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Research article

Unified pricing model for corporate stocks and bonds under bankruptcy reorganization observation period and discrete bankruptcy time

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Abstract: This paper investigates the impact of a finite bankruptcy reorganization observation period and discrete bankruptcy timing on corporate asset pricing and operational strategies. Discrete bankruptcy times are modeled by the jump moments of a Poisson process with intensity ρ, and a bankruptcy reorganization mechanism based on strategic debt payments is incorporated. Within the structural credit risk framework, the problem is formulated as a nonlinear optimal stopping problem with a penalty term. By employing Itô's lemma and partial differential equation methods, explicit analytical solutions for the values of corporate equity, debt, and firm value are derived, together with expressions for the optimal bankruptcy boundary and the optimal coupon level. Numerical results show that, compared with the continuous bankruptcy model, the discrete bankruptcy mechanism significantly increases the optimal bankruptcy boundary and the optimal coupon level. While the overall impact on firm value is relatively small, it substantially changes the distribution of value between equity and debt: Equity value decreases by approximately 6%, whereas debt value increases by about 51%. Furthermore, when the bankruptcy reorganization observation period extends from 0.5 years to 1.5 years, the divergence between equity and debt values becomes more pronounced. The proposed model provides theoretical insights for corporate bankruptcy reorganization strategies, capital structure optimization, and the pricing of corporate stocks and bonds under uncertain bankruptcy environments.

Keywords: bankruptcy reorganization; finite observation period; Poisson process; asset pricing; optimal stopping problem
Mathematics Subject Classification: 91G40, 91G80

1. Introduction

The valuation of corporate equity and debt constitutes an important foundation for investment planning, risk management, and capital allocation. Its accuracy depends critically on an appropriate characterization of bankruptcy timing and bankruptcy reorganization mechanisms. In modern corporate finance practice, bankruptcy reorganization has become an important institutional arrangement through which financially distressed firms can avoid liquidation and restore firm value, and it has been widely observed in capital markets. For example, in the U.S. market, Hertz filed for Chapter 11 protection in 2020 after the COVID-19 pandemic caused a sharp decline in car rental demand and subsequently completed debt restructuring and equity conversion in 2021, returning to profitability and relisting. In the same year, Chesapeake Energy entered bankruptcy protection due to collapsing energy prices and later completed its reorganization through capital structure adjustment. In 2023, Bed Bath & Beyond entered bankruptcy proceedings owing to declining sales and cash flow pressure and maintained part of its operations through brand restructuring. Similar cases have also occurred in China. For instance, in 2022, the Shenzhen-based retailer Xinyijia implemented bankruptcy reorganization under court approval and resumed operations in 2023. In 2024, Zhongtian Financial successfully emerged from financial distress after the court approved its reorganization plan and strategic investors were introduced. In the same year, Sichuan Trust mitigated its risks and optimized its asset structure through a bankruptcy reorganization plan. These cases indicate that, when firms face severe financial distress, bankruptcy reorganization has become an important mechanism for restoring operations, relieving debt pressure, and preserving firm value.

At the same time, in actual economic environments, corporate bankruptcy is often triggered by sudden external shocks or major financial events. For example, in 2022, Revlon filed for bankruptcy protection because of supply chain disruptions and concentrated debt maturities. In 2023, a certain electric-vehicle start-up filed for bankruptcy and sold its core assets after the breakdown of a key technology partnership and a funding shortage. In 2024, Zhongzhi Enterprise Group entered bankruptcy proceedings following changes in management and the exposure of off-balance-sheet liabilities. These cases suggest that corporate bankruptcy not only involves a continuing process of reorganization and bargaining but also exhibits pronounced suddenness and randomness.

These real-world cases reveal that the bankruptcy process typically has two important features. On the one hand, once bankruptcy proceedings are initiated, the firm usually enters a finite reorganization observation period during which it attempts to restore operations and rebuild value through debt restructuring, asset adjustment, and the redistribution of claims. On the other hand, bankruptcy is often triggered by sudden external shocks or major financial events, so bankruptcy timing is inherently discrete and random in practice. However, the existing literature typically focuses on only one of these two aspects: Either bankruptcy reorganization and capital structure are studied in a continuous-time framework, or bankruptcy timing is modeled through random jump processes. Relatively little work has incorporated both a finite bankruptcy reorganization observation period and discrete bankruptcy timing within a unified framework. This separated treatment limits the ability of existing models to capture the actual bankruptcy process of firms. Therefore, developing a unified pricing framework that simultaneously incorporates a finite bankruptcy reorganization observation period and discrete bankruptcy timing is of considerable theoretical importance for more realistically characterizing firm value dynamics and the pricing of equity and debt.

Motivated by this observation, this paper develops a unified structural credit risk model in which a finite bankruptcy reorganization observation period is combined with a Poisson jump mechanism for discrete bankruptcy timing. Based on this framework, we establish a pricing model for corporate equity and debt and further analyze its implications for the optimal bankruptcy boundary and the optimal coupon level.

The main contributions of this paper are threefold. First, we incorporate both a finite bankruptcy reorganization observation period and discrete bankruptcy timing into a unified framework and establish a structural pricing model for corporate equity and debt; mathematically, the problem can be formulated as a nonlinear optimal stopping problem with a penalty term. Second, by using partial differential equation methods and Itô's formula, we derive explicit expressions for the values of equity, debt, and the firm as a whole and further obtain analytical solutions for the optimal bankruptcy boundary and the optimal coupon level. Third, based on the analytical solutions, we conduct a systematic numerical analysis of how the finite observation period and discrete bankruptcy timing affect equity value, debt value, and the optimal bankruptcy strategy, thereby providing theoretical implications for capital structure decisions and bankruptcy policy design.

The remainder of this paper is organized as follows: Section 2 reviews the related literature. Section 3 presents the theoretical framework and model assumptions. Section 4 analyzes firm value during bankruptcy reorganization. Section 5 derives the pricing formulas for corporate equity and debt and studies the optimal bankruptcy boundary and the optimal coupon level. Section 6 reports numerical results and discusses the effects of key parameters. Section 7 concludes.

2. Literature review

From the theoretical perspective, the early literature mainly studies bankruptcy and capital structure problems within continuous-time structural models. Fan and Sundaresan (2000) [1] were among the first to systematically characterize strategic debt service and asset reorganization under bankruptcy protection law, thereby laying the foundation for the pricing of corporate equity and debt under asset reorganization. However, their model assumes an infinite bankruptcy observation period, meaning that firms can continue reorganization bargaining indefinitely, which differs from actual bankruptcy institutions. To capture the finite duration of reorganization in practice, Francois and Morellec (2004) [2] extended this framework to a finite bankruptcy observation period and analyzed its effects on capital structure and bond value. Nevertheless, both studies focus only on perpetual corporate equity and debt and do not consider the finite-maturity case. Subsequently, under a finite bankruptcy observation framework, Broadie and Kaya (2007) [3] employed a binomial tree approach to construct a discrete pricing model for finite-maturity corporate claims, whereas Dai et al. (2013) [4] developed a continuous model for finite-maturity corporate claims by using partial differential equation methods and optimal stopping techniques.

From the empirical perspective, Gupta (2024) [5] showed that, for large U.S. firms, leverage is significantly positively correlated with the probability of emerging from bankruptcy, highlighting the important role of debt structure in the reorganization process. In addition, statistics reported by S&P Global (2025) [6] indicated that approximately 62.7% of U.S. bankrupt firms in 2024 resolved their financial distress through reorganization, reflecting the practical importance of bankruptcy reorganization from a macro-level perspective. Moreover, Mazur (2022) [7] studied the effects of bankruptcy law on corporate investment decisions and found that bankruptcy institutions significantly affect firms' investment discipline and capital allocation behavior.

On the other hand, recent studies have increasingly paid attention to the discreteness of bankruptcy timing. Dupuis and Wang (2002) [8] were among the first to model the timing of sudden events faced by firms by means of a Poisson process, thereby establishing a discrete bankruptcy-time model and providing an important theoretical basis for introducing random jump mechanisms into corporate bankruptcy pricing. Building on this idea, subsequent studies further extended the modeling of discrete bankruptcy timing. For example, Liang and Sun (2019) [9] examined the impact of discrete bankruptcy triggers on credit spreads within a jump-diffusion framework. Furthermore, Palmowski et al. (2020) [10] introduced Poisson observation times into the Leland–Toft model [11] and investigated how limited observation frequency affects the optimal bankruptcy threshold, capital structure, and bond value.

In summary, the existing literature has mainly developed along two separate lines: bankruptcy reorganization mechanisms and discrete bankruptcy timing. In reality, however, the bankruptcy process of firms often involves both features simultaneously. Therefore, simultaneously incorporating a finite bankruptcy reorganization observation period and discrete bankruptcy timing into a unified theoretical framework remains an important gap in the literature*.

To fill the above gap, this paper develops a unified model, within a structural credit risk framework, that combines a finite bankruptcy reorganization observation period with a Poisson-based discrete bankruptcy trigger mechanism. Under a strategic debt service reorganization mechanism, we establish pricing models for corporate equity and debt. By using partial differential equation methods and Itô's formula, we derive analytical expressions for equity and debt values and further solve for the optimal bankruptcy boundary and the optimal coupon level, thereby providing a systematic analysis of the joint effects of a finite reorganization observation period and discrete bankruptcy timing on corporate capital structure and asset pricing.

3. Theoretical framework and model assumptions

To characterize the effects of a finite bankruptcy reorganization observation period and discrete bankruptcy timing on corporate asset pricing, we establish a theoretical model within a structural credit risk framework. This section first presents the basic assumptions of the model, then describes the bankruptcy reorganization mechanism, and finally explains the Nash bargaining allocation between shareholders and creditors in the reorganization state.

3.1. Model assumptions

We assume that the risk-free interest rate r is constant and that there is no arbitrage in the market. The firm finances itself by issuing equity E and perpetual debt D. Before bankruptcy is declared, creditors receive a coupon payment C per unit time, while the firm obtains a tax shield benefit of γC per unit time from debt issuance, where γ $(0 < \gamma < 1)$ denotes the proportion of tax shield generated by one unit of coupon payment. Shareholders receive income in the form of dividends.

*To present the development of the related literature more clearly, this paper summarizes representative studies on bankruptcy reorganization models and discrete bankruptcy-time models in Table A.1 (see Table A.1 in the Appendix).

On the probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, Q)$, where Ω denotes the sample space, $\mathcal{F}$ is the associated σ-algebra, Q is the equivalent martingale measure, and $\mathcal{F}_t$ represents the information generated by the market over the time interval $[0, t]$, the firm value process V_t follows a geometric Brownian motion:

$$\frac{dV_t}{V_t} = (r - \delta)\, dt + \sigma\, dW_t, \tag{3.1}$$

where δ is the total cash payout rate of the firm, σ is the volatility, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion.

The bankruptcy time of the firm is determined by the jump times $\{T_n\}_{n\geq 1}$ of a Poisson process $\{N_t\}_{t\geq 0}$ with intensity ρ. These jump times correspond to critical moments at which the firm experiences sudden events or major economic losses. At such times, shareholders are granted the option to declare bankruptcy. The Poisson process and the Brownian motion are assumed to be independent. Specifically,

$$\mathcal{F}_t = \mathcal{F}_{W,t} \vee \mathcal{F}_{N,t},$$

where $\mathcal{F}_{W,t} = \sigma(W_s : s \leq t)$ and $\mathcal{F}_{N,t} = \sigma(N_s : s \leq t)$.

3.2. Bankruptcy reorganization mechanism

Under the strategic debt service reorganization regime, when shareholders choose to declare bankruptcy at a bankruptcy time $\tau = T_n$, the corresponding firm value V_τ represents the firm value at the bankruptcy time and serves as the bankruptcy observation threshold. At that moment, shareholders and creditors determine the reorganization plan according to a bargaining mechanism.

When the firm value satisfies $V_t < V_\tau$, the firm suspends the original coupon payment C to creditors and instead adopts a new coupon payment scheme determined through bargaining between the two parties, until the firm value recovers to the threshold level V_τ and the firm resumes normal operation.

At the same time, the reorganization agreement specifies a liquidation trigger. If the firm value V_t remains below the bankruptcy observation threshold V_τ for a continuous period exceeding the observation horizon G, the firm is forced into liquidation. In the liquidation process, the asset loss rate is α with $0 < \alpha < 1$. In addition, during the reorganization period when $V_t < V_\tau$, the firm temporarily loses the tax shield benefit and incurs a reorganization cost of ϕV_t per unit time, where $0 < \phi < \alpha\delta$. This parameter restriction ensures that bankruptcy reorganization can still generate a potential net benefit for the firm.

3.3. Nash bargaining allocation

Suppose that shareholders choose to declare bankruptcy at the bankruptcy time $\tau = T_n$, at which the firm value is V_τ. If the firm does not enter the reorganization procedure, it is immediately liquidated. In the liquidation process, the asset loss rate is α, shareholders receive nothing, and creditors obtain the liquidation payoff $(1 - \alpha)V_\tau$.

If the firm enters the bankruptcy reorganization stage, shareholders and creditors renegotiate the allocation of the total firm value generated after bankruptcy, denoted by $\bar{v}(V, \cdot; V_\tau)$, according to the Nash bargaining principle. When the firm value satisfies $V \leq V_\tau$, the equity and debt values are given by

$$\bar{E}(V, \cdot; V_\tau) = \theta \bar{v}(V, \cdot; V_\tau), \qquad \bar{D}(V, \cdot; V_\tau) = (1 - \theta)\bar{v}(V, \cdot; V_\tau), \tag{3.2}$$

where θ denotes the shareholders' share of the allocation.

According to the Nash bargaining principle, the optimal allocation ratio θ^* is determined by

$$\theta^* = \arg\max[\theta\bar{v}(V,\cdot;V_\tau)]^\eta[(1-\theta)\bar{v}(V,\cdot;V_\tau)-(1-\alpha)V]^{1-\eta},$$

which yields

$$\theta^* = \frac{\eta[\bar{v}(V,\cdot;V_\tau)-(1-\alpha)V]}{\bar{v}(V,\cdot;V_\tau)}. \tag{3.3}$$

Here, $\eta \in [0,1]$ denotes the bargaining power of shareholders in the bankruptcy reorganization negotiation. A larger η implies stronger bargaining power for shareholders, and hence a larger share of firm value, whereas a smaller η implies a more advantageous position for creditors.

Moreover, since the Nash product

$$[\theta\bar{v}(V,\cdot;V_\tau)]^\eta[(1-\theta)\bar{v}(V,\cdot;V_\tau)-(1-\alpha)V]^{1-\eta}$$

is strictly concave in θ over the interval $(0,1)$, the optimal solution is unique. Therefore, the Nash bargaining allocation ratio θ^* is uniquely determined.

4. Firm value under bankruptcy reorganization

In order to determine the payoffs received by shareholders and creditors at the bankruptcy declaration time, we first model and solve the total firm value under the bankruptcy reorganization regime, denoted by $\bar{v}(V;V_\tau)$.

Define $g_t = t - \sup\{0 \le s \le t : V_s \ge V_\tau\}$ and $\beta = \inf\{s \ge 0 : g_s \ge G\}$.

Here, g_t represents the length of time that the firm's asset value has continuously remained below the bankruptcy boundary V_τ prior to time t, and β denotes the time at which the firm is ultimately liquidated when the condition $g_t = G$ is satisfied.

According to the strategic debt service bankruptcy reorganization mechanism, under the martingale measure Q, for any bankruptcy declaration time $\tau = T_n$, the total firm value $\bar{v}(V_t, g_t; V_\tau)$ is given by

$$\begin{aligned}\bar{v}(V_t,g_t;V_\tau) = \mathbb{E}^Q\bigg[\int_t^\beta & e^{-r(s-t)}\Big(\delta V_s + \gamma C\mathbf{1}_{\{V_s\ge V_\tau\}} - \phi V_s\mathbf{1}_{\{V_s<V_\tau\}}\Big)ds \\ & + (1-\alpha)e^{-r(\beta-t)}V_\beta \,\Big|\, \mathcal{F}_t\bigg].\end{aligned} \tag{4.1}$$

Equation (4.1) describes the composition of the firm's total value. Specifically, the firm value is equal to the sum of two discounted components: The total cash flows generated from time t until the liquidation time β and the remaining asset value at the liquidation time after accounting for liquidation losses.

More precisely, the first component of cash flows consists of three key elements: The basic cash flow generated by the firm's assets per unit time, δV_s; the tax shield benefit γC obtained when the asset value satisfies $V_s \ge V_\tau$; and the restructuring negotiation cost ϕV_s incurred when the asset value lies in the region $V_s < V_\tau$.

By applying Itô's formula and the martingale method, the function $\bar{v}(V,g;V_\tau)$ satisfies the following boundary value problem P1:

$$\begin{cases} \mathcal{L}\bar{v}(V,0;V_\tau) + \delta V + \gamma C = 0, & V > V_\tau, g = 0, \\ \frac{\partial \bar{v}}{\partial g} + \mathcal{L}\bar{v}(V,g;V_\tau) + \delta V - \phi V = 0, & 0 < V < V_\tau,\ 0 \le g < G. \end{cases} \tag{4.2}$$

The continuity conditions at $V = V_\tau$ are given by

$$\begin{cases} \bar{v}(V_\tau - 0, g; V_\tau) = \bar{v}(V_\tau + 0, 0; V_\tau), & 0 \le g < G, \\ \frac{\partial \bar{v}}{\partial V}(V_\tau - 0, 0; V_\tau) = \frac{\partial \bar{v}}{\partial V}(V_\tau + 0, 0; V_\tau). \end{cases} \tag{4.3}$$

The boundary conditions are

$$\begin{cases} \lim\limits_{V\uparrow\infty} \left\{ \bar{v}(V,0;V_\tau) - \left[V + \frac{\gamma C}{r} \right] \right\} = 0, \\ \bar{v}(V,G;V_\tau) = (1-\alpha)V,\ V > 0, \end{cases} \tag{4.4}$$

where

$$\mathcal{L}\bar{v}(V,g;V_\tau) = \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 \bar{v}}{\partial V^2} + (r-\delta)V\frac{\partial \bar{v}}{\partial V} - r\bar{v}.$$

By solving the above boundary value problem P1, the analytical expression for the firm's total value function can be obtained as stated in the following theorem.

Theorem 1. *Suppose the firm declares bankruptcy at time τ and the corresponding asset value is V_τ. Under the strategic debt service bankruptcy reorganization mechanism, the total firm value $\bar{v}(V,g;V_\tau)$ is given by*

$$\bar{v}(V,g;V_\tau) = \begin{cases} V + \frac{\gamma}{r}C + AV^{\lambda_1}, & V > V_\tau, g = 0, \\ h(A;V_\tau) + e^{ax+b(G-g)} w(x,g)\Big|_{x=\ln\frac{V_\tau}{V}}, & 0 < V \le V_\tau, 0 \le g \le G, \end{cases} \tag{4.5}$$

where

$$a = \frac{r-\delta-\frac{1}{2}\sigma^2}{\sigma^2}, \quad b = -\frac{(r-\delta-\frac{1}{2}\sigma^2)^2}{2\sigma^2} - r,$$

$$\lambda_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left[\frac{1}{2} - \frac{(r-\delta)}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}}.$$

The explicit expressions of A, $h(A;V_\tau)$, and $w(x,g)$ are given in (A.9), (A.2), and (A.8), respectively. The detailed derivation can be found in the Appendix.

Combining Eqs (3.2), (3.3), and (4.5), the payoffs obtained by shareholders and creditors at the bankruptcy declaration time $V = V_\tau$ under bankruptcy reorganization are given by

$$\Phi(V) = \eta[K(G)V + \frac{\gamma C}{r}(1 - B(G))],$$

$$\Psi(V) = \left[(1-\eta)K(G) + (1-\alpha)\right]V + \frac{(1-\eta)\gamma C(1-B(G))}{r}.$$

Here, $K(G)$ and $B(G)$ are functions determined by the bankruptcy reorganization observation period G, which characterize the impact of the reorganization phase on the firm's value structure. Specifically,

$K(G)$ represents the effective retention ratio of the firm's asset value during the reorganization period when the observation horizon is G. It reflects the proportion of firm value that can be preserved through continued operations during the restructuring process.

In contrast, $B(G)$ captures the discounted attenuation of tax shield benefits during the reorganization observation period. When the observation period G becomes longer, the firm has more time to adjust its assets and restructure its liabilities, which increases the recovery rate of asset value; therefore, $K(G)$ generally increases with G. Meanwhile, because coupon payments and tax shield benefits may be suspended or reduced during the restructuring stage, $B(G)$ reflects the discounted loss of tax shield benefits in this phase.

Therefore, $K(G)$ and $B(G)$ jointly characterize the impact of the bankruptcy reorganization observation period on the firm's value structure. Together with key parameters such as the restructuring cost ϕ and the liquidation loss rate α, they determine the allocation of firm value between shareholders and creditors. The explicit expressions of $K(G)$ and $B(G)$ are given in (A.10) and (A.11) in the Appendix.

5. Pricing of corporate equity and debt

5.1. Mathematical model and pricing formula for corporate equity

Under the basic assumptions of the model and the martingale measure Q, the pricing model for the corporate equity value E is given by

$$E(V_t) = \max_{\tau\in\mathcal{T}} \mathbb{E}^Q \left[\int_t^{\tau\wedge\infty} e^{-r(s-t)}(\delta V_s + \gamma C - C)ds + e^{-r(\tau\wedge\infty-t)}\Phi(V_\tau) \middle| \mathcal{F}_t \right]. \tag{5.1}$$

Let $\mathcal{T} = \{T_1, T_2, \cdots, T_n, \cdots\}$ denote the set of bankruptcy declaration times available to shareholders. The first term on the right-hand side of Eq (5.1) represents the discounted value of dividend payments received by shareholders, while the second term corresponds to the discounted payoff $\Phi(V_\tau)$ obtained by shareholders when bankruptcy reorganization is declared.

Different from the traditional assumption that "shareholders may declare bankruptcy at any time" (which implicitly implies that the shareholder value $E(V_t)$ is always no less than the bankruptcy payoff $\Phi(V_t)$), in the model developed in this paper, the optimal bankruptcy time τ^* must satisfy

$$E(V_t) = \mathbb{E}^Q \left[\int_t^{\tau^*\wedge\infty} e^{-r(s-t)}(\delta V_s + \gamma C - C)ds + e^{-r(\tau^*\wedge\infty-t)} \max\{\Phi(V_{\tau^*}), E(V_{\tau^*})\} \middle| \mathcal{F}_t \right]. \tag{5.2}$$

The economic interpretation of the second term on the right-hand side of Eq (5.2) is as follows: If the payoff from bankruptcy reorganization, $\Phi(V_{\tau^*})$, exceeds the current equity value $E(V_{\tau^*})$, then shareholders will choose to declare bankruptcy reorganization at time τ^*; otherwise, they will choose to continue operating the firm. This means that at any admissible jump time T_n, shareholders can obtain at least the maximum of $E(V_{\tau^*})$ and $\Phi(V_{\tau^*})^\dagger$.

By Itô's formula and the dynamic programming principle, the corporate equity value $E(V)$ satisfies

†For a detailed explanation, see Theorem 2 in Dupuis and Wang (2002).

the following boundary value problem P2:

$$\begin{cases} \tilde{\mathcal{L}}E + \rho(\Phi(V) - E(V))^+ = -(\gamma - 1)C - \delta V, \quad 0 < V < \infty, \\ \lim\limits_{V\to 0} E(V) \text{ is bounded}, \\ \lim\limits_{V\to\infty} \left[E(V) - \left(V + \frac{(\gamma-1)C}{r}\right)\right] = 0, \end{cases} \tag{5.3}$$

where

$$\tilde{\mathcal{L}}E(V) = \frac{1}{2}\sigma^2 V^2 \frac{\mathrm{d}^2 E(V)}{\mathrm{d}V^2} + (r - \delta)V\frac{\mathrm{d}E(V)}{\mathrm{d}V} - rE(V).$$

Here, boundary condition 1 indicates that when the firm's asset value approaches zero, the equity value remains bounded and does not explode. Boundary condition 2 implies that when the firm's asset value is sufficiently large, the corporate debt approaches a risk-free bond with value converging to $\frac{C}{r}$, while the equity value converges to $V + \frac{(\gamma-1)C}{r}$.

Theorem 2. *The corporate equity value and the optimal bankruptcy boundary are given by*

$$E(V) = \begin{cases} V + \frac{(\gamma-1)C}{r} + C_1\left(\frac{V}{V^*}\right)^{\lambda_1}, & V \geq V^*, \\ \frac{\delta+\rho\eta K(G)}{\rho+\delta}V + \frac{(\gamma-1)C+\rho\cdot\frac{\eta\gamma C}{r}(1-B(G))}{r+\rho} + C_3\left(\frac{V}{V^*}\right)^{\lambda_4}, & V < V^*, \end{cases} \tag{5.4}$$

$$V^* = \frac{\lambda_1\lambda_4}{(1-\lambda_1)(1-\lambda_4)} \cdot \frac{\rho+\delta}{(\rho+r)r} \cdot \frac{[\eta\gamma C(1-B(G)) + (1-\gamma)C]}{1-\eta K(G)}. \tag{5.5}$$

Here, λ_1 is given in Theorem 1 and

$$\lambda_4 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2(r+\rho)}{\sigma^2}},$$

$$C_1 = \frac{1}{\lambda_4 - \lambda_1}\left[V^*(1-\lambda_4)\frac{(1-\eta K(G))\rho}{\rho+\delta} + \rho\lambda_4\frac{\eta\gamma C(1-B(G)) + (1-\gamma)C}{(\rho+r)r}\right],$$

$$C_3 = \frac{1}{\lambda_4 - \lambda_1}\left[V^*(1-\lambda_1)\frac{(1-\eta K(G))\rho}{\rho+\delta} + \rho\lambda_1\frac{\eta\gamma C(1-B(G)) + (1-\gamma)C}{(\rho+r)r}\right].$$

The detailed derivation is provided in the Appendix.

5.2. Mathematical model and pricing formula for corporate debt

Based on the optimal bankruptcy boundary V^* obtained above, the optimal bankruptcy time can be expressed as

$$\tau^* = \inf\{T_n : n \geq 1, V_{T_n} \leq V^*\}.$$

The pricing model for the corporate debt value D is given by

$$D(V_t) = \mathbb{E}^Q\left[\int_t^{\tau^*\wedge\infty} e^{-r(s-t)}Cds + e^{-r(\tau^*\wedge\infty-t)}\left(I_{\{\Phi(V_{\tau^*})\geq E(V_{\tau^*})\}}\Psi(V_{\tau^*}) + I_{\{\Phi(V_{\tau^*})<E(V_{\tau^*})\}}D(V_{\tau^*})\right)\middle|\mathcal{F}_t\right]. \tag{5.6}$$

The first term on the right-hand side of Eq (5.6) represents the discounted coupon payments received by creditors. The economic interpretation of the second term is that, if the shareholders' payoff from bankruptcy reorganization $\Phi(V_{\tau^*})$ exceeds the current equity value $E(V_{\tau^*})$, then shareholders will declare bankruptcy reorganization at time τ^*, and creditors will correspondingly obtain $\Psi(V_{\tau^*})$; otherwise, shareholders will continue operating the firm, in which case creditors' continuation payoff is $D(V_{\tau^*})$.

By Itô's formula and the dynamic programming principle, the corporate debt value $D(V)$ satisfies the following boundary value problem P4:

$$\begin{cases} \tilde{\mathcal{L}}D + \rho[\Psi(V) - D(V)]I_{\{E(V)\leq\Phi(V)\}} = -C, \quad 0 < V < \infty, \\ \lim\limits_{V\to 0} D(V) \text{ is bounded}, \\ \lim\limits_{V\to\infty} D(V) = \frac{C}{r}. \end{cases}$$

Here, boundary condition 1 indicates that when the firm's asset value approaches zero, the debt value remains bounded. Boundary condition 2 implies that when the firm's asset value is sufficiently large, the corporate debt approaches a risk-free bond, and its value converges to $\frac{C}{r}$.

Theorem 3. *The corporate debt value is given by*

$$D(V) = \begin{cases} \frac{C}{r} + C_4\left(\frac{V}{V^*}\right)^{\lambda_1}, & V \geq V^*, \\ \frac{\rho\left[(1-\eta)K(G)+(1-\alpha)\right]}{\rho+\delta}V + \frac{Cr+\rho(1-\eta)\gamma C(1-B(G))}{r(r+\rho)} + C_6\left(\frac{V}{V^*}\right)^{\lambda_4}, & V < V^*. \end{cases} \tag{5.7}$$

Here, λ_1 and λ_4 are given in Theorem 2 and

$$C_4 = \frac{\lambda_4}{\lambda_1 - \lambda_4}\left[\frac{C}{r} - \frac{\lambda_4 - 1}{\lambda_4}\cdot\frac{\rho\left[(1-\eta)K(G) + (1-\alpha)\right]V^*}{\rho+\delta} - \frac{C\left[\rho(1-\eta)\gamma(1-B(G)) + r\right]}{r(\rho+r)}\right],$$

$$C_6 = \frac{\lambda_1}{\lambda_1 - \lambda_4}\left[\frac{C}{r} - \frac{\lambda_1 - 1}{\lambda_1}\cdot\frac{\rho\left[(1-\eta)K(G) + (1-\alpha)\right]V^*}{\rho+\delta} - \frac{C\left[\rho(1-\eta)\gamma(1-B(G)) + r\right]}{r(\rho+r)}\right].$$

The detailed derivation is given in the Appendix.

5.3. Firm value and the optimal coupon

By issuing debt, the firm obtains a tax shield benefit of γC per unit time, but at the same time, it must pay coupon payments of C per unit time to creditors, which increases the firm's bankruptcy risk. Therefore, how to optimize the firm's leverage ratio becomes a key issue. In this paper, we determine the optimal leverage ratio by selecting the optimal coupon level C^* so as to maximize the total firm value v.

Based on Eqs (5.4), (5.5), and (5.7), the explicit expression of the total firm value $v(V)$ can be obtained. Then, regarding $v(V)$ as a function of the coupon C, when $V \geq V^*$, the optimal coupon C^* is determined by solving $\frac{\partial v(V;C)}{\partial C} = 0$. Following this line of reasoning, we summarize the result in the following theorem.

Theorem 4. *The total firm value and the optimal coupon C^* are given by*

$$v(V) = \begin{cases} V + \frac{\gamma C}{r} + (C_1 + C_4)\left(\frac{V}{V^*}\right)^{\lambda_1}, & V \geq V^*, \\ \frac{\delta+\rho[K(G)+(1-\alpha)]}{\rho+\delta}V + \frac{\gamma C\left[r+\rho(1-B(G))\right]}{r(r+\rho)} + (C_3 + C_6)\left(\frac{V}{V^*}\right)^{\lambda_4}, & V < V^*, \end{cases} \tag{5.8}$$

$$C^* = \frac{V}{K_1}\left(-\frac{\gamma}{rK_2}\right)^{-\frac{1}{\lambda_1}}. \tag{5.9}$$

Here, $\lambda_1, \lambda_4, C_1, C_3, C_4$ and C_6 are given in Theorems 2 and 3 and

$$K_1 = \frac{\lambda_1\lambda_4}{(1-\lambda_1)(1-\lambda_4)} \cdot \frac{\rho+\delta}{(\rho+r)r} \cdot \frac{[\eta\gamma(1-B(G)) - (\gamma-1)]}{1-\eta K(G)},$$

$$K_2 = \frac{\lambda_4(1-\lambda_1)}{(\lambda_1-\lambda_4)} \cdot \frac{\rho}{(\rho+r)r} \left\{ \frac{[\eta\gamma(1-B(G)) - (\gamma-1)]}{1-\eta K(G)} \times \right.$$
$$\left. \frac{\lambda_1\left[(1-\eta)K(G) + (1-\alpha)\right] + \eta K(G) - 1}{1-\lambda_1} + \left[1-(1-\eta)\gamma(1-B(G))\right] \right\}.$$

The detailed derivation is provided in the Appendix.

Theorem 5. *As $\rho \to \infty$, the analytical expressions of $E(V)$, $D(V)$, V^*, and C^* all converge to the corresponding results in the model of François and Morellec [2], that is,*

$$\lim_{\rho\to\infty} E(V;\rho) = \bar{E}(V), \quad \lim_{\rho\to\infty} D(V;\rho) = \bar{D}(V),$$

$$\lim_{\rho\to\infty} V^*(\rho) = \bar{V}, \quad \lim_{\rho\to\infty} C^*(\rho) = \bar{C}.$$

Here, $\bar{E}(V)$, $\bar{D}(V)$, $\bar{V}$, and $\bar{C}$ denote the corresponding pricing results in François and Morellec [2], respectively.

6. Numerical analysis and financial implications

Based on the pricing formulas for corporate equity, debt, and total firm value derived in Eqs (5.4), (5.7), and (5.8), together with the analytical expressions for the optimal bankruptcy boundary and the optimal coupon level in Eqs (5.5) and (5.9), this section numerically examines the effects of key model parameters on firm valuation and operating strategies when the firm follows the optimal bankruptcy policy and the optimal coupon policy. In particular, we focus on the impact of the Poisson jump intensity ρ on corporate equity value, debt value, and total firm value. We further investigate the sensitivity of the model to the bankruptcy reorganization observation period G, the firm-specific risk parameters δ and σ, and the market interest rate r under the discrete bankruptcy-time mechanism, so as to explore the financial implications embedded in the model. To facilitate comparison of value changes under different scenarios, define

$$P_E = \frac{E-\bar{E}}{\bar{E}},\ P_D = \frac{D-\bar{D}}{\bar{D}},\ P_v = \frac{v-\bar{v}}{\bar{v}},$$

which denote the relative change ratios of corporate equity value, debt value, and total firm value, respectively. Under the benchmark case, the model parameters are set as $\rho = 0.05,\ G = 1,\ r = 0.04,\ \gamma = 0.02,\ \delta = 0.03,\ \sigma = 0.2,\ \eta = 0.6,\ \phi = 0.01$ and $\alpha = 0.5$ ‡.

‡The above parameter values are chosen with reference to the typical calibration ranges used in the structural credit risk literature, such as Leland and Toft (1996), Fan and Sundaresan (2000), and Francois and Morellec (2004).

All numerical results reported in this section are obtained through numerical computations based on the analytical solutions derived in the theoretical model, rather than from empirical data. The simulations are implemented in Matlab.

First, the benchmark values of corporate equity, debt, and total firm value, as well as the optimal bankruptcy boundary and the optimal coupon level, are computed. Then, keeping all other parameters fixed, one-factor sensitivity analyses are performed with respect to the Poisson jump intensity, the bankruptcy reorganization observation period, the payout ratio, volatility, and the risk-free interest rate. The corresponding figures and tables are generated automatically in Matlab to ensure the reproducibility of the results.

6.1. Effects of ρ on C^*, V^*, $E(V)$, $D(V)$, and $v(V)$

By comparing our results with the corresponding pricing results in François and Morellec (FM) [2], we examine the effects of the jump intensity ρ on the firm's operating strategy and valuation.

Figure 1 shows that the optimal bankruptcy boundary is higher than that in the FM model. Indeed, if $V^* < \bar{V}$, then there exists an interval $V^* < V < \bar{V}$. From boundary value problem P3, one has $\tilde{\mathcal{L}}E = -(\gamma-1)C - \delta V$, whereas, because $V < \bar{V}$, one has $\tilde{\mathcal{L}}\bar{E} < -(\gamma-1)C - \delta V$. Since $\lim_{\rho\to\infty} \tilde{\mathcal{L}}E = \tilde{\mathcal{L}}\bar{E}$, a contradiction arises, and hence, $V^* \geq \bar{V}$.

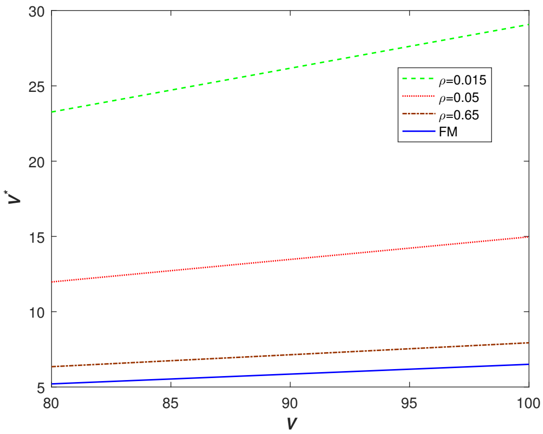

Figure 1. Relationship between the optimal bankruptcy boundary and the jump intensity ρ.

Figure 2 shows that the optimal coupon is also higher than that in the FM model. Since the optimal coupon is chosen to maximize total firm value by balancing the interests of shareholders and creditors, the increase in debt value implies that the corresponding optimal coupon level is also higher.

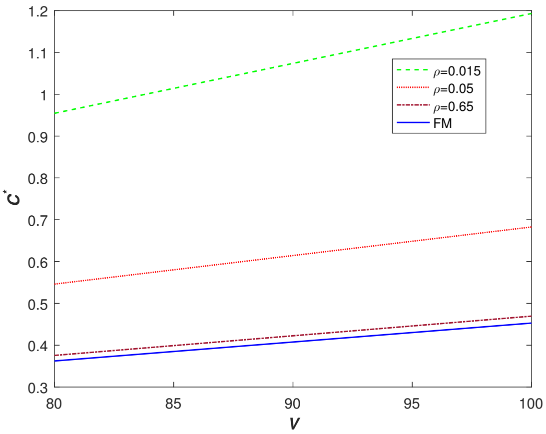

Figure 2. Relationship between the optimal coupon and the jump intensity ρ.

Figures 3–5 show that introducing Poisson-type bankruptcy timing has only a limited effect on total firm value, but it substantially affects the allocation of value between equity and debt. Compared with the FM model, corporate equity value decreases, whereas corporate debt value increases. The reason is that, relative to the FM framework in which shareholders may declare bankruptcy at any time, the present model restricts shareholders to a smaller set of admissible bankruptcy times, namely only the jump times of the Poisson process. Therefore, the optimal strategy in our model is only the optimal strategy over a subset of the bankruptcy times available in the FM model, which leads to a reduction in equity value. Since the effect on total firm value is relatively small, the decrease in shareholders' claims naturally implies an increase in creditors' claims.

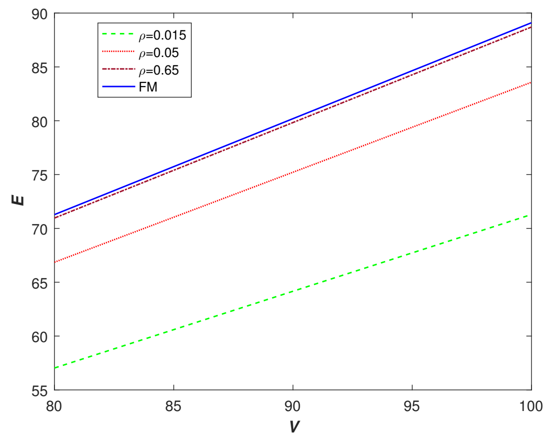

Figure 3. Relationship between corporate equity value and the jump intensity ρ.

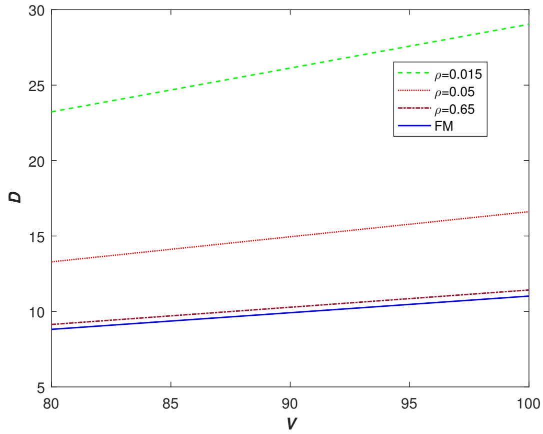

Figure 4. Relationship between corporate debt value and the jump intensity ρ.

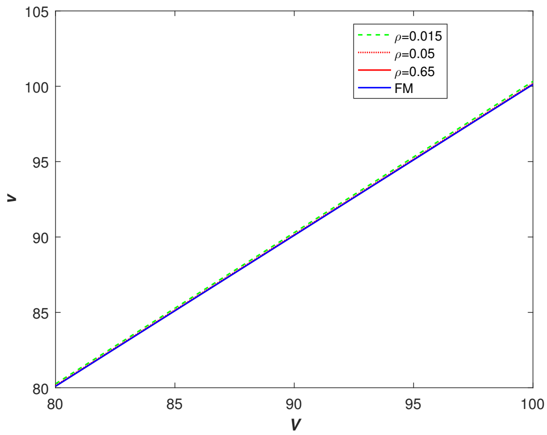

Figure 5. Relationship between total firm value and the jump intensity ρ.

Finally, as ρ increases, all results gradually converge to the corresponding results in the FM model.

In Sections 6.2–6.4 below, compared with the FM pricing results, we examine the sensitivity of the model to various factors through the change ratios of corporate equity value, debt value, and total firm value, denoted respectively by P_E, P_D, and P_v.

6.2. *Sensitivity analysis of pricing with respect to the bankruptcy reorganization observation period G*

Figures 6–8 show that, compared with the pricing results of the FM model, the introduction of discrete bankruptcy timing leads to a noticeable change in the firm's value structure. Among the three components, corporate debt value is affected the most, with its relative change ratio reaching as high as about 51%; corporate equity value is affected to a lesser extent, with a change ratio of about 6%; the change in total firm value is relatively small, remaining below 0.007%. These results indicate that the change in the bankruptcy-timing mechanism primarily affects the allocation of value between equity and debt, whereas its impact on overall firm value is relatively limited.

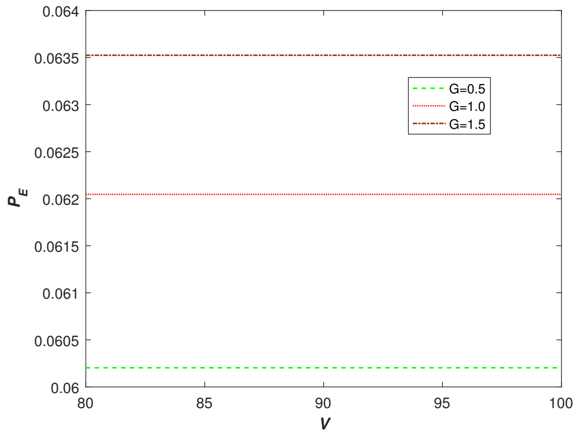

Figure 6. Relationship between the relative change ratio of corporate equity value and the bankruptcy observation period.

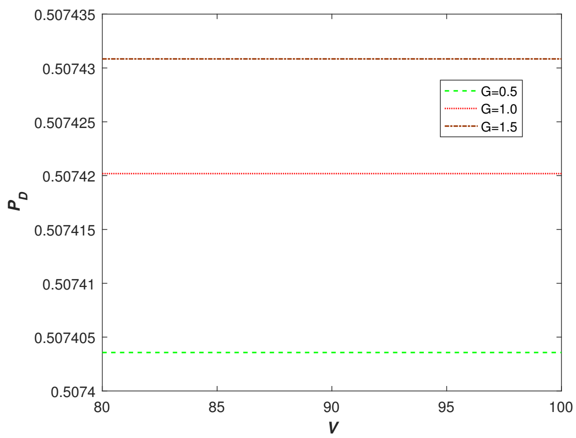

Figure 7. Relationship between the relative change ratio of corporate debt value and the bankruptcy observation period.

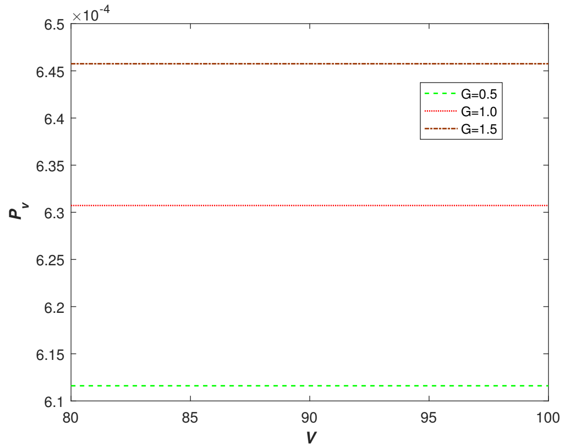

Figure 8. Relationship between the relative change ratio of total firm value and the bankruptcy observation period.

A further examination of the bankruptcy reorganization observation period G shows that all three value change ratios increase slightly as G increases. When G rises from 0.5 to 1.5, the average change ratio of corporate equity value increases by about 0.33%, while that of corporate debt value increases by about 0.0028%, and the variation in total firm value remains at a relatively low level.

From a financial perspective, a longer bankruptcy reorganization observation period G provides the firm with more time for debt restructuring and operational adjustment, thereby reducing the probability of immediate liquidation and increasing creditors' expected recovery rates. As a result, debt value rises significantly. Meanwhile, although shareholders face a lower immediate bankruptcy risk, the improvement in equity value is relatively limited because the firm must bear additional reorganization costs and operating constraints during the restructuring stage. Overall, an extension of the bankruptcy reorganization observation period mainly enhances the firm's ability to mitigate financial distress, benefiting creditors more substantially, while the changes in equity value and total firm value remain relatively moderate.

6.3. Sensitivity analysis of the firm-specific risk factors δ and σ

Figures 9–14 show that the relative change ratios of corporate equity value P_E and total firm value P_v are both monotonically decreasing in δ and σ, whereas the relative change ratio of corporate debt value P_D is monotonically increasing in δ and σ. Numerical results indicate that when the payout ratio δ increases from 0.025 to 0.035, the average change ratio of equity value decreases by about 15%, while the average change ratio of debt value increases by about 3.5%. Similarly, when the asset volatility σ rises from 0.15 to 0.25, the average change ratio of equity value decreases by about 3%, whereas that of debt value increases by about 43%.

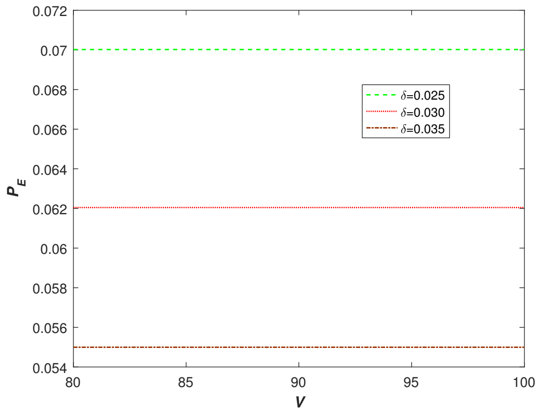

Figure 9. Relationship between the relative change ratio of corporate equity value and the payout ratio δ.

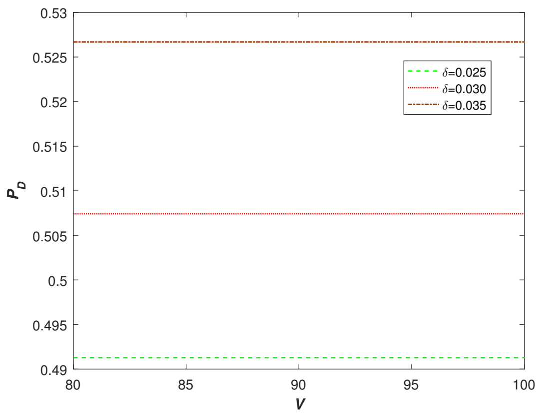

Figure 10. Relationship between the relative change ratio of corporate debt value and the payout ratio δ.

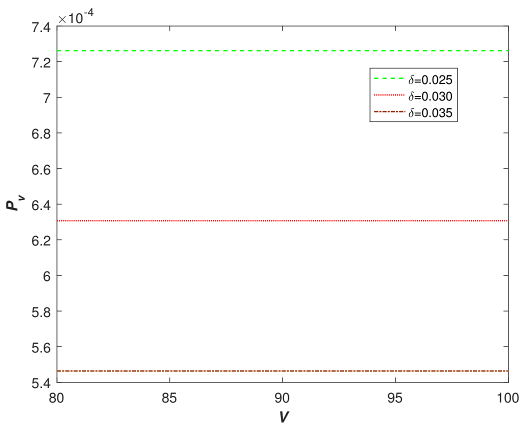

Figure 11. Relationship between the relative change ratio of total firm value and the payout ratio δ.

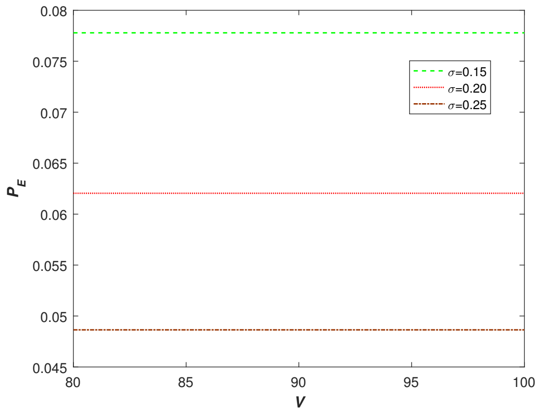

Figure 12. Relationship between the relative change ratio of corporate equity value and the volatility σ.

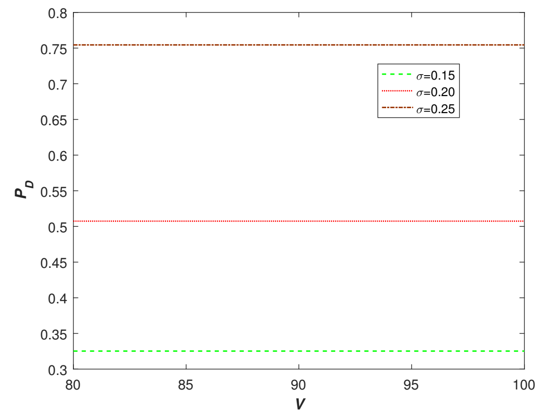

Figure 13. Relationship between the relative change ratio of corporate debt value and the volatility σ.

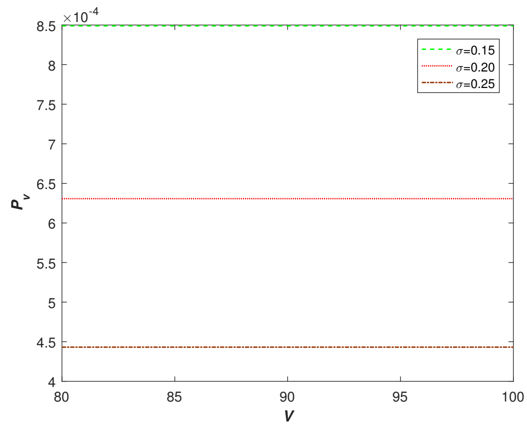

Figure 14. Relationship between the relative change ratio of total firm value and the volatility σ.

From a financial perspective, an increase in the payout ratio δ means that the firm allocates more cash flow to dividends or expenditures, thereby weakening internal capital accumulation and reducing the firm's resilience to risk. As a consequence, shareholders become more sensitive to bankruptcy risk, leading to a decline in equity value. At the same time, since debt cash flows are relatively stable, creditors' expected payoffs improve. On the other hand, a higher asset volatility σ increases the probability of bankruptcy triggers, causing shareholders to bear greater risk and thus reducing equity value. Under a finite bankruptcy reorganization observation period, however, creditors' expected recovery rates rise, and debt value correspondingly increases.

6.4. Sensitivity analysis of the market factor r

Figures 15–17 show that the risk-free interest rate r has a significant effect on the relative change ratios of corporate equity value, debt value, and total firm value. As r increases from 0.035 to 0.045, the relative change ratio of equity value rises by about 35%, whereas that of debt value declines by about 0.57%. This is because a higher risk-free rate changes investors' required returns on risky assets, making equity value more sensitive to interest-rate changes. At the same time, an increase in interest rates reduces the discounted present value of future debt cash flows, and therefore the relative change ratio of debt value exhibits a downward trend. As for total firm value, since equity value and debt value move in opposite directions, there is a certain offsetting effect between them, so that the relative change ratio of total firm value changes only moderately as r increases.

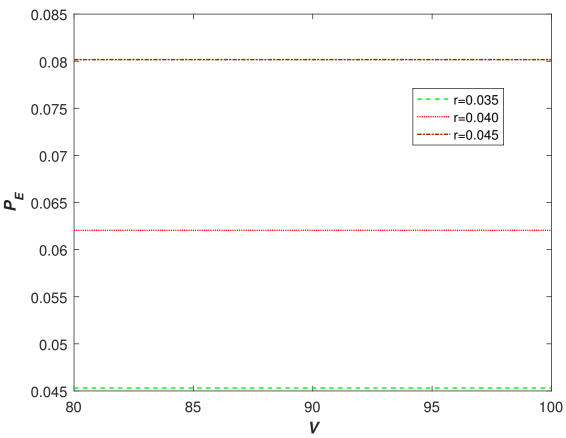

Figure 15. Relationship between the relative change ratio of corporate equity value and the risk-free interest rate r.

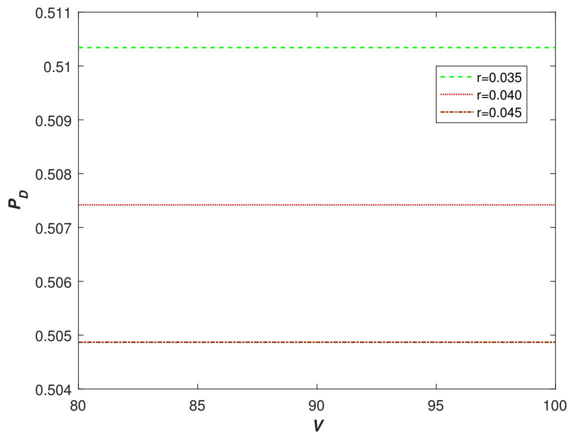

Figure 16. Relationship between the relative change ratio of corporate debt value and the risk-free interest rate r.

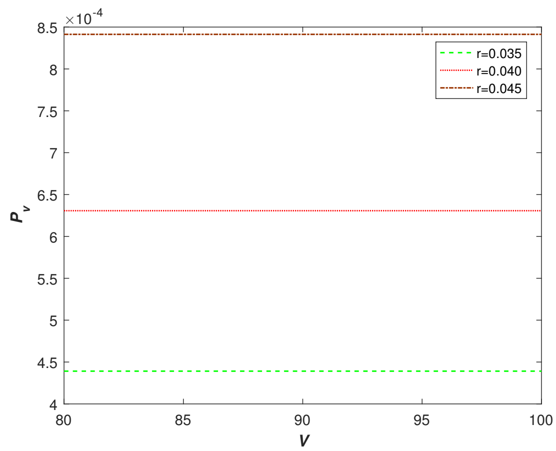

Figure 17. Relationship between the relative change ratio of total firm value and the risk-free interest rate r.

To present more intuitively the effects of key parameter changes on firm value, Table 1 summarizes the quantitative results corresponding to the major parameter variations.

Table 1. Quantitative effects of key parameter changes on P_E, P_D, and P_v.

Parameter interval change	Relative change of P_E	Relative change of P_D	Relative change of P_v
$G : 0.5 \to 1.5$	increase by about 0.33%	increase by about 0.0028%	increase by about 0.0034%
$\delta : 0.025 \to 0.035$	decrease by about 15%	increase by about 3.5%	decrease by about 0.0181%
$\sigma : 0.15 \to 0.25$	decrease by about 3%	increase by about 43%	decrease by about 0.04%
$r : 0.035 \to 0.045$	increase by about 35%	decrease by about 0.57%	increase by about 0.04%

7. Conclusions

This paper addresses the realistic feature that corporate bankruptcy decisions are often characterized simultaneously by a finite bankruptcy reorganization observation period and a discrete bankruptcytriggering mechanism. Building on the model of François and Morellec (2004) [2], we extend the assumption that firms may declare bankruptcy at any instant to the case in which bankruptcy can only be declared at the jump times of a Poisson process with intensity ρ. In this way, we construct a unified pricing model for corporate stocks and bonds that incorporates both a finite bankruptcy reorganization observation period and discrete bankruptcy timing. Within the structural credit risk framework, we establish mathematical pricing models for corporate equity and debt and derive explicit analytical expressions for equity value, debt value, and total firm value by means of the dynamic programming principle and partial differential equation methods. At the same time, explicit solutions for the optimal bankruptcy boundary and the optimal coupon level are obtained. Furthermore, we prove that, as $\rho \to \infty$, the pricing results of our model reduce to those of the continuous bankruptcy model in François and Morellec (2004).

The results show that the introduction of a discrete bankruptcy-time mechanism significantly alters the firm's optimal bankruptcy strategy. Compared with the FM model, the optimal bankruptcy boundary in our model is substantially higher, and the optimal coupon level is significantly increased. This indicates that, under discrete bankruptcy timing, shareholders face restrictions on their choice of bankruptcy timing and therefore have a stronger incentive to trigger bankruptcy earlier and to raise the coupon level, so as to achieve a new balance between tax shield benefits and financial risk. Numerical results further show that the discrete bankruptcy mechanism has a relatively limited impact on total firm value, but it significantly changes the distribution of value between equity and debt, namely, equity value decreases while debt value increases. In addition, extending the bankruptcy reorganization observation period G further strengthens this divergence effect and thereby increases debt value. The sensitivity analysis also indicates that the payout ratio, asset volatility, and market interest rate all exert significant effects on corporate asset pricing.

From an economic perspective, the present model simultaneously characterizes the finite bankruptcy reorganization observation period and the discrete bankruptcy-time mechanism within a unified framework, allowing it to reflect more realistically corporate bankruptcy and reorganization behavior under financial distress. It also reveals the important influence of uncertainty in bankruptcy timing on corporate capital structure decisions and risk allocation mechanisms. The model is particularly suitable for studying firms with relatively high financial leverage, elevated bankruptcy risk, and the potential for debt restructuring through reorganization procedures, such as firms in capital-intensive or cyclical industries.

It should be noted that the present model still has certain limitations. First, the model assumes a constant risk-free interest rate and does not take into account the effects of macroeconomic factors, such as interest rate fluctuations or inflation changes, on corporate bankruptcy decisions and asset pricing. Second, the model does not consider heterogeneity across firms in terms of growth opportunities, asset structure, and operating risk. In addition, real-world bankruptcy procedures often involve multi-stage reorganization processes, whereas this paper considers only a single-stage bankruptcy reorganization mechanism. Future research may be extended in several directions. On the one hand, heterogeneity in creditors' bargaining power may be introduced so as to characterize more realistically the allocation of interests among different types of creditors in bankruptcy reorganization. On the other hand, reorganization costs may be modeled as random variables, making it possible to analyze the effects of market fluctuations and firm-specific operating risks on reorganization costs and asset pricing. Such extensions would further enhance the practical applicability of the model and provide more valuable decision-making references for corporate managers and investors.

Author contributions

Jianwei Lin: Conceptualization, Methodology, Writing–original draft; Yuan Chen: Methodology, Validation, Writing–review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors used AI tools (e.g., ChatGPT) for language editing and manuscript preparation. The scientific content and conclusions are entirely the responsibility of the authors.

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Conflict of interest

The authors declare no conflicts of interest.

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A. Appendix

Proof of Theorem 1:

From the equation satisfied by the firm value in (4.2) for $V > V_\tau$ and $g = 0$, together with boundary condition 1 in (4.5), it follows that, for $V > V_\tau$, the general solution of $\bar{v}(V, 0; V_\tau)$ has the form

$$\bar{v}(V, 0; V_\tau) = V + \frac{\gamma C}{r} + AV^{\lambda_1},$$

where A is an undetermined constant and

$$\lambda_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}}.$$

Introduce the transformation

$$x = \ln\frac{V_\tau}{V}, \qquad \bar{g} = G - g, \qquad \bar{v}(V, g; V_\tau) - h(A, V_\tau) = e^{ax+b\bar{g}} w(x, \bar{g}), \tag{A.1}$$

where

$$h(A, V_\tau) = V_\tau + \frac{\gamma C}{r} + AV_\tau^{\lambda_1}. \tag{A.2}$$

Then, on the region $0 < V \le V_\tau$, $0 \le g \le G$, the boundary value problem for $\bar{v}(V, g; V_\tau)$ is transformed into

$$\begin{cases} \dfrac{\partial w}{\partial \bar{g}} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 w}{\partial x^2} \\ = \left[(\delta-\phi)V_\tau e^{-x} - rh(A, V_\tau)\right] e^{-ax-b\bar{g}}, \quad 0 < x < \infty, \ 0 \le \bar{g} < G, & \text{(A.3)} \\ w(x, 0) = \left[(1-\alpha)V_\tau e^{-x} - h(A, V_\tau)\right] e^{-ax}, & \text{(A.4)} \\ w(0, \bar{g}) = 0, & \text{(A.5)} \\ \lim\limits_{x\to 0} e^{bG}\left[aw(x, G) + \dfrac{\partial w(x, G)}{\partial x}\right] = -(V_\tau + A\lambda_1 V_\tau^{\lambda_1}), & \text{(A.6)} \end{cases}$$

with

$$a = \frac{r - \delta - \frac{\sigma^2}{2}}{\sigma^2}, \qquad b = -r - \frac{(r - \delta - \frac{\sigma^2}{2})^2}{2\sigma^2}.$$

Denote

$$\Theta(x, \bar{g}; A, V_\tau) = \left[(\delta - \phi)V_\tau e^{-x} - rh(A, V_\tau)\right] e^{-ax-b\bar{g}},$$
$$\Upsilon(x; A, V_\tau) = [(1 - \alpha)V_\tau e^{-x} - h(A, V_\tau)]\, e^{-ax}.$$

By the method of images, define the odd extensions of $\Theta(x, \bar{g}; A, V_\tau)$ and $\Upsilon(x; A, V_\tau)$ as follows:

$$\Theta(x, \bar{g}; A, V_\tau) = \begin{cases} [(\delta - \phi)V_\tau e^{-x} - rh(A, V_\tau)]e^{-ax-b\bar{g}}, & x > 0, \\ -[(\delta - \phi)V_\tau e^{x} - rh(A, V_\tau)]e^{ax-b\bar{g}}, & x < 0, \end{cases}$$

$$\Upsilon(x; A, V_\tau) = \begin{cases} [(1 - \alpha)V_\tau e^{-x} - h(A, V_\tau)]e^{-ax}, & x > 0, \\ -[(1 - \alpha)V_\tau e^{x} - h(A, V_\tau)]e^{ax}, & x < 0. \end{cases}$$

Therefore, the boundary value problems (A.3)–(A.6) are reduced to the standard Cauchy problem:

$$\begin{cases} \dfrac{\partial w}{\partial \bar{g}} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 w}{\partial x^2} = \Theta(x, \bar{g}; A, V_\tau), & x \in \mathbb{R}, \ \ 0 \le \bar{g} < G, \\ w(x, 0) = \Upsilon(x; A, V_\tau), & x \in \mathbb{R}. \end{cases}$$

Using $\bar{g} = G - g$ and Poisson's formula, we obtain

$$w(x, g) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-\xi)^2}{2\sigma^2(G-g)}}}{\sigma\sqrt{2\pi(G-g)}} \Upsilon(\xi; A, V_\tau)\, \mathrm{d}\xi$$
$$+ \int_g^G \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-\xi)^2}{2\sigma^2(\widehat{\eta}-g)}}}{\sigma\sqrt{2\pi(\widehat{\eta}-g)}} \Theta(\xi, G - \widehat{\eta}; A, V_\tau)\, \mathrm{d}\xi\, \mathrm{d}\widehat{\eta}. \tag{A.7}$$

After calculation, $w(x, g)$ can be expressed as

$$\begin{aligned} w(x, g) = {} & V_\tau(1 - \alpha)[M_1(x, g; G, a + 1) - M_1(-x, g; G, a + 1)] \\ & - h(A; V_\tau)[M_1(x, g; G, a) - M_1(-x, g; G, a)] \\ & + V_\tau(\delta - \phi) \int_g^G [M_1(x, g; \tilde{g}, a + 1) - M_1(-x, g; \tilde{g}, a + 1)] d\tilde{g} \\ & - rh(A; V_\tau) \int_g^G [M_1(x, g; \tilde{g}, a) - M_1(-x, g; \tilde{g}, a)] d\tilde{g}, \end{aligned} \tag{A.8}$$

where

$$M_1(x, g; \xi, a) = e^{-xa + \frac{x^2}{2}\sigma^2(\xi - g)} N\left(\frac{x - a\sigma^2(\xi - g)}{\sigma\sqrt{\xi - g}}\right) e^{-b(G-\xi)},$$
$$M_2(a, G) = -2e^{bG + \frac{a^2\sigma^2}{2}G} aN(-a\sigma\sqrt{G}) + \frac{2e^{bG}}{\sigma\sqrt{2\pi G}},$$
$$M_3(a, G) = \frac{-4a}{2b + a^2\sigma^2}\left[e^{\frac{2b + a^2\sigma^2}{4}G} N(-a\sigma\sqrt{G}) - N(0) + \frac{a\sigma}{\sqrt{-2b}}(N(\sqrt{-2bG}) - N(0))\right]$$

$$+\frac{4}{\sigma\sqrt{-2b}}[N(\sqrt{-2bG})-N(0)],$$

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{\xi^2}{2}}d\xi.$$

Then, from (A.6) and (A.8), the undetermined constant A can be obtained as

$$A=-\left\{\phi M_3(a+1,G)V_\tau+\alpha M_2(a+1,G)V_\tau+\frac{\gamma C}{r}[M_2(a,G)+rM_3(a,G)]\right\}/ \{[M_2(a,G)+rM_3(a,G)-\lambda_1]\}(V_\tau)^{\lambda_1}. \tag{A.9}$$

Finally, from

$$\bar{v}(V,g;V_\tau)=h(A;V_\tau)+e^{ax+b(G-g)}w(x,g)\Big|_{x=\ln\frac{V_\tau}{V}},$$

we obtain the expression of $\bar{v}$ on the region $0<V\le V_\tau, 0\le g\le G$. This completes the proof of Theorem 1.

Expressions of $K(G)$ and $B(G)$

Since

$$\begin{cases}\Phi(V_\tau)=\eta\left[\bar{v}(V_\tau,0;V_\tau)-(1-\alpha)V_\tau\right],\\ \Psi(V_\tau)=(1-\eta)\bar{v}(V_\tau,0;V_\tau)+\eta(1-\alpha)V_\tau,\end{cases}$$

combining these with the expression of firm value after bankruptcy in (4.5), we have

$$\Phi(V)=\eta\left[K(G)V+\frac{\gamma C}{r}(1-B(G))\right],$$

and

$$\Psi(V)=\left[(1-\eta)K(G)+(1-\alpha)\right]V+\frac{(1-\eta)\gamma C(1-B(G))}{r}.$$

Here,

$$\begin{cases}K(G)=\dfrac{-\{\phi M_3(a+1,G)+\alpha M_2(a+1,G)\}}{M_2(a,G)+rM_3(a,G)-\lambda_1}+\alpha, & \text{(A.10)}\\[2ex] B(G)=\dfrac{[M_2(a,G)+rM_3(a,G)]}{M_2(a,G)+rM_3(a,G)-\lambda_1}. & \text{(A.11)}\end{cases}$$

Proof of Theorem 2:

Assume that $E(V)-\Phi(V)$ is strictly increasing in V. Then, there exists a unique V^* such that $E(V)>\Phi(V)$ for $V>V^*$, $E(V)<\Phi(V)$ for $V<V^*$, and $E(V)=\Phi(V)$ for $V=V^*$. Under this assumption, boundary value problem P2 is reduced to the following problem P3:

$$\begin{cases}\tilde{\mathcal{L}}E=-(\gamma-1)C-\delta V, & V>V^*,\\ \tilde{\mathcal{L}}E+\rho(\Phi(V)-E(V))=-(\gamma-1)C-\delta V, & V<V^*.\end{cases}$$

$$\begin{cases}\lim\limits_{V\to 0}E(V)\text{ is bounded},\\ \lim\limits_{V\to\infty}\left[E(V)-\left(V+\frac{(\gamma-1)C}{r}\right)\right]=0,\\ \lim\limits_{V\to V^{*+0}}E(V)=\lim\limits_{V\to V^{*-0}}E(V),\\ \lim\limits_{V\to V^{*+0}}\frac{dE(V)}{dV}=\lim\limits_{V\to V^{*-0}}\frac{dE(V)}{dV},\\ \lim\limits_{V\to V^{*+0}}\frac{d^2E(V)}{dV^2}=\lim\limits_{V\to V^{*-0}}\frac{d^2E(V)}{dV^2}.\end{cases}$$

The last matching condition is the second-order smooth-fit condition, which is used to determine the optimal bankruptcy boundary.

Solving the above system of ordinary differential equations together with the two boundary conditions, we obtain the general expression

$$E(V)=\begin{cases} V+\dfrac{(\gamma-1)C}{r}+C_1\left(\dfrac{V}{V^*}\right)^{\lambda_1}, & V\geq V^*,\\ \dfrac{\delta+\rho\eta K(G)}{\rho+\delta}V+\dfrac{r(\gamma-1)C+\rho\eta\gamma C(1-B(G))}{r(r+\rho)}+C_3\left(\dfrac{V}{V^*}\right)^{\lambda_4}, & V<V^*,\end{cases}$$

where C_1 and C_3 are undetermined constants and

$$\lambda_4=\frac{1}{2}-\frac{r-\delta}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2+\frac{2(r+\rho)}{\sigma^2}}.$$

Using the continuity conditions

$$\begin{cases} \lim\limits_{V\to V^*+0}E(V)=\lim\limits_{V\to V^*-0}E(V),\\ \lim\limits_{V\to V^*+0}\dfrac{dE(V)}{dV}=\lim\limits_{V\to V^*-0}\dfrac{dE(V)}{dV},\end{cases}$$

we obtain

$$C_1=\frac{1}{\lambda_4-\lambda_1}\left[V^*(1-\lambda_4)\frac{(1-\eta K(G))\rho}{\rho+\delta}+\rho\lambda_4\frac{\eta\gamma C(1-B(G))+(1-\gamma)C}{(\rho+r)r}\right],$$

$$C_3=\frac{1}{\lambda_4-\lambda_1}\left[V^*(1-\lambda_1)\frac{(1-\eta K(G))\rho}{\rho+\delta}+\rho\lambda_1\frac{\eta\gamma C(1-B(G))+(1-\gamma)C}{(\rho+r)r}\right].$$

Then, by the second-order smooth-fit condition

$$\lim_{V\to V^*+0}\frac{d^2E(V)}{dV^2}=\lim_{V\to V^*-0}\frac{d^2E(V)}{dV^2},$$

the explicit expression of V^* is obtained as

$$V^*=\frac{\lambda_1\lambda_4}{(1-\lambda_1)(1-\lambda_4)}\frac{\rho+\delta}{(\rho+r)r}\frac{\eta\gamma C(1-B(G))+(1-\gamma)C}{1-\eta K(G)}.$$

It remains to verify that $E(V)-\Phi(V)$ is strictly increasing in V.

For $V>V^*$,

$$\frac{d[E(V)-\Phi(V)]}{dV}=1-\eta K(G)+C_1\lambda_1\left(\frac{V}{V^*}\right)^{\lambda_1-1}\frac{1}{V^*},$$

$$\frac{d^2[E(V)-\Phi(V)]}{dV^2}=C_1\lambda_1(\lambda_1-1)\left(\frac{V}{V^*}\right)^{\lambda_1-2}\frac{1}{(V^*)^2}.$$

Using the expressions of C_1 and V^*, C_1 can be rewritten as

$$C_1=\frac{\lambda_4}{(\lambda_4-\lambda_1)(1-\lambda_1)}\frac{\rho\left[\eta\gamma C(1-B(G))+(1-\gamma)C\right]}{r(r+\rho)}.$$

Since $\lambda_1 < 0$, $\lambda_4 > 1$, and $0 \le B(G) \le 1$, we have $C_1 > 0$, and therefore

$$\frac{d^2[E(V) - \Phi(V)]}{dV^2} > 0.$$

Moreover,

$$\begin{aligned}\left.\frac{d[E(V) - \Phi(V)]}{dV}\right|_{V=V^*} &= 1 - \eta K(G) + C_1\lambda_1\frac{1}{V^*} = \frac{V^*(1-\eta K(G)) + C_1\lambda_1}{V^*}\\ &= \frac{\lambda_1\lambda_4}{(1-\lambda_1)(\lambda_4-1)}\left[\rho\frac{\lambda_1 - 1}{\lambda_4 - \lambda_1} - \delta\right]\frac{\eta\gamma C(1-B(G)) + (1-\gamma)C}{r(r+\rho)}\frac{1}{V^*} > 0.\end{aligned}$$

Together with $\frac{d^2[E(V) - \Phi(V)]}{dV^2} > 0$, this implies that

$$\frac{d[E(V) - \Phi(V)]}{dV} > 0 \qquad \text{for } V > V^*.$$

For $V < V^*$,

$$\frac{d[E(V) - \Phi(V)]}{dV} = \frac{\delta(1-\eta K(G))}{\rho + \delta} + C_3\lambda_4\left(\frac{V}{V^*}\right)^{\lambda_4 - 1}\frac{1}{V^*}.$$

Using the expressions of C_3 and V^*, we further obtain

$$C_3 = \frac{\lambda_1}{(\lambda_4 - \lambda_1)(1-\lambda_4)}\frac{\rho\left[\eta\gamma C(1-B(G)) + (1-\gamma)C\right]}{r(r+\rho)} > 0.$$

Since $0 \le K(G) \le 1$, it follows directly that

$$\frac{d[E(V) - \Phi(V)]}{dV} > 0.$$

Finally, one can verify that

$$\begin{aligned}&\lim_{V\to 0}[E(V) - \Phi(V)] = -\frac{\eta\gamma C(1-B(G)) + (1-\gamma)C}{r+\rho} < 0,\\ &\lim_{V\to +\infty}[E(V) - \Phi(V)] \to +\infty,\end{aligned}$$

which ensures that $0 < V^* < +\infty$. This completes the proof of Theorem 2.

Proof of Theorem 3:

Based on boundary value problem P4, from the system of ordinary differential equations satisfied by the debt value $D(V)$ together with the two boundary conditions, the general expression of $D(V)$ is

$$D(V) = \begin{cases} \dfrac{C}{r} + C_4\left(\dfrac{V}{V^*}\right)^{\lambda_1}, & V \ge V^*,\\ \dfrac{\rho\left[(1-\eta)K(G) + (1-\alpha)\right]V}{\rho+\delta} + \dfrac{C\left[r + \rho(1-\eta)\gamma(1-B(G))\right]}{r(r+\rho)} + C_6\left(\dfrac{V}{V^*}\right)^{\lambda_4}, & V < V^*.\end{cases}$$

Here, C_4 and C_6 are undetermined constants.

Using the matching conditions

$$\begin{cases} \lim\limits_{V\to V^*+0} D(V) = \lim\limits_{V\to V^*-0} D(V), \\ \lim\limits_{V\to V^*+0} \dfrac{dD(V)}{dV} = \lim\limits_{V\to V^*-0} \dfrac{dD(V)}{dV}, \end{cases}$$

we obtain

$$C_4 = \frac{\lambda_4}{\lambda_1-\lambda_4}\left[\frac{C}{r} - \frac{\lambda_4-1}{\lambda_4}\cdot\frac{\rho\left[(1-\eta)K(G)+(1-\alpha)\right]V^*}{\rho+\delta} - \frac{C\left[r+\rho(1-\eta)\gamma(1-B(G))\right]}{r(\rho+r)}\right],$$

$$C_6 = \frac{\lambda_1}{\lambda_1-\lambda_4}\left[\frac{C}{r} - \frac{\lambda_1-1}{\lambda_1}\cdot\frac{\rho\left[(1-\eta)K(G)+(1-\alpha)\right]V^*}{\rho+\delta} - \frac{C\left[r+\rho(1-\eta)\gamma(1-B(G))\right]}{r(\rho+r)}\right].$$

Thus, the proof of Theorem 3 is complete.

Proof of Theorem 4:

From (5.4), (5.5), and (5.7), the explicit expression of total firm value $v(V)$ in (5.8) follows directly. We now focus on the derivation of the optimal coupon C^*.

For $V \geq V^*$,

$$\frac{\partial v(V;C)}{\partial C} = \frac{\gamma}{r} + \left(\frac{V}{V^*}\right)^{\lambda_1}\left[\frac{d(C_1+C_4)}{dC} - (C_1+C_4)\lambda_1\frac{\frac{dV^*}{dC}}{V^*}\right].$$

From the expressions of C_1, C_4 and V^*, we obtain

$$(C_1+C_4) = \frac{\lambda_4}{(\lambda_1-\lambda_4)(\rho+r)r}\rho C\left\{\begin{array}{l} \dfrac{\left[\eta\gamma(1-B(G))-(\gamma-1)\right]}{1-\eta K(G)}\cdot \\ \dfrac{\lambda_1\left[(1-\eta)K(G)+(1-\alpha)\right]+\eta K(G)-1}{1-\lambda_1}+ \\ \left[1-(1-\eta)\gamma(1-B(G))\right] \end{array}\right\}.$$

Let $V^* = K_1 C$. Noting that

$$\frac{d(C_1+C_4)}{dC} = \frac{(C_1+C_4)}{C}, \qquad \frac{dV^*}{dC} = \frac{V^*}{C},$$

we have

$$\left[\frac{d(C_1+C_4)}{dC} - (C_1+C_4)\lambda_1\frac{\frac{dV^*}{dC}}{V^*}\right] = (1-\lambda_1)\frac{(C_1+C_4)}{C} = K_2,$$

where

$$K_1 = \frac{\lambda_1\lambda_4}{(1-\lambda_1)(1-\lambda_4)}\frac{\rho+\delta}{(\rho+r)r}\frac{\left[\eta\gamma(1-B(G))-(\gamma-1)\right]}{1-\eta K(G)},$$

$$K_2 = \frac{\lambda_4(1-\lambda_1)}{(\lambda_1-\lambda_4)}\cdot\frac{\rho}{(\rho+r)r}\left\{\frac{\left[\eta\gamma(1-B(G))-(\gamma-1)\right]}{1-\eta K(G)}\times\right.$$
$$\left.\frac{\lambda_1\left[(1-\eta)K(G)+(1-\alpha)\right]+\eta K(G)-1}{1-\lambda_1} + \left[1-(1-\eta)\gamma(1-B(G))\right]\right\}.$$

Setting $\dfrac{\partial v(V;C)}{\partial C} = 0$, we obtain

$$\frac{\partial v(V;C)}{\partial C} = \frac{\gamma}{r} + \left(\frac{V}{K_1 C}\right)^{\lambda_1} K_2 = 0.$$

Hence, the expression of C^* in (5.9) follows. This completes the proof of Theorem 4.
Proof of Theorem 5:
(i) As $\rho \to \infty$, we have $\lambda_4 \to \infty$. From the expression of V^* in (5.5), it follows that

$$\lim_{\rho\to\infty} V^*(\rho) = \frac{-\lambda_1}{1-\lambda_1} \cdot \frac{1}{r} \cdot \frac{\eta\gamma C\,[1-B(G)] - (\gamma-1)C}{1-\eta K(G)}.$$

Define

$$C_1(G) = \frac{M_2(a+1,G)}{M_2(a,G) + rM_3(a,G) - \lambda_1}, \qquad C_2(G) = \frac{\delta M_3(a+1,G) + M_2(a+1,G)}{M_2(a,G) + rM_3(a,G) - \lambda_1}.$$

Using (A.10), $K(G)$ can be written as

$$K(G) = \alpha\,[1 - C_1(G)] - \frac{\phi}{\delta}\,[C_2(G) - C_1(G)]\,.$$

Therefore,

$$\lim_{\rho\to\infty} V^*(\rho) = \bar{V}.$$

(ii) As $\rho \to \infty$, $\lambda_4 \to \infty$ and $V^* \to \bar{V}$. Using (5.4), for $V \geq V^*$, we obtain

$$\begin{aligned}
\lim_{\rho\to\infty} E(V;\rho) &= V - \bar{V}\cdot\left(\frac{V}{\bar{V}}\right)^{\lambda_1} + \frac{(\gamma-1)C}{r} + \left[\eta K(G)\bar{V} + \frac{\eta\gamma C\,[1-B(G)] - (\gamma-1)C}{r}\right]\left(\frac{V}{\bar{V}}\right)^{\lambda_1}\\
&= V - \bar{V}\left(\frac{V}{\bar{V}}\right)^{\lambda_1} - \left[1 - \left(\frac{V}{\bar{V}}\right)^{\lambda_1}\right]\frac{C(1-\gamma)}{r} + \eta\left[K(G)\bar{V} + \frac{\gamma C\,[1-B(G)]}{r}\right]\left(\frac{V}{\bar{V}}\right)^{\lambda_1}\\
&= V - \bar{V}\left(\frac{V}{\bar{V}}\right)^{\lambda_1} - \left[1 - \left(\frac{V}{\bar{V}}\right)^{\lambda_1}\right]\frac{C(1-\gamma)}{r} + \eta R(G)\left(\frac{V}{\bar{V}}\right)^{\lambda_1},
\end{aligned}$$

where

$$R(G) = \alpha\,[1 - C_1(G)]\,\bar{V} - \frac{\phi}{\delta}\,[C_2(G) - C_1(G)]\,\bar{V} + \frac{\gamma C\,[1-B(G)]}{r}.$$

Hence,

$$\lim_{\rho\to\infty} E(V;\rho) = \bar{E}(V).$$

For $V < V^*$, since $\lambda_4 \to \infty$, we have $C_3 \to 0$, and it is easy to verify that

$$\lim_{\rho\to\infty} E(V;\rho) = \bar{E}(V).$$

(iii) As $\rho \to \infty$, $\lambda_4 \to \infty$ and $V^* \to \bar{V}$. Using (5.7), for $V \geq V^*$, we obtain

$$\begin{aligned}
\lim_{\rho\to\infty} D(V;\rho) &= \frac{C}{r}\left[1-\left(\frac{V}{\bar{V}}\right)^{\lambda_1}\right]+(1-\alpha)\bar{V}\left(\frac{V}{\bar{V}}\right)^{\lambda_1} \\
&\quad -\frac{\lambda_1}{1-\lambda_1}\cdot\frac{C}{r}\left\{\frac{(1-\eta)K(G)\left[\eta\gamma(1-B(G))-(\gamma-1)\right]}{1-\eta K(G)}\right. \\
&\quad \left. +(1-\eta)\gamma(1-B(G))\cdot\frac{\lambda_1-1}{\lambda_1}\right\}\left(\frac{V}{\bar{V}}\right)^{\lambda_1} \\
&= \frac{C}{r}\left[1-\left(\frac{V}{\bar{V}}\right)^{\lambda_1}\right]+(1-\alpha)\bar{V}\left(\frac{V}{\bar{V}}\right)^{\lambda_1}+(1-\eta)R(G)\left(\frac{V}{\bar{V}}\right)^{\lambda_1}.
\end{aligned}$$

Therefore,

$$\lim_{\rho\to\infty} D(V;\rho) = \bar{D}(V).$$

For $V < V^*$, since $\lambda_4 \to \infty$, we have $C_6 \to 0$, and thus

$$\lim_{\rho\to\infty} D(V;\rho) = \bar{D}(V).$$

(iv) As $\rho \to \infty$, $\lambda_4 \to \infty$. Using the expression of C^* in (5.9), after calculation, we obtain

$$\begin{aligned}
\lim_{\rho\to\infty} C^*(\rho) &= V\cdot\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{r\left[\eta K(G)-1\right]}{\eta\gamma\left(1-B(G)\right)-(\gamma-1)} \\
&\quad \cdot\left\{\frac{\left[\eta\gamma\left(1-B(G)\right)-(\gamma-1)\right]\cdot\left\{\lambda_1\left[(1-\eta)K(G)+(1-\alpha)\right]+\eta K(G)-1\right\}}{\gamma\left(1-\eta K(G)\right)}\right. \\
&\quad \left.+\frac{\left(1-\eta K(G)\right)\cdot\left(1-\lambda_1\right)\left[1-(1-\eta)\gamma(1-B(G))\right]}{\gamma\left(1-\eta K(G)\right)}\right\}^{\frac{1}{\lambda_1}} \\
&= V\cdot\frac{(1-\lambda_1)}{\lambda_1}\cdot\frac{r\cdot\left[\eta K(G)-1\right]}{\eta\gamma(1-B(G))-(\gamma-1)} \\
&\quad \cdot\left\{\frac{(1-\lambda_1)B(G)\gamma(1-\eta K(G))-\lambda_1[\eta\gamma(1-B(G))-(\gamma-1)](\alpha-K(G))}{\gamma(1-\eta K(G))}\right\}^{\frac{1}{\lambda_1}} \\
&= V(1-\lambda_1)^{\frac{1}{\lambda_1}}\left\{B(G)\left[\frac{-\lambda_1}{1-\lambda_1}\cdot\frac{1}{r}\cdot\frac{1-\gamma(1-\eta(1-B(G)))}{1-\eta K(G)}\right]^{-\lambda_1}\right. \\
&\quad \left.+\frac{r}{\gamma}\cdot\left[\frac{-\lambda_1}{1-\lambda_1}\cdot\frac{1}{r}\cdot\frac{1-\gamma(1-\eta(1-B(G)))}{1-\eta K(G)}\right]^{1-\lambda_1}(\alpha-K(G))\right\}^{\frac{1}{\lambda_1}} = \bar{C}.
\end{aligned}$$

This completes the proof of Theorem 5.

Table A.1. Review of the literature on bankruptcy reorganization and bankruptcy timing.

Reference	Main content	Reorganization	Bankruptcy timing	Both reorganization and discreteness
Reorganization-related studies				
Fan & Sundaresan (2000)	Develops a strategic debt service model with asset reorganization and an infinite bankruptcy observation period, highlighting the role of reorganization in capital structure decisions.	✓ (infinite)	Continuous	×
François & Morellec (2004)	Introduces a finite bankruptcy observation period into capital structure models and shows its significant impact on optimal capital structure and bond valuation.	✓ (finite)	Continuous	×
Broadie & Kaya (2007)	Applies a binomial tree framework to analyze the impact of bankruptcy reorganization on credit spreads and corporate security valuation.	✓ (finite)	Continuous	×
Dai et al. (2013)	Uses PDE and optimal stopping methods to study corporate stock and bond pricing under a finite bankruptcy observation period.	✓ (finite)	Continuous	×
Gupta (2024)	Empirically studies the relationship between leverage and successful emergence from bankruptcy among large U.S. firms.	✓ (finite)	Continuous	×
S&P Global (2025)	Analyzes U.S. corporate bankruptcies in 2024 and reports that 62.7% of firms chose reorganization, the highest level in the 21st century.	✓ (finite)	Continuous	×
Discreteness-related studies				
Dupuis & Wang (2002)	Introduces a Poisson-process framework to model bankruptcy occurring at discrete random times.	×	Discrete (Poisson jumps)	×
Liang & Sun (2019)	Models discrete bankruptcy triggers using a jump-diffusion framework and shows that jump risk significantly increases credit spreads.	×	Discrete (jump diffusion)	×
Palmowski et al. (2020)	Introduces Poisson observation times into the Leland–Toft framework and studies their effects on bankruptcy thresholds and capital structure.	×	Discrete (Poisson observation)	×
Mazur (2022)	Uses a discrete choice model to examine the impact of bankruptcy law on corporate investment behavior.	×	Discrete (Poisson jumps)	×

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