elements of A, that is,

$$\overline{A} = (\overline{a_{ij}})_{1\leq i\leq m, 1\leq j\leq n}.$$

An $n \times n$ matrix H over F_{q^2} is said to be Hermitian, if $\overline{H}^t = H$, where $\overline{H}^t$ is the transpose of $\overline{H}$. Any $n \times n$ nonsingular Hermitian matrix over F_{q^2} is necessarily cogredient to the $n \times n$ identity matrix, and it is also cogredient to

$$H_0 = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix} \quad \text{or} \quad H_1 = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{pmatrix},$$

when $n = 2\nu$ is even or $n = 2\nu + 1$ is odd, respectively, where $I^{(\nu)}$ is the $\nu \times \nu$ identity matrix. In order to cover these two cases, we use the notation $n = 2\nu + \delta$ and H_δ, where $\delta = 0$ or 1. Sometimes, H_δ is also simplified as H. Now, let H be an $n \times n$ nonsingular Hermitian matrix over F_{q^2}. An $n \times n$ matrix T over F_{q^2} is called a unitary matrix with respect to H if $TH\overline{T}^t = H$. All $n \times n$ unitary matrices with respect to a nonsingular Hermitian matrix H form a group with respect to matrix multiplication called the unitary group of degree n with respect to H over F_{q^2} and are denoted by $U_n(F_{q^2}, H)$. Let $S = \{a \in F_{q^2} | a\overline{a} = 1\}$ and $Z_n = \{aI^{(n)} | a \in S\}$, and let $U_n(F_{q^2}, H)/Z_n$ be denoted by $PU_n(F_{q^2}, H)$, which is called the projective unitary group of degree n over F_{q^2}. Let

$$F_{q^2}^{(n)} = \{(a_1, \ldots, a_n) | a_i \in F_{q^2}, i = 1, \ldots, n\}$$

be the n-dimensional row vector space over F_{q^2}. There is an action of $U_n(F_{q^2}, H)$ on $F_{q^2}^{(n)}$ defined as follows:

$$F_{q^2}^{(n)} \times U_n(F_{q^2}, H) \to F_{q^2}^{(n)},$$
$$((x_1, x_2, ..., x_n), T) \mapsto (x_1, x_2, ..., x_n)T.$$

The vector space $F_{q^2}^{(n)}$, with the above action of the unitary group $U_n(F_{q^2}, H)$, is called the n-dimensional unitary space over F_{q^2}. Throughout this paper, let $\mathbb{U}_n$ be n-dimensional unitary space over a finite field F_{q^2}. Let U be an m-dimensional vector subspace of $\mathbb{U}_n$, and we use the same letter U to denote a matrix representation of the vector subspace U. For an $n \times n$ nonsingular Hermitian matrix H, it is clear that $UH\overline{U}^t$ is Hermitian. If $UH\overline{U}^t$ is of rank r, we say that U is a subspace of type (m, r) with respect to H. In particular, subspaces of type $(m, 0)$ with respect to H are called m-dimensional totally isotropic subspaces with respect to H, and subspaces of type (m, m) with respect to H are called m-dimensional nonisotropic subspaces with respect to H. Then, U is called a totally isotropic subspace if and only if $UH\overline{U}^t = 0$ and is nonisotropic if and only if $UH\overline{U}^t$ is nonsingular. A vector $\alpha \in \mathbb{U}_n$ is called totally isotropic if and only if $\alpha H\overline{\alpha}^t = 0$. By [1, Corollary 5.9], totally isotropic subspaces of $\mathbb{U}_n$ are of dimension $\leq \nu = [n/2]$, and ν-dimensional totally isotropic subspaces are called maximal totally isotropic subspaces.

By [1, Lemma 5.12], let P_1 and P_2 be two $m \times n$ matrices of rank m; then, there exists an element $T \in U_n(F_{q^2}, H)$ such that $P_1 = P_2 T$ if and only if $P_1 H\overline{P_1}^t = P_2 H\overline{P_2}^t$.

Denote by $M(m, r; n)$ the set of subspaces of $\mathbb{U}_n$ of type (m, r) with respect to H, and put $N(m, r; n) = |M(m, r; n)|$. By [1, Theorem 5.19], the number of subspaces of type (m, r) in the n-dimensional unitary space over F_{q^2} is given by

$$N(m, r; n) = q^{r(n+r-2m)} \frac{\prod_{i=n+r-2m+1}^{n} (q^i - (-1)^i)}{\prod_{i=1}^{r} (q^i - (-1)^i) \prod_{i=1}^{m-r} (q^{2i} - 1)}.$$

In particular, by [1, Corollary 5.20], the number of m-dimensional totally isotropic subspaces in the n-dimensional unitary space over F_{q^2} is

$$N(m,0;n) = \frac{\prod_{i=n-2m+1}^{n}(q^i-(-1)^i)}{\prod_{i=1}^{m}(q^{2i}-1)}.$$

Let U be a fixed subspace of type (m,r) in $\mathbb{U}_n$. Denote by $\mathcal{M}(m_1,r_1;m,r;n)$ the set of subspaces of type (m_1,r_1) contained in U. Let $N(m_1,r_1;m,r;n) = |\mathcal{M}(m_1,r_1;m,r;n)|$. Then, by [1, Theorem 5.28], we obtain

$$N(m_1,r_1;m,r;n) = \sum_{k=\max\left\{0,\left\lfloor\frac{2m_1-r-r_1+1}{2}\right\rfloor\right\}}^{\min\{m-r,m_1-r_1\}} q^{r_1(r+r_1-2m_1+2k)+2(m_1-k)(m-r-k)} \times \frac{\prod_{i=r+r_1-2m_1+2k+1}^{r}(q^i-(-1)^i)\cdot\prod_{i=m-r-k+1}^{m-r}(q^{2i}-1)}{\prod_{i=1}^{r_1}(q^i-(-1)^i)\cdot\prod_{i=1}^{m_1-r_1-k}(q^{2i}-1)\cdot\prod_{i=1}^{k}(q^{2i}-1)}.$$

Suppose $r_1 = r = 0$. It follows that the number of m_1-dimensional totally isotropic subspaces in $\mathbb{U}_n$ contained in U is

$$N(m_1,0;m,0;n) = \frac{\prod\limits_{i=m-m_1+1}^{m}(q^{2i}-1)}{\prod\limits_{i=1}^{m_1}(q^{2i}-1)}.$$

Now, let U_1 be a fixed subspace of type (m_1,r_1) in $\mathbb{U}_n$. Denote by $\mathcal{M}'(m_1,r_1;m,r;n)$ the set of subspaces of type (m,r) containing U_1. Let $N'(m_1,r_1;m,r;n) = |\mathcal{M}'(m_1,r_1;m,r;n)|$. Then, by [1, Theorem 5.37], we have

$$N'(m_1,r_1;m,r;n) = \sum_{k=\max\{0,\lfloor\frac{2m_1-r-r_1+1}{2}\rfloor\}}^{min\{m-r,m_1-r_1\}} q^{(n-2m+r)(r+r_1-2m_1+2k)+2(n-m-k)(m_1-r_1-k)} \times \frac{\prod_{i=r+r_1-2m_1+2k+1}^{n-2m_1+r_1}(q^i-(-1)^i)\cdot\prod_{i=m_1-r_1-k+1}^{m_1-r_1}(q^{2i}-1)}{\prod_{i=1}^{n-2m+r}(q^i-(-1)^i)\cdot\prod_{i=1}^{m-r-k}(q^{2i}-1)\cdot\prod_{i=1}^{k}(q^{2i}-1)}.$$

Suppose $r_1 = r = 0$. It follows that the number of m-dimensional totally isotropic subspaces in $\mathbb{U}_n$ containing U_1 is

$$N'(m_1,0;m,0;n) = \frac{\prod\limits_{i=n-2m+1}^{n-2m_1}(q^i-(-1)^i)}{\prod\limits_{i=1}^{m-m_1}(q^{2i}-1)}.$$

Generally, determining the full automorphisms of a graph is an important and yet difficult problem both in graph theory and in algebraic theory. Searching the literature, we find that a few results have been known on the automorphisms of a graph associated to a vector space; see [2–4]. In particular, we consider automorphisms of the graphs related to the finite classical spaces. For symplectic spaces, see [5]. For orthogonal spaces, see [6–8]. For unitary spaces, see [9,10]. In [11], Wan et al. introduced the concept of unitary graphs over finite fields and determined their automorphism groups. In [12], the authors introduced the symplectic totally isotropic subspace inclusion graphs over finite fields and determined their automorphism groups. Motivated by previous studies, we introduce a new graph on nontrivial unitary spaces over a finite field for studying the interplay between properties of unitary totally isotropic subspaces and the structure of graphs.

The totally isotropic subspace inclusion graph relative to $\mathbb{U}_n$ over F_{q^2}, denoted by $\mathcal{I}n(\mathbb{U}_n)$, is the graph defined on all totally isotropic subspaces of $\mathbb{U}_n$ with an edge between vertices U_1 and U_2, denoted by $U_1 \sim U_2$ if and only if $U_1 \subset U_2$ or $U_2 \subset U_1$. The set of all vertices and all edges of $\mathcal{I}n(\mathbb{U}_n)$ is denoted by $V(\mathcal{I}n(\mathbb{U}_n))$ and $E(\mathcal{I}n(\mathbb{U}_n))$, respectively. When $n = 2$ or 3, we know the dimension of a totally isotropic subspace of $\mathbb{U}_n$ is 1. Clearly, $\mathcal{I}n(\mathbb{U}_n)$ is a null graph (has no edge), and any permutation of vertices of $\mathcal{I}n(\mathbb{U}_n)$ is an automorphism of $\mathcal{I}n(\mathbb{U}_n)$. Thus, the automorphism group of $\mathcal{I}n(\mathbb{U}_n)$ is isomorphic to a symmetric group. Next, we always assume that $n \geq 4$. In this paper, we consider the automorphisms of $\mathcal{I}n(\mathbb{U}_n)$ for the case when $n \geq 4$. We first introduce two classes of standard automorphisms of $\mathcal{I}n(\mathbb{U}_n)$, and then we prove that every automorphism of $\mathcal{I}n(\mathbb{U}_n)$ can be generated by some standard automorphisms.

- **Unitary automorphism**
  Let U be an m-dimensional totally isotropic subspace and $T \in U_n(F_{q^2}, H)$. We define σ_T from $V(\mathcal{I}n(\mathbb{U}_n))$ to itself by
  $$U \mapsto UT.$$
  Obviously, $U_1 \subset U_2$ if and only if $\sigma_T(U_1) \subset \sigma_T(U_2)$. It follows that σ_T is an automorphism of $\mathcal{I}n(\mathbb{U}_n)$, which is called a unitary automorphism of $\mathcal{I}n(\mathbb{U}_n)$.

Example 1.1. *Let $q = 2$ so that $F_{q^2} = F_4 = \{0, 1, \omega, \omega^2\}$ with $\omega^2 + \omega + 1 = 0$. Let $\mathbb{U}_4$ be the 4-dimensional unitary space over F_4 and*

$$H = \begin{pmatrix} 0 & I^{(2)} \\ I^{(2)} & 0 \end{pmatrix}.$$

Take

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is easy to check that $TH\overline{T}^t = H$, and thus, T is a unitary matrix. Consider the subspaces

$$P_1 = [e_2 + e_4],\ \ P_2 = [e_2 + e_4,\ \ e_1 + e_2 + e_4],\ \ Q_1 = [e_1 + e_3],\ \ Q_2 = [e_1 + e_3,\ \ e_1 + e_2 + e_3],$$

all of which are clearly totally isotropic. Moreover, we have

$$\sigma_T(P_1) = P_1 T = Q_1,\ \ \sigma_T(P_2) = P_2 T = Q_2,$$

together with the inclusions $P_1 \subset P_2$ and $Q_1 \subset Q_2$. Consequently, the map σ_T induced by T is a unitary automorphism.

- **Field automorphism**

  Let π be an automorphism of the base field F_{q^2}, and let σ_π be the mapping on $\mathbb{U}_n$ defined by

$$[\sum_{i=1}^{n} a_i e_i] \mapsto [\sum_{i=1}^{n} \pi(a_i) e_i].$$

  Then, we define a mapping on the vertex set of $\mathcal{I}n(\mathbb{U}_n)$, also written as σ_π, by

$$\sigma_\pi(U) = \{\sigma_\pi(\alpha) | \alpha \in U\}, \quad \forall\, U \in V(\mathcal{I}n(\mathbb{U}_n)).$$

  We can easily see that $U_1 \subset U_2$ if and only if $\sigma_\pi(U_1) \subset \sigma_\pi(U_2)$, which implies that the mapping σ_π on the vertices of $\mathcal{I}n(\mathbb{U}_n)$ is an automorphism of $\mathcal{I}n(\mathbb{U}_n)$. Sometimes, we also call σ_π to be the field automorphism of $\mathcal{I}n(\mathbb{U}_n)$.

Example 1.2. *Keep the same unitary space $\mathbb{U}_4$ over F_4. Let*

$$\pi : F_4 \to F_4, \; \pi(a) = a^2$$

be the Frobenius automorphism. It is easy to check that

$$P_1 = [e_1 + \omega e_2], \;\; Q_1 = [e_1 + \omega^2 e_2], \;\; P_2 = [e_1 + \omega e_2, \;\; e_3 + \omega e_4], \;\; Q_2 = [e_1 + \omega^2 e_2, \;\; e_3 + \omega^2 e_4]$$

are totally isotropic subspaces. Moreover, we have

$$\sigma_\pi(P_1) = Q_1, \;\; \sigma_\pi(P_2) = Q_2$$

together with the inclusions $P_1 \subset P_2$ and $Q_1 \subset Q_2$. Consequently, the map σ_π induced by π is a field automorphism.

2. Automorphisms of $\mathcal{I}n(\mathbb{U}_n)$

Theorem 2.1. *Let σ be an automorphism of $\mathcal{I}n(\mathbb{U}_n)$. Then, a mapping σ on $V(\mathcal{I}n(\mathbb{U}_n))$ is an automorphism of $\mathcal{I}n(\mathbb{U}_n)$ if and only if σ can be uniquely decomposed as $\sigma = \sigma_T \circ \sigma_\pi$, where σ_T and σ_π are two standard automorphisms defined as above.*

We now give a proof of Theorem 2.1. The sufficiency condition is obvious. For the necessity, we complete the proof by establishing several lemmas. In [13], the degree of $x \in G$ is defined as the number of edges of the form $x \sim y$ in G and is denoted by $\deg(x)$.

Lemma 2.2. *Let U be an m-dimensional totally isotropic subspace. Then, the degree of U in $\mathcal{I}n(\mathbb{U}_n)$ is*

$$\deg(U) = \sum_{k=1}^{m-1} \frac{\prod\limits_{i=m-k+1}^{m} (q^{2i} - 1)}{\prod\limits_{i=1}^{k} (q^{2i} - 1)} + \sum_{k=m+1}^{[\frac{n}{2}]} \frac{\prod\limits_{i=n-2k+1}^{n-2m} (q^{i} - (-1)^{i})}{\prod\limits_{i=1}^{k-m} (q^{2i} - 1)}.$$

Proof. It is easily seen that $\deg(U)$ is the number of vertices adjacent to U in $\mathcal{I}n(\mathbb{U}_n)$, that is, the number of totally isotropic subspaces with inclusion relation with U in $\mathbb{U}_n$. From the above preliminaries, we obtain that the number of totally isotropic subspaces of $\mathbb{U}_n$ properly contained in U is

$$\sum_{k=1}^{m-1} N(k,0;m,0;n) = \sum_{k=1}^{m-1} \frac{\prod\limits_{i=m-k+1}^{m}(q^{2i}-1)}{\prod\limits_{i=1}^{k}(q^{2i}-1)}.$$

Moreover, we obtain that the number of totally isotropic subspaces of $\mathbb{U}_n$ properly containing U is

$$\sum_{k=m+1}^{[\frac{n}{2}]} N'(m,0;k,0;n) = \sum_{k=m+1}^{[\frac{n}{2}]} \frac{\prod\limits_{i=n-2k+1}^{n-2m}(q^{i}-(-1)^{i})}{\prod\limits_{i=1}^{k-m}(q^{2i}-1)}.$$

Therefore,

$$\begin{aligned}\deg(U) &= \sum_{k=1}^{m-1} N(k,0;m,0;n) + \sum_{k=m+1}^{[\frac{n}{2}]} N'(m,0;k,0;n)\\ &= \sum_{k=1}^{m-1} \frac{\prod\limits_{i=m-k+1}^{m}(q^{2i}-1)}{\prod\limits_{i=1}^{k}(q^{2i}-1)} + \sum_{k=m+1}^{[\frac{n}{2}]} \frac{\prod\limits_{i=n-2k+1}^{n-2m}(q^{i}-(-1)^{i})}{\prod\limits_{i=1}^{k-m}(q^{2i}-1)}.\end{aligned}$$

This completes the proof. □

The following lemma follows from Lemma 2.2 immediately.

Lemma 2.3. *Let σ be an automorphism of $\mathcal{I}n(\mathbb{U}_n)$ and U be a 1-dimensional totally isotropic subspace of $\mathcal{I}n(\mathbb{U}_n)$. Then, $\dim(\sigma(U)) = 1$.*

Proof. Suppose that N is an m-dimensional ($m \neq 1$) totally isotropic subspace of $\mathcal{I}n(\mathbb{U}_n)$. By Lemma 2.2, we obtain

$$\deg(U) = \sum_{k=2}^{\nu} \frac{\prod\limits_{i=n-2k+1}^{n-2}(q^{i}-(-1)^{i})}{\prod\limits_{i=1}^{k-1}(q^{2i}-1)}$$

and

$$\deg(N) = \sum_{k=1}^{m-1} \frac{\prod\limits_{i=m-k+1}^{m}(q^{2i}-1)}{\prod\limits_{i=1}^{k}(q^{2i}-1)} + \sum_{k=m+1}^{\nu} \frac{\prod\limits_{i=n-2k+1}^{n-2m}(q^{i}-(-1)^{i})}{\prod\limits_{i=1}^{k-m}(q^{2i}-1)}.$$

We only prove the assertion for the case when $2 \le m < (\nu+1)/2$. The proof for the case when $(\nu+1)/2 \le m \le \nu$ is similar and thus omitted. Let $2 \le m < (\nu+1)/2$. Then,

$$\begin{aligned}
\deg(U)-\deg(N) &= \sum_{k=2}^{\nu} \frac{\prod_{i=n-2k+1}^{n-2}(q^i-(-1)^i)}{\prod_{i=1}^{k-1}(q^{2i}-1)} - \sum_{k=1}^{m-1} \frac{\prod_{i=m-k+1}^{m}(q^{2i}-1)}{\prod_{i=1}^{k}(q^{2i}-1)} - \sum_{k=m+1}^{\nu} \frac{\prod_{i=n-2k+1}^{n-2m}(q^i-(-1)^i)}{\prod_{i=1}^{k-m}(q^{2i}-1)} \\
&= \frac{(q^{n-3}+1)(q^{n-2}-1)-(q^{2m}-1)-(q^{n-2m-1}+1)(q^{n-2m}-1)}{q^2-1} \\
&\quad + \frac{\prod_{i=n-5}^{n-2}(q^i-(-1)^i) - \prod_{i=m-1}^{m}(q^{2i}-1) - \prod_{i=n-2m-3}^{n-2m}(q^i-(-1)^i)}{(q^2-1)(q^4-1)} + \cdots \\
&\quad + \frac{\prod_{i=n-2m+1}^{n-2}(q^i-(-1)^i) - \prod_{i=2}^{m}(q^{2i}-1) - \prod_{i=n-4m+3}^{n-2m}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(m-1)}-1)} \\
&\quad + \frac{\prod_{i=n-2m+1}^{n-2}(q^i-(-1)^i) - \prod_{i=n-4m+1}^{n-2m}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2m}-1)} + \cdots + \frac{\prod_{i=2m-1}^{n-2}(q^i-(-1)^i) - \prod_{i=n-2\nu+1}^{n-2m}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-m)}-1)} \\
&\quad + \frac{\prod_{i=2m-3}^{n-2}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-m+1)}-1)} + \cdots + \frac{\prod_{i=n-2\nu+1}^{n-2}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-1)}-1)}. \\
&= \sum_{k=1}^{m-1} \frac{\prod_{i=n-2k-1}^{n-2}(q^i-(-1)^i) - \prod_{i=m-k+1}^{m}(q^{2i}-1) - \prod_{i=n-2m+1-2k}^{n-2m}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2k}-1)} \\
&\quad + \sum_{k=m}^{\nu-m} \frac{\prod_{i=n-2k-1}^{n-2}(q^i-(-1)^i) - \prod_{i=n-2m+1-2k}^{n-2m}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2k}-1)} \\
&\quad + \frac{\prod_{i=2m-3}^{n-2}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-m+1)}-1)} + \cdots + \frac{\prod_{i=n-2\nu+1}^{n-2}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-1)}-1)} \\
&= \sum_{k=1}^{m-1} \frac{A-B-C}{(q^2-1)(q^4-1)\cdots(q^{2k}-1)} + \sum_{k=m}^{\nu-m} \frac{A-C}{(q^2-1)(q^4-1)\cdots(q^{2k}-1)} + D,
\end{aligned}$$

where

$$A = \prod_{i=n-2k-1}^{n-2}(q^i-(-1)^i), \quad B = \prod_{i=m-k+1}^{m}(q^{2i}-1), \quad C = \prod_{i=n-2m+1-2k}^{n-2m}(q^i-(-1)^i),$$

and

$$D = \frac{\prod_{i=2m-3}^{n-2}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-m+1)}-1)} + \cdots + \frac{\prod_{i=n-2\nu+1}^{n-2}(q^i-(-1)^i)}{(q^2-1)(q^4-1)\cdots(q^{2(\nu-1)}-1)}.$$

It follows immediately that $D > 0$ with $q \ge 2$.

Next, we aim to show that $A-B-C > 0$ for $1 \le k \le m-1$ and that $A-C > 0$ for $m \le k \le \nu-m$. We first determine the highest power of q appearing in each product. For A, B, and C, it is easy to see that the highest powers of q are $D_A = k(2n-2k-3)$, $D_B = k(2m-k+1)$, and $D_C = k(2n-4m-2k+1)$, respectively. Because $2 \le m < (\nu+1)/2$, and $\nu = [n/2]$, we have $n > 4m$. From $1 \le k \le m-1$, we also conclude that

$$\begin{aligned} D_A - D_B &= k(2n-2k-3) - k(2m-k+1) \\ &= k(2n-2m-k-4) > k(8m-2m-m+1-4) \\ &= k(5m-3) \ge 7, \\ D_A - D_C &= k(2n-2k-3) - k(2n-4m-2k+1) \\ &= k(4m-4) \ge 4. \end{aligned}$$

Furthermore, we establish upper and lower bounds for A, B, and C in terms of powers of q.

Lower bound for A: Pairing the factors in A as $(q^{2i-1}+1)(q^{2i}-1)$, we observe that

$$(q^{2i-1}+1)(q^{2i}-1) > q^{4i-1}.$$

Multiplying this inequality over all k pairs yields $A > q^{D_A}$.

Upper bound for B: It is clear that $B < q^{D_B}$. Because $D_B < D_A - 7$, we obtain $B < q^{D_A-7}$.

Upper bound for C: For each factor in C, we have

$$(q^{2i-1}+1)(q^{2i}-1) > q^{4i-1} + q^{2i} \le q^{4i-1}(1+\frac{1}{q}) \le 1.5q^{4i-1},$$

where $q \ge 2$. Multiplying this estimate over all k factors gives $C < 1.5^k q^{D_C}$. From the earlier degree comparison, $D_A - D_C \ge 4k$. Hence,

$$C < 1.5^k q^{D_A-4k} = (\frac{1.5}{q^4})^k q^{D_A} \le (\frac{3}{32})^k q^{D_A} \le \frac{3}{32} q^{D_A}.$$

Combining these bounds, we obtain

$$B + C < q^{D_A-7} + \frac{3}{32} q^{D_A} = (\frac{1}{q^7} + \frac{3}{32}) q^{D_A} > (\frac{1}{128} + \frac{3}{32}) q^{D_A} = \frac{13}{128} q^{D_A}.$$

Because $A > q^{D_A}$, it follows that $B + C < A$, and consequently, $A - B - C > 0$.

On the other hand, it is easy to see that A and C are each products of $2k$ factors. The rth factor of A is $q^{n-2k-2+r} - (-1)^{n-2k-2+r}$, and the rth factor of C is $q^{n-2m-2k+r} - (-1)^{n-2m-2k+r}$. Then, we conclude that

$$(q^{n-2k-2+r} - (-1)^{n-2k-2+r}) - (q^{n-2m-2k+r} - (-1)^{n-2m-2k+r}) = q^{n-2k+r}(q^{-2} - q^{-2m}) > 0$$

for each $1 \le r \le 2k$. Because every factor in A is strictly greater than its corresponding positive factor in C, the product A is greater than the product C. Therefore, we obtain $A - C > 0$.

Then, we deduce that

$$\deg(U) - \deg(N) = \sum_{k=1}^{m-1} \frac{A-B-C}{(q^2-1)(q^4-1)\cdots(q^{2k}-1)} + \sum_{k=m}^{\nu-m} \frac{A-C}{(q^2-1)(q^4-1)\cdots(q^{2k}-1)} + D > 0,$$

which implies that only the 1-dimensional totally isotropic subspace has the same degree as U. Therefore, we have $dim(\sigma(U)) = 1$. This completes the proof. □

Lemma 2.4. *Let σ be an automorphism of $\mathcal{I}n(\mathbb{U}_n)$ and U be an m-dimensional totally isotropic subspace of $\mathcal{I}n(\mathbb{U}_n)$. Then, σ sends U to a totally isotropic subspace of the same dimension for $1 \leq m \leq \nu$.*

Proof. Obviously, the lemma holds for $m = 1$. Suppose $2 \leq m \leq \nu$. For any $\alpha \in U$, we have $[\alpha] \sim U$. After applying σ to it, we obtain $\sigma([\alpha]) \sim \sigma(U)$. By Lemma 2.3, it holds that $\sigma([\alpha]) \subset \sigma(U)$. This indicates that the number of one dimensional totally isotropic subspaces of $\sigma(U)$ is greater than or equal to the number of one dimensional totally isotropic subspaces of U. Then, we conclude that $dim(\sigma(U)) \geq dim(U) = m$.

Let W be a maximal totally isotropic subspace containing U. Clearly, $U \sim W$, and $\sigma(U) \sim \sigma(W)$. The discussion of the above paragraph implies that $dim(\sigma(W)) \geq dim(W) = \nu$. Moreover, totally isotropic subspaces of $\mathbb{U}_n$ are of dimension $\leq \nu$. Hence, we have $dim(\sigma(W)) = \nu$. It follows that $\sigma(U) \subset \sigma(W)$, which further implies that the number of maximal totally isotropic subspaces containing U is not greater than the number of maximal totally isotropic subspaces containing $\sigma(U)$. That is, $dim(\sigma(U)) \leq dim(U) = m$. Therefore, $dim(\sigma(U)) = m$.

This completes the proof. □

Lemma 2.5. *Let σ be an automorphism of $\mathcal{I}n(\mathbb{U}_n)$ and U be an m-dimensional totally isotropic subspace spanned by $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $1 \leq m \leq \nu$. Then,*

$$\sigma(U) = [\sigma(\alpha_1), \sigma(\alpha_2), \ldots, \sigma(\alpha_m)].$$

Proof. Clearly, the result holds when $m = 1$. Next, we consider $2 \leq m \leq \nu$. As $[\alpha_i] \subset U$, it holds that $\sigma([\alpha_i]) \subset \sigma(U)$ for each $i = 1, 2, \ldots, m$. That is, $[\sigma([\alpha_1]), \sigma([\alpha_2]), \ldots, \sigma([\alpha_m])] \subset \sigma(U)$. We assert that $\{\sigma([\alpha_1]), \sigma([\alpha_2]), \ldots, \sigma([\alpha_m])\}$ are linearly independent, and thus,

$$dim([\sigma([\alpha_1]), \sigma([\alpha_2]), \ldots, \sigma([\alpha_m])]) = dim(\sigma(U)) = m,$$

thus completing the proof. □

Indeed, if $\sigma([\alpha_1]), \sigma([\alpha_2]), \ldots, \sigma([\alpha_m])$ are linearly dependent, then there exist elements $k_1, k_2, \ldots, k_m$ such that

$$k_1\sigma([\alpha_1]) + k_2\sigma([\alpha_2]) + \cdots + k_m\sigma([\alpha_m]) = 0,$$

where $k_i \in F_{q^2}$ is not all zero for $i = 1, \ldots, m$. Without loss of generality, let $k_1 \neq 0$. Then, we get

$$\sigma([\alpha_1]) = -k_1{}^{-1}k_2\sigma([\alpha_2]) - k_1{}^{-1}k_3\sigma([\alpha_3]) - \cdots - k_1{}^{-1}k_m\sigma([\alpha_m]).$$

It follows that $[\sigma([\alpha_1])] \subset [\sigma([\alpha_2]), \sigma([\alpha_3]), \cdots, \sigma([\alpha_m])]$.

Moreover, we also have $[\sigma([\alpha_i])] \subset [\sigma([\alpha_2]), \sigma([\alpha_3]), \cdots, \sigma([\alpha_m])]$ for all $i = 2, 3, \ldots, m$. By applying σ^{-1}, we conclude that $[\alpha_1, \alpha_2, \ldots, \alpha_m] \subset [\alpha_2, \alpha_3, \ldots, \alpha_m]$. This is a contradiction. Consequently,

$$\sigma(U) = [\sigma(\alpha_1), \sigma(\alpha_2), \ldots, \sigma(\alpha_m)].$$

Lemma 2.6. *Let σ be an automorphism of $\mathcal{I}n(\mathbb{U}_n)$ so that there exists a matrix $T_n \in U_n(F_{q^2}, H)$ such that $\sigma_{T_n} \circ \sigma([e_i]) = [e_i]$ in $\mathcal{I}n(\mathbb{U}_n)$, where $i = 1, 2, \ldots, 2\nu$.*

Proof. By Lemma 2.3, we can assume that $\sigma([e_i]) = [\alpha_i]$ for $i = 1, 2, \ldots, 2\nu$, where α_i is a nonzero isotropic vector of $\mathbb{U}_n$. It is obvious that $\{\alpha_1, \alpha_2, \ldots, \alpha_{2\nu}\}$ are linearly independent. Moreover, $\alpha_i H \overline{\alpha_j}^t = 0$ for $|i-j| \neq \nu$, and $\alpha_i H \overline{\alpha_j}^t \neq 0$ for $|i-j| = \nu$. It follows that

$$\begin{pmatrix} e_1 \\ \vdots \\ e_\nu \\ e_{\nu+1} \\ \vdots \\ e_{2\nu} \end{pmatrix} H \overline{\begin{pmatrix} e_1 \\ \vdots \\ e_\nu \\ e_{\nu+1} \\ \vdots \\ e_{2\nu} \end{pmatrix}}^t = \begin{pmatrix} \frac{\alpha_1}{\alpha_1 H \overline{\alpha_{\nu+1}}^t} \\ \vdots \\ \frac{\alpha_\nu}{\alpha_\nu H \overline{\alpha_{2\nu}}^t} \\ \alpha_{\nu+1} \\ \vdots \\ \alpha_{2\nu} \end{pmatrix} H \overline{\begin{pmatrix} \frac{\alpha_1}{\alpha_1 H \overline{\alpha_{\nu+1}}^t} \\ \vdots \\ \frac{\alpha_\nu}{\alpha_\nu H \overline{\alpha_{2\nu}}^t} \\ \alpha_{\nu+1} \\ \vdots \\ \alpha_{2\nu} \end{pmatrix}}^t .$$

By [1, Lemma 5.12], there exists $T_n \in U_n(F_{q^2}, H)$ such that

$$\begin{pmatrix} e_1 \\ \vdots \\ e_\nu \\ e_{\nu+1} \\ \vdots \\ e_{2\nu} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{\alpha_1 H \overline{\alpha_{\nu+1}}^t} \\ \vdots \\ \frac{\alpha_\nu}{\alpha_\nu H \overline{\alpha_{2\nu}}^t} \\ \alpha_{\nu+1} \\ \vdots \\ \alpha_{2\nu} \end{pmatrix} T_n.$$

Therefore, there exists a unitary matrix T_n such that $\sigma_{T_n} \circ \sigma$ fixes each $[e_i]$ of $\mathcal{I}n(\mathbb{U}_n)$ for $i = 1, 2, \ldots, 2\nu$. Thus, the proof is completed. □

In the following, we denote $\sigma_{T_n} \circ \sigma$ by σ_1, and we will denote $[\sum_{i=1}^{n} a_i e_i]$ by $[a_1, a_2, \ldots, a_n]$. Note that σ_1 sends every totally isotropic subspace to a totally isotropic subspace of equal dimension. Next, we will consider the action of an automorphism on any one dimensional totally isotropic subspace of $\mathbb{U}_n$.

Lemma 2.7. *Let $[\sum_{i=1}^{n} a_i e_i]$ in $\mathcal{I}n(\mathbb{U}_n)$, and suppose that*

$$\sigma_1([\sum_{i=1}^{n} a_i e_i]) = [\sum_{i=1}^{n} b_i e_i].$$

Then, for each $j = 1, 2, \ldots, 2\nu$, it holds that $a_j = 0$ if and only if $b_j = 0$.

Proof. If $a_j = 0$, then $[e_{\nu+j}, \sum_{i=1}^{n} a_i e_i]$ is totally isotropic. By Lemmas 2.5 and 2.6, we have

$$\sigma_1([e_{\nu+j}, \sum_{i=1}^{n} a_i e_i]) = [e_{\nu+j}, \sum_{i=1}^{n} b_i e_i],$$

which is also totally isotropic. It follows that $b_j = 0$. If $b_j = 0$, then $[e_{\nu+j}, \sum_{i=1}^{n} b_i e_i]$ is totally isotropic. By considering σ_1, we deduce that $[e_{\nu+j}, \sum_{i=1}^{n} a_i e_i]$ is also totally isotropic. Hence, $a_j = 0$. □

Lemma 2.8. *When $n = 2\nu + 1$, if $[a_1, a_2, \ldots, a_{2\nu+1}] \in V(\mathcal{I}n(\mathbb{U}_{2\nu+1}))$, and $[a_1, a_2, \ldots, a_{2\nu}] \in V(\mathcal{I}n(\mathbb{U}_{2\nu}))$, then $a_{2\nu+1} = 0$.*

Proof. Because $[a_1, a_2, \ldots, a_{2\nu+1}] \in V(\mathcal{I}n(\mathbb{U}_{2\nu+1}))$ and $[a_1, a_2, \ldots, a_{2\nu}] \in V(\mathcal{I}n(\mathbb{U}_{2\nu}))$, it follows that

$$a_1\overline{a_{\nu+1}} + a_2\overline{a_{\nu+2}} + \cdots + a_\nu\overline{a_{2\nu}} + a_{\nu+1}\overline{a_1} + \cdots + a_{2\nu}\overline{a_\nu} = 0,$$

and

$$a_1\overline{a_{\nu+1}} + a_2\overline{a_{\nu+2}} + \cdots + a_\nu\overline{a_{2\nu}} + a_{\nu+1}\overline{a_1} + \cdots + a_{2\nu}\overline{a_\nu} + a_{2\nu+1}\overline{a_{2\nu+1}} = 0,$$

which together suggest that $a_{2\nu+1}\overline{a_{2\nu+1}} = {a_{2\nu+1}}^{q+1} = 0$. In a finite field, no positive integer power of a nonzero element can be zero, so we have $a_{2\nu+1} = 0$. □

For $1 \leq i < j \leq 2\nu$, $a, b \in F_{q^2}$, Lemmas 2.7 and 2.8 show that σ_1 sends $[e_i + ae_j]$ to a vertex of the form $[e_i + be_j]$. Thus, we can define a function π_i on F_{q^2} such that $\pi_i(0) = 0$, and $\sigma_1([e_{\nu+1} + ae_i]) = [e_{\nu+1} + \pi_i(a)e_i]$, where $i = 2, \ldots, \nu, \nu + 2, \ldots, 2\nu$. Next, we will study the other properties of π_i.

We consider first the case $n = 2\nu$.

Lemma 2.9. *Suppose that $[e_1 + \sum_{j=2}^{\nu} a_je_j + \sum_{j=\nu+2}^{2\nu} a_je_j] \in V(\mathcal{I}n(\mathbb{U}_{2\nu}))$, and*

$$\sigma_1([e_1 + \sum_{j=2}^{\nu} a_je_j + \sum_{j=\nu+2}^{2\nu} a_je_j]) = [e_1 + \sum_{j=2}^{\nu} b_je_j + \sum_{j=\nu+2}^{2\nu} b_je_j].$$

For $2 \leq k \leq \nu$, we have

$$b_k = -\overline{\pi_{\nu+k}(-\overline{a_k}^{-1})}^{-1} \quad \text{if} \;\; a_k \neq 0 \tag{2.1}$$

and

$$b_{\nu+k} = -\overline{\pi_k(-\overline{a_{\nu+k}}^{-1})}^{-1} \quad \text{if} \;\; a_{\nu+k} \neq 0. \tag{2.2}$$

Proof. If $a_k \neq 0$ for some $2 \leq k \leq \nu$, then $[e_1 + \sum_{j=2}^{2\nu} a_je_j, e_{\nu+1} - \overline{a_k}^{-1}e_{\nu+k}]$ is a totally isotropic subspace. Applying σ_1 to it, we deduce that

$$[e_1 + \sum_{j=2}^{2\nu} b_je_j, e_{\nu+1} + \pi_{\nu+k}(-\overline{a_k}^{-1})e_{\nu+k}]$$

is also a totally isotropic subspace, which means that

$$b_k = -\overline{\pi_{\nu+k}(-\overline{a_k}^{-1})}^{-1}.$$

If $a_{\nu+k} \neq 0$ for some $2 \leq k \leq \nu$, then $[e_1 + \sum_{j=2}^{2\nu} a_je_j, e_{\nu+1} - \overline{a_{\nu+k}}^{-1}e_k]$ is totally isotropic, which suggests that

$$[e_1 + \sum_{j=2}^{2\nu} b_je_j, e_{\nu+1} + \pi_k(-\overline{a_{\nu+k}}^{-1})e_k]$$

is also totally isotropic. That is,

$$b_{\nu+k} = -\overline{\pi_k(-\overline{a_{\nu+k}}^{-1})}^{-1}.$$

Therefore, the lemma follows. □

Lemma 2.10. *Suppose* $[e_i + \sum_{j=i+1}^{\nu} a_j e_j + \sum_{j=\nu+2}^{2\nu} a_j e_j] \in V(\mathcal{I}n(\mathbb{U}_{2\nu}))$*, and*

$$\sigma_1([e_i + \sum_{j=i+1}^{\nu} a_j e_j + \sum_{j=\nu+2}^{2\nu} a_j e_j]) = [e_i + \sum_{j=i+1}^{\nu} b_j e_j + \sum_{j=\nu+2}^{2\nu} b_j e_j].$$

Then,

$$b_k = -\frac{\pi_k(a_k)}{\pi_i(-1)} \ \textit{for} \ \ i+1 \le k \le \nu \tag{2.3}$$

and

$$b_{\nu+k} = -\frac{\pi_{\nu+k}(a_{\nu+k})}{\pi_i(-1)} \ \textit{for} \ \ 2 \le k \ne i \le \nu. \tag{2.4}$$

Proof. If $a_k = 0$, then Eq (2.3) holds automatically. Now, we consider $a_k \ne 0$ for some $i+1 \le k \le \nu$. It is easily seen that $[e_i + \sum_{j=i+1}^{2\nu} a_j e_j, e_1 + e_{\nu+i} - \overline{a_k}^{-1} e_{\nu+k}]$ is totally isotropic. Using (2.2) and σ_1, we deduce that

$$[e_i + \sum_{j=i+1}^{2\nu} b_j e_j, e_1 - \overline{\pi_i(-1)}^{-1} e_{\nu+i} - \overline{\pi_k(a_k)}^{-1} e_{\nu+k}]$$

is also totally isotropic, which suggests that

$$\overline{(-\overline{\pi_i(-1)}^{-1})} + b_k \cdot \overline{(-\overline{\pi_k(a_k)}^{-1})} = 0.$$

Hence, we obtain

$$b_k = -\frac{\pi_k(a_k)}{\pi_i(-1)}.$$

If $a_{\nu+k} = 0$, then Eq (2.4) is clearly true. Let $a_{\nu+k} \ne 0$ for some $2 \le k \le \nu$, $k \ne i$. As $[e_i + \sum_{j=i+1}^{2\nu} a_j e_j, e_1 - \overline{a_{\nu+k}}^{-1} e_k + e_{\nu+i}]$ is totally isotropic, it holds that

$$[e_i + \sum_{j=i+1}^{2\nu} b_j e_j, e_1 - \overline{\pi_{\nu+k}(a_{\nu+k})}^{-1} e_k - \overline{\pi_i(-1)}^{-1} e_{\nu+i}]$$

is also totally isotropic, which indicates that

$$b_{\nu+k} \cdot \overline{(-\overline{\pi_{\nu+k}(a_{\nu+k})}^{-1})} + \overline{(-\overline{\pi_i(-1)}^{-1})} = 0,$$

and thus,

$$b_{\nu+k} = -\frac{\pi_{\nu+k}(a_{\nu+k})}{\pi_i(-1)}.$$

This completes the proof. $\square$

Lemma 2.11. *Let* $[e_{\nu+1} + \sum_{i=2}^{\nu} a_{\nu+i} e_{\nu+i}] \in V(\mathcal{I}n(\mathbb{U}_{2\nu}))$*, and*

$$\sigma_1([e_{\nu+1} + \sum_{i=2}^{\nu} a_{\nu+i} e_{\nu+i}]) = [e_{\nu+1} + \sum_{i=2}^{\nu} b_{\nu+i} e_{\nu+i}].$$

Then, $b_{\nu+k} = \pi_{\nu+k}(a_{\nu+k})$ *for* $2 \le k \le \nu$.

Proof. Obviously, the lemma holds for $a_{\nu+k} = 0$. If $a_{\nu+k} \neq 0$ for some $2 \leq k \leq \nu$, then $[e_{\nu+1} + \sum_{i=2}^{\nu} a_{\nu+i}e_{\nu+i}, e_1 - \overline{a_{\nu+k}}^{-1}e_k]$ is a totally isotropic subspace. By Lemma 2.9, we conclude that

$$[e_{\nu+1} + \sum_{i=2}^{\nu} b_{\nu+i}e_{\nu+i}, e_1 - \overline{\pi_{\nu+k}(a_{\nu+k})}^{-1}e_k]$$

is also a totally isotropic subspace, which implies that

$$1 + b_{\nu+k} \cdot \overline{(-\overline{\pi_{\nu+k}(a_{\nu+k})}^{-1})} = 0.$$

That is, $b_{\nu+k} = \pi_{\nu+k}(a_{\nu+k})$. This completes the proof. □

Lemma 2.12. *Let* $[e_{\nu+i} + \sum_{j=i+1}^{\nu} a_{\nu+j}e_{\nu+j}] \in V(\mathcal{I}n(\mathbb{U}_{2\nu}))$*, and*

$$\sigma_1([e_{\nu+i} + \sum_{j=i+1}^{\nu} a_{\nu+j}e_{\nu+j}]) = [e_{\nu+i} + \sum_{j=i+1}^{\nu} b_{\nu+j}e_{\nu+j}]$$

for $2 \leq i \leq \nu - 1$. *Then,*

$$b_{\nu+k} = -\frac{\pi_{\nu+k}(-1)}{\pi_{\nu+i}(a_{\nu+k}{}^{-1})} \quad \text{if } a_{\nu+k} \neq 0, \quad i+1 \leq k \leq \nu. \tag{2.5}$$

Proof. If $a_{\nu+k} \neq 0$ for some $i + 1 \leq k \leq \nu$, then

$$[e_{\nu+i} + \sum_{j=i+1}^{\nu} a_{\nu+j}e_{\nu+j}, e_1 - \overline{a_{\nu+k}}e_i + e_k]$$

is a totally isotropic subspace. From Lemma 2.9 and σ_1, we deduce that

$$[e_{\nu+i} + \sum_{j=i+1}^{\nu} b_{\nu+j}e_{\nu+j}, e_1 - \overline{\pi_{\nu+i}(a_{\nu+k}{}^{-1})}^{-1}e_i - \overline{\pi_{\nu+k}(-1)}^{-1}e_k]$$

is also a totally isotropic subspace. It follows that

$$\overline{(-\overline{\pi_{\nu+i}(a_{\nu+k}{}^{-1})}^{-1})} + b_{\nu+k} \cdot \overline{(-\overline{\pi_{\nu+k}(-1)}^{-1})} = 0.$$

Hence, we obtain

$$b_{\nu+k} = -\frac{\pi_{\nu+k}(-1)}{\pi_{\nu+i}(a_{\nu+k}{}^{-1})}.$$

This completes the proof. □

Lemma 2.13. *For any* $a, b \in F_{q^2}^*$, $2 \leq i, j \leq \nu$, *we have*

(*i*) $\pi_i(a)\overline{\pi_{\nu+i}(1)} = \pi_j(a)\overline{\pi_{\nu+j}(1)} = \pi_{\nu+i}(a)\overline{\pi_i(1)} = \pi_{\nu+j}(a)\overline{\pi_j(1)}$.

(*ii*) $\pi_i(-a) = -\pi_i(a), \ \pi_{\nu+i}(-a) = -\pi_{\nu+i}(a)$.

(*iii*) $\pi_i(a^{-1}) = \pi_i(a)^{-1}\pi_i(1)^2, \ \pi_{\nu+i}(a^{-1}) = \pi_{\nu+i}(a)^{-1}\pi_{\nu+i}(1)^2$.

(*iv*) $\pi_i(ab) = \pi_i(a)\pi_i(b)\pi_i(1)^{-1}, \ \ \pi_{\nu+i}(ab) = \pi_{\nu+i}(a)\pi_{\nu+i}(b)\pi_{\nu+i}(1)^{-1}$.

(*v*) $\pi_i(a + b) = \pi_i(a) + \pi_i(b), \ \pi_{\nu+i}(a + b) = \pi_{\nu+i}(a) + \pi_{\nu+i}(b)$.

Proof of Lemma 2.13. We see at once that

$$[e_1 - \overline{a}^{-1}e_i - \overline{a}^{-1}e_{\nu+i+1}, e_1 + e_{i+1} - e_{\nu+i}],$$

$$[e_1 - \overline{a}^{-1}e_i - \overline{a}^{-1}e_{i+1}, e_1 - e_{\nu+i} + e_{\nu+i+1}]$$

and

$$[e_1 - \overline{a}^{-1}e_{\nu+i} - \overline{a}^{-1}e_{\nu+i+1}, e_1 - e_i + e_{i+1}]$$

are totally isotropic. Applying σ_1 and (2.1), (2.2), it holds that

$$[e_1 - \overline{\pi_{\nu+i}(a)}^{-1}e_i - \overline{\pi_{i+1}(a)}^{-1}e_{\nu+i+1}, e_1 - \overline{\pi_{\nu+i+1}(-1)}^{-1}e_{i+1} - \overline{\pi_i(1)}^{-1}e_{\nu+i}],$$

$$[e_1 - \overline{\pi_{\nu+i}(a)}^{-1}e_i - \overline{\pi_{\nu+i+1}(a)}^{-1}e_{i+1}, e_1 - \overline{\pi_i(1)}^{-1}e_{\nu+i} - \overline{\pi_{i+1}(-1)}^{-1}e_{\nu+i+1}]$$

and

$$[e_1 - \overline{\pi_i(a)}^{-1}e_{\nu+i} - \overline{\pi_{i+1}(a)}^{-1}e_{\nu+i+1}, e_1 - \overline{\pi_{\nu+i}(1)}^{-1}e_i - \overline{\pi_{\nu+i+1}(-1)}^{-1}e_{i+1}]$$

are also totally isotropic. This shows that

$$\overline{\pi_{\nu+i}(a)}^{-1}\pi_i(1)^{-1} + \overline{\pi_{i+1}(a)}^{-1}\pi_{\nu+i+1}(-1)^{-1} = 0, \tag{2.6}$$

$$\overline{\pi_{\nu+i}(a)}^{-1}\pi_i(1)^{-1} + \overline{\pi_{\nu+i+1}(a)}^{-1}\pi_{i+1}(-1)^{-1} = 0 \tag{2.7}$$

and

$$\overline{\pi_i(a)}^{-1}\pi_{\nu+i}(1)^{-1} + \overline{\pi_{i+1}(a)}^{-1}\pi_{\nu+i+1}(-1)^{-1} = 0. \tag{2.8}$$

Hence, from Eqs (2.6)–(2.8), we deduce that

$$\pi_{\nu+i}(a)\overline{\pi_i(1)} = \pi_i(a)\overline{\pi_{\nu+i}(1)} = -\pi_{i+1}(a)\overline{\pi_{\nu+i+1}(-1)} = -\pi_{\nu+i+1}(a)\overline{\pi_{i+1}(-1)}. \tag{2.9}$$

Similarly,

$$[e_1 - \overline{a}^{-1}e_{i+1} - \overline{a}^{-1}e_{\nu+i}, e_1 + e_i - e_{\nu+i+1}],$$

$$[e_1 - \overline{a}^{-1}e_i - \overline{a}^{-1}e_{i+1}, e_1 + e_{\nu+i} - e_{\nu+i+1}]$$

and

$$[e_1 - \overline{a}^{-1}e_{\nu+i} - \overline{a}^{-1}e_{\nu+i+1}, e_1 + e_i - e_{i+1}]$$

are totally isotropic subspaces. In the same manner, we can see that

$$[e_1 - \overline{\pi_{\nu+i+1}(a)}^{-1}e_{i+1} - \overline{\pi_i(a)}^{-1}e_{\nu+i}, e_1 - \overline{\pi_{\nu+i}(-1)}^{-1}e_i - \overline{\pi_{i+1}(1)}^{-1}e_{\nu+i+1}],$$

$$[e_1 - \overline{\pi_{\nu+i}(a)}^{-1}e_i - \overline{\pi_{\nu+i+1}(a)}^{-1}e_{i+1}, e_1 - \overline{\pi_i(-1)}^{-1}e_{\nu+i} - \overline{\pi_{i+1}(1)}^{-1}e_{\nu+i+1}]$$

and

$$[e_1 - \overline{\pi_i(a)}^{-1}e_{\nu+i} - \overline{\pi_{i+1}(a)}^{-1}e_{\nu+i+1}, e_1 - \overline{\pi_{\nu+i}(-1)}^{-1}e_i - \overline{\pi_{\nu+i+1}(1)}^{-1}e_{i+1}]$$

are also totally isotropic subspaces, which means that

$$\pi_{\nu+i+1}(a)\overline{\pi_{i+1}(1)} + \pi_i(a)\overline{\pi_{\nu+i}(-1)} = 0, \tag{2.10}$$

$$\pi_{\nu+i}(a)\overline{\pi_i(-1)} + \pi_{\nu+i+1}(a)\overline{\pi_{i+1}(1)} = 0 \tag{2.11}$$

and

$$\pi_i(a)\overline{\pi_{\nu+i}(-1)} + \pi_{i+1}(a)\overline{\pi_{\nu+i+1}(1)} = 0. \tag{2.12}$$

Using (2.10)–(2.12), we get

$$\pi_i(a)\overline{\pi_{\nu+i}(-1)} = \pi_{\nu+i}(a)\overline{\pi_i(-1)} = -\pi_{\nu+i+1}(a)\overline{\pi_{i+1}(1)} = -\pi_{i+1}(a)\overline{\pi_{\nu+i+1}(1)}. \tag{2.13}$$

Moreover, because $[e_{\nu+1} + ae_{\nu+i} + ae_{\nu+i+1}, e_i - e_{i+1}]$ is a totally isotropic subspace, it follows that $[e_{\nu+1} + \pi_{\nu+i}(a)e_{\nu+i} + \pi_{\nu+i+1}(a)e_{\nu+i+1}, e_i - \pi_i(-1)^{-1}\pi_{i+1}(-1)e_{i+1}]$ is also a totally isotropic subspace. Hence,

$$\pi_{\nu+i}(a) + \pi_{\nu+i+1}(a)\overline{(-\pi_i(-1)^{-1}\pi_{i+1}(-1))} = 0$$

holds. That is,

$$\pi_{\nu+i}(a)\overline{\pi_i(-1)} = \pi_{\nu+i+1}(a)\overline{\pi_{i+1}(-1)}. \tag{2.14}$$

Combining (2.9), (2.13), and (2.14) yields

$$\pi_i(a)\overline{\pi_{\nu+i}(1)} = \pi_j(a)\overline{\pi_{\nu+j}(1)} = \pi_{\nu+i}(a)\overline{\pi_i(1)} = \pi_{\nu+j}(a)\overline{\pi_j(1)}. \tag{2.15}$$

This completes the proof of (i).

As $[e_1 + \overline{a}^{-1}e_2 - \overline{a}^{-1}e_{\nu+3}, e_1 + e_3 + e_{\nu+2}]$ is totally isotropic, it holds that $[e_1 - \overline{\pi_{\nu+2}(-a)}^{-1}e_2 - \overline{\pi_3(a)}^{-1}e_{\nu+3}, e_1 - \overline{\pi_{\nu+3}(-1)}^{-1}e_3 - \overline{\pi_2(-1)}^{-1}e_{\nu+2}]$ is also totally isotropic, which further implies that

$$\overline{\pi_3(a)}\pi_{\nu+3}(-1) + \overline{\pi_{\nu+2}(-a)}\pi_2(-1) = 0.$$

Note that we have actually proved that $\pi_i(-1) = -\pi_i(1)$, $\pi_{\nu+i}(-1) = -\pi_{\nu+i}(1)$, so

$$\pi_{\nu+2}(-a)\overline{\pi_2(1)} = -\pi_3(a)\overline{\pi_{\nu+3}(1)}. \tag{2.16}$$

By (2.15) and (2.16), we obtain

$$\pi_{\nu+2}(-a) = \frac{-\pi_3(a)\overline{\pi_{\nu+3}(1)}}{\overline{\pi_2(1)}} = \frac{-\pi_{\nu+2}(a)\overline{\pi_2(1)}}{\overline{\pi_2(1)}} = -\pi_{\nu+2}(a).$$

Then, we see that

$$\pi_i(-a) = \frac{\pi_{\nu+2}(-a)\overline{\pi_2(1)}}{\overline{\pi_{\nu+i}(1)}} = \frac{-\pi_{\nu+2}(a)\overline{\pi_2(1)}}{\overline{\pi_{\nu+i}(1)}} = \frac{-\pi_i(a)\overline{\pi_{\nu+i}(1)}}{\overline{\pi_{\nu+i}(1)}} = -\pi_i(a), \tag{2.17}$$

$$\pi_{\nu+i}(-a) = \frac{\pi_{\nu+2}(-a)\overline{\pi_2(1)}}{\overline{\pi_i(1)}} = \frac{-\pi_{\nu+2}(a)\overline{\pi_2(1)}}{\overline{\pi_i(1)}} = \frac{-\pi_{\nu+i}(a)\overline{\pi_i(1)}}{\overline{\pi_i(1)}} = -\pi_{\nu+i}(a). \tag{2.18}$$

This completes the proof of (ii).

Let $3 \leq i \leq \nu$. It is clear that $[e_1 + be_2 + abe_{\nu+i}, e_i - \overline{a}e_{\nu+2}]$ is a totally isotropic subspace. Then, $[e_1 - \overline{\pi_{\nu+2}(-\overline{b}^{-1})}^{-1}e_2 - \overline{\pi_i(-\overline{ab}^{-1})}^{-1}e_{\nu+i}, e_i - \pi_i(-1)^{-1}\pi_{\nu+2}(-\overline{a})e_{\nu+2}]$ is also a totally isotropic subspace, which suggests that

$$\overline{\pi_{\nu+2}(-\overline{b}^{-1})}^{-1}\pi_i(-1)^{-1}\pi_{\nu+2}(-\overline{a}) - \overline{\pi_i(-\overline{ab}^{-1})}^{-1} = 0.$$

By (*ii*), we obtain

$$\pi_i(\overline{ab}^{-1}) = \pi_{\nu+2}(\overline{b}^{-1})\pi_i(1)\pi_{\nu+2}(\overline{a})^{-1}.$$

Writing a for $\overline{a}^{-1}$ and b for $\overline{b}^{-1}$, then

$$\pi_i(ab) = \pi_{\nu+2}(b)\pi_{\nu+2}(a^{-1})^{-1}\pi_i(1). \tag{2.19}$$

Taking $b = 1$, we have

$$\pi_{\nu+2}(a^{-1}) = \pi_i(a)^{-1}\pi_{\nu+2}(1)\pi_i(1). \tag{2.20}$$

Substituting (2.20) into (2.19), we get

$$\pi_i(ab) = \pi_{\nu+2}(b)\pi_i(a)\pi_{\nu+2}(1)^{-1}. \tag{2.21}$$

Using (2.15) and (2.20), we conclude that

$$\pi_i(a^{-1}) = \frac{\pi_{\nu+2}(a^{-1})\overline{\pi_2(1)}}{\overline{\pi_{\nu+i}(1)}} = \frac{\pi_i(a)^{-1}\pi_{\nu+2}(1)\pi_i(1)\overline{\pi_2(1)}}{\overline{\pi_{\nu+i}(1)}} = \pi_i(a)^{-1}\pi_i(1)^2. \tag{2.22}$$

As $[e_1 + abe_{\nu+2} + be_i, e_2 - \overline{a}e_{\nu+i}]$ is totally isotropic, it holds that

$$[e_1 - \overline{\pi_2(-\overline{ab}^{-1})}^{-1} e_{\nu+2} - \overline{\pi_{\nu+i}(-\overline{b}^{-1})}^{-1} e_i, e_2 - \pi_2(-1)^{-1}\pi_{\nu+i}(-\overline{a})e_{\nu+i}]$$

is also totally isotropic, which leads to

$$\pi_2(\overline{ab}^{-1}) = \pi_{\nu+i}(\overline{b}^{-1})\pi_2(1)\pi_{\nu+i}(\overline{a})^{-1}.$$

Taking $b = 1$, we have

$$\pi_2(\overline{a}^{-1}) = \pi_{\nu+i}(1)\pi_2(1)\pi_{\nu+i}(\overline{a})^{-1}.$$

This equality together with (2.15) proves

$$\pi_{\nu+i}(a^{-1}) = \frac{\pi_2(a^{-1})\overline{\pi_{\nu+2}(1)}}{\overline{\pi_i(1)}} = \frac{\pi_{\nu+i}(a)^{-1}\pi_2(1)\pi_{\nu+i}(1)\overline{\pi_{\nu+2}(1)}}{\overline{\pi_i(1)}} = \pi_{\nu+i}(a)^{-1}\pi_{\nu+i}(1)^2. \tag{2.23}$$

Thus, the result of (*iii*) follows.

For any $a, b \in F_{q^2}^*$, $2 \le i \le \nu$, using (2.15) and (2.21), we have

$$\pi_i(ab) = \frac{\pi_i(a)\overline{\pi_{\nu+i}(1)}\pi_i(b)}{\overline{\pi_2(1)}\pi_{\nu+2}(1)} = \pi_i(a)\pi_i(b)\pi_i(1)^{-1}. \tag{2.24}$$

Using (2.15) and (2.24), we also have

$$\begin{aligned}
\pi_{\nu+i}(ab) &= \frac{\pi_2(ab)\overline{\pi_{\nu+2}(1)}}{\overline{\pi_i(1)}} \\
&= \frac{\pi_2(a)\pi_2(b)\pi_2(1)^{-1}\overline{\pi_{\nu+2}(1)}}{\overline{\pi_i(1)}} \\
&= \frac{\pi_{\nu+i}(a)\pi_{\nu+i}(b)\overline{\pi_i(1)}}{\overline{\pi_{\nu+2}(1)}\pi_2(1)} \\
&= \pi_{\nu+i}(a)\pi_{\nu+i}(b)\pi_{\nu+i}(1)^{-1}.
\end{aligned} \tag{2.25}$$

This proves (iv).

It is easy to check that $[e_1 + e_2 + \overline{(a+b)}e_3, e_{\nu+1} + a^{-1}be_{\nu+2} - a^{-1}e_{\nu+3}]$ is a totally isotropic subspace. Then,

$$[e_1 - \overline{\pi_{\nu+2}(-1)}^{-1}e_2 - \overline{\pi_{\nu+3}(-(a+b)^{-1})}^{-1}e_3, e_{\nu+1} + \pi_{\nu+2}(a^{-1}b)e_{\nu+2} + \pi_{\nu+3}(-a^{-1})e_{\nu+3}]$$

is also a totally isotropic subspace, which means that

$$1 + \pi_{\nu+2}(1)^{-1}\pi_{\nu+2}(a^{-1}b) - \pi_{\nu+3}((a+b)^{-1})^{-1}\pi_{\nu+3}(a^{-1}) = 0.$$

This equality together with (2.23) and (2.25) shows

$$\begin{aligned}
\pi_{\nu+3}(a+b) &= \pi_{\nu+3}(a) + \pi_{\nu+3}(a)\pi_{\nu+2}(a^{-1})\pi_{\nu+2}(b)\pi_{\nu+2}(1)^{-2} \\
&= \pi_{\nu+3}(a) + \pi_{\nu+3}(a)\pi_{\nu+2}(a)^{-1}\pi_{\nu+2}(b) \\
&= \pi_{\nu+3}(a) + \frac{\pi_{\nu+3}(a)\pi_{\nu+3}(b)\overline{\pi_3(1)}}{\overline{\pi_2(1)}\pi_{\nu+2}(a)} \\
&= \pi_{\nu+3}(a) + \pi_{\nu+3}(b).
\end{aligned}$$

Hence,

$$\pi_i(a+b) = \frac{\pi_{\nu+3}(a+b)\overline{\pi_3(1)}}{\overline{\pi_{\nu+i}(1)}} = \frac{\pi_{\nu+3}(a)\overline{\pi_3(1)} + \pi_{\nu+3}(b)\overline{\pi_3(1)}}{\overline{\pi_{\nu+i}(1)}} = \pi_i(a) + \pi_i(b)$$

and

$$\pi_{\nu+i}(a+b) = \frac{\pi_{\nu+3}(a+b)\overline{\pi_3(1)}}{\overline{\pi_i(1)}} = \frac{\pi_{\nu+3}(a)\overline{\pi_3(1)} + \pi_{\nu+3}(b)\overline{\pi_3(1)}}{\overline{\pi_i(1)}} = \pi_{\nu+i}(a) + \pi_{\nu+i}(b).$$

This completes the proof of (v).

The proof of Lemma 2.13 is completed. □

Lemma 2.14. *Let* $\pi = \pi_2/\pi_2(1)$. *Then,*

(1) $\pi = \pi_i/\pi_i(1) = \pi_{\nu+i}/\pi_{\nu+i}(1)$, $2 \leq i \leq \nu$.

(2) π *is an automorphism of* F_{q^2}.

Proof. By (i) of Lemma 2.13, it follows immediately that

$$\begin{aligned}
\pi &= \frac{\pi_2}{\pi_2(1)} = \frac{\pi_i\overline{\pi_{\nu+i}(1)}}{\overline{\pi_{\nu+2}(1)}\pi_2(1)} = \frac{\pi_i\overline{\pi_{\nu+i}(1)}}{\overline{\pi_{\nu+i}(1)}\pi_i(1)} = \frac{\pi_i}{\pi_i(1)}, \\
\pi &= \frac{\pi_2}{\pi_2(1)} = \frac{\pi_{\nu+i}\overline{\pi_i(1)}}{\overline{\pi_{\nu+2}(1)}\pi_2(1)} = \frac{\pi_{\nu+i}\overline{\pi_i(1)}}{\pi_{\nu+i}(1)\overline{\pi_i(1)}} = \frac{\pi_{\nu+i}}{\pi_{\nu+i}(1)}.
\end{aligned}$$

This proves (i).

From (iv), (v), We see at once that

$$\pi(a+b) = \frac{\pi_2(a+b)}{\pi_2(1)} = \frac{\pi_2(a) + \pi_2(b)}{\pi_2(1)} = \pi(a) + \pi(b)$$

and

$$\pi(ab) = \frac{\pi_2(ab)}{\pi_2(1)} = \frac{\pi_2(a)\pi_2(b)\pi_2(1)^{-1}}{\pi_2(1)} = \frac{\pi_2(a)\pi_2(b)}{\pi_2(1)\pi_2(1)} = \pi(a)\pi(b),$$

which is clear from (iv) and (v) of Lemma 2.13. Thus, the lemma holds. □

Substituting $a = 1$ into (i) of Lemma 2.13, we get

$$\pi_i(1)\overline{\pi_{\nu+i}(1)} = \pi_j(1)\overline{\pi_{\nu+j}(1)} = \pi_{\nu+i}(1)\overline{\pi_i(1)} = \pi_{\nu+j}(1)\overline{\pi_j(1)}.$$

This shows that $\pi_i(1)\overline{\pi_{\nu+i}(1)}$ is a constant in $F^*_{q^2}$ for all $2 \le i \le \nu$. Therefore, we may assume that

$$\pi_i(1)\overline{\pi_{\nu+i}(1)} = \pi_{\nu+i}(1)\overline{\pi_i(1)} = d\,\overline{d},$$

where $d \in F^*_{q^2}$.

Applying the above lemmas, one can easily obtain the following result.

Lemma 2.15. *Let* $[\sum_{i=1}^{2\nu} a_i e_i]$ *be any one dimensional, totally isotropic subspace of* $\mathcal{I}n(\mathbb{U}_{2\nu})$*, where* $a_i \in F_{q^2}$ *for* $i = 1, 2, \dots, 2\nu$*. Then,*

$$\sigma_1([\sum_{i=1}^{2\nu} a_i e_i]) = [\overline{d}\pi(a_1)e_1 + \sum_{j=2}^{\nu} \frac{\pi_j(1)}{d}\pi(a_j)e_j + \frac{1}{d}\pi(a_{\nu+1})e_{\nu+1} + \sum_{j=2}^{\nu} \frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j}].$$

Proof. We first consider the cases $\sigma_1([e_{\nu+1} + \sum_{j=2}^{\nu} a_{\nu+j}e_{\nu+j}])$. Let

$$\sigma_1([e_{\nu+1} + \sum_{j=2}^{\nu} a_{\nu+j}e_{\nu+j}]) = [e_{\nu+1} + \sum_{j=2}^{\nu} b_{\nu+j}e_{\nu+j}].$$

By Lemmas 2.11 and 2.14, it holds that $b_{\nu+k} = \pi(a_{\nu+k})\pi_{\nu+k}(1)$ for $2 \le k \le \nu$. Hence, we obtain

$$\sigma_1([e_{\nu+1} + \sum_{j=2}^{\nu} a_{\nu+j}e_{\nu+j}]) = [\frac{1}{d}e_{\nu+1} + \sum_{j=2}^{\nu} \frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j}]. \tag{2.26}$$

For the case $\sigma_1([e_{\nu+i} + \sum_{j=i+1}^{\nu} a_{\nu+j}e_{\nu+j}])$, let

$$\sigma_1([e_{\nu+i} + \sum_{j=i+1}^{\nu} a_{\nu+j}e_{\nu+j}]) = [e_{\nu+i} + \sum_{j=i+1}^{\nu} b_{\nu+j}e_{\nu+j}].$$

By Lemmas 2.12 and 2.13, it follows easily that

$$b_{\nu+k} = \frac{\pi_{\nu+k}(1)\pi(a_{\nu+k})}{\pi_{\nu+i}(1)}$$

for $i + 1 \le k \le \nu$. Then,

$$\sigma_1([e_{\nu+i} + \sum_{j=i+1}^{\nu} a_{\nu+j}e_{\nu+j}]) = [\frac{\pi_{\nu+i}(1)}{d}e_{\nu+i} + \sum_{j=i+1}^{\nu} \frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j}]. \tag{2.27}$$

Now, we discuss the general cases $\sigma_1([e_1 + \sum_{j=2}^{2\nu} a_j e_j])$. Let

$$\sigma_1([e_1 + \sum_{j=2}^{2\nu} a_j e_j]) = [e_1 + \sum_{j=2}^{2\nu} b_j e_j].$$

We see at once that

$$b_k = \frac{\pi_k(1)\pi(a_k)}{d\overline{d}} \qquad \text{and} \qquad b_{\nu+k} = \frac{\pi_{\nu+k}(1)\pi(a_{\nu+k})}{d\overline{d}} \tag{2.28}$$

for $2 \leq k \leq \nu$, which is clear from Lemmas 2.9, 2.13, and 2.14. What is left is to show that

$$b_{\nu+1} = \frac{\pi(a_{\nu+1})}{d\overline{d}}.$$

If $a_{\nu+1} = 0$, then the conclusion is clearly true. In the following, we assume that $a_{\nu+1} \neq 0$.

If there exists some k such that $a_k \neq 0$ for $2 \leq k \leq \nu$, then $[e_1 + \sum_{j=2}^{2\nu} a_j e_j, e_1 - \overline{a_k}^{-1}\overline{a_{\nu+1}} e_{\nu+k}]$ is totally isotropic. It holds that $[e_1 + \sum_{j=2}^{2\nu} b_j e_j, e_1 - (\pi(\overline{a_k}^{-1}\overline{a_{\nu+1}})\pi_{\nu+k}(1)/d\overline{d})e_{\nu+k}]$ is also totally isotropic, which implies that

$$b_{\nu+1} = b_k \cdot \overline{\frac{\pi(\overline{a_k}^{-1}\overline{a_{\nu+1}})}{\pi_k(1)}} = \frac{\pi(a_k)\pi_k(1)}{d\overline{d}} \cdot \overline{\frac{\pi(\overline{a_k}^{-1}\overline{a_{\nu+1}})}{\pi_k(1)}} = \frac{\pi(a_{\nu+1})}{d\overline{d}}.$$

If there exists some k such that $a_{\nu+k} \neq 0$ for $2 \leq k \leq \nu$, then $[e_1 + \sum_{j=2}^{2\nu} a_j e_j, e_1 - \overline{a_{\nu+k}}^{-1}\overline{a_{\nu+1}} e_k]$ is a totally isotropic subspace. From (2.28), we deduce that $[e_1 + \sum_{j=2}^{2\nu} b_j e_j, e_1 - (\pi_j(1)\pi(\overline{a_{\nu+k}}^{-1}\overline{a_{\nu+1}})/d\overline{d})e_k]$ is also a totally isotropic subspace, which suggests that

$$b_{\nu+1} = b_{\nu+k} \cdot \overline{(\frac{\pi_j(1)\pi(\overline{a_{\nu+k}}^{-1}\overline{a_{\nu+1}})}{d\overline{d}})} = \frac{\pi_{\nu+k}(1)\pi(a_{\nu+k})}{d\overline{d}} \cdot \frac{\pi(a_{\nu+k})^{-1}\pi(a_{\nu+1})}{\pi_{\nu+j}(1)} = \frac{\pi(a_{\nu+1})}{d\overline{d}}.$$

If $a_k = 0, a_{\nu+k} = 0$ for all $k = 2, \ldots, \nu$, then

$$[e_1 + \sum_{j=2}^{2\nu} a_j e_j] = [e_1 + a_{\nu+1} e_{\nu+1}].$$

It is clear that $[e_1 + a_{\nu+1} e_{\nu+1}, e_1 + e_2 - \overline{a_{\nu+1}} e_{\nu+1}]$ is totally isotropic. By (2.28), we conclude that $[e_1 + b_{\nu+1} e_{\nu+1}, e_1 + (\pi_2(1)\pi(1)/d\overline{d})e_2 - (\pi(\overline{a_{\nu+1}})/d\overline{d})e_{\nu+1}]$ is also totally isotropic, which means that

$$b_{\nu+1} = \frac{\pi(a_{\nu+1})}{d\overline{d}}.$$

Hence, we have

$$\sigma_1([e_1 + \sum_{j=2}^{2\nu} a_j e_j]) = [\overline{d} e_1 + \sum_{j=2}^{\nu} \frac{\pi_j(1)}{d}\pi(a_j)e_j + \frac{1}{d}\pi(a_{\nu+1})e_{\nu+1} + \sum_{j=2}^{\nu} \frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j}]. \tag{2.29}$$

Finally, for the case $\sigma_1([e_i + \sum_{j=i+1}^{2\nu} a_j e_j])$, let

$$\sigma_1([e_i + \sum_{j=i+1}^{2\nu} a_j e_j]) = [e_i + \sum_{j=i+1}^{2\nu} b_j e_j].$$

Lemmas 2.10 and 2.14 make it obvious that

$$b_k = \frac{\pi_k(1)\pi(a_k)}{\pi_i(1)} \ \text{ for } \ i+1 \le k \le \nu, \tag{2.30}$$

$$b_{\nu+k} = \frac{\pi_{\nu+k}(1)\pi(a_{\nu+k})}{\pi_i(1)} \ \text{ for } \ 2 \le k \ne i \le \nu. \tag{2.31}$$

Now, let's prove

$$b_{\nu+1} = \frac{\pi(a_{\nu+1})}{\pi_i(1)}.$$

If $a_{\nu+1} = 0$, then the result holds. If $a_{\nu+1} \ne 0$, then $[e_i + \sum_{j=i+1}^{2\nu} a_j e_j, e_1 - \overline{a_{\nu+1}} e_{\nu+i}]$ is a totally isotropic subspace. Using (2.28), we deduce that $[e_i + \sum_{j=i+1}^{2\nu} b_j e_j, e_1 - \frac{\pi(\overline{a_{\nu+1}})}{\pi_i(1)} e_{\nu+i}]$ is also a totally isotropic subspace, which leads to

$$b_{\nu+1} = \frac{\pi(a_{\nu+1})}{\pi_i(1)}.$$

Next, we consider $a_{\nu+i}$. If $a_{\nu+i} = 0$, then

$$b_{\nu+i} = \frac{\pi_{\nu+k}(1)\pi(a_{\nu+i})}{\pi_i(1)}$$

holds automatically. If $a_{\nu+i} \ne 0$, then we distinguish the following cases:

(1) If there exists some k such that $a_k \ne 0$ for $i+1 \le k \le \nu$, then $[e_i + \sum_{j=i+1}^{2\nu} a_j e_j, e_i - \overline{a_k}^{-1}\overline{a_{\nu+i}} e_{\nu+k}]$ is a totally isotropic subspace. From (2.31), it follows easily that $[e_i + \sum_{j=i+1}^{2\nu} b_j e_j, e_i - \frac{\pi(\overline{a_k}^{-1}\overline{a_{\nu+i}})\pi_{\nu+k}(1)}{\pi_i(1)} e_{\nu+k}]$ is also a totally isotropic subspace, which arrives at

$$b_{\nu+i} = b_k \cdot \frac{\pi(a_{\nu+i})\overline{\pi_{\nu+k}(1)}}{\pi(a_k)\overline{\pi_i(1)}} = \frac{\pi_k(1)\pi(a_k)}{\pi_i(1)} \cdot \frac{\pi(a_{\nu+i})\overline{\pi_{\nu+k}(1)}}{\pi(a_k)\overline{\pi_i(1)}} = \frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi_i(1)}.$$

(2) If $a_{\nu+1} \ne 0$, then $[e_i + \sum_{j=i+1}^{2\nu} a_j e_j, e_1 - \overline{a_{\nu+1} a_{\nu+i}}^{-1} e_i]$ is totally isotropic. Applying σ_1 and (2.28), we conclude that $[e_i + \sum_{j=i+1}^{2\nu} b_j e_j, e_1 + (\pi(-\overline{a_{\nu+1} a_{\nu+i}}^{-1})/\overline{\pi_{\nu+i}(1)}) e_i]$ is also totally isotropic, which suggests that

$$b_{\nu+i} = b_{\nu+1} \cdot \frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi(a_{\nu+1})} = \frac{\pi(a_{\nu+1})}{\pi_i(1)} \cdot \frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi(a_{\nu+1})} = \frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi_i(1)}.$$

(3) If there exists some k such that $a_{\nu+k} \ne 0$ for $2 \le k \le i-1$, then $[e_i + \sum_{j=i+1}^{2\nu} a_j e_j, e_k - \overline{a_{\nu+i}}^{-1}\overline{a_{\nu+k}} e_i]$ is a totally isotropic subspace. By σ_1 and (2.30), it follows immediately that $[e_i + \sum_{j=i+1}^{2\nu} b_j e_j, e_k - (\pi(\overline{a_{\nu+i}}^{-1}\overline{a_{\nu+k}})\pi_i(1)/\pi_k(1)) e_i]$ is also a totally isotropic subspace, which leads to

$$b_{\nu+k} + b_{\nu+i} \cdot \overline{(\frac{\pi(\overline{a_{\nu+i}}^{-1}\overline{a_{\nu+k}})\pi_i(1)}{\pi_k(1)})} = 0.$$

Hence, we have

$$b_{\nu+i}=\frac{\pi_{\nu+k}(1)\pi(a_{\nu+k})}{\pi_i(1)}\cdot\frac{\overline{\pi_k(1)}}{\pi(a_{\nu+i})^{-1}\pi(a_{\nu+k})\overline{\pi_i(1)}}=\frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi_i(1)}.$$

(4) If there exists some k such that $a_{\nu+k}\neq 0$ for $i+1\le k\le \nu$, then $[e_i+\sum_{j=i+1}^{2\nu}a_je_j, e_i-\overline{a_{\nu+k}}^{-1}\overline{a_{\nu+i}}e_k]$ is a totally isotropic subspace. From σ_1 and (2.30), it follows immediately that $[e_i+\sum_{j=i+1}^{2\nu}b_je_j, e_i-(\pi(\overline{a_{\nu+k}}^{-1}\overline{a_{\nu+i}})\pi_k(1)/\pi_i(1))e_k]$ is also a totally isotropic subspace, which implies that

$$b_{\nu+i}=b_{\nu+k}\cdot\frac{\pi(a_{\nu+k}{}^{-1})\pi(a_{\nu+i})\overline{\pi_k(1)}}{\overline{\pi_i(1)}}=\frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi_i(1)}.$$

(5) If $a_k=0$ for all $k=i+1,\ldots,\nu$ and $a_{\nu+k}=0$ for all $k=1,\ldots,\nu, k\neq i$, then

$$[e_i+\sum_{j=i+1}^{2\nu}a_je_j]=[e_i+a_{\nu+i}e_{\nu+i}].$$

Because $[e_i+a_{\nu+i}e_{\nu+i}, e_1+e_i-\overline{a_{\nu+i}}e_{\nu+i}]$ is a totally isotropic subspace, it follows that $[e_i+b_{\nu+i}e_{\nu+i}, e_1+\pi(1)/\overline{\pi_{\nu+i}(1)}e_i+\pi(-\overline{a_{\nu+i}})/\overline{\pi_i(1)}e_{\nu+i}]$ is also a totally isotropic subspace, which leads to

$$b_{\nu+i}=\frac{\pi_{\nu+i}(1)\pi(a_{\nu+i})}{\pi_i(1)}.$$

Hence, we obtain

$$\sigma_1([e_i+\sum_{j=i+1}^{2\nu}a_je_j])=[\frac{\pi_i(1)}{d}e_i+\sum_{j=i+1}^{\nu}\frac{\pi_j(1)}{d}\pi(a_j)e_j+\frac{1}{d}\pi(a_{\nu+1})e_{\nu+1}+\sum_{j=2}^{\nu}\frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j}].\quad(2.32)$$

Therefore, the lemma follows from (2.26), (2.27), (2.29), and (2.32). □

Lemma 2.16. *There exists a unitary matrix $A_{2\nu}$ such that $\sigma_{\pi^{-1}}\circ\sigma_{A_{2\nu}}\circ\sigma_1$ fixes each one dimensional totally isotropic subspace of $\mathbb{U}_{2\nu}$.*

Proof. Let $[\sum_{i=1}^{2\nu}a_ie_i]$ be any one dimensional totally isotropic subspace of $\mathbb{U}_{2\nu}$. By Lemma 2.15, we have

$$\sigma_1([\sum_{i=1}^{2\nu}a_ie_i])=[\bar{d}\pi(a_1)e_1+\sum_{j=2}^{\nu}\frac{\pi_j(1)}{d}\pi(a_j)e_j+\frac{1}{d}\pi(a_{\nu+1})e_{\nu+1}+\sum_{j=2}^{\nu}\frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j}].$$

Let

$$A_{2\nu}=diag(\frac{1}{\bar{d}},\frac{d}{\pi_2(1)},\ldots,\frac{d}{\pi_\nu(1)},d,\frac{d}{\pi_{\nu+2}(1)},\ldots,\frac{d}{\pi_{2\nu}(1)}).$$

It is easy to check that

$$A_{2\nu}H_0\overline{A_{2\nu}}^t=H_0,\quad\text{and}\quad\sigma_{A_{2\nu}}\circ\sigma_1([\sum_{i=1}^{2\nu}a_ie_i])=[\sum_{i=1}^{2\nu}\pi(a_i)e_i].$$

Then, we conclude that

$$\sigma_{\pi^{-1}} \circ \sigma_{A_{2\nu}} \circ \sigma_1([\sum_{i=1}^{2\nu} a_i e_i]) = [\sum_{i=1}^{2\nu} a_i e_i].$$

Therefore, there is a unitary matrix $A_{2\nu} \in U_{2\nu}(F_{q^2}, H_0)$ such that $\sigma_{\pi^{-1}} \circ \sigma_{A_{2\nu}} \circ \sigma_1$ fixes each one dimensional totally isotropic subspace. □

We now show that for $n = 2\nu + 1$, there exists a unitary matrix $A_{2\nu+1} \in U_{2\nu+1}(F_{q^2}, H_1)$ such that $\sigma_{\pi^{-1}} \circ \sigma_{A_{2\nu+1}} \circ \sigma_1$ fixes each one dimensional totally isotropic subspace of $\mathbb{U}_{2\nu+1}$.

By the same method of the proof of Lemma 2.15 and taking Lemma 2.8 into account, we can prove that the vertex $[\sum_{i=1}^{2\nu+1} a_i e_i]$ is mapped into

$$[\overline{d}\pi(a_1)e_1 + \sum_{j=2}^{\nu}(\pi_j(1)/d)\pi(a_j)e_j + (1/d)\pi(a_{\nu+1})e_{\nu+1} + \sum_{j=2}^{\nu}(\pi_{\nu+j}(1)/d)\pi(a_{\nu+j})e_{\nu+j} + b_{2\nu+1}e_{2\nu+1}]$$

under σ_1, where $b_{2\nu+1}$ is to be determined. Because $[\sum_{i=1}^{2\nu+1} a_i e_i]$ is a totally isotropic subspace, it holds that

$$a_1\overline{a_{\nu+1}} + \cdots + a_\nu\overline{a_{2\nu}} + a_{\nu+1}\overline{a_1} + \cdots + a_{2\nu}\overline{a_\nu} + a_{2\nu+1}\overline{a_{2\nu+1}} = 0. \tag{2.33}$$

Similarly, we obtain

$$\begin{aligned}
\pi(a_1)\overline{\pi(a_{\nu+1})} &+ \frac{\pi_2(1)\pi(a_2)}{d}\frac{\overline{\pi_{\nu+2}(1)\pi(a_{\nu+2})}}{\overline{d}} + \cdots + \frac{\pi_\nu(1)\pi(a_\nu)}{d}\frac{\overline{\pi_{2\nu}(1)\pi(a_{2\nu})}}{\overline{d}} \\
&+ \pi(a_{\nu+1})\overline{\pi(a_1)} + \frac{\pi_{\nu+2}(1)\pi(a_{\nu+2})}{d}\frac{\overline{\pi_2(1)\pi(a_2)}}{\overline{d}} + \cdots \\
&+ \frac{\pi_{2\nu}(1)\pi(a_{2\nu})}{d}\frac{\overline{\pi_\nu(1)\pi(a_\nu)}}{\overline{d}} + b_{2\nu+1}\overline{b_{2\nu+1}} = 0.
\end{aligned} \tag{2.34}$$

Combining (2.33) with (2.34), we have

$$b_{2\nu+1} = \frac{\pi(\overline{a_{2\nu+1}})}{\overline{b_{2\nu+1}}}\pi(a_{2\nu+1}).$$

As

$$\frac{\pi(\overline{a_{2\nu+1}})}{\overline{b_{2\nu+1}}} = \frac{b_{2\nu+1}}{\pi(a_{2\nu+1})},$$

it holds that

$$\frac{\pi(\overline{a_{2\nu+1}})}{\overline{b_{2\nu+1}}} \cdot \overline{(\frac{\pi(\overline{a_{2\nu+1}})}{\overline{b_{2\nu+1}}})} = 1,$$

that is, $\pi(\overline{a_{2\nu+1}})/\overline{b_{2\nu+1}} \in S$.

Lemma 2.17. *There exists a unitary matrix $A_{2\nu+1} \in U_{2\nu+1}(F_{q^2}, H_1)$ such that $\sigma_{\pi^{-1}} \circ \sigma_{A_{2\nu+1}} \circ \sigma_1$ fixes each one dimensional totally isotropic subspace of $\mathbb{U}_{2\nu+1}$.*

Proof. Suppose $[\sum_{i=1}^{2\nu+1} a_i e_i]$ is any one-dimensional totally isotropic subspace of $\mathcal{I}n(\mathbb{U}_{2\nu+1})$. The above discussions shows that

$$\sigma_1([\sum_{i=1}^{2\nu+1} a_i e_i]) = [\bar{d}\pi(a_1)e_1 + \sum_{j=2}^{\nu} \frac{\pi_j(1)}{d}\pi(a_j)e_j + \frac{1}{d}\pi(a_{\nu+1})e_{\nu+1} + \sum_{j=2}^{\nu} \frac{\pi_{\nu+j}(1)}{d}\pi(a_{\nu+j})e_{\nu+j} + \frac{\pi(\overline{a_{2\nu+1}})}{\overline{b_{2\nu+1}}}\pi(a_{2\nu+1})e_{2\nu+1}].$$

Set

$$A_{2\nu+1} = diag(\frac{1}{\bar{d}}, \frac{d}{\pi_2(1)}, \ldots, \frac{d}{\pi_\nu(1)}, d, \frac{d}{\pi_{\nu+2}(1)}, \ldots, \frac{d}{\pi_{2\nu}(1)}, \frac{\pi(\overline{a_{2\nu+1}})}{\overline{b_{2\nu+1}}}).$$

Obviously, $A_{2\nu+1} \in U_{2\nu+1}(F_{q^2}, H_1)$. Then, we deduce that

$$\sigma_{\pi^{-1}} \circ \sigma_{A_{2\nu+1}} \circ \sigma_1([\sum_{i=1}^{2\nu+1} a_i e_i]) = [\sum_{i=1}^{2\nu+1} a_i e_i].$$

This completes the proof. □

With the above lemmas in hands, we can easily prove the main result of this paper.

Lemma 2.18. *$\sigma_{\pi^{-1}} \circ \sigma_{A_n} \circ \sigma_1$ fixes every vertex of $\mathcal{I}n(\mathbb{U}_n)$.*

Proof. Let $U = [\alpha_1, \alpha_2, \ldots, \alpha_m]$ be an m-dimensional totally isotropic subspace of $\mathcal{I}n(\mathbb{U}_n)$, where $1 \leq m \leq \nu$. From Lemmas 2.5, 2.16, and 2.17, we obtain

$$\begin{aligned}\sigma_{\pi^{-1}} \circ \sigma_{A_n} \circ \sigma_1(U) &= [\sigma_{\pi^{-1}} \circ \sigma_A \circ \sigma_1(\alpha_1), \sigma_{\pi^{-1}} \circ \sigma_{A_n} \circ \sigma_1(\alpha_2), \ldots, \sigma_{\pi^{-1}} \circ \sigma_A \circ \sigma_1(\alpha_m)] \\ &= [\alpha_1, \alpha_2, \ldots, \alpha_m] = U.\end{aligned}$$

This indicates that $\sigma_{\pi^{-1}} \circ \sigma_{A_n} \circ \sigma_1$ fixes every vertex of $\mathcal{I}n(\mathbb{U}_n)$. □

Lemma 2.18 shows that $\sigma_{\pi^{-1}} \circ \sigma_{A_n} \circ \sigma_{T_n} \circ \sigma$ is an identical automorphism of $\mathcal{I}n(\mathbb{U}_n)$, which implies that $\sigma = \sigma_T \circ \sigma_\pi$, where $T = {A_n}^{-1}{T_n}^{-1}$. We now prove the uniqueness of the decomposition of σ. Suppose that

$$\sigma = \sigma_T \circ \sigma_\pi = \sigma_{T'} \circ \sigma_{\pi'},$$

where $T^{'} \in U_n(F_{q^2}, H), \pi^{'} \in Aut(F_{q^2})$. From the equality above, we obtain

$$\sigma_{T'^{-1}} \circ \sigma_T = \sigma_{\pi'} \circ \sigma_{\pi^{-1}}.$$

Because $\sigma_{\pi'} \circ \sigma_{\pi^{-1}}$ fixes the vertices $[e_i]$ for all $i = 1, 2, \ldots, 2\nu$, it follows that

$$\sigma_{T'^{-1}} \circ \sigma_T([e_i]) = ([e_i]).$$

Consequently, we have $e_i T = k_i e_i T^{'}$ for some $k_i \in F_{q^2}^*$ and for each $i = 1, 2, \ldots, 2\nu$. Moreover, the same automorphism $\sigma_{T'^{-1}} \circ \sigma_T$ also fixes the vertices $[e_1 + e_2], [e_2 + e_3] \cdots [e_{2\nu-1} + e_{2\nu}]$. From these conditions, we deduce that $k_1 = k_2 = \cdots = k_{2\nu} = k$ for some $k \in F^*_{q^2}$. Let us write

$$T = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \text{and} \quad T^{'} = \begin{pmatrix} \alpha^{'}_1 \\ \alpha^{'}_2 \\ \vdots \\ \alpha^{'}_n \end{pmatrix}.$$

If $n = 2\nu$, then we have $T = kT^{'}$. If $n = 2\nu + 1$, then we have $\alpha_i = k\alpha^{'}_i$ for $i = 1, 2, \ldots, 2\nu$. Furthermore, since $TH_1T^t = T^{'}H_1T^{'t} = H_1$, a block matrix computation yields $\alpha_{2\nu+1} = k\alpha^{'}_{2\nu+1}$ and $k\overline{k} = 1$. Hence, we also obtain $T = kT^{'}$. In both cases, we have shown that $\sigma_T = \sigma_{T'}$ and $\sigma_\pi = \sigma_{\pi'}$. Thus, the decomposition is unique, and the proof of Theorem 2.1 is completed.

Corollary 2.1. *Let $\mathbb{U}_n$ be an n-dimensional unitary space over the finite field F_{q^2}. Then, the automorphism group of $\mathcal{I}n(\mathbb{U}_n)$ is isomorphic to $PU_n(F_{q^2}, H) \rtimes \mathbb{Z}_{2m}$.*

Proof. The proof will be divided into three steps:

(1) The automorphism group of F_{q^2} is a cyclic group of order $2m$, which is isomorphic to the additive group of $\mathbb{Z}_{2m}$, where $q = p^m$.

(2) $\sigma_{T_1} = \sigma_{T_2} \Leftrightarrow T_1 = kT_2$, where $k \in S$.

Obviously, $T_1 = kT_2$, $k \in S$ shows $\sigma_{T_1} = \sigma_{T_2}$. Conversely, assume that $\sigma_{T_1} = \sigma_{T_2}$. Then, for any $[\alpha] \in V(\mathcal{I}n(\mathbb{U}_n))$, we have $\alpha T_1 = k\alpha T_2$ for some $k \in F^*_{q^2}$, where k depends on $[\alpha]$.

When $n = 2\nu$, let

$$[\alpha] = [e_1], [e_2], \ldots, [e_{2\nu}],$$

so we obtain $T_1 = diag(k_1, k_2, \ldots, k_{2\nu})T_2$ for some $k_1, k_2, \ldots, k_{2\nu} \in F^*_{q^2}$. Taking

$$[\alpha] = [e_1 + e_2], [e_2 + e_3], \ldots, [e_{2\nu-1} + e_{2\nu}],$$

it follows that $k_1 = k_2 = \cdots = k_{2\nu}$.

When $n = 2\nu + 1$, let

$$[\alpha] = [e_1], [e_2], \ldots, [e_{2\nu}], [e_1 + \lambda e_{\nu+1} + e_{2\nu+1}],$$

where $\lambda \in F^*_{q^2}$, and $\lambda + \overline{\lambda} + 1 = 0$. We then obtain

$$T_1 = \begin{pmatrix} diag(k_1, k_2, \ldots, k_{2\nu}) & \\ \beta & k_{2\nu+1} \end{pmatrix} T_2,$$

where $k_1, k_2, \ldots, k_{2\nu} \in F^*_{q^2}$, and $\beta = (k_{2\nu+1} - k_1)e_1 - \lambda(k_{2\nu+1} - k_{\nu+1})e_{\nu+1}$. Taking

$$[\alpha] = [e_1 + e_2], [e_2 + e_3], \ldots, [e_{2\nu-1} + e_{2\nu}], \quad [\alpha] = [e_1 + e_2 + \lambda e_{\nu+1} + e_{2\nu+1}],$$

we deduce that $k_1 = k_2 = \cdots = k_{2\nu} = k_{2\nu+1}$.

Thus, $T_1 = kT_2$, and it holds that $kI = T_1T_2^{-1} \in U_n(F_{q^2}, H)$, which further implies that $(kI)H\overline{(kI)}^t = H$. Therefore, $k\overline{k}H = H$, and hence, $k\overline{k} = 1$, that is, $k \in S$.

Therefore, the subgroup of $Aut(\mathcal{I}n(\mathbb{U}_n))$ consisting of all σ_T with $T \in U_n(F_{q^2}, H)$ is isomorphic to $PU_n(F_{q^2}, H)$.

(3) The subgroup of $Aut(\mathcal{I}n(\mathbb{U}_n))$ consisting of all σ_T with $T \in U_n(F_{q^2}, H)$ is a normal subgroup of $Aut(\mathcal{I}n(\mathbb{U}_n))$ because $\sigma_\pi \circ \sigma_T \circ \sigma_{\pi^{-1}} = \sigma_{\pi(T)}$.

Then, the proof is completed. □

3. Conclusions

In this study, we extended the previous research on graphs of finite classical spaces by introducing the inclusion graph of totally isotropic subspaces in the unitary space. We constructed two types of standard automorphisms of the inclusion graph of unitary totally isotropic subspaces and proved that any automorphism of this graph can be generated by these two types of standard automorphisms.

Author contributions

Jianmei He: conceived the research concept, proved the theorems, wrote the original draft, and revised the manuscript; Lanxin Chen: conducted the literature review and verified the correctness of the results; Zhengyu Guo: supervised the research, polished the manuscript, and acquired the funding; All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest related to this work.

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